The Feynman Integrand for the Perturbed Harmonic Oscillator as a Hida Distribution

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Abstract

We review some basic notions and results of White Noise Analysis that are used in the construction of the Feynman integrand as a generalized White Noise functional. We show that the Feynman integrand for the harmonic oscillator in an external potential is a Hida distribution.

Keywords: Functional integration; Quantum theory; White noise analysis.

1 Introduction

Path integrals are a useful tool in many branches of theoretical physics including quantum mechanics, quantum field theory and polymer physics. We are interested in a rigorous treatment of such path integrals. As our basic example we think of a quantum mechanical particle.

On one hand it is possible to represent solutions of the heat equation by a path integral representation, based on the Wiener measure in a mathematically rigorous way. This is stated by the famous Feynman Kac formula. On the other hand there have been a lot of attempts to write solutions of the Schrödinger equation as a Feynman (path) integral in a useful mathematical sense. The methods are always more involved and less direct than in the euclidean (i.e. Feynman Kac) case. Among them are analytic continuation, limits of finite dimensional approximations and Fourier transform. We are not interested in giving full reference on various theories of Feynman integrals (a brief survey can be found in [2]) but we

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like to mention the method in [1] using Fresnel integrals. Here we have chosen a white noise approach.

White noise analysis is a mathematical framework which offers various generalizations of concepts known from finite dimensional analysis, among them are differential operators and Fourier transform. Although we will give a brief introduction to white noise calculus in section 2 the reader unfamiliar with this topic is recommended to the monographs [5], [15], [4] and the introductory articles [12], [17], [19], [21].

The idea of realizing Feynman integrals within the White Noise framework goes back to [6]. The "average over all paths" is performed with a Hida distribution as the weight (instead of a measure). The existence of such Hida distributions corresponding to Feynman integrands has been established in [3]. In [8] Khandekar and Streit moved beyond the existence theorem by giving an explicit construction for a large class of potentials including singular ones. Basically they constructed a strong Dyson series for the Feynman integrand in the space of Hida Distributions. This approach only works for one space dimension. Their construction was generalized to (one dimensional) time-dependent potentials of noncompact support in [13].

In this work, which originated in the White Noise workshop on Madeira in August 1993, we carry those ideas over to perturbations of the harmonic oscillator. Hence instead of constructing a Dyson series around the free particle Feynman integrand we expand around the Feynman integrand of the harmonic oscillator as obtained in [3]. The external potentials to which the oscillator is submitted correspond to the wide class of time-dependent singular potentials treated in [13].

In [1, chap 5] the path integral of the anharmonic oscillator is defined within the theory of Fresnel integrals. Compared to our ansatz this procedure has the advantage of being manifestly independent of the space dimension. Despite the lack of a generalization to higher dimensional quantum systems our construction has some interesting features:

- The admissible potentials may be very singular.
- We are not restricted to smooth initial wave functions and may thus study the propagator directly.
- Instead of giving a meaning to the Feynman *integral* we define the Feynman *integrand* as a Hida distribution. By taking expectation we get the propagator. On the other hand one may now use the toolbox of white noise analysis and apply differential operators to derive variational relations or Ehrenfest's theorem, see [5, chap 12], [18].

2 White Noise Analysis

The starting-point of White Noise Analysis is the real Gel'fand triple

$$\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R}),$$

where $\mathcal{S}'(\mathbf{R})$ denotes the real Schwartz space. Using Minlos' theorem we construct the White Noise measure space $(\mathcal{S}'(\mathbf{R}), \mathcal{B}, \mu)$ by fixing the characteristic functional in the following way:

$$C(f) = \int_{\mathcal{S}'} \exp i \langle \omega, f \rangle \; d\mu(\omega) = \exp\left(-\frac{1}{2}\int_{\mathbf{R}} f^2(\tau) \; d\tau\right) \;, \; f \in \mathcal{S}(\mathbf{R}) \;.$$

We denote by $\langle \cdot, \cdot \rangle$ the bilinear pairing between $\mathcal{S}'(\mathbf{R})$ and $\mathcal{S}(\mathbf{R})$ and by $|\cdot|_0$ the norm on $L^2(\mathbf{R})$.

Within this formalism a version of Wiener's Brownian motion is given by

$$B(t) := \left\langle \omega, \mathbf{1}_{[0,t)} \right\rangle = \int_0^t \omega(s) \; ds \;.$$

We now consider the space (L^2), which is defined to be the complex Hilbert space $L^2(\mathcal{S}'(\mathbf{R}), \mathcal{B}, \mu)$. For applications the space (L^2) is often too small. A convenient way to solve this problem is to introduce a space of test functionals in (L^2) and to use its larger dual space.

We like to work with the space of test functions $(\mathcal{S})$. So we review the standard construction of $(\mathcal{S})$ due to [11]. For a more detailed discussion see [5], [10]. Take one system of Hilbertian norms $\left\{|\cdot|_p\right\}$ topologizing $\mathcal{S}(\mathbf{R})$ which grows sufficiently fast. Then $\mathcal{S}(\mathbf{R})$ is realized as a projective limit of Hilbert spaces $\mathcal{S}_p(\mathbf{R})$:

$$\mathcal{S}(\mathbf{R}) = \bigcap_{p \geq 0} \mathcal{S}_p(\mathbf{R}) \;,$$

where $\mathcal{S}_p(\mathbf{R})$ denotes the completion of $\mathcal{S}(\mathbf{R})$ with respect to $|\cdot|_p$. Then the space of tempered distributions is

$$\mathcal{S}'(\mathbf{R}) = \bigcup_{p \geq 0} \mathcal{S}_{-p}(\mathbf{R}) \;,$$

where the dual norm $|\cdot|_{-p}$ topologizes the Hilbert space $\mathcal{S}_{-p}(\mathbf{R})$.

One convenient choice is

$$|f|_p := |A^p f|_0 \,, \quad f \in \mathcal{S}(\mathbf{R}), \tag{1}$$

where

$$Af(t) = -f''(t) + \left(t^2 + 1\right) f(t)$$

is the Hamiltonian of the harmonic oscillator. Since (L^2) is Segal isomorphic to the complex symmetric Fock space $\Gamma(L^2)$ of $L^2(\mathbf{R})$, we can identify the Fock space $\Gamma(\mathcal{S}_p)$ with a subspace $(\mathcal{S})_p$ of (L^2) and define the nuclear space

$$(\mathcal{S}) = \bigcap_{p\geq 0} (\mathcal{S})_p \ .$$

Thus we arrive at the Gel'fand triple:

$$(\mathcal{S}) \subset \left(L^2\right) \subset (\mathcal{S})^* .$$

Elements of the space $(\mathcal{S})^*$ are called Hida distributions (or generalized Brownian functionals). It is possible to characterize the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ by their S- or T-transforms $(\Phi \in (\mathcal{S})^*\,,\ f \in \mathcal{S}(\mathbf{R}))$:

$$T\Phi(f) \equiv \langle\!\langle \Phi, \exp(i\langle \cdot, f\rangle)\rangle\!\rangle = \int_{\mathcal{S}'(\mathbf{R})} \exp(i\langle \omega, f\rangle)\,\Phi(\omega)\,d\mu(\omega)\,, \tag{2}$$

$$S\Phi(f) \equiv \langle\!\langle \Phi, :\exp\langle \cdot, f\rangle :\rangle\!\rangle\,,$$

here $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ denotes the bilinear pairing between $(\mathcal{S})$ and $(\mathcal{S})^*$ and we have used the traditional notation:

$$:\exp\langle \cdot, f\rangle : \ \equiv C(f)\exp(\langle \cdot, f\rangle)\,,\ f \in \mathcal{S}(\mathbf{R})\ . \tag{3}$$

We denote by $\mathbf{E}(\Phi) \equiv \langle\!\langle \Phi, 1\rangle\!\rangle$ the expectation of a Hida distribution Φ. S- and T-transform have extensions to the complex Schwartz space $\mathcal{S}_{\mathbf{C}}(\mathbf{R})$ and are related by the following formula:

$$S\Phi(f) = C(f)\ T\Phi(-if)\,,\ f \in \mathcal{S}_{\mathbf{C}}(\mathbf{R}) \tag{4}$$

Let us now quote the above mentioned characterization theorem, which is due to Potthoff and Streit [16] and has been generalized in various ways (see e.g. [9], [14], [20]). For a full proof of a generalized version see [10].

Theorem 2.1
The following statements are equivalent:

1. $F : \mathcal{S}(\mathbf{R}) \to \mathbf{C}$ *is*
   - *(A)* **Ray-entire**, *i.e. for all* $g, f \in \mathcal{S}(\mathbf{R})$ *the mapping* $\mathbf{C} \ni z \mapsto F(zf+g)$ *is entire.*
   - *(B) and uniformly of order two, i.e. there exist constants* $K_1, K_2 > 0$ *such that*

     $$|F(zf)| \leq K_1 \exp\left(K_2\,|z|^2\,|f|^2\right), \qquad f \in \mathcal{S}(\mathbf{R})\,.$$

     for some continuous norm $|\cdot|$ *on* $\mathcal{S}(\mathbf{R})$.

2. *F is the S- transform of a unique Hida distribution* $\Phi \in (\mathcal{S})^*$.

3. *F is the T- transform of a unique Hida distribution* $\overset{\wedge}{\Phi} \in (\mathcal{S})^*$.

A functional satisfying 1. is usually called a U-functional.

As an example of an application of this theorem we consider Donsker‘s delta function. Consider the composition $\delta_a \circ B(t)$ of the Dirac distribution δ_a at $a \in \mathbf{R}$ with Brownian motion $B(t)$, $t > 0$:

$$\Phi = \delta\left(B(t) - a\right)$$

$$\Phi = \delta\left(\left\langle \cdot, \mathbf{1}_{[0,t)} \right\rangle - a\right), \; a \in \mathbf{R}. \tag{5}$$

The S-transform of Φ is calculated to be [5]:

$$S\Phi(f) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\int_0^t f(s)\, ds - a\right)^2\right)$$

and theorem 2.1 gives immediately that Φ is a well defined element in $(\mathcal{S})^*$.

Now we want to mention some important consequences of theorem 2.1. The first one concerns the convergence of sequences of Hida distributions and can be found in [5], [16], [10].

Theorem 2.2

Let $\{F_n\}_{n \in \mathbf{N}}$ *denote a sequence of U-functionals with the following properties:*

1. *For all* $f \in \mathcal{S}(\mathbf{R})$, $\{F_n(f)\}_{n \in \mathbf{N}}$ *is a Cauchy sequence,*

2. *There exist* $K_1, K_2 > 0$ *such that the bound*

$$|F_n(zf)| \leq K_1 \exp\left(K_2 |z|^2 |f|^2\right) \;, \; f \in \mathcal{S}(\mathbf{R})$$

holds for almost all $n \in \mathbf{N}$ *in a continuous norm* $|\cdot|$ *on* $\mathcal{S}(\mathbf{R})$.

Then there is a unique $\Phi \in (\mathcal{S})^*$ *such that* $T^{-1}F_n$ *converges strongly to* Φ.

This theorem is also valid for S-transforms.

Another corollary of theorem 2.1 deals with the integration of Hida distributions which depend on an additional parameter (see [5], [8], [10]).

Theorem 2.3

Let (Ω, B, m) *denote a measure space and* $\lambda \mapsto \Phi(\lambda)$ *a mapping from* Ω *to* $(\mathcal{S})^*$. *Let* $F(\lambda)$ *denote the T-transform of* $\Phi(\lambda)$ *which satisfies the following conditions:*

1. $\lambda \mapsto F(\lambda, f)$ *is a measurable function for all* $f \in \mathcal{S}(\mathbf{R})$,
2. *There exists a continuous norm* $|\cdot|$ *on* $\mathcal{S}(\mathbf{R})$ *such that*

$$|F(\lambda, zf)| \leq K_1(\lambda) \exp\left(K_2(\lambda) |z|^2 |f|^2\right), \qquad f \in \mathcal{S}(\mathbf{R})$$

with $K_1 \in L^1(\Omega, m)$ *and* $K_2 \in L^\infty(\Omega, m)$.

Then Φ *is Bochner integrable in some* $(\mathcal{S})_{-q}$ *and thus*

$$\int_\Omega \Phi(\lambda)\, dm(\lambda) \in (\mathcal{S})^*.$$

T-transform and integration commute

$$T\left(\int_\Omega \Phi(\lambda) \;\, dm(\lambda)\right)(f) = \int_\Omega T(\Phi(\lambda))(f) \;\, dm(\lambda).$$

Again the same theorem holds for the S-transform.

Example

$$\delta(B(t) - a) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i\lambda(B(t)-a)} d\lambda$$

in the sense of Bochner integration (see, e.g., [5]).

Remark

For later use we have to define pointwise products of a Hida distribution Φ with a Donsker-Delta function

$$\delta(\langle \omega, g \rangle - a)$$

If the mapping $\lambda \longmapsto T\Phi(f + \lambda g)$ is integrable on $\mathbf{R}$ the following formula may be used to define the product $\Phi \cdot \delta$

$$T(\Phi \cdot \delta(\langle \omega, g \rangle - a))(f) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-i\lambda a} \; T\Phi(f + \lambda g) \;\, d\lambda, \tag{6}$$

in case the right hand integral is indeed a U-functional.

3 The Feynman Integrand as a Hida Distribution

We follow [3] and [6] in viewing the Feynman integral as a weighted average over Brownian paths. These paths are modeled within the White Noise framework according to

$$x(t) \equiv x_0 + \sqrt{\frac{\hbar}{m}} \int_{t_0}^{t} \omega(\tau)\ d\tau,$$

in the sequel we set $\hbar = m = 1$.

In [3] the (distribution-valued) weight for the free quantum mechanical propagation from $x(t_0) = x_0$ to $x(t) = x$ is constructed from a kinetic energy factor $\exp\left(\frac{i}{2}\int_{t_0}^{t} \omega^2(\tau)\ d\tau\right)$ and a Donsker delta function $\delta(x(t) - x)$. Furthermore a factor $\exp\left(\frac{1}{2}\int_{t_0}^{t} \omega^2(\tau)\ d\tau\right)$ is introduced to compensate the Gaussian fall-off of the White Noise measure in order to mimic Feynman's non-existing "flat" measure $D^\infty x$. Thus in [3] the Feynman integrand for the free motion reads (the N indicates appropriate normalization):

$$I_0 = \mathrm{N}\exp\left(\frac{i+1}{2}\int_{t_0}^{t} \omega^2(\tau)\ d\tau\right)\delta\left(x(t) - x\right).$$

As in [3] I_0 is a Hida distribution, with T- transform given by

$$TI_0(f) = \frac{1}{\sqrt{2\pi i\,|t-t_0|}} \exp\left(-\frac{i}{2}|f_\Delta|^2 - \frac{1}{2}|f_{\Delta^c}|^2 + \frac{i}{2|t-t_0|}\left(\int_{t_0}^{t} f(\tau)\,d\tau + x - x_0\right)^2\right),$$

where $\Delta = [t_0, t]$ and f_Δ, f_{Δ^c} denote the restrictions of f to Δ and its complement Δ^c respectively. Furthermore the Feynman integral $\mathbf{E}(I_0) = TI_0(0)$ is indeed the free particle propagator $\frac{1}{\sqrt{2\pi i|t-t_0|}} \exp\left[\frac{i}{2|t-t_0|}\ (x-x_0)^2\right]$. Not only the expectation but also the T-transform has a physical meaning. By a formal integration by parts

$$TI_0(f) = \mathbf{E}\Big(I_0\ e^{-i\int_{t_0}^{t} x(\tau)\dot{f}(\tau)\,d\tau}\Big)\ e^{\,ixf(t)-ix_0f(t_0)}\ e^{-\frac{1}{2}|f_{\Delta^c}|^2}.$$

The term $e^{-i\int_{t_0}^{t} x(\tau)\dot{f}(\tau)\,d\tau}$ would thus arise from a time-dependent potential $W(x,t) = \dot{f}(\tau)x$. And indeed it is straightforward to verify that

$$\theta(t-t_0)\,TI_0(f) = K_0^{(\dot{f})}(x,t|x_0,t_0)\ e^{\,ixf(t)-ix_0f(t_0)}\ e^{-\frac{1}{2}|f_{\Delta^c}|^2}, \tag{7}$$

where

$$K_0^{(\dot{f})}(x,t|x_0,t_0) = \frac{\theta(t-t_0)}{\sqrt{2\pi i\,|t-t_0|}} \times$$

$$\exp\left(ix_0 f\left(t_0\right) - ixf\left(t\right) - \frac{i}{2}\left|f_\Delta\right|^2 + \frac{i}{2\left|t-t_0\right|}\left(\int_{t_0}^t f\left(\tau\right)d\tau + x - x_0\right)^2\right)$$

is the Green's function corresponding to the potential W, i.e. $K_0^{(\dot{f})}$ obeys the Schrödinger equation

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{f}\left(t\right)x\right)K_0^{(\dot{f})}\left(x,t|x_0,t_0\right) = i\,\delta\left(t-t_0\right)\,\delta\left(x-x_0\right).$$

More generally one calculates

$$\theta(t-t_0)T\left(I_0\prod_{i=1}^{n+1}\delta\left(x\left(t_i\right)-x_i\right)\right)(f) = e^{-\frac{1}{2}|f_{\Delta^c}|^2}e^{\,ixf(t)-ix_0f(t_0)}\prod_{i=1}^{n+1}K_0^{(\dot{f})}\left(x_i,t_i|x_{i-1},t_{i-1}\right). \tag{8}$$

Here $t_0 < t_1 < ... < t_n < t_{n+1} \equiv t$ and $x_{n+1} \equiv x$.

In order to pass from the free motion to more general situations, one has to give a rigorous definition of the heuristic expression

$$I = I_0\exp\left(-i\int_{t_0}^t V\left(x\left(\tau\right)\right)\;d\tau\right).$$

In [8] Khandekar and Streit accomplished this by perturbative methods in case V is a finite signed Borel measure with compact support. This construction was generalized in [13] to a wider class of potentials by allowing time-dependent potentials and a Gaussian fall-off instead of a bounded support.
The starting point is a power series expansion of $\exp\left(-i\int_{t_0}^t V\left(x\left(\tau\right),\tau\right)d\tau\right)$ using $V\left(x\left(\tau\right),\tau\right) = \int dx\,V\left(x,\tau\right)\,\delta\left(x\left(\tau\right)-x\right)$:

$$\exp\left(-i\int_{t_0}^t V\left(x\left(\tau\right),\tau\right)d\tau\right) = \sum_{n=0}^{\infty}(-i)^n\int_{\Lambda_n}d^nt\prod_{i=1}^n\int dx_i\,V\left(x_i,t_i\right)\delta\left(x\left(t_i\right)-x_i\right)$$

where $\Lambda_n = \{(t_1,...,t_n)\,|\,t_0 < t_1 < ... < t_n < t\}$.

In order to consider singular potentials V is no longer taken to be a function V but a measure ν. Under suitable conditions on ν it is proven in [8] and [13] that

$$I_V = I + \sum_{n=1}^{\infty}(-i)^n\int_{\mathbf{R}^n}\int_{\Lambda_n}\left(\prod_{j=1}^n\nu(dx_j,dt_j)\right)I_0\prod_{j=1}^n\delta\left(x\left(t_j\right)-x_j\right)$$

exists as a well-defined element of $(\mathcal{S})^*$ using theorems 2.2 and 2.3.

4 The unperturbed harmonic oscillator

In this section we first review some results of [3] which are necessary for the formulation and proof of our main result. Then we prepare a proposition on which we base our perturbative method.

To define the Feynman integrand

$$I_h = I_0 \exp\left(-i \int_{t_0}^{t} U\left(x(\tau)\right)\, d\tau\right),\, U(x) = \frac{1}{2}k^2x^2$$

of the harmonic oscillator, at least two things have to be done.

First we have to justify the pointwise multiplication of I_0 with the interaction term and secondly it has to be shown that $\mathbf{E}(I_h)$ solves the Schrödinger equation for the harmonic oscillator. Both has been done in [3]. There the T-transform of I_h has been calculated and shown to be a U-Functional. Thus $I_h \in (\mathcal{S})^*$. Later we will use the following modified version of their result:

$$TI_h\left(f\right) = \sqrt{\frac{k}{2\pi i \sin k\left|\Delta\right|}} \exp\left(-\frac{i}{2}\left|f_\Delta\right|^2 - \frac{1}{2}\left|f_{\Delta^c}\right|^2\right) \exp\left\{\frac{ik}{2\sin k\left|\Delta\right|}\left[\left(x_0^2 + x^2\right)\cdot\right.\right.$$

$$\cdot \cos k\left|\Delta\right| - 2x_0x + 2x\int_{t_0}^{t} dt' f\left(t'\right)\cos k\left(t'-t_0\right) - 2x_0\int_{t_0}^{t} dt' f\left(t'\right)\cos k\left(t-t'\right) +$$

$$\left.\left. +2\int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 f\left(s_1\right) f\left(s_2\right)\cos k\left(t-s_1\right)\cos k\left(s_2-t_0\right)\right]\right\}, \tag{9}$$

with $0 < k\left|\Delta\right| < \dfrac{\pi}{2}$, which is easily seen to be a U-functional.

For our purposes it is convenient to introduce

$$K_h^{(\dot f)}(x,t \mid x_0,t_0) = \theta(t-t_0)\; TI_h(f)\cdot \exp\tfrac{1}{2}\left|f_{\Delta}c\right|^2 \cdot \exp\left(ix_0 f(t_0) - ixf(t)\right)\;,$$

which is the propagator of a particle in a time dependent potential $\frac{1}{2}k^2x^2 + x\dot f(t)$. This allows for an independent check on the correctness of the above result. In advanced textbooks of quantum mechanics such as [7] the propagator for an harmonic oscillator coupled to a source j (forced harmonic oscillator) is worked out. Upon setting $j = \dot f$ their result is easily seen to coincide with the formula given above.

Proceeding exactly as in the free case (see [8], [13]) we first have to define the (pointwise) product

$$I_h \prod_{j=1}^{n} \delta\left(B(t_j) - x_j\right)$$

in $(\mathcal{S})^*$. The expectation of this object can be interpreted as the propagator of a particle in a harmonic potential, where the paths all are "pinned" such that $B(t_j) = x_j$, $1 \leq j \leq n$. Following the ideas of the remark at the end of the section 2 we will have to apply (6) repeatedly. But due to the form of $TI_h(f)$, which contains f only in the exponent up to second order, all these integrals are expected to be Gaussian.

Using this we arrive at the following

Proposition 1 *For $x_0 < x_j < x$, $1 \leq j \leq n$, $\quad t_0 < t_j < t_{j+1} < t$, $1 \leq j \leq n-1$, $I_h \prod_{j=1}^n \delta\left(B(t_j) - x_j\right)$ is a Hida distribution and its T-transform is given by*

$$T\Big(I_h \prod_{j=1}^n \delta\left(B(t_j) - x_j\right)\Big)(f) = e^{-\frac{1}{2}|f\Delta c|^2} e^{i(xf(t)-x_o f(t_o))} \prod_{j=1}^{n+1} K_h^{(\dot{f})}\left(x_{j-1}, t_{j-1}|x_j, t_j\right) \ .$$

Proof. For $n = 1$ we may check the assertion by direct computation using formula (6). To perform induction one needs the following

Lemma 2 *Let $[t_0, t] \subset [t_0^{'}, t^{'}]$ then*

$$K_h^{\left(\left(f+\lambda \mathbf{1}_{[t_0^{'}, t^{'}]}\right)^{\cdot}\right)}\left(x_0, t_0|x, t\right) = K_h^{(\dot{f})}\left(x_0, t_0|x, t\right), \ \forall \lambda \in \mathbf{R} \ .$$

The Lemma is also proven by a lengthy but straightforward computation. On a formal level the assertion of the lemma is obvious as both sides of the equation are solutions of the same Schrödinger equation if $[t_0, t] \subset [t_0^{'}, t^{'}]$. ■

The proposition states what one intuitively expects, ordinary propagation from one intermediate position to the next.

5 The Feynman integrand for the harmonic oscillator in an external potential

In this section we construct the Feynman integrand for the harmonic oscillator in an external potential V(x,t). Thus we have to define

$$I_V = I_h \cdot \exp\left(-i \int_{t_0}^{t} V(x(\tau), \tau) d\tau\right) \ .$$

As for the free particle we introduce the perturbation V via the series expansion of the exponential. Hence we have to find conditions for V such that the following object exists in $(\mathcal{S})^*$

$$I_V = I_h + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbf{R}^n} d^n x \int_{\Lambda_n} d^n t \prod_{j=1}^{n} V\left(x_j, t_j\right) \delta\left(x\left(t_j\right) - x_j\right) I_h \ .$$

Since we want to study singular time-dependent potentials, we consider ν a finite signed Borel measure on $\mathbf{R}\times\Delta$. Let ν_x denote the marginal measure

$$\nu_x\left(A\in\mathcal{B}\left(\mathbf{R}\right)\right)\equiv\nu\left(A\times\Delta\right)$$

and similarly

$$\nu_t\left(B\in\mathcal{B}\left(\Delta\right)\right)\equiv\nu\left(\mathbf{R}\times B\right).$$

The following theorem contains conditions under which the Feynman integrand I_V exists as a Hida distribution.

Theorem 3 *Let $\nu\equiv\nu_+-\nu_-$ be a finite signed Borel measure on $\mathbf{R}\times\Delta$ where the marginal measures $|\nu|_x:=(\nu_++\nu_-)_x$ and $|\nu|_t$ satisfy*

i) $\exists R>0,\ \forall r>R:\qquad |\nu|_x\left(\{x:|x|>r\}\right)<\exp\left(-\beta r^2\right)$ *for some* $\beta>0$;
ii) $|\nu|_t$ *has a* L^∞ *density.*

Then

$$I_V=I_h+\sum_{n=1}^{\infty}(-i)^n\int_{\mathbf{R}^n}\int_{\Lambda_n}\left(\prod_{j=1}^{n}\nu(dx_j,dt_j)\right)I_h\prod_{j=1}^{n}\delta\left(x\left(t_j\right)-x_j\right)\tag{10}$$

is a Hida distribution.

Remark: Conditions *i*) and *ii*) allow for some rather singular potentials, e.g. $\sum e^{-n^2}\delta_n$. For a cut-off interaction, i.e. compactly supported ν_x, condition *i*) is of course valid. Note also that ν is not supposed to be a product measure, hence the time dependence can be more intricate than simple multiplication by a function of time.

Proof.
1. part: In the first part of the proof we have to perform some technicalities which are necessary to establish the central estimate (11). We have to use a very careful procedure to achieve that (11) survives n-fold integration and summation in the second part of the proof.

From proposition 4.1 and the explicit formula 9 we find

$$\left|T\Big(I_h\prod_{j=1}^{n}\delta\left(B(t_j)-x_j\right)\Big)\left(zf\right)\right|\le e^{\frac{|z|^2}{2}|f|_0^2}\left(\prod_{j=1}^{n+1}\sqrt{\frac{1}{4|\Delta_j|}}\right)\exp\left(\left(|x_{n+1}|+|x_o|\right)\frac{\pi}{2}|z|\sup_{\Delta}|f|\right)\cdot$$

$$\cdot\left|\exp\left(\left\{\sum_{j=1}^{n}ikzx_j\Big[\frac{1}{\sin k\left|\Delta_j\right|}\int_{\Delta_j}dtf\left(t\right)\cos k\left(t-t_{j-1}\right)\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{\sin k\left|\Delta_{j+1}\right|}\int_{\Delta_j+1}dtf\left(t\right)\cos k\left(t-t_{j+1}\right)\Big]\right\}\right)\right|\cdot$$

$$\cdot \exp\left\{\sum_{j=1}^{n} \frac{\pi\left|z\right|^2}{2\left|\Delta_j\right|}\int_{\Delta_j} ds_1 \int_{\Delta_j} ds_2 \left|f\left(s_1\right)\right|\left|f\left(s_2\right)\right|\right\}$$

We define

$$X = \sup_{0\leq j\leq n+1} |x_j|$$

and

$$\|f\| \equiv \sup_{\Delta}|f| + \sup_{\Delta}|f'| + |f|_o$$

With these

$$\left|T\Big(I_h \prod_{j=1}^{n} \delta\left(B(t_j)-x_j\right)\Big)\left(zf\right)\right| \leq e^{\frac{|z|^2}{2}\|f\|^2}\left(\prod_{j=1}^{n+1}\sqrt{\tfrac{1}{4|\Delta_j|}}\right)\exp\left(X\pi\left|z\right|\|f\| + \frac{\pi\left|z\right|^2}{2}\left|\Delta\right|\|f\|^2\right)\cdot$$

$$\cdot\Bigg|\exp\Bigg(\Bigg\{\sum_{j=1}^{n} ikzx_j\Big[\frac{1}{\sin k\left|\Delta_j\right|}\int_{\Delta_j} dtf\left(t\right)\cos k\left(t-t_{j-1}\right)$$

$$-\frac{1}{\sin k\left|\Delta_{j+1}\right|}\int_{\Delta_j+1} dtf\left(t\right)\cos k\left(t-t_{j+1}\right)\Big]\Bigg\}\Bigg)\Bigg|\,.$$

To estimate the last factor we proceed as follows:

$$\left|\sum_{j=1}^{n} ikzx_j\left[\frac{1}{\sin k\left|\Delta_{j+1}\right|}\int_{\Delta_{j+1}} dtf\left(t\right)\cos k\left(t-t_j\right) - \frac{1}{\sin k\left|\Delta_{j+1}\right|}\int_{\Delta_j+1} dtf\left(t\right)\cos k\left(t-t_{j+1}\right)\right]\right|$$

$$\begin{aligned}
&\leq \sum_{j=1}^{n} k\left|z\right|X\frac{1}{\sin k\left|\Delta_{j+1}\right|}\left|\int_{\Delta_{j+1}} dtf\left(t\right)\int_{t_j}^{t_{j+1}} k\sin k\left(t-\tau\right)d\tau\right| \\
&\leq \sum_{j=1}^{n} k\left|z\right|X\sup_{\Delta}|f|\,\frac{\pi}{2}\left|\Delta_{j+1}\right| \\
&\leq \frac{\pi}{2}\left|z\right|Xk\,\|f\|\left|\Delta\right|
\end{aligned}$$

To obtain a bound for the remaining term

$$\left|\sum_{j=1}^{n} ikzx_j\left[\frac{1}{\sin k\left|\Delta_j\right|}\int_{\Delta_j} dtf\left(t\right)\cos k\left(t-t_{j-1}\right) - \frac{1}{\sin k\left|\Delta_{j+1}\right|}\int_{\Delta_j+1} dtf\left(t\right)\cos k\left(t-t_j\right)\right]\right|$$

we expand $F(t_{j-1}) = \int_{t_{j-1}}^{t_j} dt\, f(t)\cos k\left(t-t_{j-1}\right)$ and $G(t_{j+1}) = \int_{t_j}^{t_{j+1}} dt\, f(t)\cos k\left(t-t_j\right)$ around t_j. This yields with $\eta_j \in \Delta_j$ and $\eta_{j+1} \in \Delta_{j+1}$

$$\leq k\left|z\right|X\left|\sum_{j=1}^{n} f\left(t_j\right)\left[\frac{\left|\Delta_j\right|}{\sin k\left|\Delta_j\right|} - \frac{\left|\Delta_{j+1}\right|}{\sin k\left|\Delta_{j+1}\right|}\right]\right| +$$

$$+k\,|z|\,X\sum_{j=1}^{n}\left[\frac{(t_{j-1}-t_j)^2}{2\sin k\,|\Delta_j|}\left(-f'(\eta_j)-k^2\int_{\eta_j}^{t_j}dtf(t)\cos k(t-\eta_j)\right)\right]-$$

$$-k\,|z|\,X\sum_{j=1}^{n}\left[-\frac{(t_{j+1}-t_j)^2}{2\sin k\,|\Delta_{j+1}|}\left(f'(\eta_{j+1})\cos k(\eta_{j+1}-t_j)-kf(\eta_{j+1})\sin k(\eta_{j+1}-t_j)\right)\right]$$

Since $\sup\limits_{0\le x\le \frac{\pi}{2}}\left(\frac{x}{\sin x}\right)'=1$ then the first term above is bounded by

$$2k\,|\Delta|\,|z|\,X\,\|f\|$$

For the second term we obtain the bound

$$|z|\,X\frac{|\Delta|}{4}\sup_{\Delta}|f'|+\frac{k^2\,|\Delta|^2}{4}\sup_{\Delta}|f|+\frac{\pi}{4}\,|\Delta|\sup_{\Delta}|f'|+\frac{\pi k}{4}\,|\Delta|\sup_{\Delta}|f|\le$$

$$\le\frac{|z|\,X\,|\Delta|\,\pi}{4}\left(\|f\|\left(2+k^2\,|\Delta|+k\right)\right)$$

Putting all of this together we finally arrive at

$$\left|T\Big(I_h\prod_{j=1}^{n}\delta\left(B(t_j)-x_j\right)\Big)(zf)\right|\le\left(\prod_{j=1}^{n+1}\sqrt{\tfrac{1}{4|\Delta_j|}}\right)\exp\left(L\,|z|\,X\,\|f\|+\left(\frac{\pi}{2}\,|\Delta|+\frac{1}{2}\right)|z|^2\,\|f\|^2\right)$$

where $L=\pi+\frac{3}{4}\pi k\,|\Delta|+2k\,|\Delta|+\frac{\pi}{4}\,|\Delta|\left(2+k^2\,|\Delta|\right)$ is a constant.

Hence we have the following estimate

$$\left|T\Big(I_h\prod_{j=1}^{n}\delta\left(B(t_j)-x_j\right)\Big)(zf)\right|\le\left(\prod_{j=1}^{n+1}\sqrt{\tfrac{1}{4|\Delta_j|}}\right)\exp\left(X^2\gamma\right)\exp\left[|z|^2\,\|f\|^2\left(\frac{1}{2}+\frac{\pi}{2}\,|\Delta|+\frac{L^2}{2\gamma}\right)\right]\tag{11}$$

where $\gamma>0$.

2. part: In this final step we use the method developed in [13] to control the convergence of (10). Although the slight modification to our case is easy we give the basic steps for the convenience of the reader.

In order to apply theorem 2.3 to perform the integration we need to show that

$$\left(\prod_{j=1}^{n+1}\sqrt{\tfrac{1}{4|\Delta_j|}}\right)\exp\left(X^2\gamma\right)$$

is integrable with respect to ν. To this end we choose $q>2$ and $0<\gamma<\frac{\beta}{q}$. With this choice of γ the property i) of ν yields that $\exp\left(\gamma X^2\right)\in L^q\left(\mathbf{R}^n\times\Lambda_n,|\nu|\right)$ and with

$$Q\equiv\left(\int_{\mathbf{R}}\int_{\Delta}|\nu|\,(dx,dt)\exp\left(\gamma qx^2\right)\right)^{\frac{1}{q}}$$

we have

$$\left(\int_{\mathbf{R}^n}\int_{\Lambda_n}\prod_{j=1}^{n}|\nu|\left(dx_j,dt_j\right)\exp\left(\gamma qX^2\right)\right)^{\frac{1}{q}}\leq\exp\left(\gamma\left(x_o^2+x^2\right)\right)Q^n<\infty.$$

Now we choose p such that $\frac{1}{p}+\frac{1}{q}=1$. Using the property ii) of ν and the formula

$$\int_{\Lambda_n}d^nt\prod_{j=1}^{n+1}\left(\frac{1}{4\left|t_j-t_{j-1}\right|}\right)^{\alpha}=\left(\frac{\Gamma\left(1-\alpha\right)}{4^{\alpha}}\right)^{n+1}\frac{|\Delta|^{n(1-\alpha)-\alpha}}{\Gamma\left(\left(n+1\right)\left(1-\alpha\right)\right)},\ \alpha<1$$

we obtain the following bound

$$\left[\int_{\mathbf{R}^n}\int_{\Lambda_n}\prod_{j=1}^{n}|\nu|\left(dx_j,dt_j\right)\prod_{j=1}^{n+1}\left(\frac{1}{4\left|t_j-t_{j-1}\right|}\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\leq$$

$$\leq|\nu|_{t\infty}^{n/p}\frac{\Gamma\left(\frac{2-p}{2}\right)^{\frac{n+1}{p}}|\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{4^{\frac{n+1}{2}}\Gamma\left(\left(n+1\right)\frac{2-p}{2}\right)^{\frac{1}{p}}}<\infty$$

$|\nu|_{t\infty}$ is shorthand notation for the essential supremum of the L^{∞}-density of $|\nu|_t$ which exists due to condition *ii*).

Finally an application of Hölder's inequality gives

$$\left|\left(\prod_{j=1}^{n+1}\sqrt{\frac{1}{4|t_j-t_{j-1}|}}\right)\exp\left(\gamma X^2\right)\right|_1\leq$$

$$\leq\exp\left(\gamma x_o^2+\gamma x^2\right)Q^n\,|\nu|_{t\infty}^{n/p}\frac{\Gamma\left(\frac{2-p}{2}\right)^{\frac{n+1}{p}}|\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{2^{n+1}\Gamma\left(\left(n+1\right)\frac{2-p}{2}\right)^{\frac{1}{p}}}\equiv C_n<\infty$$

Hence theorem 2.3 yields

$$I_n\equiv\int_{\mathbf{R}^n}\int_{\Lambda_n}\prod_{j=1}^{n}\nu\left(dx_j,dt_j\right)\left(I_h\prod_{j=1}^{n}\delta\left(B\left(t_j\right)-x_j\right)\right)\in\left(\mathcal{S}\right)^*.$$

As the C_n are rapidly decreasing in n the hypotheses of theorem 2.2 are fulfilled and hence

$$I_V=\sum_{n=0}^{\infty}I_n\in\left(\mathcal{S}\right)^*\ .$$

■

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References

[1] Albeverio, S.A., Høegh-Krohn, R. (1976), *Mathematical Theory of Feynman Integrals.* LNM **523**, Springer Verlag.

[2] Exner, P. (1985), *Open Quantum Systems and Feynman Integrals.* Reidel, Dordrecht.

[3] Faria, M. de, Potthoff, J. and Streit, L. (1991), *The Feynman Integrand as a Hida Distribution.* J. Math. Phys. **32**, 2123.

[4] Hida, T. (1980), *Brownian Motion,* Applications of Mathematics **11**, Springer Verlag, Berlin.

[5] Hida, T., Kuo, H.H., Potthoff, J. and Streit, L. (1993), *White Noise: An Infinite Dimensional Calculus.* Kluwer Academic Publishers, Dordrecht.

[6] Hida, T. and Streit, L. (1983), *Generalized Brownian Functionals and the Feynman Integral.* Stoch. Proc. Appl. **16**, 55.

[7] Holstein, B.R. (1992), *Topics in advanced Quantum mechanics.* Redwood City, Ca., Addison-Wesley.

[8] Khandekar, D.C. and Streit, L.(1992), *Constructing the Feynman Integrand.* Ann. Physik **1**, 49-55.

[9] Kondratiev, Yu.G. and Streit, L. (1992), *Spaces of White Noise Distributions. Constructions, Descriptions, Applications I.* To appear in Rep. Math. Phys. **33**.

[10] Kondratiev, Yu.G., Leukert, P., Potthoff, J., Streit, L., Westerkamp, W. (1994), *Generalized Functionals in Gaussian Spaces - The Characterization Theorem Revisited* - Preprint Univ. Mannheim Nr. 175/94.

[11] Kubo, I. and Takenaka, S. (1980), *Calculus on Gaussian White Noise I+II.* Proc. Japan Acad. 56A, 376-380 and 411-416.

[12] Kuo, H.H. (1991), *Lectures on White Noise Analysis.* Soochow J. Math. **18** 229-300.

[13] Lascheck, A., Leukert, P., Streit, L. and Westerkamp, W. (1993), *Quantum Mechanical Propagators in Terms of Hida Distributions.* Rep. Math. Phys. **33**, 221-232.

[14] Meyer, P.A. and Yan, J.–A. (1990), *Les "fontions caractéristiques" des distribuitions sur l'éspace de Wiener.* Seminaire de Probabilites XXV, ed.: P. A. Meyer, M. Yor, Springer, p. 61-78.

[2] Program of Junta Nacional de Investigacao Cientifica e Technologica (Portugal)

[15] Obata, N. (1994), *White Noise Calculus and Fock Space.* LNM **1577**. Springer, Berlin.

[16] Potthoff, J. and Streit, L. (1991), *A Characterization of Hida Distributions.* J. Funct. Anal. **101**, 212-229.

[17] Potthoff, J. (1991), *Introduction to White Noise Analysis.* Baton Rouge Preprint.

[18] Streit, L. (1993), *The Feynman Integral - Recent Results.* In: Dynamics of complex and Irregular Systems. Ph. Blanchard et al., eds. World Scientific.

[19] Streit, L. (1994), *White Noise Analysis and Applications in Quantum Physics.* In: Stochastic Analysis and Applications in Physics. Ed.: A.I. Cardoso et al.; Kluwer Dordrecht, in print.

[20] Streit, L. and Westerkamp, W. (1993), *A generalization of the characterization theorem for generalized functionals of White Noise.* In: Dynamics of complex and Irregular Systems. Ph. Blanchard et al., eds. World Scientific.

[21] Westerkamp, W. (1993), *A Primer in White Noise Analysis.* In: Dynamics of complex and Irregular Systems. Ph. Blanchard et al., eds. World Scientific.