Differential Geometry on Configuration Spaces[1]

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Abstract

On these notes we give an introduction into the geometry of configuration spaces. The emphasis is given to the "lifting" procedure which turns out to coincide with the direct approach introduced in [AKR98a]. Since in applications we also need much more general spaces of configurations than the ones introduced in [AKR98a], therefore we treat the marked configuration spaces also.

Contents

1 Introduction and motivation

The aim of these notes is to give a detailed account on the geometry of configuration spaces. On the one hand there is not a unique way to introduce the geometry on the configuration spaces, on the other hand the structure of configuration spaces is getting more rich such that at the first glance its seems that we have a lot of structure concerning configuration spaces but they are not connected. Hence I plan to explore these facts and try to see how can one relate the different kinds of structure that appear.

In all my considerations I will be concern with the so-called Internal Geometry on configuration spaces, that means the geometry inherited from the underlying manifold. This is fundamental for the strategy how we handle geometry in configuration spaces. Usually one starts by giving the definition of a measure on some nice linear space of distributions, e.g. $\mathcal{D}'(\mathbb{R}^d)$ or $S'(\mathbb{R}^d)$ and by additional considerations this measure has support on a subset of this linear space, the configuration space. As an example the Poisson measure given by Minlos' theorem on $\mathcal{D}'(\mathbb{R}^d)$. Then one consider the L^2 space with respect to the introduced measure say $L^2(\mu)$ and obtain the Segal isomorphism to the Fock space. Thus one can introduce in $L^2(\mu)$ the images under the isomorphism of the Fock space operators, e.g., creation annihilation etc. The geometry arising in this way we call external one. Therefore it is natural to ask wether or not these geometries are connected. This question will not be treated on that talk and we refer to [Sil98] for more details.

2 Configuration Spaces

2.1 Underlying spaces

In this section we will give a detailed description of configuration spaces which appear in applications. Before that we need to fix the underlying space on which each configuration will sit. The most typical examples for the underlying space are

1. $X = \mathbb{R}^d, d \in \mathbb{N}$.
2. $X = \mathbb{R}^d \times \mathbb{R}^l$, $d, l \in \mathbb{N}$
3. X Riemannian manifold.
4. $X = M \times S$, M Riemannian manifold, S -Complete separable metric space.
5. $X = M \times S$, M Riemannian manifold and $S = \{1, \ldots, q\}$.
6. $X = M \times S$, M Riemannian manifold and $S = \mathcal{L}^\theta(\mathbb{R}^d)$ the Banach space of all continuous functions $s : [0, \theta] \to \mathbb{R}^d$ with $s(0) = s(\theta)$.

7. $X = M \times S$, $M = \mathbb{R}^d$ and $S = \mathbb{T}$ one dimensional torus.

8. $X = M \times S$, M, S Standard Borel Spaces.

Of course depends of the applications we have in mind we choose a more convenient underlying space. The most general case when M, S are standard Borel spaces is possible because we still can apply Kolmogorov's theorem to the projective limit of standard Borel spaces see e.g. [Par67]. However in applications we do not need such general framework till now.

We will choose X as being a Riemannian manifold.

Denote by $\mathcal{O}(X)$ the family of all open subsets of X and $\mathcal{B}(X)$ the corresponding Borel σ-algebra on X. By $\mathcal{B}_c(X)$ we mean

$$\mathcal{B}_c(X) = \{Y \in \mathcal{B}(X) | \, Y \text{ bounded} \Rightarrow Y \text{ has compact closure}\}.$$

2.2 Finite configuration space

Let $Y \in \mathcal{B}(X)$ be given, then we define the space of n-point configuration $\Gamma_{0,Y}^{(n)}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ by

$$\Gamma_{0,Y}^{(n)} := \{\eta \subset Y | |\eta| = n\}, \Gamma_{0,Y}^{(0)} := \{\emptyset\},$$

where $|A|$ denotes the cardinality of the set A. Let us now introduce the following mappings which play a basic rule in the sequel. Let $\Lambda \in \mathcal{B}_c(X)$ be such that $\Lambda \subset Y \in \mathcal{B}(X)$, then we define N_Λ by

$$N_\Lambda : \Gamma_{0,Y}^{(n)} \to \mathbb{N}_0, \eta \mapsto |\eta \cap \Lambda| = |\eta_\Lambda|.$$

Next we would like to introduce more structure in $\Gamma_{0,Y}^{(n)}$, more precisely, we would like to prove that $\Gamma_{0,Y}^{(n)}$ has a structure of a manifold. To this end consider the symmetization mapping $\mathrm{sym}_Y^{(n)}$ given for any $n \in \mathbb{N}$, $Y \in \mathcal{B}(X)$ by

$$\mathrm{sym}_Y^{(n)} : \tilde{Y}^n \to \Gamma_{0,Y}^{(n)}, \ (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\},$$

where

$$\tilde{Y}^n = \{(x_1, \ldots, x_n) \in Y^n | x_x \neq x_j \text{ if } i \neq j\}.$$

Thus if we consider S_n as the permutation group of $\{1, \ldots, n\}$, then we can identify, using $\mathrm{sym}_Y^{(n)}$ the space of n-point configurations with $\tilde{Y}^n / S_n$. Therefore $\Gamma_{0,Y}^{(n)}$ inherits the structure of a $n \times \dim(X)$-dimensional manifold. A basis for the topology in $\Gamma_{0,X}^{(n)}$ denoted by $\mathcal{O}(\Gamma_{0,X}^{(n)})$, is given by

$$U_1 \hat{\times} \cdots \hat{\times} U_n := \{\eta \in \Gamma_{0,X}^{(n)} | N_{U_1}(\eta) = 1, ..., N_{U_n}(\eta) = 1\},$$

where $U_i \in \mathcal{O}_c(X)$ with $U_i \cap U_j = \varnothing$ if $i \neq j$. A chart of $\Gamma_{0,X}^{(n)}$ can be constructed as follows: let $(U_i, \varphi_1), \ldots, (U_n, \varphi_n)$ be n charts in X such that $U_i \cap U_j = \varnothing$ if $i \neq j$. Then

$$\varphi_1 \hat{\times} \cdots \hat{\times} \varphi_n : U_i \hat{\times} \cdots \hat{\times} U_n \to \varphi_1(U_i) \times \ldots \times \varphi_n(U_n),\ \{x_1, \ldots, x_n\} \mapsto (\varphi_1(x_{i_1}), \ldots, \varphi_n(x_{i_n})),$$

where $x_{i_k} \in U_k$ is a chart in $\Gamma_{0,X}^{(n)}$.

The Borel σ-algebra on $\Gamma_{0,X}^{(n)}$ is denoted by $\mathcal{B}\left(\Gamma_{0,X}^{(n)}\right)$ and it coincides with the σ-algebra generated by the mapping N_Λ, i.e.,

$$\mathcal{B}\left(\Gamma_{0,X}^{(n)}\right) = \sigma\left(N_\Lambda, \Lambda \in B_c(X)\right).$$

Finally, we define the space of finite configurations $\Gamma_{0,X}$ by

$$\Gamma_{0,X} = \bigsqcup_{n=0}^{\infty} \Gamma_{0,X}^{(n)}$$

equipped with the topology $\mathcal{O}(\Gamma_{0,X})$ of the disjoint union, i.e., the strongest topology on $\Gamma_{0,X}$ such that all the embedding

$$i_n : \Gamma_{0,X}^{(n)} \hookrightarrow \Gamma_{0,X}$$

are continuous. $B(\Gamma_{0,X})$ stands for the Borel σ-algebra on $\Gamma_{0,X}$.

Hence de natural manifold structure of $\Gamma_{0,X}^{(n)}$ gives us also the possibility to endow $\Gamma_{0,X}$ with a geometrical structure. We will came back later to this geometry.

Bounded sets in $\Gamma_{0,X} \equiv \Gamma_0$ may be defined as follows: $B \in \mathcal{B}(\Gamma_0)$ is called bounded iff $\exists N \in \mathbb{N}, \Lambda \in \mathcal{B}_c(X)$ such that $B \subset \sqcup_{n=0}^{N} \Gamma_{0,\Lambda}^{(n)}$. B is called compact iff $B \cap \Gamma_0^{(n)} = \emptyset$ for $n > N$ and $B \cap \Gamma_0^{(n)}$ is compact for $n \leq N$. $B \cap \Gamma_0^{(n)}$ is compact iff $(\mathrm{sym}_X^n)^{-1}(B)$ is compact in $\tilde{X}^n$.

2.3 Configuration space

The configuration space is defined as the set of all subsets of X which are locally finite, i.e.,

$$\Gamma_X = \{\gamma \subset X \| \gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(X)\}.$$

In $\Gamma_X \equiv \Gamma$ we introduce the vague topology $\mathcal{O}(\Gamma)$, i.e., the weakest topology on Γ such that all the mappings

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

are continuous for all $f \in C_{bs}(X)$ (the set of all continuous functions on X with bounded support). A basis for $\mathcal{O}(\Gamma)$ is given by the sets of the form

$$\{\gamma \in \Gamma || \gamma_\Lambda| = n\}, \; n \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(X).$$

If we denote the Borel σ-algebra on Γ by $\mathcal{B}(\Gamma)$, then

$$\mathcal{B}(\Gamma) = \sigma(N_\Lambda | \Lambda \in \mathcal{B}_c(X)).$$

We can describe Γ in a different way. Normally, for any $\Lambda \in \mathcal{O}_c(X)$ we define

$$\Gamma_\Lambda := \{\gamma \in \Gamma | \gamma \cap (X/\Lambda) = \emptyset\}.$$

In Γ_Λ we introduce the topology $\mathcal{O}(\Gamma_\Lambda)$ with a subbase of open sets

$$\{\gamma \in \Gamma || \gamma \cap \tilde{\Lambda}| = n, \tilde{\Lambda} \subset \Lambda\}.$$

For any $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(X)$ with $\Lambda_{1,} \subset \Lambda_2$ we can consider the natural projections

$$p_{\Lambda_2, \Lambda_1} : \Gamma_{\Lambda_2} \to \Gamma_{\Lambda_1}, \; \gamma \mapsto \gamma_{\Lambda_1}.$$

Then the space of configuration Γ_X is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))_{\Lambda \in \mathcal{B}_c(X)}$ with respect to the above projections and the continuous mappings

$$p_\Lambda : \Gamma \to \Gamma_{\Lambda,} \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda.$$

2.4 Marked configuration space

In applications the configuration space introduced above is not sufficient, therefore we need more elaborated underlying spaces. Namely, if we are interested in studying configurations with some additionally degree of freedom, then the underlying space X should be modified and consequently the configuration space on it.

Thus we consider X as before and additionally we suppose given a complete metric space S. The space X describes the positions of the particles and the elements of S we call marks.

Local sets in $X \times S$ now is given by

$$\Im = \{B \in \mathcal{B}(X) \times \mathcal{B}(S) | \exists \Lambda \in \mathcal{B}_c(X) \text{ with } B \subset \Lambda \times S\}$$

Then we obtain the following generalizations:

$$\Omega_{0,X}^{(n)}(S) := \{\hat{\eta} = \{(x_1, s_1), \ldots, (x_n, s_n)\} \in \Gamma_{0,X \times S}^{(n)} | x_i \neq x_j \text{ if } i \neq j\}, \; \Omega_{0,X}^{(0)}(S) = \{\emptyset\},$$

which is the n-point marked configuration space. The space of finite marked configurations Ω_0 is given by

$$\Omega_0 := \bigsqcup_{n=0}^{\infty} \Omega_{0,X}^{(n)}.$$

Finally the marked configuration space is defined by

$$\Omega_X(S) \equiv \Omega := \{\omega := \{(x, s_x) | x \in \gamma\} \in \Gamma_{X\times S} | \gamma \in \Gamma_X,\ s_x \in S \text{ for all } x \in \gamma\}.$$

Again we have that Ω coincides, with the projective limit of $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))_{\Lambda\in\mathcal{B}_c(X)}$, where

$$\Omega_\Lambda(S) \equiv \Omega_\Lambda := \{\omega \in \Omega | \omega \subset \Lambda \times S\}$$

and the projective limit with respect to the projections

$$p_{\Lambda_1,\Lambda_2} : \Omega_\Lambda \to \Omega_{\Lambda_2},\ \Lambda_1 \supset \Lambda_2, \quad p_\Lambda : \Omega \to \Omega_\Lambda.$$

2.5 Configurations over fiber bundles

The marked configuration space Ω defined above and the related analysis which we will derive later on still is not very satisfactory, namely its geometric aspect, see e.g., [KSS98]. Therefore all the results given in this talk are valid independent of the position space we choose but in the case of the marked space we need an additional careful analysis in order to overcome some problems. This I hope will be the contents of the second talk on this meeting.

3 Measures on configurations spaces

The introduction of measures on the configuration space mostly is connected with the applications we have in mind. Therefore, since we are interested in the geometrical aspects the measures introduced below are not the most general although they are very important in applications.

3.1 Lebesgue-Poisson measure

Before the construction of any kind of measure on the configuration space we need first of all to fix on intensity measure σ on the underlying space X. Thus we assume given σ which is a non-atomic Radon measure on X, i.e.,

$$\sigma(\Lambda) < \infty, \quad \forall \Lambda \in \mathcal{B}_c(X), \quad \text{and} \quad \sigma(\{x\}) = 0,\ \forall x \in X.$$

The most interesting cases arises when $\sigma(X) = \infty$.

Having an underlying measure σ on X we can consider $\sigma^{\otimes n}$ as a measure on $\widetilde{X}$. We denote by $\sigma^{(n)} \upharpoonright \Gamma_0^{(n)} := \sigma^{\otimes n} \left(\mathrm{sym}_X^n\right)^{-1}$ the corresponding measure on $\Gamma_0^{(n)}$and $\sigma^{(0)}\left(\{\emptyset\}\right) := 1$.

The Lebesgue-Poisson measure λ_σ on $\mathcal{B}\left(\Gamma_0\right)$ is defined such that it coincides on each $\Gamma_0^{(n)}$ with the measure $\frac{1}{n!}\sigma^n \upharpoonright \Gamma_0^{(n)}$, i.e.,

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^n \upharpoonright \Gamma_0^{(n)}.$$

Let us compute $\lambda_\sigma\left(\Gamma_\Lambda\right), \Lambda \in \mathcal{B}_c\left(X\right)$. Since Γ_Λ is represented by

$$\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_{0,\Lambda}^{(n)},$$

then we have

$$\begin{aligned}
\lambda_\sigma(\Gamma_\Lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\sigma^{(n)} \upharpoonright \Gamma_{0,\Lambda}^{(n)}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^{\otimes n_0}\left(\mathrm{sym}_\Lambda^n\right)^{-1}\left(\Gamma_{0,\Lambda}^{(n)}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^{\otimes n}(\tilde{\Lambda}) \\
&= e^{\sigma(\Lambda)}
\end{aligned}$$

because $\sigma^{\otimes n}(\Lambda^n \backslash \tilde{\Lambda}^n) = 0$.

3.2 Poisson measure

Having defined the Lebesgue-Poisson measure on Γ_0 and taking into account that $\lambda_\sigma\left(\Gamma_\Lambda\right) = e^{\sigma(\Lambda)}$, we can define a probability measure on Γ_Λ putting

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)}\lambda_\sigma$$

We observe that the family $\{\pi_\sigma^\Lambda \Lambda \in \mathcal{B}_c\left(X\right)\}$ is a consistent family of probabilities measures, therefore by a version of Kolmogorov's theorem for the projective limit spaces such family determines uniquely a measure on $\mathcal{B}\left(\Gamma\right)$ such that $\pi_\sigma^\Lambda = \pi_\sigma \circ p_\Lambda^{-1}$. The measure π_σ is called Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$.

It is possible to compute in close form the Laplace transform of π_σ: let $f \in C_{bs}(X)$ be such that $\operatorname{supp} f \subset \Lambda$ for some $\Lambda \in \mathcal{O}_c(X)$. Then

$$\begin{aligned}\int_\Gamma e^{\langle f,\gamma\rangle} d\pi_\sigma(\gamma) &= \int_{\Gamma_\Lambda} e^{\langle f,\gamma_\Lambda\rangle} d\pi_\sigma^\Lambda(\gamma_\Lambda) \\ &= e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \exp\left(\sum_{n=0}^{n} f(x_n)\right) d\sigma(x_1) \ldots d\sigma(x_n) \\ &= e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_\Lambda e^{f(x)} d\sigma(x)\right)^n \\ &= e^{-\sigma(\Lambda)} \exp\left(\int_\Lambda e^{f(x)} d\sigma(x)\right) \\ &= \exp\left(\int_\Lambda \left(e^{f(x)} - 1\right) d\sigma(x)\right).\end{aligned}$$

Thus, for all $f \in C_{bs}(X)$ we have

$$\int_\Gamma e^{\langle f,\gamma\rangle} d\sigma(x) = \exp\left(\int_X \left(e^{f(x)} - 1\right) d\sigma(x)\right). \quad (1)$$

Formula (1) also can be used as an equivalent definition of the Poisson measure π_σ via Minlos' theorem to produce a measure on a linear space of generalized functions on X.

3.3 Marked Poisson measure

Let us now concentrate on the case when additional to the position space X we have also the marked space S.

Suppose that S is furnished with a σ-finite measure τ on $\mathcal{B}(S)$. We denote the product measure on $X \times S$ by $\hat{\sigma} := \sigma \times \tau$. In applications some times we need a bit more general framework. Namely, instead of τ we consider a kernel

$$\tau : X \times \mathcal{B}(S) \to \mathbb{R}, (x, B) \mapsto \tau(x, B)$$

such that $\forall x \in X$ $\tau(x, .)$ is a finite measure on $(S, \mathcal{B}(S))$ and for any $A \in \mathcal{B}(S)$ $\tau(., A)$ is $\mathcal{B}(X)$-measurable.

The condition

$$\int_\Lambda \tau(x, s)\, d\sigma(x) < \infty,\ \Lambda \in \mathcal{B}_c(X)$$

reflects the different roles of mark and position variables.

Then one consider the measure on $X \times S$ as

$$\sigma^{\tau}(dx, ds) := \tau(x, ds)\, \sigma(dx).$$

The marked Poisson measure is constructed in a similar way as the Poisson measure. Hence we consider $(\sigma^{\tau})^{\otimes n}$ as a measure in $\left(\widetilde{X \times S}\right)^{n}$ and denote the image measure under $\mathrm{sym}^n : (\widetilde{X \times S}) \to \Omega_0^n$ by $(\sigma^{\tau})^{(n)} \restriction \Omega_0^n := (\sigma^{\tau})^{\otimes n} \circ (\mathrm{sym}^n)^{-1}$and $(\sigma^{\tau})^{(0)}(\{\emptyset\}) := 1$. Then the so called marked Lebesgue-Poisson measure $\lambda_{\sigma^{\tau}}$ on $\mathcal{B}(\Omega_0)$ is defined such that it coincides on each Ω_0^n with the measure $\frac{1}{n!}(\sigma^{\tau})^{(n)} \restriction \Omega_0^n$ as follows

$$\lambda_{\sigma^{\tau}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\sigma^{\tau})^{(n)} \restriction \Omega_0^n.$$

We can compute the measure of $\Omega_{0,\Lambda}$ as before to be

$$\lambda_{\sigma^{\tau}}(\Omega_{0,\Lambda}) = e^{\sigma\tau(\Delta \times S)}$$

$$\sigma^{\tau}(\Lambda \times S) = \int_{\Lambda} \tau(x, S)\, \sigma(dx)$$

then we define a probability measure on Ω_{Λ} by

$$\pi_{\sigma}^{\tau,\Lambda} := e^{-\sigma\tau(\Delta \times S)} \lambda_{\sigma^{\tau}}$$

which has the following property

$$\pi_{\sigma}^{\tau,\Lambda}(\Omega_{\Lambda}^{(n)}) = \frac{1}{n!} (\sigma^{\tau}(\Lambda \times S))^{n}\, e^{-\sigma_{\tau}(\Lambda \times S)}.$$

Since the family $\left\{\pi_{\sigma}^{\tau,\Lambda}, \Lambda \in \mathcal{B}_c(X)\right\}$ is consistent, then by Kolmogorov's theorem for the projective limit space Ω, there exists a unique measure π_{σ}^{τ} on $\mathcal{B}(\Omega)$ such that $\pi_{\sigma}^{\tau,\Lambda} = \pi_{\sigma}^{\tau} \circ p_{\Lambda}$. We call π_{σ}^{τ} the marked Poisson measure. As in the case of Poisson measure we can compute the Laplace transform of π_{σ}^{τ}: let $f : X \times S \to \mathbb{R}$ be such that $\mathrm{supp}\, f \subset \Lambda \times S$ for some $\Lambda \in \mathcal{B}_c(X)$. Then for any $\omega \in \Omega, \langle f, \omega\rangle = \langle f, \omega_{\Lambda}\rangle$, therefore

$$\begin{aligned}
\int_{\Omega} e^{\langle f,\omega\rangle} d\pi_{\sigma}^{\tau}(\omega) &= \int_{\Omega\Lambda} e^{\langle f,\omega\rangle} d\pi_{\sigma}^{\tau,\Lambda}(\omega_{\Lambda}) \\
&= e^{\sigma^{\tau}(\Lambda \times S)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Lambda \times S)^n} \exp\left(\sum_{n=0}^{n} f(x_n, s_{x_n})\right) d\sigma^{\tau}(x,s)_1^n \\
&= \exp\left(\int_X \int_S (e^{f(x,s)} - 1)\tau(x, ds)\, \sigma(dx)\right).
\end{aligned}$$

In the particular case when $\sigma^\tau(dx, ds) = \tau(ds)\,\sigma(dx)$, and $f(x, s) = s\varphi(x)$, then we find the most standard case of compound Poisson measure with Laplace transform given by

$$\int_\Omega e^{\langle f,\omega\rangle} d\pi_\sigma^\tau(\omega) = \exp\left(\int_X \int_S \left(e^{s\varphi(x)} - 1\right) \tau(ds)\,\sigma(dx)\right).$$

The results derived below are valid for such measures except the case when the measure τ is given be a kernel. More precisely, concerning the geometry on the different kinds of configuration spaces it is known for the measures introduced above with the expectation of σ^τ when τ is a kernel. The results concerning the compound Poisson measure turns out to be not sufficient for applications. Hence, in order to overcome this problem we will need a modification of our framework. This subject will be treated on the next talk, see Section 6.

4 Poisson and Lebesgue-Poisson spaces

4.1 Fock space isomorphism

Having defined configuration spaces as well as measures on them we are ready to go further on into the analysis arising from the above measures. Hence we will consider the L^2 spaces originated by them, i.e., the Poisson space $L^2(\Gamma, \pi_\sigma)$ and the Lebesgue-Poisson space $L^2(\Gamma\lambda_\sigma)$. It is well known that the Poisson space is isomorphic to the Fock space

$$I_\pi : L^2(\pi_\sigma) \to \mathrm{Exp}L^2(X, \sigma) \equiv \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}\left(X^n, n!\sigma^{\otimes n}\right)$$

where the pre-image of the coherent states $\mathrm{Exp}(f) = (f^{\otimes n}/n!)_{n=0}^{\infty}$, $f \in B_{ls}(X)$ is given by

$$I_\pi^{-1}(\mathrm{Exp}(f)) = e_\pi(f, \gamma) = \prod_{x\in\gamma}(1 + f(x)) \exp\left(-\int_X f(x)\,d\sigma(x)\right).$$

If we additional assume that $f > -1$ then we can express $e_\pi(f, \gamma)$ in a suitable form for linear theory

$$e_\pi(f; \gamma) = e^{\langle \log(1+f), \gamma\rangle - \langle f\rangle_\sigma},$$

where $\langle f\rangle_\sigma$ is the short notation for $\int_X f(x)d\sigma(x)$. The function $e_\pi(f; \gamma)$ is the generating function of the Charier polynomials in $L^2(\pi_\sigma)$, i.e.,

$$e_\pi(f; \gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} < C_n^\sigma(f), f^{\otimes n} >, \quad C_n^\sigma(\gamma) \in \mathcal{D}'(X)^{\hat{\otimes} n}$$

and

$$\int_{\Gamma}\langle C_n^{\sigma}, f^{(n)}\rangle\langle C_m^{\sigma}(\gamma), g^{(m)}\rangle d\pi_{\sigma}(\gamma) = n!\left(f^{(n)}, g^{(n)}\right)\delta_{nm}.$$

In the case of the Lebesgue-Poisson space this isomorphism is natural:

$$I_{\lambda} : L^2(\Gamma_0, \lambda_{\sigma}) \to \mathrm{Exp}(L^2(X, \sigma)),\ G \mapsto (g^{(n)})_{n=0}^{\infty},$$

where $g^{(n)}(x_1, \ldots, x_n) = \frac{1}{n!}G(\{x_1, \ldots, x_n\})$, and $g^0 = G(\emptyset)$.

If $G, F, \in L^2(\Gamma_0, \lambda_{\sigma})$, then we have

$$\begin{aligned}
(G, F)_{L^2(\Gamma_0, \lambda_{\sigma})} &= \int_{\Gamma_0} G(\eta)F(\eta)\,\lambda_{\sigma}(d\eta) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{x^n} G(\{x_{1,} \ldots, x_n\})\,F(\{x_1, \ldots, x_n\})\,d\sigma(x)_1^n \\
&= \sum_{n=0}^{\infty}\left(g^{(n)}, f^{(n)}\right)_{L^2_{\mathrm{sym}}(X^n, n!\sigma^{\otimes n})} \\
&= ((g^n)_{n=0}^{\infty}, (f^n)_{n=0}^{\infty})_{\mathrm{Exp}(L^2(X,\sigma))}.
\end{aligned}$$

This proves that I_{λ} is an unitary isomorphism.

Let us define the following function $e_{\lambda}(f, .)$, $f \in L^0(X)$ (the space of measurable functions over X) on Γ_0 by

$$e_{\lambda}(f, .) : \Gamma_0 \to \mathbb{C},\ \eta \mapsto e_{\lambda}(f, \eta) := \prod_{x \in \eta} f(x)\,.$$

We have that

$$|e_{\lambda}(f, .)|^2_{L^2(\lambda_{\sigma})} = \exp(|f|^2_{L^2(X,\sigma)}).$$

Indeed, applying the definition of λ_{σ} we obtain

$$\begin{aligned}
|e_{\lambda}(f, .)|^2_{L^2(\lambda_{\sigma})} &= \int_{\Gamma_0}|e_{\lambda}(f, \eta)|^2 d\lambda\sigma(\eta) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n}|e_{\lambda}(f, \{x_1, \ldots, x_n\})|^2 \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n}\prod_{i=1}^{n} f^2(x_i)\,d\sigma(x)_1^n \\
&= \exp(|f|^2_{L^2(X,\sigma)}).
\end{aligned}$$

This result is clear from the Lebesgue-Poisson-Fock isomorphism.

The function $e_\lambda(f,.)$ is called the (Lebesgue-Poisson) coherent state corresponding to the one particle function f. The name is justified because $e_\lambda(f,.)$ is the pre-image of the usual coherent state on the Fock space under I_λ, i.e.,

$$I_\lambda^{-1}\left(\left(f^{\otimes n}/n!\right)_{n=0}^{\infty}\right) = e_\lambda(f,.) .$$

Hence we have the following diagram

4.2 The K-transform

There is a useful transform between functions on the Poisson space and the Lebesgue-Poisson space, the so-called K-transform. It is defined by

$$K : L^0_{ls}(\Gamma_0) \to \mathcal{F}L^0(\Gamma), G \mapsto (KG)(\gamma) := \sum_{\eta\subset\gamma, |\eta|<\infty} G(\eta)$$

which is well defined because the sum is finite. There is many properties and applications of this transform to statistical physics but we are interested on its use on the geometry.

We only notice the following property: $f \in B_{bs}(X)$ with $\operatorname{supp} f \subset \Lambda \in \mathcal{B}_c(X)$, then

$$(Ke_\lambda(f,.))(\gamma) = \prod_{x\in\gamma_\Lambda}(1+f(x)) .$$

In fact by definition we have

$$(Ke_\lambda(f,.))(\gamma) = \sum_{\eta\subset\gamma,|\eta|<\infty} e_\lambda(f,\eta) = \sum_{\eta\subset\gamma,|\eta|<\infty}\prod_{x\in\eta} f(x) = \prod_{x\in\gamma_\Lambda}(1+f(x)).$$

5 Geometry on configuration spaces

In this section we describe in details how appear geometry on the configuration space. Since the underlying space has a geometrical structure so we expect that the configurations inherit similar objects, e.g., derivations, gradients, etc. We would like to stress that one difficulty on configurations spaces is the absence of linear structure, therefore most of the objects can not be defined directly as in a linear space.

There is essentially two approaches to introduce geometry on the configuration space. The first one we define the directional derivation of a function on a non flat space. Then, using the directional derivation we could try to find out the tangent space as well as the gradient. Further more, having the gradient and the tangent space as a Hilbert space we define the divergence and so on.

5.1 Lifting of geometry

We start by introducing the natural geometry (or inherited or lifting) on the configuration space. Let us recall that the finite configuration space has a natural geometry. More explicitly, on X^n we have the product structure, hence for example to each point $(x_1, \ldots, x_n) \in X^n$ is attach a tangent space,$T_{(x_1,\ldots,x_n)}X^n \simeq \oplus_{n=1}^n T_{x_n}X$. Therefore by analogy and from the definition of $\Gamma_X^{(n)}$, we have

$$T_\eta \Gamma_X^{(n)} = \bigoplus_{x\in\eta} T_x X,\ \eta \in \Gamma_X^{(n)}$$

If $(v_x)_{x\in\eta}, (\omega_x)_{x\in\eta}$ are two element in $T_\eta\Gamma_X^{(n)}$, then we have

$$\left\langle (v_x)_{x\in\eta}, (\omega_x)_{x\in\eta} \right\rangle_{T_\eta\Gamma_X^{(n)}} = \sum_{x\in\eta} \langle v_x, \omega_x\rangle_{T_xX} = \langle\langle v_\cdot, \omega_\cdot\rangle_{TX}, \eta\rangle.$$

Therefore, we see that the scalar product is nothing but the lifting of the correspondent scaler product of the underlying space.

Let $v \in V_{bs}(X)$ be a smooth vector field on X with bounded support. Then this vector field originate a one-parameter subgroup of diffeomorphism on X. More explicitly: $\psi_t^v : X \to X,\ t \in \mathbb{R}$ such that

$$\begin{cases} \frac{d}{dt}\psi_t^v = v\left(\psi_t^v(x)\right) \\ \psi_0^v(x) = (x) \end{cases}.$$

We have also that $\psi_t^v \circ \psi_s^v = \psi_{t+s}^v$.

We can extend the follow associated to v onto Γ as follows:

$$\psi_t^v : \Gamma \to \Gamma, \gamma \mapsto \psi_t^v(\gamma) = \{\psi_t^v(x), x \in \gamma\}.$$

Then we consider a vector field on $\Gamma_0, \mathrm{Exp}\,(v)$ which coincides on each $\Gamma_X^{(n)}$with the vector field $\oplus_{i=1}^n v$. Denote the corresponding flow by $\psi_t^{\mathrm{Exp}(v)}$. This flows acts naturally on functions $G : \Gamma_0 \to \mathbb{R}$ as

$$(\psi_t^{\mathrm{Exp}(v)})^* G(\eta) := G\left(\psi_t^{\mathrm{Exp}(v)}(\eta)\right).$$

Then the directional derivative along $\mathrm{Exp}\,(v)$ is defined by

$$\left(\nabla^{\Gamma_0}_{\mathrm{Exp}(v)} G\right)(\eta) := \frac{d}{dt} G\left(\psi_t^{\mathrm{Exp}(v)}(\eta)\right)|_{t=0}$$

for $G \in BC^1_{bs}(\Gamma_0)$. We have the following result.

Theorem 1 *Theorem for any $G \in \mathcal{B}\ C^1_{bs}(\Gamma_0)$ the following result holds*

$$K\left(\nabla^{\Gamma_0}_{\mathrm{Exp}(v)} G\right) =: \nabla^{\Gamma}_v KG.$$

Proof. The proof is based on the following fact.

$$\begin{aligned} K\left((\psi_t^{\mathrm{Exp}(v)} * G\right)(\gamma) &= \sum_{\eta\subset\gamma,|\eta|<\infty} G(\{\psi_t^v(x)|\ x \in \eta\}) \\ &= \sum_{\eta\subset\psi_t^v(\gamma),|\eta|\subset\infty} (KG)\left(\psi_t^v(\eta)\right) \\ &= \left(\left(\psi_t^v\right)^* KG\right)(\gamma). \end{aligned}$$

Then the result follows since the sum is finite. ∎

The tangent space $T_\gamma\Gamma, \gamma \in \Gamma$ is defined as the set of all $(v_x)_{x\in\gamma}$with $v_x \in T_xX, \forall x \in \gamma$ and $\sum_{x\in\gamma} \langle v_x, v_x\rangle_{T_xX} < \infty$ equipped with the scalar product

$$\left\langle \left(v_x^1\right)_{x\in\gamma,} \left(v_x^2\right)_{x\in\gamma} \right\rangle_{T_\gamma\Gamma} = \sum_{x\in\gamma} \left\langle v_x^1, v_x^2\right\rangle_{T_xX}.$$

Now we deduce an expression for the gradient in Γ in terms of the gradient in Γ_0. The above theorem gives us that

$$\begin{aligned} \left\langle \mathrm{grad}^\Gamma KG(\gamma), (v(x))_{x\in\gamma}\right\rangle_{T\gamma\Gamma} &= \left(K\left\langle \mathrm{grad}^{\Gamma_0} G, \mathrm{Exp}(v)\right\rangle\right)(\gamma) \\ &= \sum_{\eta\subset\gamma,|\eta|<\infty} \left\langle \mathrm{grad}^{\Gamma_0} G(\eta), (v,x)_{x\in\eta}\right\rangle_{T\eta\Gamma_0} \\ &= \sum_{\eta\subset\gamma,|\eta|<\infty} \sum_{x\in\eta} \left\langle \mathrm{grad}^{\Gamma_0} G(\eta, x), v_x\right\rangle_{T_xX} \\ &= \sum_{x\in\gamma} \sum_{\substack{\eta\subset\gamma,|\eta|<\infty \\ x\in\eta}} \left\langle \mathrm{grad}^{\Gamma_0} G(\eta, x), v_x\right\rangle_{T_xX} \\ &= \sum_{x\in\gamma} \sum_{\substack{\eta\subset\gamma,|\eta|<\infty \\ x\in\eta}} \left\langle \mathrm{grad}^{\Gamma_0} G(\eta, x), v_x\right\rangle_{T_xX}. \end{aligned}$$

This implies that

$$\mathrm{grad}^\Gamma KG(\gamma) = \sum_{\substack{\eta\subset\gamma,|\eta|<\infty \\ x\in\eta}} \mathrm{grad}^{\Gamma_0} G(\eta, x).$$

There is much more objects which can be introduced in a similar form, i.e., divergence, Laplace-Beltrami operator etc. Instead of reproducing here the full collection of such objects we prefer to mention some applications of them. Before we do that we would like to mention what is the situation concerning the other spaces.

Remark 2 *Everything what was done to Γ can be "generalized" to the compound Poisson space with a special subgroup of diffeomorphism. Concerning the marked Poisson space the situation still is not clarified although in the next talk I will give hints in order to clarify it.*

5.2 Direct construction of geometry

There is a direct way to introduce geometry on the configuration space. The starting point is to consider a vector field on X with bounded support, i.e., $v \in \mathrm{Vect}_{bs}(X)$. Again we denote the flow generated by v as ψ_t^v.

Definition 3 *The directional derivative of a function $F : \Gamma \to \mathbb{R}$ in the direction of v is given by the Lie derivative of F with respect to v. Its given by*

$$\left(\nabla_v^\Gamma F\right)(\gamma) = \frac{d}{dt} F\left(\psi_t^v(\gamma)\right)|_{t=0},$$

if the right hand side exists.

This definition applies in a special set of functions, the so-called cylinder functions: $F : \Gamma \to \mathbb{R}$ is cylinder iff admits the following representation

$$F(\gamma) = g_F\left(\langle\varphi_{1,\gamma}\rangle, .., \langle\varphi_n\gamma\rangle\right), \gamma \in \Gamma.$$

$\varphi_i \in C_b^\infty(\mathbb{R}^n)$. In the case $F(\gamma) = g_F(\langle\varphi, \gamma\rangle)$, then we can compute the directional derivative as

$$\begin{aligned}
\left(\nabla_v^\Gamma F\right)(\gamma) &= g_F'\left(\langle\varphi, \gamma\rangle\right) \frac{d}{dt}\left\langle\varphi, \psi_t^v(\gamma)\right\rangle|_{t=0} \\
&= g_F'\left(\langle\varphi, \gamma\rangle\right) \frac{d}{dt}\left\langle\varphi, \psi_t^v(\gamma)\right\rangle|_{t=0} \\
&= g_F'\left(\langle\varphi, \gamma\rangle\right) \frac{d}{dt} \sum_{x\in\gamma} \varphi\left(\psi_t^v(\gamma)\right)|_{t=0} \\
&= g_F'\left(\langle\varphi, \gamma\rangle\right) \sum_{x\in\gamma} \nabla^X \varphi(x) \frac{d}{dt}\psi_t^v(x)|_{t=0} \\
&= g_F'\left(\langle\varphi, \gamma\rangle\right) \sum_{x\in\gamma} \left\langle\nabla_v^X \varphi(x), v(x)\right\rangle_{T_xX} \\
&= \sum_{x\in\gamma} \left\langle g_F'\left(\langle\varphi, \gamma\rangle\right) \nabla^X \varphi(x), v(x)\right\rangle_{T_xX} \\
&= \int_X \left\langle g_F'\left(\langle\varphi, \gamma\rangle\right) \nabla^X \varphi(x), v(x)\right\rangle_{T_xX} \gamma(dx) \\
&= \left\langle\nabla^\Gamma F(\gamma, .), v(.)\right\rangle_{L_2(X,TX;\gamma)}.
\end{aligned}$$

This implies two things: first we see that the gradient is

$$\nabla^{\Gamma} F(\gamma, x) = g'_F(\langle \varphi, \gamma \rangle) \nabla^X \varphi(x).$$

Second the tangent space at $\gamma \in \Gamma$ is

$$L^2(X, TX, \gamma) = T_\gamma \Gamma.$$

Hence if $v^1, v^2 \in T_\gamma \Gamma$ we have

$$\int_X \langle v^1(x), v^2(x) \rangle_{T_x X} \, d\gamma(x) = \sum_{x \in \gamma} \langle v^1(x), v^2(x) \rangle_{T_x X}.$$

and

$$\sum_{x \in \gamma} |v_x|^2_{T_x X} < \infty.$$

This shows that the two approaches we introduced coincide.

5.3 Some applications

Before finish this section I would like to mention some applications of the above introduced geometry.

Since the Poisson measure (as well as compound Poisson) is quasi-invariant with respect to the group of diffeomorphism with bounded support, then it is possible to define a unitary representation of $\mathrm{Diff}_0(X)$ (diffeomorphism with compact support) on the Poisson (respectively, compound Poisson) space $L^2(\pi_\sigma)$ by

$$(V_{\pi_\sigma}(\phi) F)(\gamma) := F(\phi(\gamma)) \sqrt{\frac{d\pi_\sigma(\phi(\gamma))}{d\pi_\sigma(\gamma)}}.$$

Another very interesting applications of this geometry is the integration by parts and its characterization. The gradient defined on functions over Γ, grad^Γ gives us the possibility to introduced also the Dirichlet form on $L^2(\pi_\sigma)$. It allows us to identify the diffusion process corresponding to this form. This identification uses a quite standard technique of Dirichlet forms see e.g., [MR92]. These results also works for compound Poisson measures.

Finally, I would like to mention other applications of the geometry for interacting particle systems (the case of Poisson measure corresponds to the free case). Basically this corresponds to perturb the Poisson measure by a Gibbs factor (locally) and then obtain in a certain sense the limiting measure. For this new measure, arising in this way, all the questions shown for Poisson measure also can be answered positively here. I would like to stop here concerning the details and address to the forthcoming Ph.D. of Tobias Kuna (Bonn) for the details, other applications and historical marks.

6 Generalization to fiber bundles

6.1 Introduction and motivation

Up to now we have shown the various aspects of configuration spaces and in particular the geometry on them. Even concerning geometry we have clarified in details the geometry on the simple configuration space Γ and the space of marked configurations with compound Poisson measure. For a general marked configuration space till now is not clear (or at least not clarified) that we can produce all the machinery introduced before.

The point is the following: in general the marked space is not independent of the position space i.e., every mark $s \in S$ is associated with a position on X, see e.g. the examples presented in the first talk. In the compound Poisson case the marks and the positions are taken to be independent. Another restriction we have used was the fact that the group of diffeomorphisms as being the group of diffeomorphisms only in the position space with the marks unchanged. This framework turns out to be too restrictive in applications and therefore we need a suitable framework for the marked case. Thus in this talk I plan to give the first steps on this direction which allowed us to overcome this problem although we are far of a complete and clear understanding of this situation. Therefore this talk (which is the results that I am still working out) is intend to furnish an independent and general setup in order to cover our marked space and having in mind also examples of applications.

6.2 Diffeomorphisms on marked spaces

If we consider $X \times S$ and $\mathrm{Diff}_0\left(X \times S\right)$, then the measure $\pi_{\sigma\times\tau}$ on $\Gamma_{X\times S}$ is $\mathrm{Diff}_0\left(X \times S\right)$-quasi-invariant, but the positions and marks are independent. On the other hand if we use the group

$$\mathrm{Diff}_0\left(X\right) \times \mathrm{Diff}_0\left(S\right),$$

then it follows that the support of its elements is not compact. To see this we proceed as follows:

$$\begin{aligned}(\varphi, \psi)\left(x, s\right) = \left(x, s\right) &\Leftrightarrow x \in \left(\operatorname{supp}\phi\right)^c, s \in \left(\operatorname{supp}\psi\right)^c \\ &\Leftrightarrow \left(x, s\right) \in \left(\operatorname{supp}\phi\right)^c \times S \cap X \times \left(\operatorname{supp}\psi\right)^c\end{aligned}$$

$$\begin{aligned}&\Rightarrow \left(\operatorname{supp}\left(\phi, \psi\right)\right)^c = \left(\operatorname{supp}\phi\right)^c \times S \cap X \times \left(\operatorname{supp}\psi\right)^c \\ &\Rightarrow \operatorname{supp}\left(\phi, \psi\right) = \operatorname{supp}\phi \times S \cup X \times \operatorname{supp}\psi\end{aligned}$$

which implies that (ϕ, ψ) has no local support. Hence we need another group of diffeomorphism in order to handle this situation.

Let us consider the group $(\text{Diff}_0(X), \circ)$ and $(Map(X, \text{Diff}_0(S), \circ)$ and define the semi-direct product of them as

$$\mathcal{G} = \{(\phi, f) \in \text{Diff}_0(X) \times Map(X, \text{Diff}_0(S)) \,|\, (\phi, f)(\psi, g) = (\phi\psi, f(\psi, g))\},$$

where , $f(\varphi, g)(x)(s) := f(\varphi(x), g(x, s))$. Then $\mathcal{G}$ has the following representation

$$\mathcal{G} = NH,$$

where N is a normal subgroup of $\mathcal{G}$ ($\forall g \in \mathcal{G},\ gNg^{-1} = N$) and H is a subgroup of $\mathcal{G}$. More precisely,

$$\begin{aligned} N &= \{(\phi, id_S) \,| \phi \in \text{Diff}_0(X), id_S(x)(s) = s\} \\ H &= \{(id_X, f) \,| \in Map(X, \text{Diff}_0(S))\}. \end{aligned}$$

It is not hard to check that $\mathcal{G}$ is a group with inverse element given by

$$(\phi, f)^{-1} = \left(\phi^{-1}, f^{-1}\left(\phi^{-1}, .\right)\right).$$

In fact we have

$$\begin{aligned} (\phi, f)\left(\phi^{-1}, f^{-1}\left(\phi^{-1}, .\right)\right) &= \left(\phi\phi^{-1}, f\left(\phi^{-1}, f^{-1}\left(\phi^{-1}, .\right)\right)\right) \\ &= \left(id_X, f \circ f^{-1}\left(\phi^{-1}, .\right)\right) \\ &= \left(id_X, id_S\left(\phi^{-1}, .\right)\right). \end{aligned}$$

Notice that the composition in $Map(X, \text{Diff}_0(S))$ is given by

$$(f \circ g)(x)(s) = f(x, g(x, s)).$$

Given any element $(\phi, f) \in \mathcal{G}$ we can decompose it as

$$(\phi, f) = (\phi, id_S)(id_X, f) \in NH$$

This shows that our choice of the semi-direct product has indeed the representation: $\mathcal{G} = NH$. This gives us the possibility to obtain a morphism between $\text{Diff}_0(X \times S)$ and $\mathcal{G}$. Explicitly we have

$$I : \mathcal{G} \to \text{Diff}_0(X \times S),\ (\phi, f) \mapsto [(x, s) \to (\phi(x), f(x, s))]$$

As a result we have to ask the following conditions for the group of diffeomorphism on the marked Poisson space: $(\phi, f) \in \mathcal{G}$ such that

1. $\phi \in (\text{Diff}_0(X)),\quad f(x, .) \in \text{Diff}_0(S), x \in X.$
2. $f(., s), f^{-1}(., s) \in C^\infty(X), s \in S.$
3. $\exists \Lambda \subset X$ bounded such that $f(x, .) = id_S$, $\forall x \notin \Lambda$.

6.3 Vector fields on the marked space

Having defined the group of diffeomorphism on $X \times S$ we would like to know which kind of vector fields produces these diffeomorphism. To this end suppose given a flow in $\mathcal{G}$, i.e.

$$\phi_t(x,s) = (\phi_t(x), f_t(x,s)).$$

Let V be a vector field on $X \times S$ such that

$$\begin{cases} \frac{d\Phi_t(x,s)}{dt} = V(\Phi_t(x,s)) \\ \Phi_0(x,s) = (x,s) \end{cases}.$$

Since

$$\frac{d\Phi_t(x,s)}{dt} = \begin{pmatrix} \frac{d\phi_t(x)}{dt} \\ \frac{df_t(x,s)}{dt} \end{pmatrix} = \begin{pmatrix} V^1(\Phi_t(x,s)) \\ V^2(\Phi_t(x,s)) \end{pmatrix}$$

it follows that

$$V(x,s) = \begin{pmatrix} V^1(x) \\ V^2(x,s) \end{pmatrix}.$$

Therefore the class of vectors fields on $X \times S$ should be such that:

$$\operatorname{Vect}(X \times S) = \{V = (V^1, V^2) | \frac{d}{ds} V^1 = 0, \ \operatorname{supp} V^1, \operatorname{supp}_S V^2 \text{ bounded}\}.$$

Conversely, given a vector field $V(s,x) = \binom{V^1(x)}{V^2(x,s)}$, then the follow associated to V ϕ_t has the form

$$\Phi_t^V(x,s) = (\phi_t(x), f_t(x,s)).$$

Indeed we have

$$\frac{d}{dt}\Phi_t^V(x,s) = V\left(\Phi_t^V(x,s)\right)$$

$$\Leftrightarrow \begin{cases} \frac{d\Phi_t^1(x,s)}{dt} = V^1(\Phi^1(x,s)) \\ \frac{d\Phi_t^2(x,s)}{dt} = V^2(\Phi^1(x,s), \Phi_t^2(x,s)) \\ \Phi_0^V(x,s) = (x,s) \end{cases}$$

which implies that

$$\Phi_t^1(x,s) = \phi_t(x)$$
$$\Phi_t^2(x,s) = f_t(x,s)$$

but this means that $\Phi_t^V(x,s)$ belongs to $\mathcal{G}$.

6.4 Vector fields on fiber bundle and associated flows

Till now we have introduced diffeomorphism and vector fields on the Cartesian product between the position and marked space. Nevertheless we would like to introduce a generalization of it. This is the so-called fiber bundle. A fiber bundle is an object which generalizes the well known notion of Cartesian product. More over every fiber bundle locally is represented by a Cartesian product, see figure in the next page.

Thus let us fix the notation for this framework. M is a fiber bundle, X the base space, and S a typical fiber.

Denote by $\pi : M \to X, m \to \pi(m)$ the projection. For each $m \in M$ the tangent space $T_m M \simeq T_x X \oplus T_s S$, where $m = (x, s)$. Hence we will define the vectors fields over M as

$$\mathrm{Vect}(M) := \{V | \forall m \in M \; d\pi_m (V(m)) = V^1(\pi(m))\}.$$

Locally any vector field over M may be written as

$$V(m) = \begin{pmatrix} V^1(m) \\ V^2(m) \end{pmatrix}.$$

For our class of vectors fields we have additionally information, namely:

$$d\pi_m = \begin{pmatrix} I_{n\times n} & 0_{n\times N} \end{pmatrix}$$

$$d\pi(x,s) \begin{pmatrix} V^1(x,s) \\ V^2(x,s) \end{pmatrix} = V^1(x,s)$$

$$\Leftrightarrow V^1(\pi(x,s)) = V^1(x) = V^1(x,s).$$

Therefore the vector fields over M locally are of the form

$$V(x,s) = \begin{pmatrix} V^1(x) \\ V^2(x,s) \end{pmatrix}.$$

Let $V \in \mathrm{Vect}(M)$ be given and ψ_t^V the corresponding flow. Then $\pi(\psi_t^V)$ should be a flow in X. To see that in reality this is well defined we proceed like this

$$\begin{aligned} \frac{d}{dt}\pi\left(\psi_t^V(m)\right) &= d\pi\left(\psi_t^V(m)\right)\frac{d}{dt}\psi_t^V(m) \\ &= d\pi\left(\psi_t^V(m)\right)V\left(\psi_t^V(m)\right) \\ &= V^1\left(\pi\left(\psi_t^V(m)\right)\right). \end{aligned}$$

This means

$$\frac{d}{dt}\pi\left(\psi_t^V(m)\right) = V^1\left(\pi\left(\psi_t^V(m)\right)\right)$$

and therefore

$$\pi\left(\psi_t^V(m)\right)=\phi_t^{V^1}(\pi(m)).$$

Conversely, if ψ_t^V is a flow in M such that

$$d\pi_m(V(m))=V^1(\pi(m)).$$

But this can be obtained making the following observation

$$\frac{d}{dt}\pi\left(\psi_t^V(m)\right)=\frac{d}{dt}\phi_t^{V^1}(\pi(m))$$

$$\Leftrightarrow d\pi\left(\psi_t^V(m)\right)\frac{d}{dt}\psi_t^V(m)=V^1\left(\phi_t^{V^1}(\pi(m))\right).$$

The results follows when $t=0$.

Now we would like to study the support of a vector field in our class. Hence let $V\in\mathrm{Vect}(M)$ be given and suppose that

$$\operatorname{supp}V\subset\pi^{-1}(\Lambda),\Lambda\in\mathcal{B}_c(X).$$

Then if $m\in\pi^{-1}(\Lambda)^c$ we have $V(m)=0$. On the other hand $\pi^{-1}(\Lambda)^c=\pi^{-1}(\Lambda^c)$ which implies that $\pi(m)\in\Lambda^c$. Moreover, since $d\pi(m)(V(m))=V^1(\pi(m))=0$. This gives us that $\operatorname{supp}V^1\subset\Lambda$.

Remark 4 *Let us stress that our class of vectors fields over M, $\mathrm{Vect}(M)$ is really necessary. In fact, we know that*

$$T_\eta\Omega_0=\bigoplus_{m\in\eta}T_mM$$

and if $V_\eta\in T_\eta\Omega_0$ is given by $V_\eta=\binom{V^1(m)}{V^2(m)}_{m\in\eta}$, then we define a vector field in $\mathrm{Vect}(M)$ by

$$\tilde{V}(m)=\begin{pmatrix}\tilde{V}^1(x)\\ \tilde{V}^2(x,s)\end{pmatrix}:=\begin{pmatrix}V^1(x,s)\\ V^2(x,s)\end{pmatrix}.$$

That $\tilde{V}$ is well defined is a consequence of the marked property, i.e., for any $x\in X$ there exists only one $s\in S$ such that $(x,s)\in\eta$.

6.5 Volume element on fiber bundle

Before doing something else on the configuration space over the fiber bundle Ω_M (or $\Omega_{0,M}$) we need an intensity measure on M. To this end let us recall some standard concepts from differential geometry.

Let $\mathrm{Alt}_k(M)$ be the set if all alternating forms on M, i.e.,

$$\mathrm{Alt}_k(M) := \{w : M \to \mathrm{Alt}_k(TM), m \mapsto w(m) \in \mathrm{Alt}_k(T_m M)\},$$

where for each $m \in M$, $w(m) : T_m M \times \cdots \times T_m M \to \mathbb{R}$ is k-linear and anti-symmetric form. If $(\partial_i|_m)_{i=1}^n$ is a basis for $T_m M$, then its dual basis is denoted by $(dm^i)_{i=1}^n$. Hence a basis for $\mathrm{Alt}_k(M)$ is given by

$$(dm^{i_1} \wedge \cdots \wedge dm^{i_k})_{1 \le i_1 < \ldots < i_k \le n},$$

where

$$dm^{i_1} \wedge \cdots \wedge dm^{i_k} := \frac{1}{n!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, dm^{i_{\sigma_1}} \otimes \cdots \otimes dm^{i_{\sigma_k}}.$$

For any $w \in \mathrm{Alt}_n(M)$ we can represent it locally by

$$w = dm^1 \wedge \cdots \wedge dm^n.$$

Suppose that $(d\tilde{m}^i)_{i=1}^n$ is another coordinate system at m. Then dm^i may be expressed as

$$dm^i = \sum_{j=1}^n \frac{\partial m^i}{\partial \tilde{m}^j} d\tilde{m}^j$$

and, therefore w has the following transformation rule

$$\begin{aligned} w &= \left(\sum_{j_1=1}^n \frac{\partial m^1}{\partial \tilde{m}^{j_1}} d\tilde{m}^{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_n=1}^n \frac{\partial m^n}{\partial \tilde{m}^{j_n}} d\tilde{m}^{j_n} \right) \\ &= \sum_{j_1, \cdots j_n = 1}^n \frac{\partial m^1}{\partial \tilde{m}^{j_1}} \cdots \frac{\partial m^n}{\partial \tilde{m}^{j_n}} d\tilde{m}^{j_1} \wedge \cdots \wedge d\tilde{m}^{j_n} \\ &= \left(\sum_{\sigma \in S_n} \operatorname{sgn} \sigma \frac{\partial m^1}{\partial \tilde{m}^{j_{\sigma_1}}} \cdots \frac{\partial m^n}{\partial \tilde{m}^{j_{\sigma_n}}} \right) d\tilde{m}^1 \wedge \cdots \wedge d\tilde{m}^n. \\ &\mathrm{Det} \left(\left(\frac{\partial m^i}{\partial \tilde{m}^j} \right)_{i,j=1}^n \right) d\tilde{m}^1 \wedge \cdots \wedge d\tilde{m}^n. \end{aligned}$$

The above calculations shows that the properties of forms gives the right transformations rules for the volume element. Thus, as a volume element on M we will take an orientation w_0 which is differentiable n-form different from zero.

Definition 5 *The intensity measure on M σ is defined as*

$$\sigma = \rho w_0, \quad \rho > 0.$$

6.6 Quasi-invariant measures

Proposition 6 *The intensity measure σ is absolutely continuous with respect to Diff$_0(M)$ and*

$$\rho^\sigma_\phi(m) = \sum_{k=1}^{\infty} \mathbb{1}_{\phi(M_k)}(m) \left| \frac{\rho_{h_k}(\phi^{-1}(m))}{\rho_{h_k\circ\phi^{-1}}((m))} \right| = \frac{d\sigma(\phi^{-1}(m))}{d\sigma(m)}$$

is the Radon-Nikodym density of σ for any $\phi \in$Diff$_0(M)$.

Note that ρ^σ_ϕ is finite because ϕ has compact support and therefore we can take $\operatorname{supp}\rho_{h_k}$ and $\inf \rho_{h_k\circ\phi^{-1}}$.

Proof. Here we only present a sketch of the proof. Take a covering $(M_k)_{k=1}^{\infty}$ of measurable sets from M such that each M_k is inside of some domain of a chart. Locally the intensity measure σ is written as

$$d\sigma(m) = \rho_{h_k}(m) dm^1_k \wedge \cdots \wedge dm^n_k.$$

then the result follows by definition and the change of variable formula. ■

Lemma 7 *Let $\phi \in$Diff$_0(M)$ be given and suppose that* $\operatorname{supp}\phi \subset \Lambda$ *with $\Lambda \in \mathcal{B}_c(X)$. Then*

$$\int_M (1 - \rho^\sigma_\phi)(m) d\sigma(m) = 0.$$

Proof. The proof is a straightforward calculations of the integral.

$$\begin{aligned}\int_M (1 - \rho^\sigma_\phi)(m) d\sigma(m) &= \int_\Lambda (1 - \rho^\sigma_\phi)(m) d\sigma(m) \\ &= \sigma(\Lambda) - \int_\Lambda \rho^\sigma_\phi(m) d\sigma(m) \\ &= \sigma(\Lambda) - \sigma(\phi^{-1}(\Lambda)).\end{aligned}$$

The result follows taking into account that $\phi^{-1}(\Lambda) = \Lambda$. ■

Theorem 8 *The Poisson measure π_σ is quasi-invariant with respect to Diff$_0(M)$ and for any $\phi \in$Diff$_0(M)$ we have*

$$\rho^{\pi_\sigma}_\phi(\omega) = \frac{d\pi_\sigma(\phi^{-1}(m))}{d\pi_\sigma(m)} = \prod_{m\in\omega} \rho^\sigma_\phi(m).$$

Proof. ´The Poisson measure is given on Ω by its Laplace transform by

$$\begin{aligned}\int_\Omega e^{\langle\varphi,\omega\rangle}\rho_\phi^{\pi_\sigma}(\omega)d\pi_\sigma(\omega) &= \int_\Omega e^{\langle\varphi+\log\rho_\phi^\sigma,\omega\rangle}d\pi_\sigma(\omega)\\ &= \exp\left(\int_M (e^{\varphi(m)+\log\rho_\phi^\sigma(m)}-1)d\sigma(m)\right)\exp\left(\int_M(1-\rho_\phi^\sigma)d\sigma\right)\\ &= \exp\left(\int_M (e^{\varphi(m)}-1)\rho_\phi^\sigma(m)d\sigma(m)\right)\\ &= \int_\Omega e^{\langle\varphi,\omega\rangle}d\pi_{\sigma\circ\phi^{-1}}(\omega).\end{aligned}$$

Then the result follows from the fact that $\phi^*\pi_\sigma = \pi_{\phi^*\sigma}$. ∎

7 Supplementary Bibliography

In this short note I tried to collect the most relevant aspects of the geometry of configuration spaces. Of course that a better understanding of all this material should be done with the reading of the original works on that. Hence I would like to present here the essential references on geometry on configuration spaces.

- About Poisson measure on configuration spaces there is many papers but I would like to stress the following: [AKR98a], [AKR98b], [KSS98], [Oba87], and references therein.
- Marked Poisson measures are used in statistical physics and I refer to [KKS98], [Kun99], and references therein.
- The geometry on configuration space as it is described in these notes was first introduced in [AKR98a] and after a series of generalizations and applications born on that basis. Here I mention few of them: [AKR98b], [KSS98], [Röc98], [Sil98], [Kun99].
- Finally let me mention that most of this literature is on-line via WWW, below I list some of them where everybody can find them and print at home:
  - www.uma.pt/ccm/ccm.html
  - www.physik.uni-bielefeld.de/bibos/start.html
  - www.mathematik.uni-bielefeld.de/fakultaet/content.html
  - xxx.lanl.gov
  - www.ma.utexas.edu/mp_arc/index.html
  - www.emis.de
  - www.sissa.it

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