On the stochastic transport equation of convolution type

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ABSTRACT. In this paper we study the stochastic transport equation of convolution type. For general initial condition and its coefficients we give an explicit solution which is a well defined generalized stochastic process in a suitable distribution space. Under certain assumptions on the coefficients we also write the obtained solution as a convergent series of integrals.

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1. Introduction

The aim of this work is to study the solution of the following Cauchy problem corresponding to the stochastic equation modeling the transport of a substance which is dispersing in a moving medium

$$\begin{cases} \frac{\partial}{\partial t}X(t,x,\omega) &= \frac{1}{2}\sigma^2\Delta X(t,x,\omega) + \nabla X(t,x,\omega) * V(t,x,\omega) \\ & +K(t,x,\omega) * X(t,x,\omega) + g(t,x,\omega) \\ X(0,x,\omega) &= f(x,\omega), \end{cases} \tag{1}$$

where $X(t,x,\omega)$ is the concentration of the substance at time $t \in [0,\infty)$ and at the point $x \in \mathbb{R}^r$, $r \in \mathbb{N}$, $\frac{1}{2}\sigma^2 > 0$ is the dispersion coefficient (constant), Δ (resp. ∇) is the Laplacian (resp. the gradient) with respect to the spatial variable x, $\omega = (\omega_1,\ldots,\omega_d)$ is the stochastic vector variable in the tempered Schwartz distribution space $S'_d := S'(\mathbb{R},\mathbb{R}^d)$ with the standard Gaussian measure, $d \in \mathbb{N}$, $*$ is the convolution product between generalized functions (see Subsection 2.2 below), f is the initial concentration of the substance, $V = (V_1,\ldots,V_r)$ is the vector velocity of the medium, K is the relative leakage rate and g is the source rate of the substance.

The Cauchy problem (1) was analyzed by many authors from different point of view, see e.g., [**7**], [**14**] and references therein for more details and historical remarks.

Recently Ouerdiane et al. [**12**] obtained the solution of (1) in the particular case when $V = g = 0$ in terms of the convolution exponential as a well defined generalized function in a suitable distribution space. In addition for the case when K is a positive family of generalized stochastic processes the corresponding solution is given as a limit of integrals, see [**13**] for details and also [**1**], [**2**] for related topics.

The starting point is the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}',\gamma) \supset \mathcal{F}_\theta(\mathcal{N}'),$$

where $\mathcal{N}'$ is the dual of a complex nuclear Fréchet space $\mathcal{N}$, θ is a Young function, γ is the usual Gaussian measure on $\mathcal{M}'$ which corresponds to the real part of $\mathcal{N}'$. The test function space $\mathcal{F}_\theta(\mathcal{N}')$ is defined as the space of all holomorphic functions on $\mathcal{N}'$ with an exponential growth condition of order θ. The generalized function space $\mathcal{F}'_\theta(\mathcal{N}')$ represents the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$. In the following we will choose the nuclear space $\mathcal{N} = (S_d \times \mathbb{R}^r)_\mathbb{C}$, the complexification of the real nuclear space $S_d \times \mathbb{R}^r$, which is adapted to our situation. We would like to stress that all differential operators involved in equation (1) are interpreted in the generalized sense.

Using the Laplace transform $\mathcal{L}$ we may define the convolution of two generalized functions $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi \cdot \mathcal{L}\Psi)$$

which allows us to introduce the convolution exponential of Φ denoted by $\exp^* \Phi$ as an element in $\mathcal{F}'_\varphi(\mathcal{N}')$, where the Young function $\varphi = (e^{\theta^*})^*$ and

$$\theta^*(x) := \sup_{y\geq 0}(yx - \theta(y)) \tag{2}$$

denotes the polar function associated to θ, see e.g., [**8**].

For positive generalized stochastic process $\Phi = (\Phi(t))_{t\geq 0}$ there exists a family of Radon measures $\mu = (\mu_t)_{t\geq 0}$ (see e.g., [**11**]) on $\mathcal{M}'$ which represents V such that the Fourier transform of μ_t, $t \geq 0$ is given by

$$\langle\langle \Phi(t), \exp(i\xi)\rangle\rangle = \hat{\mu}_t(\xi) = \int_{\mathcal{M}'} \exp(i\langle y,\xi\rangle) d\mu_t(y),$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ and corresponds to the extension of the inner product of $L^2(\mathcal{M}',\gamma)$.

The paper is organized as follows: in Section 2 we review some of the terminology and theory necessary for the stochastic model and its calculus. In particular, we define the Laplace transform, the convolution product on the space of generalized functions and establish some of its properties, i.e., the characterization of generalized functions and convolution exponential. These are the contents of Subsections 2.1 and 2.2. In Section 3 we give a general scheme for solving the convolution type equations as e.g., the Cauchy problem (1). Finally in Section 4 we write the solution of (1) as a limit of integrals.

2. Preliminaries

2.1. Test and generalized functions spaces . In this section we introduce the framework and tools which is suited for the applications which are intended in later sections. For a general account on the distributions spaces and convolution calculus presented here the interested reader is referred to [**5**], [**3**] and the references quoted there. We start with a separable real Hilbert $\mathcal{H}$ space which we choose to be $\mathcal{H} = L^2(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. More precisely, if $(f,x) = ((f_1,\ldots,f_d),(x_1,\ldots,x_r)) \in \mathcal{H}$, then

$$|(f,x)|^2 := \sum_{i=1}^d \int_\mathbb{R} f_i^2(u)du + \sum_{i=1}^r x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r = \mathcal{M}. \tag{3}$$

The pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle(\omega,x),(\xi,y)\rangle := (\omega,\xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x,y)_{\mathbb{R}^r}$, $(\omega,x)\in\mathcal{M}'$ and $(\xi,y)\in\mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_n(\mathbb{R},\mathbb{R}^d)\times\mathbb{R}^r = \bigcap_{n=0}^{\infty}\mathcal{M}_n,$$

where $S_n(\mathbb{R},\mathbb{R}^d)\times\mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|_{\mathbb{R}^r}^2$, see e.g., [**6**] or [**4**] and references therein. We will consider the complexification of the triple (3) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{4}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos's theorem via its characteristic functional: for every $(\xi,p)\in\mathcal{M}$

$$C_\gamma(\xi,p) = \int_{\mathcal{M}'} \exp(i\langle(\omega,x),(\xi,p)\rangle)d\gamma((\omega,x)) = \exp(-\frac{1}{2}(|\xi|^2 + |p|^2)).$$

In order to solve the Cauchy problem (1) we need to introduce an appropriate space of generalized functions. We borrow this construction from [**9**]. Let $\theta = (\theta_1,\theta_2):\mathbb{R}_+^2\to\mathbb{R}$, $(t_1,t_2)\mapsto\theta_1(t_1)+\theta_2(t_2)$ where θ_1,θ_2 are two Young functions, i.e., $\theta_i:\mathbb{R}_+\to\mathbb{R}_+$ continuous convex strictly increasing function and

$$\lim_{t\to\infty}\frac{\theta_i(t)}{t} = \infty, \quad \theta_i(0)=0, \quad i=1,2.$$

For every pair $m=(m_1,m_2)$ with $m_1,m_2\in]0,\infty[$, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n\in\mathbb{N}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f:\mathcal{N}_{-n}\to\mathbb{C},\ \text{entire},\ \|f\|_{\theta,m,n} = \sup_{z\in\mathcal{N}_{-n}}|f(z)|\exp(-\theta(m|z|_{-n}))<\infty\},$$

where for each $z=(\omega,x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Now we consider as test function space the space of entire functions on $\mathcal{N}'$ of (θ_1,θ_2)-exponential growth and minimal type

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0}\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to construct the triple of the complex Hilbert space $L^2(\mathcal{M}',\gamma)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need another condition on the pair of Young functions (θ_1,θ_2). Namely,

$$\lim_{t\to\infty}\frac{\theta_i(t)}{t^2} < \infty, \quad i=1,2. \tag{5}$$

This is enough to obtain the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}',\gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{6}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}',\gamma)$ endowed with the inductive limit topology.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. First we define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ the exponential function $\exp((\xi, p))$ is a well defined element in $\mathcal{F}_\theta(\mathcal{N}')$, see [**5**]. The Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ is defined by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\langle \Phi, \exp((\xi, p))\rangle\rangle. \tag{7}$$

We are ready to state to characterization theorem, see e.g., [**5**] and [**12**].

THEOREM 2.1. *The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g,\ |g(\xi, p)| \leq k \exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)),\ (\xi, p) \in \mathcal{N}_n.$$

2.2. The Convolution Product $*$. It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator. Let us define the convolution between a generalized and a test function. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi\rangle\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega + \eta, x + y).$$

It is not hard the see that $\Phi * \varphi \in \mathcal{F}_\theta(\mathcal{N}')$. The convolution product is given in terms of the dual pairing as $(\Phi * \varphi)(0, 0) = \langle\langle \Phi, \varphi\rangle\rangle$ for any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given. Then $\Phi * \Psi$ is defined as

$$\langle\langle \Phi * \Psi, \varphi\rangle\rangle := \langle\langle \Phi, \Psi * \varphi\rangle\rangle,\ \forall\varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{8}$$

This definition of convolution product for generalized functions will be used on Section 3 in order to write the solution of the stochastic heat equation given in (1). We have the following equality, see [**12**], Proposition 3.3:

$$\Phi * \exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p) \exp((\xi, p)),\ (\xi, p) \in \mathcal{N}.$$

As a consequence of the above equality and the definition (8) we obtain that

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi,\ \Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}') \tag{9}$$

which says that the Laplace transform maps the convolution product in $\mathcal{F}'_\theta(\mathcal{N}')$ into the usual pointwise product in the algebra of functions $\mathcal{G}_{\theta^*}(\mathcal{N})$. Therefore we may use Theorem 2.1 to define convolution product between two generalized functions as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi\mathcal{L}\Psi).$$

Relation (9) allows us to define the convolution exponential of a generalized function. In fact, for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we may easily check that $\exp(\mathcal{L}\Phi) \in \mathcal{G}_{e^{\theta^*}}(\mathcal{N})$. Using the inverse Laplace transform and the fact that any Young function θ verify the property $(\theta^*)^* = \theta$ we obtain that $\mathcal{L}^{-1}(\mathcal{G}_{e^{\theta^*}}(\mathcal{N})) = \mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$. Now we give the definition of the convolution exponential of $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$, denoted by $\exp^* \Phi$

$$\exp^* \Phi := \mathcal{L}^{-1}(\exp(\mathcal{L}\Phi)). \tag{10}$$

Notice that $\exp^* \Phi$ is well defined element in $\mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$ and therefore the distribution $\exp^* \Phi$ is given in terms of a convergent series

$$\exp^* \Phi = \delta_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi^{*n}, \tag{11}$$

where Φ^{*n} is the convolution of Φ with itself n times, $\Phi^{*0} := \delta_0$ by convention with δ_0 denoting the Dirac distribution at 0. The following property follows easily from (10) and (9): if $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ then

$$\exp^* \Phi * \exp^* \Psi = \exp^*(\Phi * \Psi). \tag{12}$$

3. Applications to the stochastic transport equation

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi(t),\ t \geq 0\} \subset \mathcal{F}'_\theta(\mathcal{N}')$. The process $\Phi(t)$ is said to be continuous if the map $t \mapsto \Phi(t)$ is continuous. In order to introduce generalized stochastic integrals, we use the characterization theorem for sequences of generalized functions, see [**10**], Theorem 3. For a given continuous generalized stochastic process $(X(t))_{t\geq 0}$ we define the stochastic generalized process

$$Y(t,x,\omega) = \int_0^t X(s,x,\omega) ds \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X(s,x,\omega)ds\right)(\xi,p) := \int_0^t \mathcal{L}X(s,p,\xi)ds. \tag{13}$$

The process $Y(t,x,\omega)$ is differentiable and we have $\frac{\partial}{\partial t}Y(t,x,\omega) = X(t,x,\omega)$. The details of the proof can be seen in [**12**], Proposition 4.11.

The results established up to now may be applied to a wide class of SPDE's of convolution type. The general procedure is the following:
Step 1: Assume that the functions involved in the SPDE can be modelled as some convolution functional and all products involved are interpreted as convolutions products.
Step 2: Apply the Laplace transform $\mathcal{L}$ to the SPDE. This produce a deterministic differential equation (with usual products) with the unknown $t \mapsto \hat{X}(t,p,\xi)$ function where $(p,\xi) \in \mathcal{N}$.
Step 3: Solve this deterministic differential equation and then by the characterization Theorem 2.1 $\hat{X}(p,\xi)$ is indeed the Laplace transform of an element $X(\cdot,\cdot) \in \mathcal{F}'_\beta(\mathcal{N}')$ for a suitable choice of β which then solves the original equation.

Let us apply this scheme to solve the Cauchy problem in (1). We recall again this problem for the reader convenience. Let f be a given generalized function in $\mathcal{F}'_\theta(\mathcal{N}')$ and V, K and g $\mathcal{F}'_\theta(\mathcal{N}')$-valued continuous generalized stochastic process. Consider the following stochastic differential equation with initial condition f

$$\begin{cases} \frac{\partial}{\partial t}X(t,x,\omega) & = \frac{1}{2}\sigma^2 \Delta X(t,x,\omega) + \nabla X(t,x,\omega) * V(t,x,\omega) \\ & +K(t,x,\omega) * X(t,x,\omega) + g(t,x,\omega) \\ X(0,x,\omega) & = f(x,\omega), \end{cases} \tag{14}$$

To solve this SPDE we apply the Laplace transform to (14) and obtain

(15)

$$\begin{cases} \frac{\partial}{\partial t}\hat{X}(t,p,\xi) &= [\frac{1}{2}\sigma^2 p^2 + \sum_{i=1}^r p_i \hat{V}_i(t,p,\xi) + \hat{K}(t,p,\xi)]\hat{X}(t,p,\xi) + \hat{g}(t,p,\xi) \\ \hat{X}(0,p,\xi) &= \hat{f}(p,\xi), \end{cases}$$

The solution of (15) is given as (using the method of variations of constants)

(16)

$$\hat{X}_t(\xi,p) = \hat{f}(p,\xi)\exp\left(\frac{1}{2}\sigma^2 p^2 t + \sum_{i=1}^r p_i \int_0^t \hat{V}_i(s,p,\xi)ds + \int_0^t \hat{K}(s,p,\xi)ds\right) + \int_0^t \hat{g}(s,p,\xi)\exp\left(\frac{1}{2}\sigma^2 p^2 (t-s) + \sum_{i=1}^r p_i \int_s^t \hat{V}_i(u,p,\xi)du + \int_s^t \hat{K}(u,p,\xi)du\right) ds.$$

Now the solution of the system (14) is given using (13), (12) and (10) and the characterization theorem, Theorem 2.1. We give it on the next proposition.

THEOREM 3.1. (1) *The Cauchy problem (1) has an unique solution $X(t)$ which is a generalized $\mathcal{F}'_\beta(\mathcal{N}')$-valued stochastic process, where the Young function β is given by $\beta = (e^{\theta^*})^*$. Moreover, the solution $X(t)$ is given explicitly by*

(17)

$$X(t,\omega,x) = f(x,\omega) * \gamma_{\sigma^2 t} * \exp^*\left(\int_0^t [\mathrm{div}V(s,x,\omega) + K(s,x,\omega)]ds\right) + \int_0^t \left[g(s,x,\omega) * \gamma_{\sigma^2(t-s)} * \exp^*\left(\int_s^t [\mathrm{div}V(u,x,\omega) + K(u,x,\omega)]du\right)\right] ds,$$

where $\gamma_{\sigma^2 t}$ is Gaussian measure on $\mathbb{R}^r$ with variance $\sigma^2 t$ and $\mathrm{div}V = \sum_{i=1}^r \frac{\partial V}{\partial x_i}$.

4. The solution of the transport equation as limit of integrals

In this section we will write the solution of the Cauchy problem (1) as a limit of convergent series of integrals. In general, if we suppose that $W = (W(t))_{t\geq 0}$ is a positive generalized stochastic process (i.e., $\forall t \geq 0$ $\langle\langle W(t),\varphi\rangle\rangle \geq 0$ for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ with $\varphi(x+i0) \geq 0$ $\forall x \in \mathcal{M}$) represented by the family of Radon measures $(\mu_t)_{t\geq 0}$, then for any $t \geq 0$

$$\langle\langle W(t),\varphi\rangle\rangle = \int_{\mathcal{M}'} \varphi(y)d\mu_t(y), \quad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

Moreover the measure μ_t verify the following integrability condition: for any $t \geq 0$ there exists $n \in \mathbb{N}$ and $m > 0$ with $\mu_t(\mathcal{M}_{-n}) = 1$ such that

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n}))d\mu_t(y) < \infty. \tag{18}$$

For each Radon measure μ on $\mathcal{M}'$ verifying (18) and all $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and $u = (x,\omega) \in \mathcal{N}' = \mathcal{M}' + i\mathcal{M}'$ we have the following equality, see [**13**] Lemma 3.2

$$((\exp^* \mu) * \varphi)(u) = \varphi(u) + \sum_{n=1}^\infty \frac{1}{n!}\int_{(\mathcal{M}')^n} \varphi(u+y_1+\ldots+y_n)d\mu(y_1)\ldots d\mu(y_n). \tag{19}$$

This equality may be generalized when φ is replaced by a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$. In fact, we have

$$(\exp^* \mu) * \Phi = \Phi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n), \tag{20}$$

where for every $n = 1, 2, \ldots$ the distribution $\int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n)$ is defined for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ as

$$\begin{aligned}
&\left\langle\!\!\left\langle \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n), \varphi \right\rangle\!\!\right\rangle \\
&= \int_{(\mathcal{M}')^n} \langle\!\langle \tau^*_{-y_1-\ldots-y_n} \Phi, \varphi \rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n) \\
&= \int_{(\mathcal{M}')^n} \langle\!\langle \Phi, \tau_{-y_1-\ldots-y_n} \varphi \rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n) \\
&= \int_{(\mathcal{M}')^n} (\Phi * \varphi)(y_1 + \ldots + y_n) d\mu(y_1)\ldots d\mu(y_n).
\end{aligned}$$

For the details of the proof see [**13**] Lemma 3.6.

Moreover if $(W(s))_{s\geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ be a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s\geq 0}$, then for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have

$$\left\langle\!\!\left\langle \int_0^t W(s) ds, \varphi \right\rangle\!\!\right\rangle = \int_0^t \left(\int_{\mathcal{M}'} \varphi(y) d\mu_s(y) \right) ds, \tag{21}$$

and consequently

$$\left\langle\!\!\left\langle \exp^* \left(\int_0^t W(s) ds \right), \varphi \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \exp^* \left(\int_0^t \mu_s ds \right), \varphi \right\rangle\!\!\right\rangle. \tag{22}$$

In fact equality (21) is nothing but the definition (13) with $\varphi = \exp((\xi, p))$. Therefore by a limit procedure we get the required result (21) for general test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$. To prove equality (22) we proceed in two steps: first we notice that for every $s \geq 0$ $W(s) * W(s)$ is represented by $\mu_s * \mu_s$. Then iterating this process we obtain

$$\langle\!\langle \exp^* W(s), \varphi \rangle\!\rangle = \langle\!\langle \exp^* \mu_s, \varphi \rangle\!\rangle. \tag{23}$$

Then equality (22) is a consequence of (21) and (23).

Combining (22) and (19) with $(W(s))_{s\geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s\geq 0}$ and any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$, $u \in \mathcal{N}'$ we have

$$\begin{aligned}
&\left(\exp^* \left(\int_0^t W(s) ds \right) * \varphi \right)(u) \\
&= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \varphi(u + y_1 + \ldots + y_n) d\mu_{s_1}(y_1)\ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n.
\end{aligned} \tag{24}$$

If instead of (19) we use (20) then the equality (24) reads as

$$\exp^* \left(\left(\int_0^t W(s) ds \right) * \Psi \right) = \Psi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Psi d\mu_{s_1}(y_1)\ldots d\mu_{s_n}(y_n) ds_1 \cdots ds_n, \tag{25}$$

for every generalized function $\Psi \in \mathcal{F}'_\theta(\mathcal{N}')$.

We are now ready to write the solution (17) of the Cauchy problem (1) as a convergent series of integrals. We will apply (25) for a suitable choice of $(W(s))_{s\geq 0}$ and Ψ.

THEOREM 4.1. *Let V, K be such that $(\mathrm{div}V(s) + K(s))_{s\geq 0}$ is a positive generalized stochastic process represented by the family of Radon measures $(\mu_s)_{<\geq 0}$ on $\mathcal{M}'$ which verify the integrability condition (18). If the initial condition f and the source rate of the substance g are generalized functions in $\mathcal{F}'_\theta(\mathcal{N}')$, then the solution of the Cauchy problem (1) is given by*

$$X(t,x) = \Psi(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Psi(t) d\mu_{s_1}(y_1) \ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n$$
$$+ \int_0^t \left(\Phi_s(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[s,t]^n} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi_s(t) d\mu_{u_1}(y_1) \ldots d\mu_{u_n}(y_n) du_1 \ldots du_n \right) ds,$$

where $\Psi(t) = f * \gamma_{\sigma^2 t} \otimes \delta_0$ and $\Phi_s(t) = g(t) * \gamma_{\sigma^2(t-s)} \otimes \delta_0$ and δ_0 is the Dirac measure on S'_d.

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