POISSON NOISE AND THE DYNAMICS OF INFINITE PARTICLE SYSTEMS

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We introduce Poisson analysis as a means to study processes of particle configurations in the continuum and present the free Kawasaki dynamics as illustration..

1. Poisson versus Gaussian White Noise - Setting the stage

Gaussian white noise is most concisely characterized by the Fourier transform of its (probability) measure. Given

$$C(f) = \exp\left(-\frac{1}{2}\int f^2(x)\,dx\right),$$

the Bochner-Minlos theorem, - see e.g. [6] - ensures the existence of a probability measure μ on distribution space such that

$$C(f) = \int_{S^*(R)} d\mu(\omega)\exp\left(i\int \omega(x)\,f(x)\,dx\right)$$

for test functions f. The L^2space w.r. to this measure

$$\left(L^2\right) \equiv L^2\left(S^*(R), d\mu\right)$$

is then the basis of Gaussian white noise analysis.

In the Poisson case we could proceed similarly, starting from the characteristic function

$$C_\pi(f) = \exp\left(\int_{R^d}\left(e^{if(x)}-1\right)dx\right) = \int_{D^*(R)} d\pi(\omega)\exp\left(i\int \omega(x)\,f(x)\,dx\right)$$

to obtain the Poisson measure π, again by the Bochner-Minlos theorem. However, it is interesting to proceed differently.

1.1. *Configurations*

We want to describe infinite systems of particles: configurations of indistinguishable point particles in R^d or in some subset $X \subseteq R^d$.

The configuration space $\Gamma := \Gamma_X$ is the set of all **locally finite subsets** of X, *i.e.*,

$$\Gamma := \{\gamma \subset X : \#(\gamma \cap K) < \infty \text{ for bounded } K \subset X\}.$$

For a given configuration $\gamma = \{x_1, x_2, \ldots\}$ we denote

$$\langle \gamma, f \rangle = \sum_{x \in \gamma} f(x) = \sum_{x \in \gamma} \int \delta(x - x') f(x') dx'.$$

This is well defined if f is continuous and zero outside a finite volume: the sum is then finite - no problem of convergence arises.

1.2. *Poisson Measures*

We begin by considering configurations in a finite volume:

$$|X| = V < \infty$$

For configurations of only one point $x \in R^d$ the obvious choice will be a probability proportional to the volume element dv. For n-point configurations, elements of $\Gamma_X^{(n)}$ we shall use

$$dm_n = \frac{1}{n!} (dv)^n$$

the combinatorial 1/n! factor for the indistinguishability of the n particles. - But we are interested in configurations of arbitrary many particles, i.e. we want a probability measure on

$$\Gamma_X = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}.$$

We first extend the measures m_n to a measure m on Γ_X, simply by setting

$$m|_{\Gamma_X^{(n)}} = m_n.$$

This is not a probability measure:

$$m(\Gamma_X) = m\left(\bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}\right) = \sum_n m\left(\Gamma_X^{(n)}\right) = \sum_n \frac{1}{n!} \left(\int_X dv\right)^n = \exp(V).$$

We must normalize it to get a probability measure on Γ

$$\pi \equiv \exp\left(-V\right) \cdot m$$

with characteristic function

$$E\left(\exp(i\left\langle \gamma, f\right\rangle)\right) = \int_{\Gamma} \exp(i\left\langle \gamma, f\right\rangle) d\pi(\gamma) = \sum_{n} \int_{\Gamma^{(n)}} \exp(i\left\langle \gamma, f\right\rangle) d\pi(\gamma)$$

$$= \exp\left(-V\right) \sum_{n} \frac{1}{n!} \left(\int_{X^n} \exp(i \sum_{k=1}^{n} f(x_k)) \prod{}_k \left(dx_k\right) \right) = e^{-V} \sum_{n} \frac{1}{n!} \left(\int_{X} \exp(i f(x)) dx \right)^n$$

$$= e^{-V} \left(\int_{X} \exp(i f(x)) dx \right) = \exp\left(\int_{X} \left(e^{i f(x)} - 1 \right) dx \right).$$

We have (re)discovered the characteristic function, i.e. the Fourier transform, of the Poisson White Noise probability measure:

$$E\left(\exp(i\left\langle \gamma, f\right\rangle)\right) = \exp\left(\int \left(e^{i f(x)} - 1 \right) dx \right) = C_{\pi}(f) = \int e^{i\left\langle \omega, f\right\rangle} d\pi(\omega)$$

Note: there is no need to restrict ourselves to a space of finite volume -

$$C_{\pi}(f) = \exp\left(\int_{X} \left(e^{i f(x)} - 1 \right) dx \right)$$

is well defined even in the limit where $X = R^d$, and we have a limiting measure

$$\pi = \lim_{X \to R^d} \pi|_{\Gamma_X}$$

Likewise for more general densities, with

$$dv = z(x) dx$$

where z is a non-negative “intensity”:

$$C_{\pi_z}(f) = \exp\left(\int_{R^d} \left(e^{i f(x)} - 1 \right) z(x) dx \right)$$

Bochner-Minlos

Recall that the Bochner-Minlos theorem guarantees the existence of a probability measure *on the space of distributions* such that

$$C_{\pi_z}(f) = \int_{D^*} e^{i\left\langle \omega, f\right\rangle} d\pi_z(\omega)$$

In our explicit construction we have used the formula

$$\langle \gamma, f \rangle = \sum_{x \in \gamma} f(x) = \sum_{x \in \gamma} \int \delta(x - x') f(x') dx'.$$

We see from this that the measure is concentrated on only those distributions which are sums of Dirac δ-functions

$$\omega = \omega_\gamma = \sum_{x \in \gamma} \delta_x.$$

1.3. *Recall Fock Space*

Consider Fock space

$$\mathcal{F} = \bigoplus_n Sym\, L^2(R^n, n! d^n t),$$

i.e.

$$\mathcal{F} = \{\Psi : \Psi = (\Psi_0, \Psi_1, \dots, \Psi_n, \dots)\}$$

with norm

$$\|\Psi\|^2_{\mathcal{F}} = \sum_{n=0}^{\infty} n! \left(\Psi_n, \Psi_n\right)_{L^2(R^n)}.$$

For $n = 0$ the zero-particle vectors Ψ_0 are just constants: $\Psi_0 = c$ with

$$\|\Psi_0\|^2 = |c|^2.$$

Annihilation operators $a(f)$ are given by

$$\left(a(f)\,\Psi\right)_n (x_1, \dots, x_n) = (n+1) \int dx\, f(x)\, \Psi_{n+1} (x, x_1, \dots, x_n)$$

We consider in particular Fock space vectors $\Psi(g)$ with $\Psi_n(g) = \frac{1}{n!} g^{\otimes n}$. They are eigenvectors of the annihilation operators

$$a(f)\Psi(g) = (f, g)\Psi(g)$$

and have the scalar product

$$\left(\Psi(f), \Psi(g)\right)_F = e^{(f,g)}.$$

(Suitably normalized, they become the "coherent states" of quantum optics.)

1.4. *Three Isomorphisms*

It is well known, and easy to verify, that in the White Noise Hilbert space $\left(L^2\right)$ the vectors $e(f)$, with

$$e\left(f,\omega\right)=\frac{e^{\int\omega(x)f(x)dx}}{E\left(e^{\int\omega(x)f(x)dx}\right)}$$

have the same scalar product:

$$\left(e\left(f\right),e\left(g\right)\right)_{\left(L^2\right)}=\left(\Psi\left(f\right),\Psi\left(g\right)\right)_F=e^{(f,g)}.$$

Now, in the Poisson Hilbert space $L^2\left(d\pi_z\right)$, consider the vectors

$$e_\pi(f,\omega)=\exp\left(\langle\omega,\ln(1+f)\rangle-\langle f\rangle\right),\quad \omega=\omega_\gamma, \tag{1}$$

with

$$\langle f\rangle=\int f(x)z(x)dx.$$

For $\omega=\omega_\gamma{=}\sum_{x\in\gamma}\delta_x$, find

$$e_\pi(f,\omega_\gamma)=\exp\left(-\langle f\rangle\right)\prod_{x\in\gamma}(1+f(x)).$$

Their scalar product is again computed directly from the characteristic function:

$$\left(e_\pi\left(f\right),e_\pi\left(g\right)\right)_{L^2(d\pi_z)}=e^{(f,g)_{L^2(dv)}}.$$

As a consequence, we have three isomorphisms

$$\begin{array}{ccc} & \mathcal{F} & \\ \nearrow & & \searrow \\ L^2\left(d\mu\right) & \leftrightarrow & L^2\left(d\pi\right)\end{array}$$

one of them the famous *Gelfand-Ito-Segal isomorphism* between the L^2-space with Gaussian measure μ and symmetric or "boson" Fock space $\mathcal{F}$.

For yet another naturally isomorphic L^2-space, on *finite* configurations ("Lebesgue-Poisson space"), see [4 5 9 10]. For an overview and much more see e.g. the thesis of M. J. Oliveira [22].

1.5. *Operators*

In Fock space there is a representation of the CCR:

$$[a(f), a(g)] = 0 = [a^*(f), a^*(g)]$$
$$[a(f), a^*(g)] = \int dx\, f(x)g(x).$$

The image D_f of $a(f)$ in Gaussian White Noise space is a directional derivative ("Hida derivative")

$$D_f\varphi(\omega) = \lim_{\varepsilon\to 0} \frac{\varphi(\omega + \varepsilon f) - \varphi(\omega)}{\varepsilon}$$

and that of its adjoint $a^*(f)$ is

$$D_f^* = -D_f + \int \omega\,(x)\, f\,(x)\, dx.$$

In Poisson space the image of $a(f)$ acts as a difference operator, adding one more particle to the configuration γ:

$$(a_\pi\,(f)\, F)\,(\gamma) = \int_X (F\,(\gamma \cup \{x\}) - F\,(\gamma))\, f\,(x)\, dx$$

(straightforward to check for eigenvectors $e_\pi(g)$, which span $L^2(d\pi)$). For the adjoint one finds

$$(a_\pi^*\,(g)\, F)\,(\gamma) = \sum_{x\in\gamma} F\,(\gamma\setminus\{x\})\, g\,(x) - \int f(y)z(y)dy \cdot F\,(\gamma)\,.$$

For later reference we finally introduce

$$e_B(f,\omega_\gamma) = \exp\langle f\rangle\, e_B(f,\omega_\gamma) = \prod_{x\in\gamma}(1 + f(x)).$$

Their expectation w.r. suitable measures μ on configuration space

$$E\,(e_B\,(f)) = \int_\Gamma e_B(f,\omega_\gamma) d\mu\,(\gamma) = \sum_n \frac{1}{n!}\left\langle k_n^\mu, f^{\otimes n}\right\rangle$$

are called Bogoliubov functionals and are the generators of the n^{th}order correlation functions k_n^μ for the distribution μ.

2. Dynamics of Infinite Particle Systems

For discrete configuration spaces there exists a vast literature on various possible dynamical processes, see e.g . [20], used for modelling a large variety of models of population dynamics in the wider sense - epidemiology, ecology, opinion formation, spreading of information in large distributed systems, ...

Examples are independent random births and deaths of particles : "*Glauber dynamics*", or simultaneous death and birth at two lattice sites: "Kawasaki dynamics", where particles hop from one site to another and particle number is conserved.

2.1. *Continuous Configuration Space*

For this type of dynamics in the continuum much less is known. Recent results can be found for Glauber dynamics e.g. in [1] [15] [16], for Kawasaki in [18] [14].

2.2. *Methods*

In terms of Markov processes the complication is - for infinite configurations (finite density, thermodynamic limit) - that infinitely many jumps occur in any finite time interval, and even without interactions, free jumps can produce infinite local densities in finite times ("explosion").

Hilbert Space methods - Dirichlet forms , evolution operators - are suitable for (approach to) equilibrium [2] [7].

2.3. *Free Kawasaki Dynamics*[13]

As a simple model, we shall focus on "free" Kawasaki dynamics:

$$\partial_t F(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, g(x-y)\, (F(\gamma\backslash x \cup y) - F(\gamma))$$

Particles are hopping from x to y, with rate $g(x-y)$, otherwise independent of the configuration γ.

In terms of creation and annihilation operators, one finds

$$H = \int dx\, z(x) \int dy\, (g(x-y) - g_0\delta(x-y))\, (a_\pi^*(x) a_\pi(y) - a_\pi(y))\,,$$

Clearly, in Fock space language this corresponds to a quadratic Hamiltonian, and time development can be calculated in closed form.

Time evolution of Bogoliubov exponentials takes on a particularly simple form:

$$e^{Ht} e_B(f) = e_B(e^{tA} f)$$

$$Af(x) := \int_{\mathbb{R}^d} dy\, g(x-y)\,(f(y) - f(x))\,.$$

Evolution of the initial (Poisson) distribution

$$\pi_z \to P_{\pi_z,t}$$

under the dual of e^{Ht} is characterized by

$$\int e_B(f,\gamma) P_{\pi_z,t}(d\gamma) = \int e_B(e^{tA} f,\gamma)\pi_z(d\gamma) = \exp\left(\int_{\mathbb{R}^d} e^{tA} f(x) z(x) dx\right).$$

Starting with a Poisson distribution the distribution at time t, $P_{\pi_z,t}(d\gamma)$ is again Poissonian, with intensity $z_t \in L^\infty(\mathbb{R}^d, dx)$, given by

$$\int_{\mathbb{R}^d} dx\, e^{tA} f(x) z(x) = \int_{\mathbb{R}^d} dx\, f(x) z_t(x), \tag{2}$$

for all $f \in L^1(\mathbb{R}^d, dx)$. Since e^{tA} is positivity preserving in $L^1(\mathbb{R}^d, dx)$, it follows that $z_t \geq 0$.

Poisson distributions are invariant under free Kawasaki dynamics iff their intensity is constant:

$H^* 1 = 0$ iff the linear annihilation term in H vanishes:

$$\int dy\, a(y) \left(\int dx z\,(x)\,(g\,(x-y) - g\,(x))\right) \overset{!}{=} 0$$

Using Fourier transforms one sees that this requires $z = const$.

For g even and constant $z > 0$, H gives rise to a symmetric Dirichlet form on $L^2(\Gamma, \pi_z)$,

$$(F, HF) = -\frac{1}{2}\int_\Gamma \pi_z(d\gamma) \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, g(x-y)\, |F(\gamma)|^2\,. \tag{3}$$

This allows to derive a Markov process on Γ with *cadlag* paths and having π_z as an invariant measure[18]. In this setting H is a negative essentially self-adjoint operator on $L^2(\Gamma, \pi_z)$, and the generator of a contraction semigroup on $L^2(\Gamma, \pi_z)$.

3. Asymptotics

3.1. *Large Time Asymptotics* $t \to \infty$

- Any Poisson state of constant intensity is invariant under the evolution (equilibrium).
- "Local Equilibrium": Poisson states with non-constant intensity $z = z(x)$. Recall that the state at $t \geq 0$ is again Poissonian, with intensity $z_t \in L^\infty(\mathbb{R}^d, dx)$, given by

$$\int_{\mathbb{R}^d} dx\, e^{tA} f(x) z(x) = \int_{\mathbb{R}^d} dx\, f(x) z_t(x).$$

One says that a function $z \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ has arithmetic mean whenever

$$\lim_{R \to +\infty} \frac{1}{\text{vol}(B(R))} \int_{|x|<R} dx\, z(x) \equiv mean(z) \tag{4}$$

exists.

If $z \geq 0$ is a bounded measurable function whose Fourier transform $\widetilde{z}$ is a signed measure, then $mean(z) = \widetilde{z}(\{0\})$,

$$\int_{\mathbb{R}^d} dx\, f(x) z_t(x) = \int_{\mathbb{R}^d} dx\, e^{tA} f(x) z(x) \to mean(z) \int_{\mathbb{R}^d} dx\, f(x),$$

and the distribution π_{z_t} converges weakly to $\pi_{\text{mean}(z)}$ as t goes to infinity, because of convergence of the characteristic functions[a].

Not all measurable bounded non-negative functions z have an arithmetic mean. Counterexamples are slowly oscillating functions such as

$$z(x) = c + \cos(\ln(1 + |x|)), \quad x \in \mathbb{R}^d,$$

where $c > 1$. Then for large R

$$\frac{1}{\text{vol}(B(R))} \int_{B(R)} z(x)\, dx \sim c + \frac{d}{\sqrt{1+d^2}}\, \sin\Big(\ln(R) + \arctan(d) \Big).$$

3.2. *The hydrodynamic limit*

Shall use the so-called "empirical field" corresponding to a $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \varphi, \mathbf{X}_t \rangle = \sum_{x \in \mathbf{X}_t} \varphi(x).$$

[a]To extend the result to all bounded z with arithmetic mean finer considerations are necessary. If the jump rate a has finite second moment we can check the requirements of [3]. General jump rates will be considered in.[19]

The first correlation function $\rho_t(x)$ is given by

$$E\left(\langle\varphi,\mathbf{X}_t\rangle\right) = E_{\pi_{z_t}}\left(\langle\varphi,\gamma\rangle\right) = \int \varphi(x)\rho_t(x)dx$$

Consider space-time scale transformation given by $\langle\varphi,\gamma\rangle \to \varepsilon^d\langle\varphi(\varepsilon\cdot),\gamma\rangle$, $t\to\varepsilon^{-\kappa}t$ for suitable $\kappa>0$, $z\to z(\varepsilon\cdot)$.

1. If

$$g_i^{(1)} := \int_{\mathbb{R}^d} dx\, x_i g(x) \neq 0,$$

then for $\kappa = 1$

$$\int_{\mathbb{R}^d} dx\,\rho_t(x)\varphi(x) = \int_{\mathbb{R}^d} dx\, z(x+tg^{(1)})\varphi(x),$$

so that, if the intensity z is smooth enough

$$\frac{\partial}{\partial t}\rho_t(x) = g^{(1)}\cdot\nabla\rho_t(x) = \mathrm{div}(g^{(1)}\cdot\rho_t(x))$$

with the initial condition $\rho_0 = z$.

2. If $g^{(1)} = 0$, and

$$g_{ij}^{(2)} := \int_{\mathbb{R}^d} dx\, x_i x_j g(x)$$

then for time rescaling with $\kappa = 2$

$$\int_{\mathbb{R}^d} dx\,\rho_t(x)\varphi(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dx\, z(x)\int_{\mathbb{R}^d} dk\, e^{ik\cdot x}e^{-\frac{t}{2}\langle g^{(2)}k,k\rangle}\hat{\varphi}(k),$$

solution of the partial differential equation

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}^{(2)}\frac{\partial^2}{\partial x_i\partial x_j}\rho_t(x).$$

3. Consider weak asymmetries, decomposing g into a sum of an even function p and an odd function q, and use the scaling

$$g_\varepsilon := p + \varepsilon q$$

and $\kappa = 2$.

The limiting density ρ_t is solution of the partial differential equation

$$\frac{\partial}{\partial t}\rho_t(x) = \mathrm{div}(g^{(1)}\rho_t(x)) + \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}^{(2)}\frac{\partial^2}{\partial x_i\partial x_j}\rho_t(x).$$

3.3. *Far from Equilibrium*

The construction of the free Kawasaki process and its scaling limits are not restricted to Poissonian initial distributions. Sufficient conditions for admissible measures can be stated in terms of their correlation functions and are in particular fulfilled for Gibbs measures at high temperatures. For more details see [13].

3.4. *Yet another Scaling Limit*

For interacting Kawasaki processes, as the jump range becomes infinite, Glauber dynamics appears as scaling limit [8]. for the free Kawasaki dynamics, in the limit $g \to 1$, we obtain as generator the image of the Fock space number operator $N = \int dx\, a^*(x)\, a(x)$.It is straghtforward to compute that this is the generator of a birth-and-death process with unit death rate and birth rate $b = z$:

$$\partial_t F(\gamma) = \int_{\mathbb{R}^d} dy\, z \ \ (F(\gamma \cup y) - F(\gamma)) + \sum_{x \in \gamma} (F(\gamma \backslash x) - F(\gamma)) \,,$$

and from its isomorphism with N in Fock space we see immediately that this generator, contrary to the original one in (3), now has a spectral gap.

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