Studies in non-Gaussian Analysis

by

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Contents

Chapter 1

Introduction

Gaussian analysis and in particular white noise analysis have developed to a useful tool in applied mathematics, Stochastics, and mathematical physics. For a detailed exposition of the theory and for many examples of applications we refer the reader to the monographs [BK95], [HKPS93], [Hid80], [Kon91], [Kuo96], and [Oba94] and the introductory articles [Kuo92], [Str94], and [Wes93]. However, in applications there are many situations where non-Gaussian analysis appear as well, see e.g., [AKR98a], [AKR98b], [AKR97] and references therein. The subject of this thesis is to study non-Gaussian analysis and its applications. For an overview of the different topics treated in this thesis we refer to the list of contents. We also refer to the introductions of the corresponding chapters for a detailed overview of the results obtained in this thesis. In addition, this thesis might serve as a guide for the reader to the following papers on the subject: [KSS97], [KSSU98], [KSSU97], [KSS98b], [KSS98a].

The first approach to study non-Gaussian analysis in infinite dimensions was recently proposed in [AKS93] and developed in [ADKS96]. For smooth probability measures on infinite dimensional linear spaces a biorthogonal decomposition is a natural extension of the orthogonal one that is well known in Gaussian analysis. This biorthogonal "Appell" system has been constructed for smooth measures also by Yu. L. Daletskii [Dal91]. For a detailed description of its use in infinite dimensional analysis we refer to [ADKS96].

Later Kondratiev et al. [KSWY95] (see also [Wes95]) considered the case of probability measures μ on the dual of a nuclear space satisfying the following two conditions:

(A.1) the measure μ has an characteristic functional which is analytic on some open neighborhood of zero,

(A.2) μ is non-degenerate, i.e., if φ is a polynomial with $\varphi = 0$ μ-a.e., then $\varphi \equiv 0$.

No further conditions such as quasi-invariance of the measure or smoothness of the logarithmic derivative was required. The main advantage of using conditions (A.1) and (A.2) (instead of the conditions used in [ADKS96]) is the possibility to cover a bigger class of measures which contains in particular the example of Poisson measures. Let us stress that in applications the latter case is of special interest. For example, using Poisson measures we can describe the free Bose gases of non-zero density, see e.g., [AKR97], and references therein.

This thesis consists of three main parts:

1. generalization of Gaussian analysis using generalized Appell systems,

2. further developments of Poisson and compound Poisson analysis,

3. introduction and study of a differentiable structure on compound configurations spaces.

For a given measure μ which satisfies conditions (A.1) and (A.2) we construct an Appell biorthogonal system $\mathbb{A}^{\mu}$ as a pair $(\mathbb{P}^{\mu}, \mathbb{Q}^{\mu})$ of Appell polynomials $\mathbb{P}^{\mu}$ and a canonical system of generalized functions $\mathbb{Q}^{\mu}$, properly associated to the measure μ. This framework enables us to obtain spaces of test and generalized functions as well as their characterizations. This will be worked out in Chapter 3.
In Chapter 4 we consider a class of measures satisfying the conditions (A.1) and (A.2). We introduce a transformation α on the nuclear space $\mathcal{N}_{\mathbb{C}}$ such that α is an invertible holomorphic functions on a neighborhood of zero with $\alpha(0) = 0$. For any such measure μ we construct an generalized Appell system $\mathbb{A}^{\mu,\alpha}$ as a pair $(\mathbb{P}^{\mu,\alpha}, \mathbb{Q}^{\mu,\alpha})$ of generalized Appell polynomials $\mathbb{P}^{\mu,\alpha}$ and a system of generalized functions $\mathbb{Q}^{\mu,\alpha}$ associated to the measure μ. It is worth emphasizing that by varying the function α one produces different generalized Appell systems. Let us give some examples which will be considered later on in this thesis.

- Define α by $\alpha(\varphi) := \log(1+\varphi)$ and the Poisson measure π; it produces the system of generalized Charlier polynomials which is orthogonal with respect to π, see Chapter 5 , Section 5.2 for more details.

- For α given by $\alpha(\varphi) := \varphi/(\varphi - 1)$ and the Gamma measure μ_G we obtain the system of generalized Laguerre polynomials and these are orthogonal polynomials with respect to μ_G, cf. Chapter 6, Section 6.4.

Let us mention that these two examples are of special type. More precisely, in these cases the biorthogonal system reduces to an orthogonal one, the $\mathbb{P}^{\mu,\alpha}$-system coincides with the $\mathbb{Q}^{\mu,\alpha}$-system. We would like to draw the reader's attention to the second choice of α given above. In fact, this special case of generalized Appell system produces concrete analysis and geometry on the Gamma space, see Section 6.4. Up to now, we are able to produce a Fock type decomposition of the L^2 space, but is still not clear how to obtain a representation of the creation and annihilation operators on Gamma space. Moreover, the differential geometry on the corresponding configuration space seems to be very interesting. This considerations will be implemented in forthcoming paper [KSU99].

The central results of Chapter 4 are:

- description and characterization of test function spaces and proof that the test function spaces are independent of the transformation α, see Theorem 4.3.9,

- description and characterization theorems for generalized functions, in particular, we prove that for fixed measure μ and all transformations α the space of generalized functions are the same, cf. Theorem 4.4.3,

- we extend the Wick product and the corresponding Wick calculus in the present setting,

- we provide formulas for the change of the generalized Appell system under a transformation of the measure.

Concerning applications for the above general theory to non-Gaussian analysis we give special emphasis to Poisson and compound Poisson spaces. In particular analysis and geometry on these spaces are developed in great detail in Chapters 5, 6, and 7. Here we would like to remark that there are many papers on Poissonian analysis from different points of view, see e.g., [BLL95], [CP90], [IK88], [NV90], [NV95], [Pri95], [Us95], and many others. In this thesis we will use the general methods developed in Chapter 4 to produce, e.g., the Fock space isomorphism for Poisson space as well as natural operations on Poisson space. Therefore, we refer to the cited references for related considerations of analysis on Poisson spaces.

In Chapter 5 we start giving a detailed construction of the Poisson measures on the configuration space Γ_X over a Riemannian manifold X, cf. Section 5.1 for details. Moreover, we also give a useful integral characterization of the Poisson measure, the so-called Mecke's identity (cf. [Mec67, Satz 3.1] or see (5.8) below) which is useful in computations in Poissonian analysis.

There are essentially two procedures to realize "geometry" over the configuration space Γ_X equipped with Poisson measure. One is the so-called *external geometry*, namely the geometry obtained by transportation via the Fock space isomorphism, see Section 5.3. The other one is the *internal geometry* corresponding to a lifting procedure of the differential geometry on the underlying manifold X, see [AKR98a], [KSSU98]. We develop the details in Section 5.4.

These two (external and internal) geometries are connected in a non trivial way, see [AKR98a, Theorem 5.2]. In Section 5.5 we investigate this connection for the case of interacting particle systems and show that even for the interacting case there is a transparent relation between the intrinsic and the extrinsic Dirichlet form, see Theorem 5.5.3.

As a consequence of the mentioned relation we prove the closability of the pre-Dirichlet form $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C^{\infty}_{b}(\mathcal{D},\Gamma))$ on $L^2(\Gamma_X,\mu)$, where μ is a tempered grand canonical Gibbs measure, see Section 5.5. The closability is crucial (for physical reasons, see [AKR97]) for applying the general theory of Dirichlet forms including the construction of a corresponding diffusion process (cf. [MR92]). It models an infinite particle system with (possibly) very singular interactions cf. [AKR98b].

Another contribution of this thesis is to clarify the analysis and differential geometry on compound Poisson spaces. Since there is no Lebesgue measure on infinite dimensional linear spaces one has the problem to define the notion "volume element" on compound Poisson spaces in a reasonable way. Following [Cha84] the volume element on X can be defined (up to constant multiples) as the unique positive Radon measure μ on X such that the gradient ∇^X and the divergence div^X become dual operators on $L^2(X,\mu)$ (with respect to $\langle\cdot,\cdot\rangle_{TX}$).

In Subsection 7.2.2 we prove that the compound Poisson measure π_σ^τ on the compound configuration space Ω_X is the right "volume element" corresponding to the introduced differential geometry on Ω_X.

In fact, the results on analysis and geometry on compound Poisson spaces are connected with the possibility to establish an unitary isomorphism between the compound Poisson space and the Poisson space, see Section 6.2 for the corresponding description. Hence, this unitary isomorphism is the bases of our considerations in Chapter 6 as well as in Chapter 7. We also give the corresponding representation of the associated Lie algebra of compactly supported vector fields and exhibit explicit formulas for the corresponding generators, see Section 7.3.

In Section 7.4 we identify the diffusion process corresponding to the Dirichlet form on compound Poisson space. It comes out that this process is nothing but the equilibrium process (or distorted Brownian motion on Ω_X) together with the corresponding marks.

Finally in Section 7.6 we prove in detail the existence of a marked Poisson measure over the marked Poisson space Ω_X^M, where M is a complete separable metric space with a probability measure. Hence all the results obtained in Chapter 7 extend with obvious changes to marked Poisson spaces.

We would like to point out that most of the results presented in this thesis have already been published, see e.g., [KSS97], [KSSU98], [KSSU97], [KSS98b], [KSS98a] as well as they have been announced in international conferences in Marseille'95, '96, '97, Kiev'96, Madeira Math Encounters X, XI, XII, XIII, XIV, XV, Stochastic Analysis and its Applications Barcelona'97 and Paris'98, and Seminar on Stochastic Analysis Bonn'98. Here we present a systematic exposition of this circle of ideas.

Finally, we would like to mention that there are other results concerning further applications of non-Gaussian analysis, see [KKS98] and [KSSU99].

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Chapter 2

Preliminaries

In this chapter we shall collect the most important properties of nuclear spaces, Fock spaces over Hilbert spaces, and some facts about holomorphic functions over a locally convex topological vector space, see Sections 2.1, 2.2, and 2.3 below, respectively. We will not reproduce full details in the mentioned subjects but rather give the convenient descriptions for our purpose. Hence, the results concerning nuclear triples and kernel theorem are best summarized in [BK95], [GV68], [HKPS93]. About topologies in co-nuclear spaces we refer to the books [Köt71], [RR73], [Sch71], and [Trè67] for details, proofs, and examples. In Section 2.2 we define the Fock space over a Hilbert space and define on it the creation, annihilation, and second quantization operators which play an important role in the further chapters. Detailed description of Fock spaces can be founded in [BK95, Chapter 2 Section 2], [HKPS93, Appendix A.2], [Oba94], and [RS75a]. The standard references for holomorphy in locally convex spaces are [Din81], [Bar85], and [Col82].

2.1 Nuclear triples and kernel theorem

We start with a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. For a given separable nuclear space $\mathcal{N}$ densely and continuously embedded into $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'. \tag{2.1}$$

The dual pairing $\langle \cdot, \cdot \rangle$ of $\mathcal{N}'$ and $\mathcal{N}$ then is realized as an extension of the inner product in $\mathcal{H}$

$$\langle f, \xi \rangle = (f, \xi), \; f \in \mathcal{H}, \; \xi \in \mathcal{N}.$$

Instead of reproducing the abstract definition of nuclear spaces (see e.g., [GV68], [RR73], [Sch71], and [Trè67]) we give a complete and convenient characterization in terms of projective limits of decreasing chains of Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$.

Theorem 2.1.1 *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p, \; p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\iota_{p,p_1} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$, $\iota_{p,p_2} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The space $\mathcal{N}$ is dense in each $\mathcal{H}_p$ for any $p \in \mathbb{N}$. The topology of $\mathcal{N}$ is given by the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ such that for all $p \in \mathbb{N}$ the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous.

The Hilbert norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$, $\forall p \in \mathbb{N}$. These norms are compatible, i.e., if a sequence converges to zero with respect to a norm $|\cdot|_p$ and is Cauchy with respect to a norm $|\cdot|_q$, then it also converges to zero with respect to $|\cdot|_q$. A basis of open neighborhoods of zero in the projective limit topology is given by the sets

$$\mathcal{U}(p, \varepsilon) = \{\xi \in \mathcal{N} \,|\, |\xi|_p < \varepsilon\}, \text{ for any } p \in \mathbb{N} \text{ and any } \varepsilon > 0.$$

Without loss of generality we always suppose that the system of norms is ordered, i.e., $\forall p \in \mathbb{N}$, $\forall \xi \in \mathcal{N} : |\xi| \leq |\xi|_p$ and that the norms are increasing

$$|\xi|_1 \leq |\xi|_2 \leq \ldots \leq |\xi|_p \leq |\xi|_{p+1} \leq \ldots, \; \xi \in \mathcal{N}.$$

In this case the nuclearity of $\mathcal{N}$ means that for any $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that the embedding $\iota_{q,p} : \mathcal{H}_q \hookrightarrow \mathcal{H}_p$ is nuclear, i.e., for any basis $(e_k)_{k \in \mathbb{N}}$ of $\mathcal{H}_q$ the sum

$$\sum_{k=1}^{\infty} |\iota_{q,p} e_k|_p < \infty.$$

It is sufficient to require for any $p \in \mathbb{N}$ the existence of $q' \in \mathbb{N}$ such that the embedding $\iota_{q',p} : \mathcal{H}_{q'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt, i.e.,

$$\|\iota_{q',p}\|_{HS}^2 = \sum_{k=1}^{\infty} |\iota_{q',p} e_k|_p^2 < \infty,$$

because the composition of two Hilbert-Schmidt operators is nuclear, see e.g., [GV68]. In the present situation $\mathcal{N}$ is a countably Hilbert space in the sense of [GV68].

By general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p},$$

with inductive limit topology τ_{ind} by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}_p', \; p \in \mathbb{N}\}$. The inductive limit topology (with respect to this family) is the finest topology on $\mathcal{N}'$ such that for all $p \in \mathbb{N}$ the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$.

Remark 2.1.2 *Let us mention that in our setting the inductive limit topology τ_{ind} coincides with the Mackey topology $\tau(\mathcal{N}', \mathcal{N})$ (i.e., the strongest topology in $\mathcal{N}'$ such that $(\mathcal{N}', \tau(\mathcal{N}', \mathcal{N}))' = \mathcal{N}$ with the projective limit topology) and the strong topology $\beta(\mathcal{N}', \mathcal{N})$ (i.e., a basis of open neighborhoods of zero is given by*

$$\mathcal{U}_\beta(0, A) = \{x \in \mathcal{N}' \,|\, |\langle x, \xi \rangle| < \varepsilon, \; \forall \xi \in A, \; A \subset \mathcal{N} \text{ bounded}\}).$$

This fact is also known as the reflexivity of countable Hilbert spaces. We refer to [BK95, Chapter 1 Section 1], [HKPS93, Appendix 5] and also [Sch71, Chapters 2, 4] for more details.

Further we want to introduce the notion of tensor powers of a nuclear space, see e.g., [RS75b] or [Gui72]. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$ of Hilbert spaces. More precisely, let $(e_j)_{j \in \mathbb{N}}$ be an orthonormal basis in $\mathcal{H}_p$. Let us construct a formal product

$$e_\alpha := e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}, \tag{2.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, in other words, we consider the ordered sequences $(e_{\alpha_1}, \ldots, e_{\alpha_n})$ and span a Hilbert space by the formal vectors (2.2), assuming that they form the orthogonal basis of this space. The separable Hilbert space obtained as a result is called the n-th tensor product of the space $\mathcal{H}_p$ and is denoted by $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$. Since there is no danger of confusion we will preserve the notation $(\cdot,\cdot)_p$, (resp. $|\cdot|_p$) and $(\cdot,\cdot)_{-p}$ (resp. $|\cdot|_{-p}$) for the inner product (resp. norm) on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$. Vectors from $\mathcal{H}_p^{\otimes n}$ have the form

$$f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha e_\alpha,\ f_\alpha \in \mathbb{R},\ |f|_p^2 = \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|^2,$$

and

$$(f, g)_p = \sum_{\alpha \in \mathbb{N}^n} f_\alpha g_\alpha,\ g = \sum_{\alpha \in \mathbb{N}^n} g_\alpha e_\alpha,\ g_\alpha \in \mathbb{R}.$$

Using the definition

$$\mathcal{N}^{\otimes n} := \operatorname*{pr\,lim}_{p \in \mathbb{N}} \mathcal{H}_p^{\otimes n},$$

one can prove (see e.g., [Sch71]) that $\mathcal{N}^{\otimes n}$ is a nuclear space which is called the n-th tensor power of $\mathcal{N}$.

The dual space of $\mathcal{N}^{\otimes n}$ can be written

$$\mathcal{N}'^{\otimes n} = \operatorname*{ind\,lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^{\otimes n}.$$

Thus we have introduced nuclear triples

$$\operatorname*{pr\,lim}_{p \in \mathbb{N}} \mathcal{H}_p^{\otimes n} = \mathcal{N}^{\otimes n} \subset \mathcal{H}^{\otimes n} \subset \mathcal{N}'^{\otimes n} = \operatorname*{ind\,lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^{\otimes n},\ n \in \mathbb{N}. \tag{2.3}$$

Most important for the applications we have in mind in the abstract non Gaussian analysis theory (cf. Chapter 3 and followings) is the following *kernel theorem*, see e.g., [BK95, Chapter 1 Section 2] and [GV68, Chapter I]. Below the kernel theorem is stated for the nuclear triples (2.3) which is sufficient later on, nevertheless this theorem holds also for Hilbert triples (or rigged Hilbert spaces), $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$, where the embedding $\mathcal{H}_+ \hookrightarrow \mathcal{H}_0$ is Hilbert-Schmidt and the pairing $\langle \cdot,\cdot \rangle$ of $\mathcal{H}_-$ and $\mathcal{H}_+$ is realized as an extension of the inner product $(\cdot,\cdot)_0$ in $\mathcal{H}_0$.

Theorem 2.1.3 *Suppose that the nuclear triples (2.3) are given. Let F_n be an n-linear form*

$$\mathcal{N} \times \ldots \times \mathcal{N} \ni (\xi_1, ..., \xi_n) \mapsto F_n(\xi_1, ..., \xi_n) \in \mathbb{R},$$

which is $\mathcal{H}_p$-continuous, i.e.,

$$|F_n(\xi_1, ..., \xi_n)| \leq C|\xi_1|_p|\xi_2|_p \dots |\xi_n|_p,$$

for some $p \in \mathbb{N}$ and $C > 0$. Then for all $p' > p$ such that the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt there exists a unique $\Phi^{(n)} \in \mathcal{H}_{-p'}^{\otimes n}$ such that

$$F_n(\xi_1, \dots, \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \dots \otimes \xi_n \rangle, \ \xi_1, ..., \xi_n \in \mathcal{N},$$

and the following norm estimate holds

$$|\Phi^{(n)}|_{-p'} \leq C \, \|i_{p',p}\|_{HS}^n$$

using the Hilbert-Schmidt norm of $i_{p',p}$.

Corollary 2.1.4 *Let $\mathcal{N} \times \dots \times \mathcal{N} \ni (\xi_1, ..., \xi_n) \mapsto F(\xi_1, ..., \xi_n)$ be a n-linear form which is $\mathcal{H}_{-p}$-continuous, i.e.,*

$$|F_n(\xi_1, \dots, \xi_n)| \leq C|\xi_1|_{-p}|\xi_2|_{-p} \dots |\xi_n|_{-p}$$

for some $p \in \mathbb{N}$ and $C > 0$. Then for all $p' < p$ such that the embedding $i_{p,p'} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is Hilbert-Schmidt there exists a unique $\Phi^{(n)} \in \mathcal{H}_{p'}^{\otimes n}$ such that

$$F_n(\xi_1, ..., \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \dots \otimes \xi_n \rangle, \ \xi_1, ..., \xi_n \in \mathcal{N}$$

and the following norm estimate holds

$$\left|\Phi^{(n)}\right|_{p'} \leq C \, \|i_{p,p'}\|_{HS}^n .$$

2.2 Fock spaces over Hilbert spaces

We now proceed to describe the Fock space. Let $\mathcal{H}$ be a real separable Hilbert space, and let $\mathcal{H}_{\mathbb{C}}$ be the complexified Hilbert space of $\mathcal{H}$ with inner product

$$(f_1, f_2)_{\mathcal{H}_{\mathbb{C}}} = (f_1, \bar{f}_2) = (g_1, g_2) + (h_1, h_2) + i(h_1, g_2) - i(g_1, h_2),$$

for $f_1, f_2 \in \mathcal{H}_{\mathbb{C}}, \ f_1 = g_1 + ih_1, \ f_2 = g_2 + ih_2, \ g_1, g_2, h_1, h_2 \in \mathcal{H}.$

First we introduce the concept of symmetric tensor powers of Hilbert spaces. For any $n \in \mathbb{N}$ we denote by $\mathfrak{S}_n$ the permutation group over

$\{1, \ldots, n\}$. Then for any $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$ we define a unitary operator $U_{\sigma,n}$ on $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ by the formula

$$U_{\sigma,n}(f_1 \otimes \ldots \otimes f_n) := f_{\sigma_1} \otimes \ldots \otimes f_{\sigma_n},$$

on a total set of elements of the form $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$. It is easy to verify directly that, for the operator

$$P_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} U_{\sigma,n},$$

we have $P_n^2 = P_n$, $P_n^* = P_n$ and, therefore, it is an orthogonal projector in $\mathcal{H}_{\mathbb{C}}^{\otimes n}$. The closed subspace of $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ onto which P_n projects is denoted by $\mathrm{Exp}_n\mathcal{H}$, the n-th symmetric tensor power of the Hilbert space $\mathcal{H}_{\mathbb{C}}$.

If in Theorem 2.1.3 (resp. Corollary 2.1.4) we start from a symmetric n-linear form F_n on $\mathcal{N}^{\otimes n}$ i.e., $F_n(\xi_{\sigma_1}, \ldots, \xi_{\sigma_n}) = F_n(\xi_1, \ldots, \xi_n)$ for any permutation $\sigma \in \mathfrak{S}_n$, then the corresponding kernel $\Phi^{(n)}$ can be localized in $\mathrm{Exp}_{p'}\mathcal{H} \subset \mathcal{H}_{p',\mathbb{C}}^{\otimes n}$. For $f_1, \ldots, f_n \in \mathcal{H}_{\mathbb{C}}$ let $\hat{\otimes}$ also denote the symmetrization of the tensor product

$$f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := P_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n}.$$

All the above quoted theorems also hold for complex spaces, in particular the complexified space $\mathcal{N}_{\mathbb{C}}$. By definition an element $\xi \in \mathcal{N}_{\mathbb{C}}$ decomposes into $\xi = \zeta + i\theta$, $\zeta, \theta \in \mathcal{N}$.

The Fock space (Boson or symmetric) $\mathrm{Exp}\mathcal{H}$ over $\mathcal{H}$ is defined as a Hilbert orthogonal sum

$$\mathrm{Exp}\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n\mathcal{H}, \; \mathrm{Exp}_0\mathcal{H} := \mathbb{C},$$

and, hence, consists of the sequences $F = (f^{(n)})_{n=0}^{\infty}$ ($f_n \in \mathrm{Exp}_n\mathcal{H}$) for which

$$\|F\|^2_{\mathrm{Exp}\mathcal{H}} = \sum_{n=0}^{\infty} n! |f^{(n)}|^2 < \infty.$$

The subspaces $\mathrm{Exp}_n\mathcal{H}$ are often called *n-particle* subspaces, $\mathrm{Exp}_0\mathcal{H}$ is called the *vacuum* subspace. There is a well known standard procedure to construct a complete orthonormal system for $\mathrm{Exp}\mathcal{H}$, since later on we mostly will be interested in L^2-analysis, we skip this procedure and refer to [BK95], [Hid80], [HKPS93], and [Oba94] for the corresponding construction.

Let us now introduce a series of objects in the Fock space which are relevant in what follows.

Denote by $\mathrm{Exp}_{fin}\mathcal{H}$ the subspace of finite vectors from $\mathrm{Exp}\mathcal{H}$, i.e., vectors of the form $F = (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots)$. Obviously, $\mathrm{Exp}_{fin}\mathcal{H}$ is dense in $\mathrm{Exp}\mathcal{H}$.

For any $f \in \mathcal{H}_{\mathbb{C}}$ and $n \in \mathbb{N}$ let $f^{\otimes n} := f \otimes \ldots \otimes f \in \mathrm{Exp}_n\mathcal{H}$ be given. The vectors

$$\mathrm{Exp} f := (1, \frac{1}{1!} f, \ldots, \frac{1}{n!} f^{\otimes n}, \ldots),$$

are called *coherent states* (or exponential vectors) corresponding to the one-particle state f. For any set $\mathcal{L} \subset \mathcal{H}_{\mathbb{C}}$ which is total in $\mathcal{H}_{\mathbb{C}}$ the set of coherent states $\{\mathrm{Exp} f \,|\, f \in \mathcal{L}\} \subset \mathrm{Exp}\mathcal{H}$ is also total in $\mathrm{Exp}\mathcal{H}$, see e.g. [Gui72, Chapter 2] and [BK95]. According to the definition of $\mathrm{Exp}\mathcal{H}$, we have

$$(\mathrm{Exp} f_1, \mathrm{Exp} f_2)_{\mathrm{Exp}\mathcal{H}} = e^{(f_1, f_2)_{\mathcal{H}_{\mathbb{C}}}},$$

and

$$\|\mathrm{Exp} f\|^2_{\mathrm{Exp}\mathcal{H}} = e^{|f|^2_{\mathcal{H}_{\mathbb{C}}}}, \;\; f_1, f_2, f \in \mathcal{H}_{\mathbb{C}}.$$

Next we introduce creation and annihilation operators on the Fock space $\mathrm{Exp}\mathcal{H}$, see e.g., [HKPS93, Appendix A.2] and [RS75a]. Consider $f^{(n)} \in \mathrm{Exp}_n\mathcal{H}$ of the form

$$f^{(n)} = f_1 \hat{\otimes} \ldots \hat{\otimes} f_n, \;\; f_i \in \mathcal{H}, \; i = 1, \ldots, n. \tag{2.4}$$

Let $h \in \mathcal{H}$. Then the *annihilation operator* $a^-(h)$ of h acts on $f^{(n)}$ as follows:

$$a^-(h) f^{(n)} := \sum_{j=1}^{n} (h, f_j) f_1 \hat{\otimes} \ldots \hat{\otimes} f_{j-1} \hat{\otimes} f_{j+1} \hat{\otimes} \ldots \hat{\otimes} f_n \in \mathrm{Exp}_{n-1}\mathcal{H}. \tag{2.5}$$

This definition is independent of the particular representation of $f^{(n)}$ in (2.4), hence $a^-(h)$ is well-defined. Then we extend (2.5) by linearity to a dense subspace of $\mathrm{Exp}_n\mathcal{H}$ consisting of finite linear combinations of elements of the form (2.4). One easily finds the following inequality for such elements (cf. [RS75a])

$$|a^-(h) f^{(n)}| \leq \sqrt{n} |h| |f^{(n)}|, \tag{2.6}$$

which shows that the extension of $a^-(h)$ to $\mathrm{Exp}_n\mathcal{H}$ as a bounded operator exists. The bound (2.6) allows us to extend $a^-(h)$, $h \in \mathcal{H}$, component-wise to $\mathrm{Exp}_{fin}\mathcal{H}$ which, therefore, give us a densely defined operator on $\mathrm{Exp}\mathcal{H}$ denoted again by $a^-(h)$. So the adjoint operator of $a^-(h)$ exists, which we denote by $a^+(h)$ and called *creation operator*. The action of the creation operator on elements $f^{(n)} \in \mathrm{Exp}_n\mathcal{H}$ is given by

$$a^+(h)f^{(n)} = h\hat{\otimes}f^{(n)} \in \mathrm{Exp}_{n+1}\mathcal{H}.$$

For the creation operator we also have an estimate

$$|a^+(h)f^{(n)}| \leq \sqrt{n+1}|h||f^{(n)}|.$$

As before, this estimate give us the possibility to deduce that $a^+(h)$ is densely defined on $\mathrm{Exp}\mathcal{H}$. Hence, $a^-(h)$ and $a^+(h)$ are closable and we use the same notation for their closures. In Chapter 4 we will obtain explicit representations for these operators on L^2 with respect to the Poisson measure.

Finally we would like to introduce the second quantization operator on $\mathrm{Exp}\mathcal{H}$. For any contraction B in $\mathcal{H}$ it is possible to define an operator $\mathrm{Exp}B$ as a contraction in $\mathrm{Exp}\mathcal{H}$ which in any n-particle subspace $\mathrm{Exp}_n\mathcal{H}$ is given by $B\otimes\cdots\otimes B$ (n times). Let A be a positive self-adjoint operator in $\mathcal{H}$ with domain of essential selfadjointness D. Furthermore, suppose that $\mathcal{N} \subset D$ and that A leaves $\mathcal{N}$ invariant. Then we have a contraction semigroup e^{-tA}, $t \geq 0$ and it is possible to introduce the *second quantization operator* $d\mathrm{Exp}A$ as the generator of the semigroup $\mathrm{Exp}(e^{-tA})$, $t \geq 0$, i.e.,

$$\mathrm{Exp}(e^{-tA}) = \exp(-td\mathrm{Exp}A),$$

see e.g., [RS75a].

2.3 Holomorphy on locally convex spaces

We shall collect some facts from the theory of holomorphic functions in locally convex topological vector spaces $\mathcal{E}$ (over the complex field $\mathbb{C}$), see e.g., [Bar85], [Col82], and [Din81].

2.3.1 Holomorphic functions

Let $\mathcal{L}(\mathcal{E}^n)$ be the space of n-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric n-linear forms. Also let $P^n(\mathcal{E})$ denote the set of all n-homogeneous polynomials on $\mathcal{E}$. There is a linear bijection

$$\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \hat{A} \in P^n(\mathcal{E}).$$

Definition 2.3.1 *Let $\mathcal{U} \subset \mathcal{E}$ be open and $F : \mathcal{U} \to \mathbb{C}$ a function. Then F is said to be G-holomorphic (or Gâteaux-holomorphic) if and only if for all $\theta_0 \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping*

$$\mathbb{C} \ni \lambda \mapsto F(\theta_0 + \lambda\theta) \in \mathbb{C},$$

is holomorphic in some neighborhood of $0 \in \mathbb{C}$.

If F is G-holomorphic, then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{d^n F(\eta)}$ such that

$$F(\theta + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\eta)}(\theta),$$

for all θ from some open neighborhood $\mathcal{V}$ of zero. Of course, $\widehat{d^n F(\eta)}(\theta)$ is the n-th partial derivative of F at η in direction θ.

Definition 2.3.2 *Let $F : \mathcal{U} \to \mathbb{C}$ be a G-holomorphic function. Then*

1. *F is said to be holomorphic, if and only if for all $\eta \in \mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that*

$$\mathcal{V} \ni \theta \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\eta)}(\theta),$$

   converges uniformly on $\mathcal{V}$ to a continuous function.

2. *F is holomorphic at θ_0 if and only if there is an open set $\mathcal{U}$ containing θ_0 such that F is holomorphic on $\mathcal{U}$.*

3. *F is called entire if and only if F is holomorphic on $\mathcal{E}$.*

Useful in applications is the following proposition which follows from the above considerations, see [Din81, Chapter 2, Lemma 2.8].

Proposition 2.3.3 *F is holomorphic if and only if it is G-holomorphic and locally bounded.*

Let us consider the special case $\mathcal{E} = \mathcal{N}_{\mathbb{C}}$. In order to have uniqueness in the characterization of distributions spaces we must not discern between different restrictions of one function. Hence we consider germs of holomorphic functions at zero, i.e., we identify F and $\tilde{F}$ if there exists an open neighborhood of zero $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ such that $F(\xi) = \tilde{F}(\xi)$ for all $\xi \in \mathcal{U}$. Thus we define

Definition 2.3.4 $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ *as the algebra of germs of complex-valued functions F on $\mathcal{N}_{\mathbb{C}}$ which are holomorphic at zero equipped with the inductive topology given by the following family of norms*

$$\mathrm{n}_{p,l,\infty}(F) = \sup_{|\theta|_p \le 2^{-l}} |F(\theta)|, \ p, l \in \mathbb{N}.$$

A direct consequence of Proposition 2.3.3 is the following corollary.

Corollary 2.3.5 *Let $F : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}$ be given. Then $F \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ if and only if there exists $p \in \mathbb{N}$, $\varepsilon > 0$, and $C > 0$ such that*

1. *for all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \le \varepsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function of one complex variable $\mathbb{C} \ni \lambda \mapsto F(\xi_0 + \lambda\xi) \in \mathbb{C}$ is holomorphic at zero, and*
2. *for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \le \varepsilon$ we have $|F(\xi)| \le C$.*

Later on we need also the space $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ of vector-valued holomorphic functions from $\mathcal{N}_{\mathbb{C}}$ to $\mathcal{N}_{\mathbb{C}}$.

Definition 2.3.6 *A mapping $F : \mathcal{N}_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}}$ belongs to* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ *if and only if it is G-holomorphic and for each η in $\mathcal{N}_{\mathbb{C}}$ there exists $p \in \mathbb{N}$ such that the function*

$$\theta \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\eta)}(\theta),$$

converges and defines a continuous function on some $|\cdot|_p$-neighbourhood of zero.

If $F : \mathcal{N}_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}}$ is holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$, then for every $\eta \in \mathcal{N}_{\mathbb{C}}$ there exists a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{d^n F(\eta)}$ such that

$$\mathcal{V} \ni \theta \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\eta)}\,(\theta)\,, \tag{2.7}$$

converges and define a continuous function on some neighborhood $\mathcal{V}$ of zero.

2.3.2 Spaces of entire functions

In this subsection we introduce some spaces of entire functions which will be useful later on. Let $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ be the set of all entire functions on $\mathcal{N}'_{\mathbb{C}}$ of a minimal type whose order of growth is at most $k \in [1,2]$. This means that, as a set, $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ consists of the functions entire on each $\mathcal{H}_{-p,\mathbb{C}}$, $p \in \mathbb{N}$ and such that for all $\varphi \in \mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$, $p \in \mathbb{N}$, and every $\varepsilon > 0$ there exists $C > 0$ such that

$$|\varphi(z)| \leq C \exp(\varepsilon |z|^k_{-p}), \text{ for all } z \in \mathcal{H}_{-p,\mathbb{C}},$$

i.e., these are the functions of a minimal type on each $\mathcal{H}_{-p,\mathbb{C}}$ with the order of growth at most k. The space $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ is endowed with the projective limit topology with respect to the countable system of norms

$$m_{-p,l,k}(\varphi) = \sup_{z \in \mathcal{H}_{-p,\mathbb{C}}} \{|\varphi(z)| \exp(-1/l |z|^k_{-p})\}, \ p, l \in \mathbb{N}, \tag{2.8}$$

which is a ordered system. Since the first axiom of countability holds for $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ with this topology, it suffices to describe convergent sequences of functions from this space in order to define the topology under consideration. It follows from (2.8) that $\varphi_n \in \mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ converges to zero in $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ if and only if the following two conditions holds:

1. for any $p \in \mathbb{N}$ and $\varepsilon > 0$ there exists $C > 0$ such that

$$|\varphi_n(z)| \leq C \exp(\varepsilon |z|^k_{-p}), \ z \in \mathcal{H}_{-p,\mathbb{C}}, \ n \in \mathbb{N},$$

2. as $n \to 0$, φ_n converges to 0 uniformly on every ball

$$\{z \in \mathcal{H}_{-p,\mathbb{C}} | |z|_{-p} \leq R\}, \ R > 0,$$

   in each $\mathcal{H}_{-p,\mathbb{C}}$, $p \in \mathbb{N}$.

Let us introduce one more space of entire functions. Denote by $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$ the set of functions on $\mathcal{N}_{\mathbb{C}}$ with the following properties:

1. each function is entire on some $\mathcal{H}_{p,\mathbb{C}}$, $p \in \mathbb{N}$,
2. as an entire function on $\mathcal{H}_{p,\mathbb{C}}$, it has finite type for the order of growth equal k.

In other words, for $\Phi \in \mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$, $\exists p > 0$ and constants $C, K > 0$ such that the following estimate should holds

$$|\Phi(\eta)| \leq C \exp(K|\eta|^k_p),\ \eta \in \mathcal{N}_{\mathbb{C}}.$$

The space $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$ is topologize by the inductive limit topology with respect to the family of norms

$$M_{p,l,k}(\Phi) = \sup_{\eta \in \mathcal{N}_{\mathbb{C}}} \{|\Phi(\eta)| \exp(-l|\eta|^k_p)\},\ p, l \in \mathbb{N}.$$

Hence a sequence $(\Phi_n)_{n=1}^{\infty} \subset \mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$ converges to zero if and only if they are entire functions on the common Hilbert space $\mathcal{H}_{p,\mathbb{C}}$, there exists a uniform bound

$$|\Phi_n(\eta)| \leq C \exp(K|\eta|^k_p),\ \eta \in \mathcal{H}_{p,\mathbb{C}},\ n \in \mathbb{N},$$

and that Φ_n converges uniformly to 0 as $n \to \infty$ on every ball in $\mathcal{H}_{p,\mathbb{C}}$.

Remark 2.3.7 *Let us mention that the space $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ (resp. $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$) admits the following representation in terms of projective limit (resp. inductive limit) of the family of normed spaces $\mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$, $p, l \in \mathbb{Z}$ (resp. $\mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}})$, $p, l \in \mathbb{Z}$). More precisely, the set of all entire functions on each $\mathcal{H}_{-p,\mathbb{C}}$ (resp. on some $\mathcal{H}_{p,\mathbb{C}}$) of growth $k \in [1, 2]$ and type 2^{-l} (resp. 2^l) with norm given by*

$$n_{p,l,k}(\varphi) := \sup_{z \in \mathcal{H}_{-p,\mathbb{C}}} |\varphi(z)| \exp(-2^{-l}|z|^k_{-p}),\ \varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}),$$

$$(\text{resp. } n_{p,l,k}(\varphi) := \sup_{z \in \mathcal{H}_{p,\mathbb{C}}} |\varphi(z)| \exp(-2^l|z|^k_p),\ \varphi \in \mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}})).$$

The space of entire functions on $\mathcal{N}'_{\mathbb{C}}$ (resp. $\mathcal{N}_{\mathbb{C}}$) of growth k and minimal type (resp. maximal type) is naturally introduced by

$$\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}}) = \operatorname*{pr\,lim}_{p,l \in \mathbb{N}} \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}),$$

$$(\text{resp. } \mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}}) = \operatorname*{ind\,lim}_{p,l \in \mathbb{N}} \mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}})).$$

We now give an equivalent description of $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$. The Cauchy inequality and Corollary 2.1.4 allow to write the Taylor coefficients in a convenient form. Let $\varphi \in \mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $z \in \mathcal{N}'_{\mathbb{C}}$, then there exist kernels $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ such that

$$\langle z^{\otimes n}, \varphi^{(n)} \rangle = \frac{1}{n!} \widehat{d^n \varphi(0)}(z),$$

i.e.,

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle. \tag{2.9}$$

Let $\mathrm{E}^{\beta}_{p,q}$ denote the space of all functions of the form (2.9) such that the following Hilbert norm

$$|||\varphi|||^2_{p,q,\beta} := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |\varphi^{(n)}|^2_p, \; p, q \in \mathbb{N}, \tag{2.10}$$

is finite for all $\beta \in [0,1]$. (By $|\varphi^{(0)}|_p$ we simply mean the complex modulus for all p). The space $\mathrm{E}^{-\beta}_{-p-,q}$ with the norm $|||\varphi|||_{-p,-q,-\beta}$ is defined analogously. The following two theorems gives a description of the above introduced spaces of entire functions $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$, and $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$. For the proof we refer to [KSWY95], [Oue91], and [Wes95].

Theorem 2.3.8 *The following topological identity holds:*

$$\operatorname*{pr\,lim}_{p,q \in \mathbb{N}} \mathrm{E}^{\beta}_{p,q} = \mathcal{E}^{\frac{2}{1+\beta}}_{\min}(\mathcal{N}'_{\mathbb{C}}).$$

The proof is based on the following two lemmata which show that the two systems of norms are in fact equivalent.

Lemma 2.3.9 *Let* $\varphi \in \mathrm{E}^{\beta}_{p,q}$ *be given, then* $\varphi \in \mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ *for* $l = \frac{q}{1+\beta}$. *Moreover*

$$n_{p,l,k}(\varphi) \leq |||\varphi|||_{p,q,\beta}, \; k = \tfrac{2}{1+\beta}. \tag{2.11}$$

Lemma 2.3.10 *For any* $p', q \in \mathbb{N}$ *there exist* $p, l \in \mathbb{N}$ *such that*

$$\mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}) \subset \mathrm{E}^{\beta}_{p',q},$$

i.e., there exists a constant $C > 0$ *such that*

$$|||\varphi|||_{p',q,\beta} \leq C n_{p,l,k}(\varphi), \; \varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}), \; k = \tfrac{2}{1+\beta}.$$

The above results also imply the following theorem, see also [Oue91, Proposition 8.6] for related results.

Theorem 2.3.11 *If $\beta \in [0,1)$ then the following topological identity holds:*

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \mathrm{E}^{-\beta}_{-p,-q} = \mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}}).$$

If $\beta = 1$ we have

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \mathrm{E}^{-1}_{-p,-q} = \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}).$$

This theorem and its proof will appear in the context of Section 3.4. The characterization of distributions in infinite dimensional analysis is strongly related to this theorem. Therefore we postpone the proof of the second part of this theorem to Section 3.4, Theorem 3.4.3. The first part is proved in [KSWY95, Theorem 37].

Chapter 3

Generalized functions in infinite dimensional analysis

Non-Gaussian analysis was already introduced in [AKS93] for smooth probability measures on infinite dimensional linear spaces. The method used was biorthogonal decomposition which is a natural extension of the chaos decomposition that is well known in Gaussian analysis. This biorthogonal "Appell" system has been constructed for smooth measures by Yu. L. Daletski [Dal91]. For a detailed description of its use in infinite dimensional analysis and for the proof of the results which were announced in [AKS93] we refer to [ADKS96]. These results are based on quasi-invariance of the measures and smoothness of the logarithmic derivatives. Here we would like to mention that this approach does not cover the important case of Poisson measures, see Chapter 5 for more details of Poisson measures.

Kondratiev et al. [KSWY95] considered the case of non-degenerate measures on the dual of a nuclear space with analytic characteristic functionals for which no further condition such as quasi-invariance of the measure or smoothness of the logarithmic derivative was required. In this case the important example of Poisson measures is now accessible. Again for a given measure μ with analytic Laplace transform they construct an Appell biorthogonal system $\mathbb{A}^\mu$ as a pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ of Appell polynomials $\mathbb{P}^\mu$ and a canonical system of generalized functions $\mathbb{Q}^\mu$, properly associated to the measure μ. Hence within this framework they obtained:

- explicit description of the test function space introduced in [ADKS96];
- the test functions space is identical for all measures that they consider;

- characterization theorems for generalized as well as test functions were obtained analogously as in Gaussian analysis, see [KLP+96] for more references;
- extension of the Wick product and the corresponding Wick calculus (see [KLS96] for this notion) as well as a full description of positive distributions (as measures).

In this chapter we will recall this construction under the same assumptions on the measure μ (cf. Assumptions 3.1.3 and 3.1.6 below) as well as the aforementioned results. In the next chapter we are going to generalize them along the lines presented here.

3.1 Measures on linear topological spaces

Given the triple (2.1) we would like to introduce probability measures on the vector space $\mathcal{N}'$. In $\mathcal{N}'$ we consider the σ-algebra $\mathcal{C}_\sigma(\mathcal{N}')$ generated by the cylinder sets on $\mathcal{N}'$. Let us describe this more precisely. Consider the following collection of finite dimensional subsets from $\mathcal{N}$:

$$\mathcal{L} := \{L | L \subset \mathcal{N},\ \dim L < \infty\}.$$

For any $L \in \mathcal{L}$ and every $A \in \mathcal{B}(L)$ (i.e., the Borel σ-algebra on L) we introduce the set

$$C(L, A) := \{x \in \mathcal{N}' | P_L(x) \in A\}, \tag{3.1}$$

which is called the *cylinder* set from $\mathcal{N}'$ with coordinate L. Here P_L is an orthogonal projector onto L defined in $\mathcal{H}$ which extends to $\mathcal{N}'$ by continuity. Then we define the σ-algebra $\mathcal{C}(L, \mathcal{N}')$ with a fixed coordinate L using the cylinder sets $C(L, A)$ by

$$\mathcal{C}(L, \mathcal{N}') := \{C(L, A) | A \in \mathcal{B}(L)\}.$$

Finally the σ-algebra of cylinder sets $\mathcal{C}_\sigma(\mathcal{N}')$ is defined by

$$\mathcal{C}_\sigma(\mathcal{N}') := \sigma\left(\bigcup_{L \in \mathcal{L}} \mathcal{C}(L, \mathcal{N}')\right).$$

Remark 3.1.1 *It is sometimes convenient to introduce cylinder sets (3.1) by employing the "coordinate" method. To this end we choose an orthonormal basis* $(e_k)_{k=1}^n$ *in* L *which is also orthonormal in* $\mathcal{N}$*. Then for any* $x \in \mathcal{N}'$ *we have*

$$P_L(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k,$$

which implies the following representation for $C(L, A)$*,* $A \in \mathcal{B}(\mathbb{R}^n)$*:*

$$C(L, A) = \{x \in \mathcal{N}' | (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \in A\}.$$

Moreover, if we replace the vectors e_k *by arbitrary vectors* $\xi_k \in \mathcal{N}$*, then the set obtained as a result*

$$\{x \in \mathcal{N}' | (\langle x, \xi_1 \rangle, \ldots, \langle x, \xi_n \rangle) \in B\},\ B \in \mathcal{B}(\mathbb{R}^n).$$

will be cylinder too.

Let us consider one more topology on $\mathcal{N}'$, the so-called *weak topology* $\mathcal{B}_\sigma(\mathcal{N}')$. It is given by the following system of base neighborhoods of zero

$$\mathcal{U}(0; \{\xi_1, \ldots, \xi_n\}) := \{x \in \mathcal{N}' || \langle x, \xi_j \rangle | < 1,\ n \in \mathbb{N},\ j = 1, \ldots n\},$$

for any $\xi_1, \ldots, \xi_n \in \mathcal{N}$ and $x \in \mathcal{N}'$. The weak topology $\mathcal{B}_\sigma(\mathcal{N}')$ is consistent with the duality of the triple (2.1) in the following sense: a set of linear functionals over $\mathcal{N}'$ coincides with $\mathcal{N}$ in this topology, i.e., an arbitrary functional x of this kind admits a representation $x(\xi) = (x, \xi)$, for some $\xi \in \mathcal{N}$, and an arbitrary vector $\xi \in \mathcal{N}$ generates (according to this formula) a linear $\mathcal{B}_\sigma(\mathcal{N}')$-continuous functional on $\mathcal{N}'$.

Remark 3.1.2 *In the case of a countable Hilbert space* $\mathcal{N}$ *(which is in fact the case we handle), we have that the* σ*-algebras* $\mathcal{B}_\sigma(\mathcal{N}')$ *and* $\mathcal{B}_\beta(\mathcal{N}')$ *generated by the weak and strong topology on* $\mathcal{N}'$*, respectively and the* σ*-algebra generated by the cylinder sets* $\mathcal{C}_\sigma(\mathcal{N}')$ *coincide, i.e.,*

$$\mathcal{C}_\sigma(\mathcal{N}') = \mathcal{B}_\sigma(\mathcal{N}') = \mathcal{B}_\beta(\mathcal{N}').$$

Thus we will consider this σ*-algebra as the natural* σ*-algebra on* $\mathcal{N}'$*. We refer to e.g., [BK95] and [HKPS93, Appendix 5] and references therein for more details and historical remarks.*

We will restrict our investigations to a special class of probability measures μ on $\mathcal{C}_\sigma(\mathcal{N}')$ which satisfy two additional assumptions, see Assumptions (3.1.3) and (3.1.6) below. The first one concerns some analyticity of the Laplace transform

$$l_\mu(\theta) := \int_{\mathcal{N}'} \exp\langle x, \theta\rangle d\mu(x) =: \mathbb{E}_\mu(\exp\langle\cdot, \theta\rangle),\ \theta \in \mathcal{N}_{\mathbb{C}}.$$

Here we also have introduced the convenient notion of expectation $\mathbb{E}_\mu$ of a μ-integrable function.

Assumption 3.1.3 *The measure μ has an analytic Laplace transform in a neighborhood of zero. That means, there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero, such that l_μ is holomorphic on $\mathcal{U}$, i.e., $l_\mu \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. This class of analytic measures is denoted by $\mathcal{M}_a(\mathcal{N}')$.*

An equivalent description of analytic measures is given by the following lemma.

Lemma 3.1.4 *The following statements are equivalent.*

1. $\mu \in \mathcal{M}_a(\mathcal{N}')$,
2. $\exists p_\mu \in \mathbb{N},\ \exists C > 0 : \left|\int_{\mathcal{N}'} \langle x, \theta\rangle^n d\mu(x)\right| \le n!\, C^n |\theta|_{p_\mu}^n,\ \theta \in \mathcal{H}_{p_\mu,\mathbb{C}}$,
3. $\exists p'_\mu \in \mathbb{N},\ \exists \varepsilon_\mu > 0 : \int_{\mathcal{N}'} \exp(\varepsilon_\mu |x|_{-p'_\mu}) d\mu(x) < \infty.$

The proof can be found in [KSW95].

For $\mu \in \mathcal{M}_a(\mathcal{N}')$ the estimate in statement 2 of the above lemma allows to define the moment kernels $M_n^\mu \in \mathcal{N}'^{\hat{\otimes} n}$ of μ. This is done by extending the above estimate by a simple polarization argument and applying the kernel theorem (cf. Theorem 2.1.3). The kernels are determined by

$$l_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M_n^\mu, \theta^{\otimes n}\rangle, \tag{3.2}$$

or equivalently

$$\langle M_n^\mu, \theta_1 \hat{\otimes} \ldots \hat{\otimes} \theta_n\rangle = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} l_\mu(t_1\theta_1 + \ldots + t_n\theta_n)|_{t_1 = \ldots = t_n = 0}. \tag{3.3}$$

Moreover, if $p > p_\mu$ is such that the embedding $\iota_{p,p_\mu} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_\mu}$ is Hilbert-Schmidt, then there exists $C > 0$ such that

$$|M_n^\mu|_{-p} \leq (nC \left\|\iota_{p,p_\mu}\right\|_{HS})^n \leq n!(eC \left\|\iota_{p,p_\mu}\right\|_{HS})^n. \tag{3.4}$$

Definition 3.1.5 *A function $\varphi : \mathcal{N}' \to \mathbb{C}$ of the form*

$$\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle,\ x \in \mathcal{N}',\ N \in \mathbb{N},$$

is called a smooth continuous polynomial (for short $\varphi \in \mathcal{P}(\mathcal{N}')$) if and only if $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $\forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, $\langle \cdot^{\otimes n}, \varphi^{(n)} \rangle$ is called smooth continuous monomial.

Now we are ready to formulate the second assumption on μ.

Assumption 3.1.6 *For all $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\varphi = 0$ μ-a.e. we have $\varphi \equiv 0$. In the following a measure with this property will be called non-degenerate.*

Remark 3.1.7

1. *Assumption 3.1.6 can be formulated as follows. For any continuous polynomial $\varphi \in \mathcal{P}(\mathcal{N}')$ with*

$$\int_A \varphi(x) d\mu(x) = 0, \text{ for all } A \in \mathcal{C}_\sigma(\mathcal{N}'),$$

we have $\varphi \equiv 0$.

2. *A sufficient condition can be obtained by regarding admissible shifts of the measure μ. If $\mu(\cdot + \xi)$ is absolutely continuous with respect to μ for all $\xi \in \mathcal{N}$, i.e., there exists the Radon-Nikodym derivative*

$$\rho_\mu(\xi, x) = \frac{d\mu(x+\xi)}{d\mu(x)} \in L^1(\mathcal{N}', \mu),\ x \in \mathcal{N}',$$

then we say that μ is $\mathcal{N}$-quasi-invariant (in other words, the set of admissible shifts contains a linear manifold dense in $\mathcal{N}$), see e.g., [GV68], [Sko74] for more details. This is sufficient to ensure Assumption 3.1.6, see e.g., [KT91], [BK95].

Example 3.1.8 (Gaussian measures) *In Gaussian Analysis (especially white noise analysis) the Gaussian measure $\gamma_{\mathcal{H}}$ corresponding to the Hilbert space $\mathcal{H}$ is considered. Its Laplace transform is given by*

$$l_{\gamma_{\mathcal{H}}}(\theta) = e^{\frac{1}{2}|\theta|^2},\ \theta \in \mathcal{N}_{\mathbb{C}},$$

hence $\gamma_{\mathcal{H}} \in \mathcal{M}_a(\mathcal{N}')$. It is well known that $\gamma_{\mathcal{H}}$ is $\mathcal{N}$-quasi-invariant (moreover $\mathcal{H}$-quasi-invariant) see e.g., [Sko74]. Due to the previous remark $\gamma_{\mathcal{H}}$ satisfies also Assumption 3.1.6.

Example 3.1.9 (Poisson measures) *Let us consider the classical (real) Schwartz triple*

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}).$$

The Poisson white noise measure π is defined as a probability measure on $\mathcal{C}_\sigma(S'(\mathbb{R}))$ with Laplace transform

$$l_\pi(\theta) = \exp\left[\int_{\mathbb{R}} (e^{\theta(t)} - 1)dt\right],\ \theta \in S(\mathbb{R}),$$

see e.g., [GV68]. It is not hard to see that l_π is a holomorphic function on $S(\mathbb{R})$, so Assumption 3.1.3 is satisfied. But to check Assumption 3.1.6, we need additional considerations. We will prove this fact in the next chapter in the context of the construction of generalized Appell systems.

Remark 3.1.10 *In Chapter 5 we will return to Poisson measures and give detailed analysis as well as its applications, for instances intrinsic and extrinsic geometry on Poisson spaces.*

3.2 Concept of generalized functions in infinite dimensional non-Gaussian analysis

In this section we will introduce a preliminary distribution theory in infinite dimensional non-Gaussian analysis. We want to point out in advance that the distribution space constructed here is in some sense too big for practical purposes. Therefore this section may be viewed as a stepping stone to introduce the more useful structures in Section 3.3 Subsection 3.3.1 and 3.3.2.

We will choose $\mathcal{P}(\mathcal{N}')$ as our minimal test function space (see, for instance, [KMP65] for a discussion in which sense this space is minimal). First we have to ensure that $\mathcal{P}(\mathcal{N}')$ is densely embedded in $L^2(\mu)$. This is fulfilled because of our Assumption 3.1.3 (see [Sko74, Section 10 Theorem 1]). The space $\mathcal{P}(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes a nuclear space, see [BK95]. The topology on $\mathcal{P}(\mathcal{N}')$ is chosen such that is becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$ see e.g., [Sch71, Chapter II-6.1, Chapter III-7.4]

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n},$$

via

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi^{(n)} \rangle \longleftrightarrow \vec{\varphi} = (\varphi^{(n)})_{n=0}^{\infty}.$$

Note that only a finite number of $\varphi^{(n)}$ are non-zero. The notion of convergence of sequences in this topology on $\mathcal{P}(\mathcal{N}')$ is the following: for any $\varphi \in \mathcal{P}(\mathcal{N}')$ such that

$$\varphi(x) = \sum_{n=0}^{N(\varphi)} \langle x^{\otimes n}, \varphi^{(n)} \rangle,$$

let

$$p_n : \mathcal{P}(\mathcal{N}') \to \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n},$$

denote the mapping p_n defined by $p_n(\varphi) := \varphi^{(n)}$. A sequence $(\varphi_j)_{j=1}^{\infty}$ of smooth continuous polynomials converge to $\varphi \in \mathcal{P}(\mathcal{N}')$ if and only if the $N(\varphi_j)$ are bounded and

$$p_n \varphi_j \underset{j\to\infty}{\longrightarrow} p_n \varphi \ \text{ in } \ \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n} \ \text{ for all } n \in \mathbb{N}.$$

Now we can introduce the dual space $\mathcal{P}'_\mu(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to $L^2(\mu)$. As a result we have constructed the triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}').$$

The (bilinear) dual pairing $\langle\langle \cdot, \cdot \rangle\rangle_\mu$ between $\mathcal{P}'_\mu(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by

$$\langle\langle \varphi, \psi \rangle\rangle_\mu = (\!(\varphi, \bar{\psi})\!)_{L^2(\mu)}, \ \varphi \in L^2(\mu), \ \psi \in \mathcal{P}(\mathcal{N}').$$

Since the constant function $1 \in \mathcal{P}(\mathcal{N}')$ we may extend the concept of expectation from random variables to generalized functions: for any $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle_\mu.$$

The main goal of this section is to provide a description for $\mathcal{P}'_\mu(\mathcal{N}')$, see Theorem 3.2.11 below (here we mention that in the next chapter we are going to present a generalization of the approach exhibited in this section where this is a special case). The simplest approach to this problem seems to be the use of so called Appell polynomials.

3.2.1 Appell polynomials associated to the measure μ

Because of the holomorphy of the Laplace transform l_μ and the fact that $l_\mu(0) = 1$, there exists a neighborhood of zero in $\mathcal{N}_\mathbb{C}$

$$\mathcal{U}_0 = \{\theta \in \mathcal{N}_\mathbb{C} | 2^{q_0} |\theta|_{p_0} < 1\},$$

where $p_0, q_0 \in \mathbb{N}$, $p_0 \geq p'_\mu$, $2^{-q_0} \leq \varepsilon_\mu$ (p'_μ, ε_μ from Lemma 3.1.4 - 3) such that $l_\mu(\theta) \neq 0$ for any $\theta \in \mathcal{U}_0$. Thus the *normalized exponential*

$$e_\mu(\theta; z) := \frac{\exp\langle z, \theta\rangle}{l_\mu(\theta)}, \text{ for any } \theta \in \mathcal{U}_0,\ z \in \mathcal{N}'_\mathbb{C}, \tag{3.5}$$

is well defined. We use the holomorphy of $\theta \mapsto e_\mu(\theta; z)$ to expand it in a power series in θ similar to the case corresponding to the construction of one dimensional Appell polynomials, see e.g., [Bou76]. We have in analogy to [AKS93], [ADKS96]

$$e_\mu(\theta; z) = \sum_{n=0}^\infty \frac{1}{n!} d^n \widehat{e_\mu(0; z)}(\theta),$$

where $d^n \widehat{e_\mu(0; z)}$ is a n-homogeneous continuous polynomial. Since $e_\mu(\theta; z)$ is not only G-holomorphic but also holomorphic we know that $\theta \mapsto e_\mu(\theta; z)$ is also locally bounded. Thus Cauchy's inequality for Taylor series (cf. [Din81, Chapter 2] may be applied. Hence we choose $\rho \leq 2^{-q_0}$, $p \geq p_0$ and estimate the n-th coefficient in Taylor's series by

$$\frac{1}{n!} |d^n \widehat{e_\mu(0; z)}(\theta)| \leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} |e_\mu(\theta; z)| |\theta|_p^n \leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} \frac{1}{l_\mu(\theta)} \exp(\rho |z|_{-p}) |\theta|_p^n, \tag{3.6}$$

for any $z \in \mathcal{H}_{-p,\mathbb{C}}$. This inequality extends by polarization [Din81, Chapter 1 Theorem 1.5] to an estimate sufficient for the kernel theorem. Thus we have a representation

$$d^n\widehat{e_\mu(0;z)}(\theta) = \langle P_n^\mu(z), \theta^{\otimes n}\rangle,$$

where

$$P_n^\mu : \mathcal{N}'_{\mathbb{C}} \longrightarrow \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}.$$

The kernel theorem really gives a little more: $P_n^\mu(z) \in \mathcal{H}^{\hat{\otimes} n}_{-p',\mathbb{C}}$ for any p' $(> p \geq p_0)$ such that the embedding operator

$$\iota_{p',p} : \mathcal{H}_{p',\mathbb{C}} \hookrightarrow \mathcal{H}_{p,\mathbb{C}},$$

is Hilbert-Schmidt. Thus we have

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle P_n^\mu(z), \theta^{\otimes n}\rangle, \text{ for any } \theta \in \mathcal{U}_0,\ z \in \mathcal{N}'_{\mathbb{C}}. \tag{3.7}$$

We will also use the notation

$$P_n^\mu(\varphi^{(n)})(\cdot) := \langle P_n^\mu(\cdot), \varphi^{(n)}\rangle,\ \varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N},$$

which is called *Appell polynomial*. Thus for any measure satisfying Assumption 3.1.3 we have defined the $\mathbb{P}^\mu$-system

$$\mathbb{P}^\mu = \{\langle P_n^\mu(\cdot), \varphi^{(n)}\rangle | \varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N}_0\}.$$

The following proposition summarize some useful properties of the Appell polynomials $P_n^\mu(\cdot)$.

Proposition 3.2.1 *The system of Appell polynomials has the following properties.*

(P1) *For any $z \in \mathcal{N}'_{\mathbb{C}}$ and $n \in \mathbb{N}$*

$$P_n^\mu(z) = \sum_{k=0}^{n} \binom{n}{k} z^{\otimes k} \hat{\otimes} P_{n-k}^\mu(0). \tag{3.8}$$

(P2) *For every $z \in \mathcal{N}'_{\mathbb{C}}$ and M_n^μ as defined in (3.3) we have*

$$z^{\otimes n} = \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(z) \hat{\otimes} M_{n-k}^\mu. \tag{3.9}$$

(P3) *For all $z, w \in \mathcal{N}'_{\mathbb{C}}$ and M_n^μ as above*

$$
\begin{aligned}
P_n^\mu(z+w) &= \sum_{k+l+m=n} \frac{n!}{k!\,l!\,m!} P_k^\mu(z) \hat{\otimes} P_l^\mu(w) \hat{\otimes} M_m^\mu \\
&= \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(z) \hat{\otimes} w^{\otimes(n-k)}. \qquad (3.10)
\end{aligned}
$$

(P4) *Further we observe*

$$
\mathbb{E}_\mu(\langle P_m^\mu(\cdot), \varphi^{(m)} \rangle) = 0, \text{ for } m \neq 0, \; \varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}. \qquad (3.11)
$$

(P5) *For all $p > p_0$ such that the embedding $\iota_{p,p_0} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0}$ is Hilbert–Schmidt and for all $\varepsilon > 0$ small enough ($\varepsilon \leq (2^{q_0} e \, \|i_{p,p_0}\|_{HS})^{-1}$) there exists a constant $C(p, \varepsilon) > 0$ with*

$$
|P_n^\mu(z)|_{-p} \leq C(p,\varepsilon) \, n! \, \varepsilon^{-n} \exp(\varepsilon |z|_{-p}), \; z \in \mathcal{H}_{-p,\mathbb{C}}. \qquad (3.12)
$$

Remark 3.2.2 *Notice that formula (3.8) together with*

$$
\frac{1}{l_\mu(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\mu(0), \theta^{\otimes n} \rangle, \; \theta \in \mathcal{N}_{\mathbb{C}}, \; |\theta|_q < \delta > 0, \; q \in \mathbb{N},
$$

can also be used as an alternative definition of the polynomials $P_n^\mu(\cdot)$.

The proof of the above proposition will be given in a more general framework in Chapter 4 (cf. Proposition 4.1.1), therefore we postpone its proof to Chapter 4.

The following lemma describes the set of polynomials $\mathcal{P}(\mathcal{N}')$.

Lemma 3.2.3 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$
\varphi(x) = \sum_{n=0}^{N} \langle P_n^\mu(x), \varphi^{(n)} \rangle, \; \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \qquad (3.13)
$$

and vice versa, any functional of the form (3.13) is a smooth continuous polynomial.

Proof. The representation from Definition 3.1.5 and equation (3.13) can be transformed into one another using (3.8) and (3.9). ■

3.2.2 The dual Appell system and description of $\mathcal{P}'_\mu(\mathcal{N}')$

To give an internal description of the type (3.13) for $\mathcal{P}'_\mu(\mathcal{N}')$ we have to construct an appropriate system of generalized functions, the $\mathbb{Q}^\mu$-system. We propose to construct the $\mathbb{Q}^\mu$-system using differential operators (this approach has the advantage to cover the important example of Poisson measures, cf. Example 3.1.9. In [ADKS96] they constructed such $\mathbb{Q}^\mu$-system but for smooth logarithmic derivative of the measure μ, therefore the Poisson measure π is not covered, see Example 3.1.9).

Definition 3.2.4 *Let $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ be given. We define a differential operator $D(\Phi^{(n)})$ (of order n and constant coefficients $\Phi^{(n)}$)*

$$D(\Phi^{(n)}) : \mathcal{P}(\mathcal{N}') \longrightarrow \mathcal{P}(\mathcal{N}'),$$

which acts on smooth continuous monomials $\langle \cdot^{\otimes m}, \varphi^{(m)} \rangle$, $\varphi^{(m)} \in \mathcal{N}^{\hat{\otimes} m}_{\mathbb{C}}$, $m \in \mathbb{N}$

$$(D(\Phi^{(n)})\langle \cdot^{\otimes m}, \varphi^{(m)} \rangle)(x) := \begin{cases} \dfrac{m!}{(m-n)!} \langle x^{\otimes (m-n)} \hat{\otimes} \Phi^{(n)}, \varphi^{(m)} \rangle & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases}, \tag{3.14}$$

$x \in \mathcal{N}'$ and extend by linearity from smooth continuous monomials to $\mathcal{P}(\mathcal{N}')$.

Lemma 3.2.5 *The operator $D(\Phi^{(n)})$ is a continuous linear operator from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$.*

Remark 3.2.6 *For $\Phi^{(1)} \in \mathcal{N}'$ we have the usual Gâteaux derivative as e.g., in white noise analysis [HKPS93]:*

$$(D(\Phi^{(1)})\varphi)(x) = (D_{\Phi^{(1)}}\varphi)(x) := \frac{d}{dt}\varphi(x + t\Phi^{(1)})|_{t=0}, \ \varphi \in \mathcal{P}(\mathcal{N}').$$

Moreover we have $D((\Phi^{(1)})^{\otimes n}) = (D_{\Phi^{(1)}})^n$, thus $D((\Phi^{(1)})^{\otimes n})$ is a differential operator of order n.

Proof of Lemma 3.2.5. By definition $\mathcal{P}(\mathcal{N}')$ is isomorphic to the topological direct sum of tensor powers $\mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$, i.e.,

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}.$$

Via this isomorphism $D(\Phi^{(n)})$ transforms each component $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m}$, $m \geq n$ by

$$\varphi^{(m)} \mapsto \frac{n!}{(m-n)!}(\Phi^{(n)}, \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}},$$

where the contraction $(\Phi^{(n)}, \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}} \in \mathcal{N}_{\mathbb{C}}^{\otimes(m-n)}$ is defined by

$$\langle x^{\otimes(m-n)}, (\Phi^{(n)}, \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}}\rangle := \langle x^{\otimes(m-n)}\hat{\otimes}\Phi^{(n)}, \varphi^{(m)}\rangle, \tag{3.15}$$

for all $x \in \mathcal{N}'$. It is easy to verify that

$$|(\Phi^{(n)}, \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}}|_q \leq |\Phi^{(n)}|_{-q}|\varphi^{(m)}|_q, \ q \in \mathbb{N},$$

which guarantees that $(\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}} \in \mathcal{N}_{\mathbb{C}}^{\otimes(m-n)}$ and shows at the same time that $D(\Phi^{(n)})$ is continuous on each component. This is sufficient to ensure the stated continuity of $D(\Phi^{(n)})$ on $\mathcal{P}(\mathcal{N}')$. ■

In view of Lemma 3.2.5 it is possible to define the adjoint operator

$$D(\Phi^{(n)})^* : \mathcal{P}'_\mu(\mathcal{N}') \longrightarrow \mathcal{P}'_\mu(\mathcal{N}'), \ \Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}.$$

Further we introduce the constant function $1 \in L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}')$ such that $1\,(x) \equiv 1$ for all $x \in \mathcal{N}'$, so

$$\langle\!\langle 1, \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x) d\mu(x) = \mathbb{E}_\mu(\varphi), \ \varphi \in \mathcal{P}(\mathcal{N}').$$

Definition 3.2.7 *For any $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}$ we define a generalized function $Q_n^\mu(\Phi^{(n)}) \in \mathcal{P}'_\mu(\mathcal{N}')$ by*

$$Q_n^\mu(\Phi^{(n)}) := D(\Phi^{(n)})^* 1.$$

We want to introduce an additional formal notation which stresses the linearity of $\Phi^{(n)} \mapsto Q_n^\mu(\Phi^{(n)}) \in \mathcal{P}'_\mu(\mathcal{N}')$:

$$\langle Q_n^\mu, \Phi^{(n)} \rangle := Q_n^\mu(\Phi^{(n)}).$$

Example 3.2.8 *The simplest non trivial case can be studied using finite dimensional real analysis. We consider the nuclear "triple"*

$$\mathbb{R} \subseteq \mathbb{R} \subseteq \mathbb{R},$$

where the dual pairing between a "test function" and a "generalized function" degenerates to multiplication. On $\mathbb{R}$ we consider a measure $d\mu(x) := \rho(x)\,dx$ where ρ is a positive C^∞-function on $\mathbb{R}$ such that Assumptions 3.1.3 and 3.1.6 are fulfilled. In this setting the adjoint of the differentiation operator is given by

$$\left(\frac{d}{dx}\right)^* f(x) = -\left(\frac{d}{dx} + \beta(x)\right) f(x),\ f \in C^\infty(\mathbb{R}),$$

where β is the logarithmic derivative of the measure μ given by

$$\beta(x) = \frac{\nabla^{\mathbb{R}}\rho(x)}{\rho(x)} = \frac{\rho'(x)}{\rho(x)}.$$

This enables us to calculate the generalized functions $\mathbb{Q}^\mu_n$. One has

$$Q^\mu_n(x) = \left(\left(\frac{d}{dx}\right)^*\right)^n 1 = (-1)^n \left(\frac{d}{dx} + \beta(x)\right)^n 1 = (-1)^n \frac{\rho^{(n)}(x)}{\rho(x)},$$

where the last equality can be seen by simple induction (for ρ not C^∞-function this construction produces generalized functions Q^μ_n even in this one dimensional case).

If $\rho(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$ is the Gaussian density, then Q^μ_n is related to the n-th Hermite polynomial:

$$Q^\mu_n(x) = 2^{-n/2} H_n\left(2^{-1/2}x\right).$$

Now we are ready to define the $\mathbb{Q}^\mu$-system.

Definition 3.2.9 *We define the $\mathbb{Q}^\mu$-system in $\mathcal{P}'_\mu(\mathcal{N}')$ by*

$$\mathbb{Q}^\mu := \{Q^\mu_n(\Phi^{(n)}) | \Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N}_0\},$$

and the pair $\mathbb{A}^\mu = (\mathbb{P}^\mu, \mathbb{Q}^\mu)$ will be called the Appell system generated by the measure μ.

The Appell system $\mathbb{A}^\mu$ has the following central property.

Theorem 3.2.10 *The system of Appell polynomials $\mathbb{P}^\mu$ and the dual Appell system $\mathbb{Q}^\mu$ are biorthogonal with respect to μ and*

$$\langle\!\langle Q^\mu_n(\Phi^{(n)}), P^\mu_m(\varphi^{(m)}) \rangle\!\rangle_\mu = \delta_{m,n} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle, \tag{3.16}$$

for any $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ and all $\varphi^{(m)} \in \mathcal{N}^{\hat{\otimes} m}_{\mathbb{C}}$.

Now we are going to characterize the space $\mathcal{P}'_\mu(\mathcal{N}')$.

Theorem 3.2.11 *For any $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ there exists a unique sequence $(\Phi^{(n)})_{n=0}^\infty$, where $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$, $n \in \mathbb{N}_0$ such that*

$$\Phi = \sum_{n=0}^\infty Q_n^\mu(\Phi^{(n)}) \equiv \sum_{n=0}^\infty \langle Q_n^\mu, \Phi^{(n)}\rangle, \tag{3.17}$$

and vice versa, every series of the form (3.17) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

We will prove this property is more general form in the next chapter (cf. Theorem 4.2.7), see also [KSWY95, Theorem 19].

3.3 Spaces of test and generalized functions

3.3.1 Test functions on a linear space with measure

In this subsection we will construct the test function space $(\mathcal{N})^1$ and study its properties. On the space of continuous polynomials $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the Appell decomposition from Lemma 3.2.3. Let

$$\varphi(\cdot) = \sum_{n=0}^N \langle P_n^\mu(\cdot), \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}'),$$

be given, then $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n}$ for each $p \geq 0$, $n \in \mathbb{N}_0$. Thus we may define for any $p, q \in \mathbb{N}$ a Hilbert norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|_{p,q,\mu}^2 := \sum_{n=0}^\infty (n!)^2 2^{nq} |\varphi^{(n)}|_p^2.$$

The completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q,\mu}$ is denoted by $(\mathcal{H}_p)_{q,\mu}^1$, i.e.,

$$(\mathcal{H}_p)_{q,\mu}^1 := \overline{\mathcal{P}(\mathcal{N}')}^{\|\cdot\|_{p,q,\mu}}.$$

Definition 3.3.1 *We define the test function space as*

$$(\mathcal{N})_\mu^1 := \operatorname*{pr\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_p)_{q,\mu}^1.$$

This space has the following properties, see [KSWY95, Section 3.3] for proofs and Chapter 4 for its generalizations.

Theorem 3.3.2 *The space of test functions $(\mathcal{N})^1_\mu$ is a nuclear space. The topology $(\mathcal{N})^1_\mu$ is uniquely defined by the topology on $\mathcal{N}$. It does not depend on the choice of the family of norms $\{|\cdot|_p,\ p \in \mathbb{N}\}$.*

Theorem 3.3.3 *There exists $p', q' > 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu} \subset L^2(\mu)$ holds.*

Corollary 3.3.4 *The space $(\mathcal{N})^1_\mu$ is continuously and densely embedded in $L^2(\mu)$.*

Theorem 3.3.5 *Any test function φ in $(\mathcal{N})^1_\mu$ has a uniquely defined extension to $\mathcal{N}'_{\mathbb{C}}$ as an element of $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.*

In this construction one surprising moment was the following.

Theorem 3.3.6 *For all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$ we have the topological identity*

$$(\mathcal{N})^1_\mu = \mathcal{E}^1_{\min}(\mathcal{N}').$$

Since this last theorem states that the space of test functions $(\mathcal{N})^1_\mu$ is isomorphic to $\mathcal{E}^1_{\min}(\mathcal{N}')$ for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$, we will drop the subscript μ. The test function space $(\mathcal{N})^1$ is the same for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

3.3.2 Generalized functions

In this subsection we will introduce and study the space of generalized functions $(\mathcal{N})^{-1}_\mu$, i.e., the dual space of the space of test functions $(\mathcal{N})^1$. The space $(\mathcal{N})^{-1}_\mu$ of generalized functions can be viewed as a subspace of $\mathcal{P}'_\mu(\mathcal{N}')$, since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ topologically, i.e.,

$$(\mathcal{N})^{-1}_\mu \subset \mathcal{P}'_\mu(\mathcal{N}').$$

Let us introduce the Hilbert subspace $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ of $\mathcal{P}'_\mu(\mathcal{N}')$ for which the norm

$$\|\Phi\|^2_{-p,-q,\mu} := \sum_{n=0}^{\infty} 2^{-qn} |\Phi^{(n)}|^2_{-p},$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}\left(\Phi^{(n)}\right) \in \mathcal{P}'_{\mu}(\mathcal{N}'),$$

from Theorem 3.2.11. The space $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ is the dual space of $(\mathcal{H}_p)^1_{q,\mu}$ with respect to $L^2(\mu)$ (because of the biorthogonality of $\mathbb{P}^{\mu}$- and $\mathbb{Q}^{\mu}$-systems). By the general duality theory

$$(\mathcal{N})^{-1}_{\mu} := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu},$$

is the dual space of $(\mathcal{N})^1$ with respect to $L^2(\mu)$. So, we have the topological nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu}.$$

The action of a generalized function

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)}) \in (\mathcal{N})^{-1}_{\mu},$$

on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu}, \varphi^{(n)}\rangle \in (\mathcal{N})^1,$$

is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

For a more detailed characterization of the singularity of the generalized functions in $(\mathcal{N})^{-1}_{\mu}$ we will introduce some subspaces in this space. For $\beta \in [0,1]$ we define

$$(\mathcal{H}_{-p})^{-\beta}_{-q,\mu} := \left\{ \mathcal{P}'_{\mu}(\mathcal{N}') \ni \Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)}) | \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |\Phi^{(n)}|^2_{-p} < \infty \right\},$$

and also

$$(\mathcal{N})^{-\beta}_{\mu} := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-\beta}_{-q,\mu}.$$

It is clear that the singularity increases with increasing β. Thus

$$(\mathcal{N})^{-0} \subset (\mathcal{N})^{-\beta_1} \subset (\mathcal{N})^{-\beta_2} \subset (\mathcal{N})^{-1}, \text{ for any } \beta_1 \le \beta_2.$$

We will also consider $(\mathcal{N})^{\beta}_{\mu}$ equipped with the natural topology.

Example 3.3.7 (Generalized Radon-Nikodym derivative) *We want to define a generalized function $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property*

$$\langle\!\langle \rho_\mu(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\,d\mu(x),\ \varphi \in (\mathcal{N})^1.$$

That means, we have to establish the continuity of $\rho_\mu(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$, $p \in \mathbb{N}$. If $p' \geq p$ is sufficiently large and $\epsilon > 0$ is small enough, there exists $q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned}\left|\int_{\mathcal{N}'} \varphi(x-z)\,d\mu(x)\right| &\leq C\,\|\varphi\|_{p',q,\mu} \int_{\mathcal{N}'} \exp(\epsilon|x-z|_{-p'})\,d\mu(x) \\ &\leq C\,\|\varphi\|_{p',q,\mu} \exp(\epsilon|z|_{-p'}) \int_{\mathcal{N}'} \exp(\epsilon|x|_{-p'})\,d\mu(x).\end{aligned}$$

If ϵ is chosen sufficiently small the last integral exists because of Lemma 3.1.4-3. Thus we have in fact $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$. It is clear that whenever the Radon-Nikodym derivative $\frac{d\mu(x+\xi)}{d\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu(z,\cdot)$ defined above. We will show that in $(\mathcal{N})^{-1}_\mu$ we have the canonical expansion

$$\rho_\mu(z,\cdot) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^\mu_n(z^{\otimes n}),\ z \in \mathcal{N}'_{\mathbb{C}}.$$

Since both sides of the above equality are in $(\mathcal{N})^{-1}_\mu$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$ we have

$$\begin{aligned}\langle\!\langle \rho_\mu(z,\cdot), \langle P^\mu_n, \varphi^{(n)}\rangle \rangle\!\rangle_\mu &= \int_{\mathcal{N}'} \langle P^\mu_n(x-z), \varphi^{(n)}\rangle\,d\mu(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{\mathcal{N}'} \langle P^\mu_k(x) \hat{\otimes} z^{\otimes(n-k)}, \varphi^{(n)}\rangle\,d\mu(x) \\ &= (-1)^n \langle z^{\otimes n}, \varphi^{(n)}\rangle \\ &= \left\langle\!\!\left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k Q^\mu_k(z^{\otimes k}), \langle P^\mu_n, \varphi^{(n)}\rangle \right\rangle\!\!\right\rangle_\mu ,\end{aligned}$$

where we have used (3.10), (3.11) and the biorthogonality of Appell system $\mathbb{A}^\mu$. In other words, we have proven that $\rho_\mu(-z,\cdot)$ is the generating function *of the $\mathbb{Q}^\mu$-system*

$$\rho_\mu(-z, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^\mu(z^{\otimes n}).$$

3.4 Characterization theorems

Gaussian Analysis has shown that for applications it is very useful to characterize test and generalized functions by integral transforms. In the non-Gaussian setting these results have been obtained by [AKS93] and [ADKS96]. Thus, first we will present the necessary integral transforms for the mentioned characterizations.

3.4.1 Normalized Laplace transform - S_μ

We first introduce the Laplace transform of a function $\varphi \in L^2(\mu)$. The global assumption $\mu \in \mathcal{M}_a(\mathcal{N}')$ guarantees the existence of $p'_\mu \in \mathbb{N}, \epsilon_\mu > 0$ such that

$$\int_{\mathcal{N}'} \exp(-\epsilon_\mu |x|_{-p_\mu}) d\mu(x) < \infty,$$

by Lemma 3.1.4. Thus $\exp\langle \cdot, \theta \rangle \in L^2(\mu)$ if $2|\theta|_{p'_\mu} < \epsilon_\mu$, $\theta \in \mathcal{H}_{p'_\mu, \mathbb{C}}$. Then by Cauchy-Schwarz inequality the Laplace transform

$$(L_\mu \varphi)(\theta) := \int_{\mathcal{N}'} \varphi(x) \exp\langle x, \theta \rangle d\mu(x),$$

is well defined for $\varphi \in L^2(\mu)$, $\theta \in \mathcal{H}_{p'_\mu, \mathbb{C}}$. Now we are interested to extend this integral transform from $L^2(\mu)$ to the space of generalized functions $(\mathcal{N})_\mu^{-1}$.

Since the construction of test and generalized functions spaces is closely related to $\mathbb{P}^\mu$- and $\mathbb{Q}^\mu$-systems, it is useful to introduce the so called S_μ-transform, in other words the normalized Laplace transform

$$(S_\mu \varphi)(\theta) := \frac{L_\mu \varphi(\theta)}{l_\mu(\theta)} = \int_{\mathcal{N}'} \varphi(x) e_\mu(\theta; x) d\mu(x).$$

The normalized exponential $e_\mu(\theta; \cdot)$ is not a test function in $(\mathcal{N})^1$, see [KSWY95, Example 6], so the definition of the S_μ-transform of a generalized functions $\Phi \in (\mathcal{N})_\mu^{-1}$ must be more careful. Every such Φ is of finite order, i.e., $\exists p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})_{-q,\mu}^{-1}$ and $e_\mu(\theta; \cdot)$ is in the corresponding dual space $(\mathcal{H}_p)^1_{q,\mu}$ if $\theta \in \mathcal{H}_{p,\mathbb{C}}$ is such that $2^q\,|\theta|_p^2 < 1$. Then we can define a consistent extension of the S_μ-transform by

$$(S_\mu \Phi)(\theta) := \langle\!\langle \Phi, e_\mu(\theta,\cdot)\rangle\!\rangle_\mu,$$

if θ is chosen in the above way. The biorthogonality of $\mathbb{P}^\mu$- and $\mathbb{Q}^\mu$-systems implies

$$(S_\mu \Phi)(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n}\rangle, \tag{3.18}$$

moreover $S_\mu \Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$, see [KSWY95, Theorem 35] and proof of Theorem 3.4.3 below.

3.4.2 Convolution - C_μ

The third integral transform we are going to introduce is more appropriate for the test function space $(\mathcal{N})^1$. We define the convolution of a function $\varphi \in (\mathcal{N})^1$ with the measure μ by

$$(C_\mu \varphi)(y) := \int_{\mathcal{N}'} \varphi(x+y)\,d\mu(x),\ y \in \mathcal{N}'.$$

For any $\varphi \in (\mathcal{N})^1$, $z \in \mathcal{N}'_\mathbb{C}$, the convolution has the representation

$$(C_\mu \varphi)(z) = \langle\!\langle \rho_\mu(-z,\cdot), \varphi\rangle\!\rangle_\mu.$$

If $\varphi \in (\mathcal{N})^1$ has the canonical $\mathbb{P}^\mu$-decomposition

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)}\rangle,$$

then

$$(C_\mu \varphi)(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)}\rangle.$$

In Gaussian analysis C_μ- and S_μ-transform coincide. It is a typical non-Gaussian effect that these two transformations differ from each other.

Now we will characterize the spaces of test and generalized functions by the integral transforms introduced before. Since these characterizations are fundamental in applications, we will give a proof borrowed from [KSWY95, Theorem 33 and 35], see also [BK95].

We will start to characterize the space $(\mathcal{N})^1$ in terms of the convolution C_μ.

Theorem 3.4.1 *The convolution C_μ is a topological isomorphism from $(\mathcal{N})^1$ on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.*

Remark 3.4.2 *Since we have identified the test function space $(\mathcal{N})^1$ and $\mathcal{E}^1_{\min}(\mathcal{N}')$ by Theorem 3.3.6, the above assertion can be restated as follows. We have*

$$C_\mu : \mathcal{E}^1_{\min}(\mathcal{N}') \to \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}),$$

as a topological isomorphism.

Proof of Theorem 3.4.1. The proof has been well prepared by Theorem 2.3.8, because the nuclear topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ is the most natural one from the point of view of the above theorem. Let $\varphi \in (\mathcal{N})^1$ with decomposition

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle.$$

From the above considerations it follows

$$(C_\mu \varphi)(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle.$$

It is obvious from (2.10) that

$$|\!|\!| C_\mu \varphi |\!|\!|_{p,q,1} = \|\varphi\|_{p,q,\mu}\,,$$

for all $p, q \in \mathbb{N}_0$, which proves the continuity of

$$C_\mu : (\mathcal{N})^1 \to \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}).$$

Conversely let $F \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ be given. Then Theorem 2.3.8 ensures the existence of a sequence of generalized kernels $\{\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} | n \in \mathbb{N}_0\}$ such that

$$F(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle, \; z \in \mathcal{N}'_{\mathbb{C}}.$$

Moreover, for all $p, q \in \mathbb{N}_0$

$$|\!|\!| F |\!|\!|^2_{p,q,1} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|_p^2,$$

is finite. Choosing

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu}, \varphi^{(n)} \rangle,$$

we have

$$\|\varphi\|_{p,q,\mu} = \|\!|F|\!\|_{p,q,1}.$$

Thus $\varphi \in (\mathcal{N})^1$. Since $C_\mu \varphi = F$ we have shown the existence and continuity of the inverse of C_μ. ■

Next Theorem characterizes generalized functions from $(\mathcal{N})_\mu^{-1}$ in terms of S_μ-transform.

Theorem 3.4.3 *The S_μ-transform is a topological isomorphism from $(\mathcal{N})_\mu^{-1}$ on* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$.

Proof. Let $\Phi \in (\mathcal{N})_\mu^{-1}$. Then there exists $p, q \in \mathbb{N}$ such that

$$\|\Phi\|_{-p,-q,\mu}^2 = \sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|_{-p}^2 < \infty.$$

It follows from (3.18) that

$$(S_\mu \Phi)(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle. \tag{3.19}$$

For $\theta \in \mathcal{N}_{\mathbb{C}}$ such that $2^q |\theta|_p^2 < 1$ we have by definition (cf. (2.10))

$$\|\!|S_\mu \Phi|\!\|_{-p,-q,-1} = \|\Phi\|_{-p,-q,\mu}.$$

By Cauchy–Schwarz inequality

$$\begin{aligned} |S_\mu \Phi(\theta)| &\leq \sum_{n=0}^{\infty} |\Phi^{(n)}|_{-p} |\theta|_p^n \\ &\leq \left(\sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|_{-p}^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} 2^{nq} |\theta|_p^{2n} \right)^{1/2} \\ &= \|\Phi\|_{-p,-q,\mu} (1 - 2^q |\theta|_p^2)^{-1/2}. \end{aligned}$$

Thus the series (3.19) converges uniformly on any closed ball $\{\theta \in \mathcal{H}_{p,\mathbb{C}} | |\theta|_p^2 \leq r,\ r < 2^{-q}\}$. Hence $S_\mu \Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ and

$$n_{p,l,\infty}(S_\mu \Phi) \leq \|\Phi\|_{-p,-q,\mu} (1 - 2^{q-2l})^{-1/2},$$

if $2l > q$. This proves that S_μ is a continuous mapping from $(\mathcal{N})_\mu^{-1}$ to $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. In the language of Section 2.3 this reads

$$\operatorname*{ind\,lim}_{p,q \in \mathbb{N}} \mathrm{E}_{-p,-q}^{-1} \subset \mathrm{Hol}_0(\mathcal{N}_\mathbb{C}),$$

topologically.

Conversely, let $F \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ be given, i.e., there exist $p, l \in \mathbb{N}$ such that $n_{p,l,\infty}(F) < \infty$. The first step is to show that there exists $p', q \in \mathbb{N}$ such that

$$\|F\|_{-p',-q,-1} < C n_{p,l,\infty}(F),$$

for sufficiently large $C > 0$. This implies immediately

$$\mathrm{Hol}_0(\mathcal{N}_\mathbb{C}) \subset \operatorname*{ind\,lim}_{p,q \in \mathbb{N}} \mathrm{E}_{-p,-q}^{-1},$$

topologically, which is the missing part in the proof of the second statement in Theorem 2.3.11.

By assumption the Taylor expansion

$$F(\theta) = \sum_{k=0}^{\infty} \frac{1}{k!} \widehat{d^k F(0)}(\theta),$$

converges uniformly on any closed ball $\{\theta \in \mathcal{H}_{p,\mathbb{C}} | |\theta|_p^2 \leq r,\ r < 2^{-l}\}$ and

$$|F(\theta)| \leq n_{p,l,\infty}(F).$$

An application of Cauchy's inequality gives

$$\frac{1}{k!} |\widehat{d^k F(0)}(\theta)| \leq 2^{l(k+1)} |\theta|_p^k \sup_{|\theta|_p \leq 2^{-l}} |F(\theta)| \leq n_{p,l,\infty}(F) 2^{kl} |\theta|_p^k.$$

Then by polarization identity we obtain

$$\frac{1}{k!} |d^k F(0)(\theta_1, \ldots, \theta_k)| \leq n_{p,l,\infty}(F) e^k 2^{kl} |\theta_1|_p |\theta_2|_p \ldots |\theta_k|_p.$$

Then by kernel theorem (cf. Theorem 2.1.3) there exist kernels $\Phi^{(k)} \in \mathcal{H}_{-p',\mathbb{C}}^{\hat{\otimes}k}$ for $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ such that

$$F(\theta) = \sum_{k=0}^{\infty} \langle \Phi^{(k)}, \theta^{\otimes k} \rangle.$$

Moreover we have the following norm estimate

$$|\Phi^{(k)}|_{-p'} \leq n_{p,l,\infty}(F)(2^l e \,\|i_{p',p}\|_{HS})^k.$$

Thus

$$\begin{aligned}
\|\!|F|\!\|^2_{-p',-q,-1} &= \sum_{k=0}^{\infty} 2^{-kq} |\Phi^{(k)}|^2_{-p'} \\
&\leq n^2_{p,l,\infty}(F) \sum_{k=0}^{\infty} (2^{2l-q} e^2 \,\|i_{p',p}\|^2_{HS})^k \\
&= n^2_{p,l,\infty}(F)(1 - 2^{2l-q} e^2 \,\|i_{p',p}\|^2_{HS})^{-1},
\end{aligned}$$

if $q \in \mathbb{N}$ is such that $\rho := 2^{2l-q} e^2 \,\|i_{p',p}\|^2_{HS} \; < 1$. So we have in fact

$$\|\!|F|\!\|_{-p',-q,-1} \leq n_{p,l,\infty}(F)(1-\rho)^{-1/2}.$$

Now the rest is simple. Define $\Phi \in (\mathcal{N})^{-1}_{\mu}$ by

$$\Phi := \sum_{n=0}^{\infty} Q^{\mu}_n(\Phi^{(n)}),$$

then $S_\mu \Phi = F$ and

$$\|\Phi\|_{-p',-q,\mu} = \|\!|F|\!\|_{-p',-q,-1}.$$

This proves the existence of a continuous inverse of the S_μ-transform. Uniqueness of Φ follows from the fact that normalized exponentials are total in any $(\mathcal{H}_p)^1_q$. ■

Chapter 4

Generalized Appell Systems

In this chapter we are going to extend the results from the last chapter. Let us explain this more precisely. As in Chapter 3 we consider the case of non-degenerate measures μ on the dual of a nuclear space with analytic Laplace transform, i.e., μ satisfies Assumption 3.1.3 and Assumption 3.1.6. Note that under these assumptions the Poisson measure is covered. The generalization step is related to the generating function for the Appell polynomials. Namely, instead of the normalized exponential $e_\mu(\cdot;\cdot)$ we use a generalization of it, called *generalized normalized exponential* $e_\mu^\alpha(\cdot;\cdot)$, where α is an invertible holomorphic function from $\mathcal{N}_\mathbb{C}$ to $\mathcal{N}_\mathbb{C}$ in a neighborhood of zero (i.e., $\alpha \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C}, \mathcal{N}_\mathbb{C})$) and $\alpha(0) = 0$. The special case $\alpha \equiv 1$ produces the normalized exponential $e_\mu(\cdot;\cdot)$. Hence using $e_\mu^\alpha(\cdot,\cdot)$ we construct a *generalized Appell system* $\mathbb{A}^{\mu,\alpha}$ as a pair $(\mathbb{P}^{\mu,\alpha}, \mathbb{Q}^{\mu,\alpha})$ of generalized Appell polynomials $\mathbb{P}^{\mu,\alpha}$ and a system of generalized functions $\mathbb{Q}^{\mu,\alpha}$ associated in a proper way with the measure μ. This framework allowed us to generalize entirely the results obtained in the last chapter. Namely,

1. we obtain an explicit description of the test function space introduced in [ADKS96], see Subsection 4.3.2, Proposition 4.3.6;

2. the spaces of test functions turns out to be the same for all $\alpha \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C}, \mathcal{N}_\mathbb{C})$ and for all measures that we consider, see Section 4.3, Theorem 4.3.9;

3. characterization theorems for generalized as well as test functions are obtained analogously as in the Gaussian case, cf. Section 4.4;

4. the spaces of distributions for a fixed measure μ are again identical for all function α satistying the above conditions, see Theorem 4.4.3;

5. the well known Wick product and the corresponding Wick calculus [KLS96] extend rather directly, cf. Section 4.5;

6. in the important case of Poisson white noise a special choice of α produces the orthogonal system of generalized Charlier polynomials, see Example 3.1.9;

7. finally we provide formulas for the re-decomposition of the generalized Appell system $\mathbb{A}^{\mu,\alpha}$ under the change of the measure μ, see Section 4.6.

Finally, let us mention that for different functions α we produce different generalized Appell systems and there are exactly 5 choices for α which produces *not* only biorthogonal but orthogonal systems. For the one dimensional case see [Mei34].

4.1 Generalized Appell polynomials

4.1.1 Definitions

Recall from the last chapter that the normalized exponential is the generating function of the $\mathbb{P}^\mu$-system, i.e., if $\theta \in \mathcal{U}_0 \subset \mathcal{N}_\mathbb{C}$ and $z \in \mathcal{N}'_\mathbb{C}$, then

$$e_\mu(\theta, z) := \frac{\exp\langle z, \theta\rangle}{l_\mu(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle P_n^\mu(z), \theta^{\otimes n}\rangle,\ P_n^\mu(z) \in \mathcal{N}'^{\hat{\otimes} n}_\mathbb{C}.$$

In view to generalize the Appell system $\mathbb{A}^\mu$ we consider a holomorphic function $\alpha \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C}, \mathcal{N}_\mathbb{C})$ such that α is invertible and $\alpha(0) = 0$. Moreover we have the following decomposition (cf. (2.7))

$$\alpha(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!}\langle \alpha^{(n)}(0), \theta^{\otimes n}\rangle,\ \theta \in \mathcal{U}_\alpha \subset \mathcal{N}_\mathbb{C}, \tag{4.1}$$

where $\alpha^{(n)}(0) \in \mathcal{N}'^{\hat{\otimes} n}_\mathbb{C} \otimes \mathcal{N}_\mathbb{C}$ since α is vector valued. Analogously for the inverse function $g_\alpha := \alpha^{-1}$, we have

$$g_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!}\langle g_\alpha^{(n)}(0), \theta^{\otimes n}\rangle,\ \theta \in \mathcal{V}_\alpha \subset \mathcal{N}_\mathbb{C}, \tag{4.2}$$

where $g_\alpha^{(n)}(0) \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}}$. Now we introduce a new normalized exponential using the function α, i.e.,

$$e_\mu^\alpha(\theta; z) := e_\mu(\alpha(\theta); z) = \frac{\exp\langle z, \alpha(\theta)\rangle}{l_\mu(\alpha(\theta))},\ \theta \in \mathcal{U}'_\alpha \subset \mathcal{U}_\alpha,\ z \in \mathcal{N}'_{\mathbb{C}}.$$

Using the same procedure as in Subsection 3.2.1 there exist $P_n^{\mu,\alpha}(z) \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ called *generalized Appell polynomial* (or *α-polynomial* for short) such that

$$e_\mu^\alpha(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), \theta^{\otimes n}\rangle,\ \theta \in \mathcal{U}'_\alpha\,,\ z \in \mathcal{N}'_{\mathbb{C}}, \tag{4.3}$$

which for fixed $z \in \mathcal{N}'_{\mathbb{C}}$ converges uniformly on some neighborhood of zero on $\mathcal{N}_{\mathbb{C}}$. Hence we have constructed the $\mathbb{P}^{\mu,\alpha}$-system

$$\mathbb{P}^{\mu,\alpha} := \{\langle P_n^{\mu,\alpha}(\cdot)\,, \varphi_\alpha^{(n)}\rangle | \varphi_\alpha^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N}\}.$$

In this case the related moments kernels of the measure μ are determined by

$$l_\mu^\alpha(\theta) := l_\mu(\alpha(\theta)) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M_n^{\mu,\alpha}, \theta^{\otimes n}\rangle,\ \theta \in \mathcal{N}_{\mathbb{C}},\ M_n^{\mu,\alpha} \in \mathcal{N}'^{\hat{\otimes} n}. \tag{4.4}$$

4.1.2 Properties and description of polynomials

Let us collect some properties of the polynomials $P_n^{\mu,\alpha}(\cdot)$.

Proposition 4.1.1 *For the generalized Appell polynomials the following properties hold.*

($P_\alpha 1$) *For any $z \in \mathcal{N}'_{\mathbb{C}}$ we have*

$$P_n^{\mu,\alpha}(z) = \sum_{m=1}^{n} \frac{1}{m!} \langle P_m^\mu(z), A_n^m\rangle, \tag{4.5}$$

where A_n^m are related to the kernels of α and are given in the proof, see (4.13) below.

($P_\alpha 2$) *For every $z \in \mathcal{N}'_{\mathbb{C}}$ and M_n^μ defined in (3.2)*

$$z^{\otimes n} = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle P_m^{\mu,\alpha}(z), B_k^m\rangle \hat{\otimes} M_{n-k}^\mu, \tag{4.6}$$

where B_k^m are related with the kernels of g_α and are given in the proof, see (4.14) below.

($\mathrm{P}_\alpha 3$) *For all* $z, w \in \mathcal{N}'_{\mathbb{C}}$ *and* $M_n^{\mu,\alpha}$ *defined in (4.4)*

$$P_n^{\mu,\alpha}(z+w) = \sum_{k+l+m=n} \frac{n!}{k!l!m!} P_k^{\mu,\alpha}(z)\hat{\otimes}P_l^{\mu,\alpha}(w)\hat{\otimes}M_m^{\mu,\alpha}. \tag{4.7}$$

($\mathrm{P}_\alpha 4$) *Given* $z, w \in \mathcal{N}'_{\mathbb{C}}$, *then*

$$P_n^{\mu,\alpha}(z+w) = \sum_{k=0}^{n} \binom{n}{k} P_k^{\mu,\alpha}(z)\hat{\otimes}P_{n-k}^{\delta_0,\alpha}(w). \tag{4.8}$$

($\mathrm{P}_\alpha 5$) *Further, we observe*

$$\mathbb{E}_\mu(\langle P_m^{\mu,\alpha}(\cdot), \varphi_\alpha^{(m)}\rangle) = 0, \text{ for } m \neq 0, \, \varphi_\alpha^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m}. \tag{4.9}$$

($\mathrm{P}_\alpha 6$) *For all* $p' > p$ *such that the embedding* $\mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ *is of Hilbert-Schmidt class and for all* $\varepsilon > 0$ *there exist* $\sigma_\varepsilon > 0$ *such that*

$$|P_n^{\mu,\alpha}(z)|_{-p'} \leq 2\, n! \sigma_\varepsilon^{-n} \, \exp(\varepsilon |z|_{-p}), \; z \in \mathcal{H}_{-p',\mathbb{C}}, \, n \in \mathbb{N}_0, \tag{4.10}$$

where σ_ε *is chosen in such a way that* $|\alpha(\theta)| \leq \varepsilon$ *and* $|l_\mu(\alpha(\theta))| \geq 1/2$ *for* $|\theta|_p = \sigma_\varepsilon$.

Proof. ($\mathrm{P}_\alpha 1$) Analogously with (3.7) we have

$$e_\mu^\alpha(\theta; z) := \frac{\exp\langle z, \alpha(\theta)\rangle}{l_\mu(\alpha(\theta))} = \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_m^\mu(z), \alpha(\theta)^{\otimes m}\rangle. \tag{4.11}$$

Using the representation from (4.1) we compute $\alpha(\theta)^{\otimes m}$

$$\begin{aligned} \alpha(\theta)^{\otimes m} &= \sum_{l=1}^{\infty} \frac{1}{l!} \langle \alpha^{(l)}(0), \theta^{\otimes l}\rangle \otimes \cdots \otimes \sum_{l=1}^{\infty} \frac{1}{l!} \langle \alpha^{(l)}(0), \theta^{\otimes l}\rangle \\ &= \sum_{l_1,\ldots,l_m=1}^{\infty} \frac{1}{l_1! \cdots l_m!} \langle \alpha^{(l_1)}(0) \otimes \cdots \otimes \alpha^{(l_m)}(0), \theta^{\otimes(l_1+\ldots+l_m)}\rangle \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \langle A_n^m, \theta^{\otimes n}\rangle, \end{aligned} \tag{4.12}$$

where

$$A_n^m := \begin{cases} \displaystyle\sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1!\cdots l_m!} \alpha^{(l_1)}(0) \otimes \cdots \otimes \alpha^{(l_m)}(0) & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases} . \quad (4.13)$$

Now we introduce (4.12) in (4.11) to obtain

$$\begin{aligned} e_\mu^\alpha(\theta; z) &= \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle P_m^\mu(z), \sum_{n=1}^{\infty} \frac{1}{n!} \langle A_n^m, \theta^{\otimes n} \rangle \right\rangle \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^{n} \frac{1}{m!} \langle P_m^\mu(z), A_n^m \rangle, \theta^{\otimes n} \right\rangle. \end{aligned}$$

By definition

$$e_\mu^\alpha(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), \theta^{\otimes n} \rangle,$$

then a comparison of coefficients gives

$$P_n^{\mu,\alpha}(z) = \sum_{m=1}^{n} \frac{1}{m!} \langle P_m^\mu(z), A_n^m \rangle.$$

($\mathrm{P}_\alpha 2$) Since $\theta = \alpha(g_\alpha(\theta))$ we have

$$e_\mu(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), g_\alpha(\theta)^{\otimes n} \rangle.$$

Having in mind (4.2) we first compute $g_\alpha(\theta)^{\otimes n}$:

$$\begin{aligned} g_\alpha(\theta)^{\otimes n} &= \sum_{l=1}^{\infty} \frac{1}{l!} \langle g_\alpha^{(l)}(0), \theta^{\otimes l} \rangle \otimes \cdots \otimes \sum_{l=1}^{\infty} \frac{1}{l!} \langle g_\alpha^{(l)}(0), \theta^{\otimes l} \rangle \\ &= \sum_{l_1,\ldots,l_n=1}^{\infty} \frac{1}{l_1!\cdots l_n!} \langle g_\alpha^{(l_1)}(0) \otimes \cdots \otimes g_\alpha^{(l_n)}(0), \theta^{\otimes (l_1+\ldots+l_n)} \rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \langle B_m^n, \theta^{\otimes m} \rangle, \end{aligned}$$

where

$$B_m^n = \begin{cases} \sum_{l_1+\ldots+l_n=m} \frac{m!}{l_1!\cdots l_n!} g_\alpha^{(l_1)}(0) \otimes \cdots \otimes g_\alpha^{(l_n)}(0) & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} . \qquad (4.14)$$

Hence

$$\begin{aligned} e_\mu(\theta; z) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\mu,\alpha}(z), \sum_{m=1}^{\infty} \frac{1}{m!} \langle B_m^n, \theta^{\otimes m} \rangle \right\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle \sum_{n=0}^{m} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), B_m^n \rangle, \theta^{\otimes m} \right\rangle . \end{aligned}$$

On the other hand

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\mu(z), \theta^{\otimes n} \rangle,$$

so we conclude that

$$P_m^\mu(z) = \sum_{n=1}^{m} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), B_m^n \rangle. \qquad (4.15)$$

The result follows using property (P2) of the polynomials $P_n^\mu(\cdot)$.
($\mathrm{P}_\alpha 3$) Let us start from the equation of the generating functions

$$e_\mu^\alpha(\theta; z+w) = e_\mu^\alpha(\theta; z) e_\mu^\alpha(\theta; w) l_\mu^\alpha(\theta).$$

This implies

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z+w), \theta^{\otimes n} \rangle = \sum_{k,l,m=0}^{\infty} \frac{1}{k!l!m!} \langle P_k^{\mu,\alpha}(z) \hat{\otimes} P_l^{\mu,\alpha}(w) \hat{\otimes} M_m^{\mu,\alpha}, \theta^{\otimes(k+l+m)} \rangle,$$

from this ($\mathrm{P}_\alpha 3$) follows immediately.
($\mathrm{P}_\alpha 4$) We note that

$$e_\mu^\alpha(\theta; z+w) = e_\mu^\alpha(\theta; z) \exp\langle w, \alpha(\theta) \rangle, \; \theta \in \mathcal{U}_0 \subset \mathcal{N}_{\mathbb{C}}.$$

Now, since $l_{\delta_0}(\theta) = 1$, we have the following decomposition

$$\exp\langle w, \alpha(\theta) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\delta_0,\alpha}(w), \theta^{\otimes n} \rangle, \qquad (4.16)$$

where for $\alpha \equiv \mathrm{id}$, $P_n^{\delta_0,\alpha}(w) = w^{\otimes n}$. The result follows as done in (P$_\alpha$3).
(P$_\alpha$5) To see this we use, $\theta \in \mathcal{N}_{\mathbb{C}}$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_\mu(\langle P_m^{\mu,\alpha}(\cdot), \theta^{\otimes n}\rangle) = \mathbb{E}_\mu(e_\mu^\alpha(\theta;\cdot)) = \frac{\mathbb{E}_\mu(\exp\langle \cdot, \alpha(\theta)\rangle)}{l_\mu(\alpha(\theta))} = 1.$$

Then the polarization identity and a comparison of coefficients give the result.
(P$_\alpha$6) Using the definition of $P_n^{\mu,\alpha}$ and Cauchy's inequality for Taylor series (see e.g., [Din81]) we have

$$\begin{aligned} |\langle P_n^{\mu,\alpha}(z), \theta^{\otimes n}\rangle| &= n! |d^n \widehat{e_\mu^\alpha(0;z)}(\theta)|_{-p} \\ &\leq n! \frac{1}{\sigma_\varepsilon^n} \sup_{|\theta|_p = \sigma_\varepsilon} \frac{\exp(|\alpha(\theta)|_p |z|_{-p})}{|l_\mu(\alpha(\theta))|} |\theta|_p^n \\ &\leq 2n! \sigma_\varepsilon^{-n} \exp(\varepsilon |z|_{-p}) |\theta|_p^n. \end{aligned}$$

The result follows by polarization and kernel theorem. ■

Let us return to Example 3.1.9 which furnishes good arguments to use the $\mathbb{P}^{\mu,\alpha}$-system.

Example 3.1.9 (continuation) Let π be a measure on $S'(\mathbb{R})$ with characteristic functional given by

$$l_\pi(\theta) = \exp\left[\int_{\mathbb{R}} (e^{\theta(t)} - 1) dt\right], \ \theta \in S(\mathbb{R}).$$

We would like to prove that π satisfies the Assumption 3.1.6. First of all we notice that for any $\xi \in S(\mathbb{R})$, $\xi \neq 0$ the measures π and $\pi(\cdot + \xi)$ are orthogonal (see [GGV75] for a detailed analysis). It means that π is not $S(\mathbb{R})$-quasi-invariant and the Remark 3.1.7 - 2 is not applicable now.

Let $\varphi \in \mathcal{P}(S'(\mathbb{R}))$, $\varphi = 0$ π-a.s. be given. We need to show that then $\varphi \equiv 0$. To this end we construct the system of generalized Appell polynomials (so-called generalized Charlier polynomials which are orthogonal) in the space $L^2(S'(\mathbb{R}), \pi)$. We define the mapping α by

$$S(\mathbb{R}) \ni \theta(\cdot) \mapsto \alpha(\theta)(\cdot) := \log(1 + \theta(\cdot)) \in S(\mathbb{R}), \ -1 < \theta \in S(\mathbb{R}),$$

which is holomorphic on a neighborhood $\mathcal{U} \subset S(\mathbb{R})$ containing zero. Then

$$e_\pi^\alpha(\theta; x) := \frac{\exp\langle x, \alpha(\theta)\rangle}{l_\pi(\alpha(\theta))} = \exp[\langle x, \alpha(\theta)\rangle - (\theta, 1)], \ \theta \in \mathcal{U}, \ x \in S'(\mathbb{R}),$$

is a holomorphic function on $\mathcal{U}$ for any $x \in S'(\mathbb{R})$. The Taylor decomposition and the kernel theorem give

$$e_\pi^\alpha(\theta; x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^{\pi,\alpha}(x), \theta^{\otimes n} \rangle,$$

where $C_n^{\pi,\alpha} : S'(\mathbb{R}) \to S'(\mathbb{R})^{\hat{\otimes} n}$ are polynomial mappings. For $\varphi^{(n)} \in S(\mathbb{R})^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, we define the *generalized Charlier polynomials* by

$$S'(\mathbb{R}) \ni x \mapsto C_n^{\pi,\alpha}(\varphi^{(n)})(x) := \langle C_n^{\pi,\alpha}(x), \varphi^{(n)} \rangle \equiv \langle C_n(x), \varphi^{(n)} \rangle \in \mathbb{R}.$$

Due to [Ito88] and [IK88] we have the following orthogonality property:

$$\int_{S'(\mathbb{R})} \langle C_n(x), \varphi^{(n)} \rangle \langle C_m(x), \psi^{(m)} \rangle d\pi(x) = \delta_{nm} n! (\varphi^{(n)}, \psi^{(n)}),$$

for any $\varphi^{(n)} \in S(\mathbb{R})^{\hat{\otimes} n}$ and any $\psi^{(m)} \in S(\mathbb{R})^{\hat{\otimes} m}$. Now the rest is simple. Any continuous polynomial φ has a uniquely defined decomposition

$$\varphi(x) = \sum_{n=0}^{N} \langle C_n(x), \varphi^{(n)} \rangle, \ x \in S'(\mathbb{R}),$$

where $\varphi^{(n)} \in S(\mathbb{R})^{\hat{\otimes} n}$. If $\varphi = 0$ π-a.e., then

$$\|\varphi\|_{L^2(\pi)}^2 = \sum_{n=0}^{N} n! (\varphi^{(n)}, \overline{\varphi^{(n)}}) = 0.$$

Hence $\varphi^{(n)} = 0$, $n = 0, \ldots, N$, i.e., $\varphi \equiv 0$. So Assumption 3.1.6 is satisfied.

The following lemma describes the set of polynomials $\mathcal{P}(\mathcal{N}')$ in terms of the generalized Appell polynomials $P_n^{\mu,\alpha}$.

Lemma 4.1.2 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$\varphi(x) = \sum_{n=0}^{N} \langle P_n^{\mu,\alpha}(x), \varphi_\alpha^{(n)} \rangle, \ \varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \tag{4.17}$$

and vice versa, any functional of the form (4.17) is a smooth continuous polynomial.

Proof. The representation from Definition 3.1.5 and equation (4.17) can be transformed into one another using (4.5) and (4.6). ■

4.2 The dual generalized Appell system

Below we give two characterizations of the dual generalized Appell system $\mathbb{Q}^{\mu,\alpha}$ which allow a representation of the generalized functions from $\mathcal{P}'_\mu(\mathcal{N}')$, see Theorem 4.2.8.

4.2.1 Definitions and properties

Let us recall that α is an invertible holomorphic function with inverse given by g_α and $\alpha\,(\theta) \in \mathcal{V}_\alpha \subset \mathcal{N}_\mathbb{C}$, for any θ from a neighborhood of zero $\mathcal{U}_\alpha$, (cf. Subsection 4.1.1).

Definition 4.2.1 *Let $\Phi_\alpha^{(n)} \in \mathcal{N}_\mathbb{C}'^{\hat{\otimes} n}$ be given. We define a generalized function $Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})$ via the S_μ-transform*

$$(S_\mu Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}))(\theta) := \langle \Phi_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n}\rangle,\ \ \theta \in \mathcal{V}_\alpha. \tag{4.18}$$

The generalized functions $Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})$ allow a representation in terms of a differential operator (of infinite order), see Theorem 4.2.4 below. We proceed giving a detailed description of this procedure.

Remember from (4.2) that the kernels of the inverse mapping of α are $g_\alpha^{(n)}(0) \in \mathcal{N}_\mathbb{C}'^{\hat{\otimes} n} \otimes \mathcal{N}_\mathbb{C}$.

Definition 4.2.2 *We define a differential operator (of infinite order) as a mapping $G_\alpha : \mathcal{P}(\mathcal{N}') \to \mathcal{P}(\mathcal{N}') \otimes \mathcal{N}_\mathbb{C}$ by*

$$G_\alpha := \sum_{n=0}^{\infty} \frac{1}{n!}\langle g_\alpha^{(n)}(0), \nabla^{\otimes n}\rangle,$$

which for any $\varphi \in \mathcal{P}(\mathcal{N}')$ and $\xi \in \mathcal{N}'_\mathbb{C}$ is given as

$$(G_\alpha^\xi \varphi)(x) := \langle \xi, (G_\alpha\varphi)(x)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \xi, \langle g_\alpha^{(n)}(0), \nabla^{\otimes n}\varphi(x)\rangle\rangle,\ x \in \mathcal{N}'.$$

We have $G_\alpha^\xi : \mathcal{P}(\mathcal{N}') \to \mathcal{P}(\mathcal{N}')$, and formally $G_\alpha := g_\alpha(\nabla)$.

Let we state the following useful lemma.

Lemma 4.2.3 *For all $\xi \in \mathcal{N}'_{\mathbb{C}}$, $x \in \mathcal{N}'$ and $\theta \in \mathcal{N}_{\mathbb{C}}$ the following equality holds*

$$\langle \xi, g_\alpha(\nabla)\rangle(\exp\langle x, \theta\rangle) = \langle \xi, g_\alpha(\theta)\rangle \exp\langle x, \theta\rangle.$$

Proof. Using the representation given in (4.2) we obtain

$$\langle \xi, g_\alpha(\nabla)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\langle g_{\alpha,\xi}^{(n)}(0), \nabla^{\otimes n}\rangle, \quad g_{\alpha,\xi}^{(n)}(0) := \langle g_\alpha^{(n)}(0), \xi\rangle \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}.$$

For simplicity we put $\Psi^{(n)} := g_{\alpha,\xi}^{(n)}(0)$. At first we compute the operator $\langle \Psi^{(n)}, \nabla^{\otimes n}\rangle$ on a continuous monomial. For given $\theta \in \mathcal{N}_{\mathbb{C}}$, $m \geq n$ we have

$$\begin{aligned}
\langle \Psi^{(n)}, \nabla^{\otimes n}\rangle\langle x, \theta\rangle^m &= \langle \Psi^{(n)}, \nabla^{\otimes n}\rangle\langle x^{\otimes m}, \theta^{\otimes m}\rangle \\
&= m(m-1)\cdots(m-n+1)\left\langle \Psi^{(n)} \hat{\otimes} x^{\otimes(m-n)}, \theta^{\otimes m}\right\rangle \\
&= m(m-1)\cdots(m-n+1)\langle x, \theta\rangle^{m-n}\langle \Psi^{(n)}, \theta^{\otimes n}\rangle,
\end{aligned}$$

where we used (3.14) in the second equality. Now expand the given function, $\exp\langle x, \theta\rangle$, in the Taylor series and applying the above result we get

$$\begin{aligned}
\langle \Psi^{(n)}, \nabla^{\otimes n}\rangle \exp\langle x, \theta\rangle &= \langle \Psi^{(n)}, \nabla^{\otimes n}\rangle \sum_{m=0}^{\infty} \frac{\langle x, \theta\rangle^m}{m!} \\
&= \sum_{m=n}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{m!}\langle \Psi^{(n)} \hat{\otimes} x^{\otimes(m-n)}, \theta^{\otimes m}\rangle \\
&= \langle \Psi^{(n)}, \theta^{\otimes n}\rangle \sum_{m=n}^{\infty} \frac{1}{(m-n)!}\langle x, \theta\rangle^{m-n} \\
&= \langle \Psi^{(n)}, \theta^{\otimes n}\rangle \exp\langle x, \theta\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \xi, g_\alpha(\nabla)\rangle(\exp\langle x, \theta\rangle) &= \sum_{n=0}^{\infty} \frac{1}{n!}\langle \Psi^{(n)}, \nabla^{\otimes n}\rangle \exp\langle x, \theta\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\langle \Psi^{(n)}, \theta^{\otimes n}\rangle \exp\langle x, \theta\rangle \\
&= \langle \xi, g_\alpha(\theta)\rangle(\exp\langle x, \theta\rangle).
\end{aligned}$$

■

Theorem 4.2.4 *The generalized functions* $Q_n^{\mu,\alpha}(\xi^{\otimes n})$ *are given by*

$$Q_n^{\mu,\alpha}(\xi^{\otimes n})(\cdot) = (\langle \xi, g_\alpha(\nabla)\rangle^{*n} 1)(\cdot). \tag{4.19}$$

Proof. Applying the S_μ-transform to the right hand side of (4.19) we have

$$\begin{aligned} S_\mu(\langle \xi, g_\alpha(\nabla)\rangle^{*n}1)(\theta) &= \langle\!\langle (\langle \xi, g_\alpha(\nabla)\rangle^{*n}1)(\cdot), e_\mu(\theta;\cdot)\rangle\!\rangle_\mu \\ &= \langle\!\langle 1(\cdot), \langle \xi, g_\alpha(\nabla)\rangle^{n} e_\mu(\theta;\cdot)\rangle\!\rangle_\mu \\ &= \frac{1}{l_\mu(\theta)} \int_{\mathcal{N}'} \langle \xi, g_\alpha(\nabla)\rangle^n \exp\langle x, \theta\rangle d\mu(x) \\ &= \frac{\langle \xi, g_\alpha(\theta)\rangle^n}{l_\mu(\theta)} \int_{\mathcal{N}'} \exp\langle x, \theta\rangle d\mu(x) \\ &= \langle \xi, g_\alpha(\theta)\rangle^n. \end{aligned} \tag{4.20}$$

On the other hand the S_μ-transform of the left hand side of (4.19), by (4.18), is the same as (4.20) which prove the theorem. ■

Example 4.2.5 *As an illustration of* G_α *we use again the Poisson measure* π *(see Example 3.1.9) and* $\alpha(\theta)(\cdot) = \log(1+\theta(\cdot))$, $\theta \in S(\mathbb{R})$. *Thus, we have*

$$g_\alpha(\theta)(\cdot) = \exp\theta(\cdot) - 1 = \sum_{n=1}^{\infty} \frac{\theta^n(\cdot)}{n!}.$$

On the other hand, from (4.2) we know that

$$g_\alpha(\theta)(\cdot) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle g_\alpha^{(n)}(0), \theta^{\otimes n}\rangle(\cdot).$$

This implies

$$g_\alpha^{(n)}(0)(t) = \delta(\cdot - t)\hat{\otimes}\cdots\hat{\otimes}\delta(\cdot - t).$$

Let $f \in \mathcal{P}(S'(\mathbb{R}))$ *and* $h \in S(\mathbb{R})$ *be given. Denote by* $\nabla_h f = (\nabla f, h)$ *the directional derivative of* f *in direction of* h *(here the tangent space is* $L^2(\mathbb{R})$*). Then it is not hard to see that*

$$(\exp(\nabla_h)f)(\cdot) = f(\cdot + h),\ f \in \mathcal{P}(S'(\mathbb{R})),\ h \in S(\mathbb{R}).$$

Hence

$$(g_\alpha(\nabla_{\delta_t}))(f(\cdot)) = (\exp(\nabla_{\delta_t}) - 1)f(\cdot) = f(\cdot + \delta_t) - f(\cdot),$$

and if $\xi \in S(\mathbb{R})$ we have

$$\langle g_\alpha(\nabla_{\delta_t}), \xi\rangle f(\cdot) = \int_{\mathbb{R}} [f(\cdot + \delta_t) - f(\cdot)]\xi(t)dt.$$

Therefore the operator G_α has the form

$$G_\alpha : f(\cdot) \longmapsto f(\cdot + \delta_t) - f(\cdot),\ f \in \mathcal{P}(S'(\mathbb{R})).$$

This mapping can be considered as a "gradient" operator on the Poisson space $(S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \pi)$. In Chapter 5 we will return to the gradient G_α from another point of view and derive some properties of it. Here we only would like to stress that G_α produces (via integration by parts formula) the system of generalized Charlier polynomials.

Definition 4.2.6 *We define the $\mathbb{Q}^{\mu,\alpha}$-system in $\mathcal{P}'_\mu(\mathcal{N}')$ by*

$$\mathbb{Q}^{\mu,\alpha} = \{Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}) | \Phi_\alpha^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N}_0\},$$

and the pair $\mathbb{A}^{\mu,\alpha} = (\mathbb{P}^{\mu,\alpha}, \mathbb{Q}^{\mu,\alpha})$ will be called the generalized Appell system generated by the measure μ for a given mapping $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$.

Now we are going to discuss the central property of the generalized Appell system $\mathbb{A}^{\mu,\alpha}$.

Theorem 4.2.7 *The generalized Appell polynomials $\mathbb{P}^{\mu,\alpha}$ and the dual system $\mathbb{Q}^{\mu,\alpha}$ are biorthogonal with respect to μ and*

$$\langle\!\langle Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}), P_m^{\mu,\alpha}(\varphi_\alpha^{(m)})\rangle\!\rangle_\mu = \delta_{nm} n! \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)}\rangle, \tag{4.21}$$

for any $\Phi_\alpha^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ and any $\varphi_\alpha^{(m)} \in \mathcal{N}^{\hat{\otimes} m}_{\mathbb{C}}$.

Proof. By definition of $Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})$ we have

$$(S_\mu Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}))(\theta) := \langle\!\langle Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}), e_\mu(\theta;\cdot)\rangle\!\rangle_\mu,$$

if we substitute $\theta = \alpha(\eta)$, then we obtain

$$\begin{aligned} (S_\mu Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}))(\alpha(\eta)) &= \langle\!\langle Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})(\cdot), e_\mu(\alpha(\eta);\cdot)\rangle\!\rangle_\mu \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \langle\!\langle Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})(\cdot), \langle P_m^{\mu,\alpha}(\cdot), \eta^{\otimes m}\rangle\rangle\!\rangle_\mu. \end{aligned}$$

Substituting θ by $\alpha(\eta)$ in (4.18) give us

$$(S_\mu Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}))(\alpha(\eta)) = \langle \Phi_\alpha^{(n)}, \eta^{\otimes n}\rangle.$$

Then a comparison of coefficients and the polarization identity give the desired result. ■

4.2.2 Description of generalized functions

Now we describe the space of generalized functions $\mathcal{P}'_\mu(\mathcal{N}')$.

Theorem 4.2.8 *For all generalized function $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ there exists a unique sequence $(\Phi_\alpha^{(n)})_{n=0}^\infty$, $\Phi_\alpha^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}$ such that*

$$\Phi = \sum_{n=0}^\infty Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}) \equiv \sum_{n=0}^\infty \langle Q_n^{\mu,\alpha}, \Phi_\alpha^{(n)}\rangle, \tag{4.22}$$

and vice versa, every series of the form (4.22) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

Proof. For $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ we can uniquely define $\Phi_\alpha^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}$ by

$$\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)}\rangle := \frac{1}{n!}\langle\!\langle \Phi, \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)}\rangle\rangle\!\rangle_\mu,\ \varphi_\alpha^{(n)} \in \mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}},$$

which is well defined since $\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$. The continuity of $\varphi_\alpha^{(n)} \mapsto \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)}\rangle$ follows from the continuity of $\varphi \mapsto \langle\!\langle \Phi, \varphi\rangle\!\rangle_\mu$, $\varphi \in \mathcal{P}(\mathcal{N}')$. This implies that

$$\varphi \mapsto \sum_{n=0}^\infty n! \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)}\rangle,$$

is continuous on $\mathcal{P}(\mathcal{N}')$. This defines a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$, which we denote by

$$\sum_{n=0}^\infty Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}).$$

In view of Theorem 4.2.7 it is easy to see that

$$\Phi = \sum_{n=0}^\infty Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}).$$

To see the converse consider a series of the form (4.22) and $\varphi \in \mathcal{P}(\mathcal{N}')$. Then there exists $\varphi_\alpha^{(n)} \in \mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}}$, $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$\varphi = \sum_{n=0}^N \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)}\rangle.$$

So we have

$$\left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}), \varphi \right\rangle\!\!\right\rangle_\mu = \sum_{n=0}^{N} n! \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle,$$

because of Theorem 4.2.7. The continuity of

$$\varphi \mapsto \left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}), \varphi \right\rangle\!\!\right\rangle_\mu ,$$

follows because $\varphi_\alpha^{(n)} \mapsto \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle$ is continuous for all $n \in \mathbb{N}$. ∎

4.3 Construction and characterization of test function spaces

4.3.1 Test function spaces

We will construct the test function space $(\mathcal{N})^1_{\mu,\alpha}$ using $\mathbb{P}^{\mu,\alpha}$-system and study some properties. On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the representation from (4.17)

$$\varphi = \sum_{n=0}^{N} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle,$$

with $\varphi_\alpha^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n}$ for each $p > 0$, $n \in \mathbb{N}$. Thus we may define for any $p, q \in \mathbb{N}$ a Hilbert norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|^2_{p,q,\mu,\alpha} := \sum_{n=0}^{N} (n!)^2 2^{nq} |\varphi_\alpha^{(n)}|_p^2 < \infty.$$

Then we define $(\mathcal{H}_p)^1_{q,\mu,\alpha}$ as the completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q,\mu,\alpha}$.

Definition 4.3.1 *We define the test function space by*

$$(\mathcal{N})^1_{\mu,\alpha} := \operatorname*{pr\,lim}_{p,q \in \mathbb{N}} (\mathcal{H}_p)^1_{q,\mu,\alpha}.$$

Theorem 4.3.2 *The test function space $(\mathcal{N})^1_{\mu,\alpha}$ is nuclear. The topology in $(\mathcal{N})^1_{\mu,\alpha}$ is uniquely defined by the topology on $\mathcal{N}$. It does not depend on the choice of the family of norms $\{|\cdot|_p, p \in \mathbb{N}\}$.*

Proof. Nuclearity of $(\mathcal{N})^1_{\mu,\alpha}$ follows essentially from that of $\mathcal{N}$. For fixed p, q choose p' such that the embedding

$$\iota_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$$

is Hilbert-Schmidt and consider the embedding

$$I_{p',q',p,q,\alpha} : (\mathcal{H}_{p'})^1_{q',\mu,\alpha} \hookrightarrow (\mathcal{H}_p)^1_{q,\mu,\alpha}.$$

Then $I_{p',q',p,q,\alpha}$ is induced by

$$I_{p',q',p,q,\alpha}(\varphi) = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \iota_{p',p}^{\otimes n} \varphi_\alpha^{(n)} \rangle, \text{ for } \varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle \in (\mathcal{H}_{p'})^1_{q',\mu,\alpha}.$$

Its Hilbert-Schmidt norm, for a given orthonormal basis of $(\mathcal{H}_{p'})^1_{q',\mu,\alpha}$, can be estimate by

$$\|I_{p',q',p,q,\alpha}\|^2_{HS} = \sum_{n=0}^{\infty} 2^{n(q-q')} \, \|i_{p',p}\|^{2n}_{HS} \,,$$

which is finite for a suitably chosen q'.

To prove the independence of the family of norms, let us assume that we are given two different systems of Hilbert norms $|\cdot|_p$ and $|\cdot|'_k$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|'_{k,l,\mu,\alpha}$ by $\|\cdot\|_{p,q,\mu,\alpha}$ for some p, q (and vice versa which is completely analogous). But for all $f \in \mathcal{N}$ we have $|f|'_k \leq C|f|_p$ for some constant C and some p, since $|\cdot|'_k$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$. That means that the injection i from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by i also its linear extension from $\mathcal{H}_{p,\mathbb{C}}$ into $\mathcal{K}_{k,\mathbb{C}}$. It follows that $i^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$ into $\mathcal{K}_{k,\mathbb{C}}^{\otimes n}$. Now we choose q such that $2^{(q-l)/2} \geq C$. Then

$$\begin{aligned} \|\cdot\|'_{k,l,\mu,\alpha} &= \sum_{n=0}^{\infty} (n!)^2 2^{nl} |\cdot|'^2_k \\ &\leq \sum_{n=0}^{\infty} (n!)^2 2^{nl} C^{2n} |\cdot|^2_p \\ &\leq \|\cdot\|_{p,q,\mu,\alpha} \,, \end{aligned}$$

which is exactly what we need. ∎

Lemma 4.3.3 *There exist $p, C, K > 0$ such that for all $n \in \mathbb{N}_0$*

$$\int |P_n^{\mu,\alpha}(z)|_{-p}^2 d\mu\,(z) \leq 4(n!)^2 C^n K. \tag{4.23}$$

Proof. We can use the estimate (4.10) and Lemma 3.1.4 to conclude the result. ■

Theorem 4.3.4 *There exists $p', q' > 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu,\alpha} \subset L^2(\mu)$ holds.*

Proof. Elements of the space $(\mathcal{N})^1_{\mu,\alpha}$ are defined as convergent series in the given topology. Now we need the convergence of the series in $L^2(\mu)$. Choose q' such that $C > 2^{q'}$ (C from estimate (4.23)). Let us take an arbitrary element $\varphi \in \mathcal{P}(\mathcal{N}')$

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle.$$

For $p > p'$ (p' from the Lemma 4.3.3) and $q > q'$ the following estimates hold

$$\begin{aligned}
\|\varphi(z)\|_{L^2(\mu)} &\leq \sum_{n=0}^{\infty} \left\| \langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)} \rangle \right\|_{L^2(\mu)} \\
&\leq \sum_{n=0}^{\infty} |\varphi_\alpha^{(n)}|_p \, \| |P_n^{\mu,\alpha}(z)|_{-p} \|_{L^2(\mu)} \\
&\leq 2K^{1/2} \sum_{n=0}^{\infty} n! 2^{nq/2} |\varphi_\alpha^{(n)}|_p (C2^{-q})^{n/2} \\
&\leq 2K^{1/2} \left(\sum_{n=0}^{\infty} (C2^{-q})^n \right)^{1/2} \left(\sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_\alpha^{(n)}|_p^2 \right)^{1/2} \\
&= 2K^{1/2} (1 - C2^{-q})^{-1/2} \|\varphi\|_{p,q,\mu,\alpha}.
\end{aligned}$$

Taking the closure the inequality extends to the whole space $(\mathcal{H}_p)^1_{q,\mu,\alpha}$. ■

Corollary 4.3.5 *The space $(\mathcal{N})^1_{\mu,\alpha}$ is continuously and densely embedded in $L^2(\mu)$.*

4.3.2 Description of test functions

Proposition 4.3.6 *Any test function* $\varphi \in (\mathcal{N})^1_{\mu,\alpha}$ *has a uniquely defined extension to* $\mathcal{N}'_{\mathbb{C}}$ *as an element of* $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.

Proof. Any element φ in $(\mathcal{N})^1_{\mu,\alpha}$ is defined as a series of the following type

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle,\ \varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},$$

such that

$$\|\varphi\|^2_{p,q,\mu,\alpha} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_\alpha^{(n)}|_p^2 < \infty,$$

for each $p, q \in \mathbb{N}$. So we need to show the convergence of the series

$$\sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)} \rangle,\ z \in \mathcal{H}_{-p,\mathbb{C}},$$

to an entire function in z. Let $\varepsilon > 0$ and $\sigma_\varepsilon > 0$ (σ_ε from Proposition 4.1.1 - ($\mathrm{P}_\alpha 6$)). We use (4.10) and estimate as follows

$$\begin{aligned}
& \sum_{n=0}^{\infty} |\langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)} \rangle| \\
\leq\ & \sum_{n=0}^{\infty} |P_n^{\mu,\alpha}(z)|_{-p} |\varphi_\alpha^{(n)}|_p \\
\leq\ & 2 \sum_{n=0}^{\infty} n! |\varphi_\alpha^{(n)}|_p \sigma_\varepsilon^{-n} \exp(\varepsilon |z|_{-p'}) \\
\leq\ & 2 \exp(\varepsilon |z|_{-p'}) \left(\sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_\alpha^{(n)}|_p^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} 2^{-nq} \sigma_\varepsilon^{-2n} \right)^{1/2} \\
\leq\ & 2 \|\varphi\|_{p,q,\mu,\alpha} (1 - 2^{-q} \sigma_\varepsilon^{-2})^{-1/2} \exp(\varepsilon |z|_{-p'}),
\end{aligned}$$

if $2^q > \sigma_\varepsilon^{-2}$ and p' is such that $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is Hilbert-Schmidt. That means, the series

$$\sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)} \rangle,$$

converges uniformly and absolutely in any neighborhood of zero of any space $\mathcal{H}_{-p,\mathbb{C}}$. Since each term $\langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)}\rangle$ is entire in z the uniform convergence implies that

$$z \longmapsto \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)}\rangle,$$

is entire on each $\mathcal{H}_{-p,\mathbb{C}}$ and hence on $\mathcal{N}'_{\mathbb{C}}$. This complete the proof. ■

The following corollary gives an explicit estimate on the growth of test functions and is a consequence of the above proposition.

Corollary 4.3.7 *For all $p > p'$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is of the Hilbert-Schmidt class and for all $\varepsilon > 0$ there exists σ_ε (σ_ε from Proposition 4.1.1 - ($P_\sigma 6$)), we can choose $q \in \mathbb{N}$ such that $2^q > \sigma_\varepsilon^{-2}$ and obtain the following bound*

$$|\varphi(z)| \leq C \, \|\varphi\|_{p,q,\mu,\alpha} \exp(\varepsilon |z|_{-p'}), \; \varphi \in (\mathcal{N})^1_{\mu,\alpha}, \; z \in \mathcal{H}_{-p,\mathbb{C}} \,,$$

where

$$C = 2(1 - 2^{-q}\sigma_\varepsilon^{-2})^{-1/2}.$$

Remark 4.3.8 *Proposition 4.3.6 states*

$$(\mathcal{N})^1_{\mu,\alpha} \subseteq \mathcal{E}^1_{\min}(\mathcal{N}'),$$

as sets, where

$$\mathcal{E}^1_{\min}(\mathcal{N}') = \{\varphi|_{\mathcal{N}'} \, \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})\}.$$

Now we are going to show that the converse also holds.

Theorem 4.3.9 *For all functions $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$, as in Subsection 4.1, and for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$, we have the topological identity*

$$(\mathcal{N})^1_{\mu,\alpha} = \mathcal{E}^1_{\min}(\mathcal{N}').$$

Proof. Let $\varphi \in \mathcal{E}^1_{\min}(\mathcal{N}')$ be given such that

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \psi^{(n)}\rangle,$$

with

$$\|\varphi\|^2_{p,q,1} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\psi^{(n)}|^2_p < \infty,$$

for each $p, q \in \mathbb{N}$. So we have

$$|\psi^{(n)}|_p \leq (n!)^{-1} 2^{-nq/2} \|\varphi\|_{p,q,1}.$$

On the other hand, we can use (4.6) to evaluate $\varphi(z)$ as

$$\begin{aligned}
\varphi(z) &= \sum_{n=0}^{\infty} \langle z^{\otimes n}, \psi^{(n)} \rangle \\
&= \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle P_m^{\mu,\alpha}(z), B_k^m \rangle \hat{\otimes} M_{n-k}^{\mu}, \psi^{(n)} \right\rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle \langle P_m^{\mu,\alpha}(z), B_k^m \rangle, (M_{n-k}^{\mu}, \psi^{(n)})_{\mathcal{H}^{\hat{\otimes}(n-k)}} \rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle P_m^{\mu,\alpha}(z), \langle B_k^m, (M_{n-k}^{\mu}, \psi^{(n)})_{\mathcal{H}^{\hat{\otimes}(n-k)}} \rangle \rangle \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} \langle P_m^{\mu,\alpha}(z), \langle B_{k+m}^m, (M_{n-k}^{\mu}, \psi^{(n+m)})_{\mathcal{H}^{\hat{\otimes}(n-k)}} \rangle \rangle \\
&= \sum_{m=0}^{\infty} \left\langle P_m^{\mu,\alpha}(z), \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} \langle B_{k+m}^m, (M_{n-k}^{\mu}, \psi^{(n+m)})_{\mathcal{H}^{\hat{\otimes}(n-k)}} \rangle \right\rangle,
\end{aligned}$$

such that, if

$$\varphi(z) = \sum_{m=0}^{\infty} \langle P_m^{\mu,\alpha}(z), \varphi_\alpha^{(m)} \rangle,$$

then we conclude that

$$\varphi_\alpha^{(m)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} \langle B_{k+m}^m, (M_{n-k}^{\mu}, \psi^{(n+m)})_{\mathcal{H}^{\hat{\otimes}(n-k)}} \rangle.$$

Now for $p \in \mathbb{N}$ we need an estimate of $|\varphi_\alpha^{(n)}|_p$ by $\|\cdot\|_{p,q,1}$ since the nuclear topology given by the norms $\|\cdot\|_{p,q,1}$, is equivalent to the projective topology induced by the norms $n_{p,l,k}$. Now we estimate $\varphi_\alpha^{(m)}$ as follows

$$\begin{aligned}
|\varphi_\alpha^{(m)}|_p &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} |B_{k+m}^m|_{\mathcal{H}_{-p}^{\hat{\otimes}(k+m)} \otimes \mathcal{H}_p^{\hat{\otimes} m}} |(M_{n-k}^{\mu}, \psi^{(n+m)})_{\mathcal{H}^{\hat{\otimes}(n-k)}}|_p \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} |B_{k+m}^m|_{\mathcal{H}_{-p}^{\hat{\otimes}(k+m)} \otimes \mathcal{H}_p^{\hat{\otimes} m}} |M_{n-k}^{\mu}|_{-p} |\psi^{(n+m)}|_p .
\end{aligned}$$

Let us, at first, estimate the norm

$$|B^m_{k+m}|_{-p,p} := |B^m_{k+m}|_{\mathcal{H}_{-p}^{\hat{\otimes}(k+m)}\otimes\mathcal{H}_p^{\hat{\otimes}m}}.$$

To this end we choose $p > p_\mu$ such that $\left\|\iota_{p,p_\mu}\right\|_{HS}$ is finite and define

$$D_{\alpha,\varepsilon} := \sup_{|\theta|_p=\varepsilon} |g_\alpha(\theta)|_p \quad \text{and} \quad \tilde{\varepsilon} := \frac{\varepsilon}{e\left\|\iota_{p,p_\mu}\right\|_{HS}}.$$

With this we obtain

$$\begin{aligned}
|B^n_m|_{-p,p} &\leq \sum_{l_1,\ldots,l_n=m} \frac{m!}{l_1!\cdots l_n!}|g_\alpha^{(l_1)}(0)|_{-p,p}\cdots|g_\alpha^{(l_n)}(0)|_{-p,p} \\
&\leq \sum_{l_1,\ldots,l_n=m} \frac{m!l_1!\cdots l_n!}{l_1!\cdots l_n!} D^n_{\alpha,\varepsilon}\tilde{\varepsilon}^{-m} \\
&\leq m!D^n_{\alpha,\varepsilon}2^m\tilde{\varepsilon}^{-m},
\end{aligned}$$

that means

$$|B^m_{k+m}|_{-p,p} \leq (k+m)!D^m_{\alpha,\varepsilon}2^{k+m}\varepsilon^{-((k+m)}.$$

Now let $q \in \mathbb{N}$ be such that $2^{q/2} > K_p$ ($K_p := eC\left\|\iota_{p,p_\mu}\right\|_{HS}$ as in (3.4)) and such that $2/(\tilde{\varepsilon}K_p) < 1$, then we estimate $|\varphi_\alpha^{(m)}|_p$ by

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}(m+k)!D^m_{\alpha,\varepsilon}\frac{2^{k+m}}{\tilde{\varepsilon}^{k+m}}(n-k)!(K_p)^{n-k}\frac{2^{-(n+m)q/2}}{(n+m)!}\|\!|\varphi|\!\|_{p,q,1} \\
\leq\ & \|\!|\varphi|\!\|_{p,q,1}\frac{2^{-mq/2}}{m!}D^m_{\alpha,\varepsilon}\sum_{n=0}^{\infty}(2^{-q/2}K_p)^n\sum_{k=0}^{n}\left(\frac{2}{\tilde{\varepsilon}K_p}\right)^k \\
\leq\ & \|\!|\varphi|\!\|_{p,q,1}\frac{2^{-mq/2}2^m}{m!\tilde{\epsilon}^m}D^m_{\alpha,\varepsilon}(1-2^{-q/2}K_p)^{-1}\frac{\tilde{\varepsilon}K_p}{\tilde{\varepsilon}K_p-2} \\
\equiv\ & L_{p,q,\alpha,\tilde{\varepsilon}}\frac{2^{-mq/2}2^m}{m!\tilde{\varepsilon}^m}D^m_{\alpha,\varepsilon}\|\!|\varphi|\!\|_{p,q,1}.
\end{aligned}$$

For $q' < q$ such that $2^2\tilde{\varepsilon}^{-2}2^{(q'-q)}D_{\alpha,\varepsilon} < 1$ this gives the following estimate

$$\begin{aligned}
\|\varphi\|^2_{p,q',\mu,\alpha} &\leq \sum_{m=0}^{\infty}(m!)^2 2^{mq'}|\varphi^{(m)}|_p^2 \\
&\leq \|\!|\varphi|\!\|^2_{p,q,1}L^2_{p,q,\alpha,\tilde{\varepsilon}}\sum_{m=0}^{\infty}(2^2\varepsilon^{-2}2^{(q'-q)}D_{\alpha,\varepsilon})^m < \infty.
\end{aligned}$$

This completes the proof. ■

Since we now have proved that the space of test functions $(\mathcal{N})^1_{\mu,\alpha}$ is isomorphic to $\mathcal{E}^1_{\min}(\mathcal{N}')$, for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$ and for all holomorphic invertible function $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ (in the conditions of Subsection 4.1.1), will now drop the subscripts μ, α. The test function space $(\mathcal{N})^1$ is the same for all measures μ and all functions α in the above conditions.

Corollary 4.3.10 *$(\mathcal{N})^1$ is an algebra under pointwise multiplication.*

Corollary 4.3.11 *$(\mathcal{N})^1$ admits 'scaling', i.e., for $\lambda \in \mathbb{C}$ the scaling operator $\sigma_\lambda : (\mathcal{N})^1 \to (\mathcal{N})^1$ defined by $\sigma_\lambda(\varphi x) := \varphi(\lambda x)$, $\varphi \in (\mathcal{N})^1$, $x \in \mathcal{N}'$ is well-defined.*

Corollary 4.3.12 *For all $z \in \mathcal{N}'_{\mathbb{C}}$ the space $(\mathcal{N})^1$ is invariant under the shift operator $\tau_z : \varphi \mapsto \varphi(\cdot + z)$.*

4.4 Characterization of generalized functions

In this section we will introduce and study the space $(\mathcal{N})^{-1}_{\mu,\alpha}$ of generalized functions corresponding to the space of test functions $(\mathcal{N})^1$ $(\equiv (\mathcal{N})^1_{\mu,\alpha})$. The goal is to prove that, for a fixed measure μ and for all function α the space $(\mathcal{N})^{-1}_{\mu,\alpha} = (\mathcal{N})^{-1}_{\mu}$, see Theorem 4.4.3 below.

Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ the space $(\mathcal{N})^{-1}_{\mu,\alpha}$ can be viewed as a subspace of $\mathcal{P}'_\mu(\mathcal{N}')$, i.e.,

$$(\mathcal{N})^{-1}_{\mu,\alpha} \subset \mathcal{P}'_\mu(\mathcal{N}').$$

Let us now introduce the Hilbert subspace $(\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha}$ of $\mathcal{P}'_\mu(\mathcal{N}')$ for which the norm

$$\|\Phi\|^2_{-p,-q,\mu,\alpha} := \sum_{n=0}^{\infty} 2^{-qn} |\Phi^{(n)}_\alpha|^2_{-p},$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu,\alpha}_n(\Phi^{(n)}_\alpha) \in \mathcal{P}'_\mu(\mathcal{N}'),$$

from Theorem 4.2.8. The space $(\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha}$ is the dual space of $(\mathcal{H}_p)^1_{q,\mu,\alpha}$ with respect to $L^2(\mu)$ (because of the biorthogonality of $\mathbb{P}^{\mu,\alpha}$- and $\mathbb{Q}^{\mu,\alpha}$-systems). By general duality theory

$$(\mathcal{N})^{-1}_{\mu,\alpha} := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha},$$

is the dual space of $(\mathcal{N})^1$ with respect to $L^2(\mu)$. As noted in Chapter 2 there exists a natural topology on co-nuclear spaces which coincides with the inductive limit topology. We will consider $(\mathcal{N})^{-1}_{\mu,\alpha}$ as a topological vector space with this topology. So we have the nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu,\alpha}.$$

The action of a distribution

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}) \in (\mathcal{N})^{-1}_{\mu,\alpha},$$

on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)}\rangle \in (\mathcal{N})^1,$$

is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} n! \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle.$$

The following example generalizes the Example 3.3.7.

Example 4.4.1 *We want to define a generalized function $\rho_\mu^\alpha(z,\cdot) \in (\mathcal{N})^{-1}_{\mu,\alpha}$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property*

$$\langle\!\langle \rho_\mu^\alpha(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x),\ \varphi \in (\mathcal{N})^1.$$

That means we have to establish the continuity of $\rho_\mu^\alpha(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$. If $p \geq p'$ is sufficiently large and $\varepsilon > 0$ small enough, Corollary 4.3.7 applies, i.e., $\exists q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x) \right| &\leq C \|\varphi\|_{p,q,\mu,\alpha} \int_{\mathcal{N}'} \exp(\varepsilon |x-z|_{-p'})\, d\mu(x) \\ &\leq C \|\varphi\|_{p,q,\mu,\alpha} \exp(\varepsilon |z|_{-p'}) \int_{\mathcal{N}'} \exp(\varepsilon |x|_{-p'})\, d\mu(x). \end{aligned}$$

If ε is chosen sufficiently small the last integral exists (cf. Lemma 3.1.4 - 3). Thus we have in fact $\rho_\mu^\alpha(z,\cdot) \in (\mathcal{N})^{-1}_{\mu,\alpha}$. It is clear that whenever the Radon-Nikodym derivative $\frac{d\mu(x+\xi)}{d\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu^\alpha(\xi,\cdot)$ defined above. We will show that in $(\mathcal{N})^{-1}_{\mu,\alpha}$ we have the canonical expansion

$$\rho_\mu^\alpha(z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \langle Q_n^{\mu,\alpha}(\cdot), P_n^{\delta_0,\alpha}(-z)\rangle,$$

where $P_n^{\delta_0,\alpha}(-z)$ is defined in (4.16). It is easy to see that the right hand side defines an element in $(\mathcal{N})^{-1}_{\mu,\alpha}$. Since both sides are in $(\mathcal{N})^{-1}_{\mu,\alpha}$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ we have

$$\begin{aligned} &\langle\!\langle \rho_\mu^\alpha(z,\cdot), \langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\rangle\rangle\!\rangle_\mu \\ = &\left\langle\!\!\left\langle \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \langle Q_k^{\mu,\alpha}(\cdot), P_k^{\delta_0,\alpha}(-z)\rangle, \langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\rangle \right\rangle\!\!\right\rangle_\mu \\ = &\langle P_n^{\delta_0,\alpha}(-z), \varphi_\alpha^{(n)}\rangle, \end{aligned}$$

where we have used the biorthogonality property of the $\mathbb{P}^{\mu,\alpha}$- and $\mathbb{Q}^{\mu,\alpha}$-systems. On the other hand

$$\begin{aligned} \langle\!\langle \rho_\mu^\alpha(z,\cdot), \langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\rangle\rangle\!\rangle_\mu &= \int_{\mathcal{N}'} \langle P_n^{\mu,\alpha}(x-z), \varphi_\alpha^{(n)}\rangle d\mu(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} \int_{\mathcal{N}'} \langle P_k^{\mu,\alpha}(x) \hat{\otimes} P_{n-k}^{\delta_0,,\alpha}(-z), \varphi_\alpha^{(n)}\rangle d\mu\,(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}_\mu(\langle P_k^{\mu,\alpha}(\cdot) \hat{\otimes} P_{n-k}^{\delta_0,,\alpha}(-z), \varphi_\alpha^{(n)}\rangle) \\ &= \langle P_n^{\delta_0,\alpha}(-z), \varphi_\alpha^{(n)}\rangle, \end{aligned}$$

where we made use of the relation (4.9). This had to be shown. In other words, we have proven that $\rho_\mu^\alpha(z,\cdot)$ is the generating function of the $\mathbb{Q}^{\mu,\alpha}$-system.

$$\rho_\mu^\alpha(-z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle Q_n^{\mu,\alpha}(\cdot), P_n^{\delta_0,\alpha}(z)\rangle.$$

Example 4.4.2 (Delta function) *For $z \in \mathcal{N}'_{\mathbb{C}}$ we define a generalized distribution by the following $\mathbb{Q}^{\mu,\alpha}$-decomposition:*

$$\delta_z^\alpha = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{\mu,\alpha}(P_n^{\mu,\alpha}(z)).$$

If $p \in \mathbb{N}$ is large enough and $\varepsilon > 0$ sufficiently small there exists $\sigma_\varepsilon > 0$ according to (4.10) such that

$$\begin{aligned} \|\delta_z^\alpha\|^2_{-p,-q,\mu,\alpha} &= \sum_{n=0}^{\infty} (n!)^{-2} 2^{-nq} |P_n^{\mu,\alpha}(z)|^2_{-p} \\ &\leq 4\exp(2\varepsilon|z|_{-p}) \sum_{n=0}^{\infty} \sigma_\varepsilon^{-2n} 2^{-nq},\ z \in \mathcal{H}_{-p,\mathbb{C}}, \end{aligned}$$

which is finite for sufficiently large $q \in \mathbb{N}$. Thus $\delta_z^\alpha \in (\mathcal{N})^{-1}_{\mu,\alpha}$. The action of δ_z^α on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle \in (\mathcal{N})^1,$$

is given by

$$\langle\!\langle \delta_z, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}(z), \varphi_\alpha^{(n)} \rangle = \varphi(z),$$

because of the biorthogonality property (cf. Theorem 4.2.7). This means that δ_z^α (in particular for z real) plays the role of a "δ-function" (evaluation map) in the calculus we discuss.

Theorem 4.4.3 *For a fixed measure μ and for all function α, as in Subsection 4.1, we have*

$$(\mathcal{N})^{-1}_{\mu,\alpha} = (\mathcal{N})^{-1}_{\mu},$$

i.e., the space of distributions is the same for all functions α in the above conditions.

Proof. Let $\Phi \in (\mathcal{N})^{-1}_{\mu,\alpha}$ be given, then by Theorem 4.2.8 there exists generalized kernels $\Phi_\alpha^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ such that Φ has the following representation

$$\Phi = \sum_{n=0}^{\infty} \langle Q_n^{\mu,\alpha}, \Phi_\alpha^{(n)} \rangle.$$

Now we use the definition of $Q_n^{\mu,\alpha}$ given in (4.18) to obtain

$$\begin{aligned}(S_\mu \Phi)(\theta) &= \sum_{n=0}^{\infty} \langle \Phi_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \rangle \\ &= S_\mu \hat{\Phi}(g_\alpha(\theta)),\ \theta \in \mathcal{N}_{\mathbb{C}}, \qquad (4.24)\end{aligned}$$

where

$$\hat{\Phi} = \sum_{n=0}^{\infty} \langle Q_n^{\mu}, \Phi_\alpha^{(n)} \rangle \in (\mathcal{N})_\mu^{-1}.$$

Hence by characterization theorem (cf. Theorem 3.4.3) $S_\mu \hat{\Phi} \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. But from (4.24) we see that

$$S_\mu \Phi = (S_\mu \hat{\Phi}) \circ g_\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}),$$

since this is the composition of two holomorphic functions (see [Din81]), again by the characterization theorem we conclude that $\Phi \in (\mathcal{N})_\mu^{-1}$. Hence $(\mathcal{N})_{\mu,\alpha}^{-1} \subseteq (\mathcal{N})_\mu^{-1}$.

Conversely, let $\Psi \in (\mathcal{N})_\mu^{-1}$ be given, i.e.,

$$\Psi = \sum_{n=0}^{\infty} \langle Q_n^{\mu}, \Psi^{(n)} \rangle,\ \Psi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}.$$

We want to prove that $\Psi \in (\mathcal{N})_{\mu,\alpha}^{-1}$. Due to (4.18) and the definition of $(\mathcal{N})_\mu^{-1}$ it is sufficient to show that

$$(S_\mu \Psi)(\theta) = \sum_{n=0}^{\infty} \langle \hat{\Psi}_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \rangle,\ \theta \in \mathcal{N}_{\mathbb{C}},$$

where $\hat{\Psi}_\alpha^{(n)}$ satisfy

$$\sum_{n=0}^{\infty} 2^{-nq} |\hat{\Psi}_\alpha^{(n)}|_{-p}^2 < \infty\ p, q \in \mathbb{N}.$$

On the other hand, for a given $\theta \in \mathcal{N}_{\mathbb{C}}$

$$(S_\mu \Psi)(\theta) = \sum_{n=0}^{\infty} \langle \Psi^{(n)}, \theta^{\otimes n} \rangle =: G\left(\theta\right),$$

hence $G \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. But we can write

$$G(\theta) = G(\alpha \circ g_\alpha(\theta)) = \hat{G}(g_\alpha(\theta)),$$

where $\hat{G} = G \circ \alpha$, with $G \circ \alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Therefore

$$\hat{G}(g_\alpha(\theta)) = \sum_{n=0}^{\infty} \langle \hat{G}_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \rangle,$$

where the coefficients $\hat{G}_\alpha^{(n)}$ verify

$$\sum_{n=0}^{\infty} 2^{-nq} |\hat{G}_\alpha^{(n)}|_{-p}^2 < \infty.$$

The result follows with $\hat{\Psi}_\alpha^{(n)} = \hat{G}_\alpha^{(n)}$, i.e., $\Psi \in (\mathcal{N})_{\mu,\alpha}^{-1}$. ■

4.5 The Wick product

Here we give the natural generalization of the Wick multiplication in the present setting.

Definition 4.5.1 *Let $\Phi, \Psi \in (\mathcal{N})_\mu^{-1}$. Then we define the Wick product $\Phi \diamond \Psi$ by*

$$S_\mu(\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi\,.$$

This is well defined because $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra and thus by the characterization theorem there exists an element $\Phi \diamond \Psi \in (\mathcal{N})_\mu^{-1}$ such that $S_\mu(\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi$.

From this it follows

$$Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}) \diamond Q_m^{\mu,\alpha}(\Psi_\alpha^{(m)}) = Q_{n+m}^{\mu,\alpha}(\Phi_\alpha^{(n)} \hat{\otimes} \Psi_\alpha^{(m)}),$$

$\Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}$ and $\Psi_\alpha^{(m)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} m}$. So in terms of $\mathbb{Q}^{\mu,\alpha}$-decomposition

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)}) \text{ and } \Psi = \sum_{m=0}^{\infty} Q_m^{\mu,\alpha}(\Psi_\alpha^{(m)}),$$

the Wick product is given by

$$\Phi \diamond \Psi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}(\Xi_\alpha^{(n)}),$$

where

$$\Xi_\alpha^{(n)} = \sum_{k=0}^{n} \Phi_\alpha^{(k)} \hat{\otimes} \Psi_\alpha^{(n-k)}.$$

This allows for a concrete norm estimate.

Proposition 4.5.2 *The Wick product is continuous on $(\mathcal{N})_\mu^{-1}$. In particular the following estimate holds for $\Phi \in (\mathcal{H}_{-p_1})^{-1}_{-q_1,\mu,\alpha}$, $\Psi \in (\mathcal{H}_{-p_2})^{-1}_{-q_2,\mu,\alpha}$ and $p = \max(p_1, p_2)$, $q = q_1 + q_2 + 1$*

$$\|\Phi \diamond \Psi\|_{-p,-q,\mu,\alpha} \leq \|\Phi\|_{-p_1,-q_1,\mu,\alpha} \|\Psi\|_{-p_2,-q_2,\mu,\alpha} .$$

Proof. We can estimate as follows

$$\begin{aligned}
\|\Phi \diamond \Psi\|^2_{-p,-q,\mu,\alpha} &= \sum_{n=0}^{\infty} 2^{-nq} |\Xi_\alpha^{(n)}|^2_{-p} \\
&= \sum_{n=0}^{\infty} 2^{-nq} \left(\sum_{k=0}^{n} |\Phi_\alpha^{(k)}|_{-p} |\Psi_\alpha^{(n-k)}|_{-p} \right)^2 \\
&\leq \sum_{n=0}^{\infty} 2^{-nq} (n+1) \sum_{k=0}^{n} |\Phi_\alpha^{(k)}|^2_{-p} |\Psi_\alpha^{(n-k)}|^2_{-p} \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-nq_1} |\Phi_\alpha^{(k)}|^2_{-p} 2^{-nq_2} |\Psi_\alpha^{(n-k)}|^2_{-p} \\
&\leq \left(\sum_{n=0}^{\infty} 2^{-nq_1} |\Phi_\alpha^{(n)}|^2_{-p} \right) \left(\sum_{n=0}^{\infty} 2^{-nq_2} |\Psi_\alpha^{(n)}|^2_{-p} \right) \\
&= \|\Phi\|^2_{-p_1,-q_1,\mu,\alpha} \|\Psi\|^2_{-p_2,-q_2,\mu,\alpha} .
\end{aligned}$$

■

Similar to the Gaussian case the special properties of the space $(\mathcal{N})_\mu^{-1}$ allow the definition of *Wick analytic functions* under very general assumptions. This has proven to be of some relevance to solve equations e.g., of the type $\Phi \diamond X = \Psi$ for $X \in (\mathcal{N})_\mu^{-1}$. See [KLS96] for the Gaussian case and [HØUZ96] for more details.

Proposition 4.5.3 *For any $n \in \mathbb{N}$ and any α as in Subsection 4.1 we have $Q_n^{\mu,\alpha} = (Q_1^{\mu,\alpha})^{\diamond n}$.*

Proof. Let $\Phi^{(1)} \in \mathcal{N}'_{\mathbb{C}}$ be given. Then, if $\theta \in \mathcal{N}_{\mathbb{C}}$, we obtain

$$\begin{aligned} S_\mu[(Q_1^{\mu,\alpha}(\Phi^{(1)}))^{\diamond n}](\theta) &= \langle \Phi^{(1)}, g_\alpha(\theta)\rangle^n \\ &= \langle (\Phi^{(1)})^{\hat{\otimes} n}, (g_\alpha(\theta))^{\otimes n}\rangle \\ &= S_\mu[Q_n^{\mu,\alpha}((\Phi^{(1)})^{\hat{\otimes} n})](\theta). \end{aligned}$$

■

Theorem 4.5.4 *Let $F : \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of the point $z_0 = \mathbb{E}(\Phi)$, $\Phi \in (\mathcal{N})_\mu^{-1}$. Then $F^\diamond(\Phi)$ defined by $S_\mu(F^\diamond(\Phi)) = F(S_\mu\Phi)$ exists in $(\mathcal{N})_\mu^{-1}$.*

Proof. By Theorems 4.4.3 and 3.4.3 we have $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Then $F(S_\mu\Phi) \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ since the composition of two analytic functions is also analytic. Again by the above mentioned theorems we find that $F^\diamond(\Phi)$ exists in $(\mathcal{N})_\mu^{-1}$. ■

Remark 4.5.5 *If $F(z)$ have the following representation*

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then the Wick series

$$\sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\diamond n},$$

(where $\Psi^{\diamond n} = \Psi \diamond \cdots \diamond \Psi$ n-times) converges in $(\mathcal{N})_\mu^{-1}$ and

$$F^\diamond(\Phi) = \sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\diamond n},$$

holds.

Example 4.5.6 *The above mentioned equation $\Phi \diamond X = \Psi$ can be solved if $\mathbb{E}_\mu(\Phi) = S_\mu\Phi(0) \neq 0$. That implies $(S_\mu\Phi)^{-1} \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Thus*

$$\Phi^{\diamond(-1)} = S_\mu^{-1}((S_\mu\Phi)^{-1}) \in (\mathcal{N})_\mu^{-1}.$$

Then $X = \Phi^{\diamond(-1)} \diamond \Psi$ is the solution in $(\mathcal{N})_\mu^{-1}$. For more instructive examples we refer the reader to [KLS96, Section 5].

4.6 Change of measure

Suppose we are given two measures $\mu, \tilde{\mu} \in \mathcal{M}_a(\mathcal{N}')$ both satisfying Assumption 3.1.6. Let a distribution $\tilde{\Phi} \in (\mathcal{N})^{-1}_{\tilde{\mu}}$ be given. Since the test function space $(\mathcal{N})^1$ is invariant under changes of measure in view of Theorem 4.3.9, the continuous mapping

$$\varphi \longmapsto \langle\!\langle \tilde{\Phi}, \varphi \rangle\!\rangle_{\tilde{\mu}}, \ \varphi \in (\mathcal{N})^1,$$

can also be represented as a distribution $\Phi \in (\mathcal{N})^{-1}_{\mu}$. So we have the implicit relation

$$\tilde{\Phi} \in (\mathcal{N})^{-1}_{\tilde{\mu}} \longleftrightarrow \Phi \in (\mathcal{N})^{-1}_{\mu},$$

defined by

$$\langle\!\langle \tilde{\Phi}, \varphi \rangle\!\rangle_{\tilde{\mu}} = \langle\!\langle \Phi, \varphi \rangle\!\rangle_{\tilde{\mu}}.$$

This section provide formulas which make this relation more explicit in terms of re-decomposition of the $\mathbb{Q}^{\mu,\alpha}$-system. First we need an explicit relation of the corresponding $\mathbb{P}^{\mu,\alpha}$-system.

Lemma 4.6.1 *Let $\mu, \tilde{\mu} \in \mathcal{M}_a(\mathcal{N}')$ be given, then*

$$P_n^{\mu,\alpha}(x) = \sum_{k+m+l=n} \frac{n!}{k!m!l!} P_k^{\tilde{\mu},\alpha}(x) \hat{\otimes} P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}. \tag{4.25}$$

Proof. Expanding each factor in the formula

$$e_\mu^\alpha(\theta; x) = e_{\tilde{\mu}}^\alpha(\theta; x)\, l_\mu^{\alpha-1}(\theta)\, l_{\tilde{\mu}}^\alpha(\theta),$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^\infty \frac{1}{n!} \langle P_n^{\mu,\alpha}(x), \theta^{\otimes n} \rangle \\
= \ & \sum_{k=0}^\infty \frac{1}{k!} \langle P_k^{\tilde{\mu},\alpha}(x), \theta^{\otimes k} \rangle \sum_{m=0}^\infty \frac{1}{m!} \langle P_m^{\mu,\alpha}(0), \theta^{\otimes m} \rangle \sum_{l=0}^\infty \frac{1}{l!} \langle M_l^{\tilde{\mu},\alpha}, \theta^{\otimes l} \rangle \\
= \ & \sum_{n=0}^\infty \frac{1}{n!} \left\langle \sum_{k+m+l=n} \frac{n!}{k!m!l!} P_k^{\tilde{\mu},\alpha}(x) \hat{\otimes} P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \theta^{\otimes n} \right\rangle.
\end{aligned}$$

A comparison of coefficients gives the above result. ■

An immediate consequence is the next reordering lemma.

Lemma 4.6.2 *Let $\varphi \in (\mathcal{N})^1$ be given. Then φ has the representation in $\mathbb{P}^{\mu,\alpha}$-system as well as $\mathbb{P}^{\tilde{\mu},\alpha}$-system:*

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle = \sum_{n=0}^{\infty} \langle P_n^{\tilde{\mu},\alpha}, \tilde{\varphi}_\alpha^{(n)} \rangle,$$

where $\varphi_\alpha^{(n)}$, $\tilde{\varphi}_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}_0$ and the following formula holds

$$\tilde{\varphi}_\alpha^{(n)} = \sum_{m,l=0}^{\infty} \frac{(n+m+l)!}{n!m!l!} (P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi_\alpha^{(n+m+l)})_{\mathcal{H}^{\hat{\otimes}(m+l)}}. \tag{4.26}$$

Proof. We use the relation (4.25) to obtain

$$\begin{aligned}
\varphi &= \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle \\
&= \sum_{n=0}^{\infty} \left\langle \sum_{k+m+l=n} \frac{n!}{k!m!l!} P_k^{\tilde{\mu},\alpha}(x) \hat{\otimes} P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi_\alpha^{(n)} \right\rangle \\
&= \sum_{k,m,l=0}^{\infty} \frac{(k+m+l)!}{k!m!l!} \langle P_k^{\tilde{\mu},\alpha}(x), (P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi_\alpha^{(k+m+l)})_{\mathcal{H}^{\hat{\otimes}(m+l)}} \rangle \\
&= \sum_{k=0}^{\infty} \left\langle P_k^{\tilde{\mu},\alpha}(x), \sum_{m,l=0}^{\infty} \frac{(k+m+l)!}{k!m!l!} (P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi_\alpha^{(k+m+l)})_{\mathcal{H}^{\hat{\otimes}(m+l)}} \right\rangle.
\end{aligned}$$

Then a comparison of coefficients give the result. ■

Now we may prove the announced theorem.

Theorem 4.6.3 *Let $\tilde{\Phi}$ be a generalized function with representation*

$$\tilde{\Phi} = \sum_{n=0}^{\infty} \langle Q_n^{\tilde{\mu},\alpha}, \tilde{\Phi}_\alpha^{(n)} \rangle.$$

Then

$$\Phi = \sum_{n=0}^{\infty} \langle Q_n^{\mu,\alpha}, \Phi_\alpha^{(n)} \rangle,$$

defined by

$$\langle\langle \Phi, \varphi \rangle\rangle_\mu = \langle\langle \tilde{\Phi}, \varphi \rangle\rangle_{\tilde{\mu}}, \ \varphi \in (\mathcal{N})^1,$$

is in $(\mathcal{N})^{-1}_{\mu}$ *and the following relation holds*

$$\Phi^{(n)} = \sum_{k+m+l=n} \frac{1}{m!l!} \tilde{\Phi}^{(k)}_{\alpha} \hat{\otimes} P^{\mu,\alpha}_{m}(0) \hat{\otimes} M^{\tilde{\mu},\alpha}_{l}.$$

Proof. We can insert formula (4.26) in the formula

$$\begin{aligned}
& \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}_{\alpha}, \varphi^{(n)}_{\alpha} \rangle \\
= & \sum_{k=0}^{\infty} k! \rangle \tilde{\Phi}^{(k)}_{\alpha}, \tilde{\varphi}^{(k)}_{\alpha} \rangle \\
= & \sum_{k=0}^{\infty} k! \left\langle \tilde{\Phi}^{(k)}_{\alpha}, \sum_{m,l=0}^{\infty} \frac{(k+m+l)!}{k!m!l!} (P^{\mu,\alpha}_{m}(0) \hat{\otimes} M^{\tilde{\mu},\alpha}_{l}, \varphi^{(k+m+l)}_{\alpha})_{\mathcal{H}^{\hat{\otimes}(m+l)}} \right\rangle \\
= & \sum_{k,m,l=0}^{\infty} \frac{(k+m+l)!}{m!l!} \langle \tilde{\Phi}^{(k)}_{\alpha} \hat{\otimes} P^{\mu,\alpha}_{m}(0) \hat{\otimes} M^{\tilde{\mu},\alpha}_{l}, \varphi^{(k+m+l)}_{\alpha} \rangle \\
= & \sum_{n=0}^{\infty} n! \left\langle \sum_{k+m+l=n} \frac{1}{m!l!} \tilde{\Phi}^{(k)}_{\alpha} \hat{\otimes} P^{\mu,\alpha}_{m}(0) \hat{\otimes} M^{\tilde{\mu},\alpha}_{l}, \varphi^{(n)}_{\alpha} \right\rangle,
\end{aligned}$$

and compare coefficients again. ■

Chapter 5

Poisson Analysis

The Analysis on pure Poisson spaces was developed in [BLL95], [CP90], [IK88], [NV95], [Pri95], [Us95] and many others from different points of view. In Chapter 4 we have developed methods for non-Gaussian analysis based on generalized Appell systems. In the case of Poisson measures, the Appell system coincides with the system of generalized Charlier polynomials, however the desirable extensions to compound Poisson and for example Gamma processes are straightforward, see Chapter 6 for details.

On the one hand in the works [AKR96a], [AKR96b], [AKR98a], and [AKR98b] analysis and geometry on configuration spaces Γ_X over a Riemannian manifold X was developed, see Section 5.4 (and also its generalizations in Chapter 7, Section 7.2 for compound configuration spaces Ω). One of the consequences of the discussed approach was a description of the well-known equilibrium process on configuration spaces. This process is nothing but the Brownian motion associated with a Dirichlet form of the Poisson measure π_σ with intensity measure σ on $\mathcal{B}(X)$. This form is canonically associated with the introduced geometry on configuration spaces and is called *intrinsic Dirichlet form* of the measure π_σ.

On the other hand there is a well-known realization (canonical isomorphism) of the Hilbert space $L^2(\Gamma_X, \pi_\sigma)$ and the corresponding Fock space

$$\mathrm{Exp}L^2(X,\sigma) := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n L^2(X,\sigma),$$

where $\mathrm{Exp}_n L^2(X,\sigma)$ denotes the n-fold symmetric tensor product of $L^2(X,\sigma)$ and $\mathrm{Exp}_0 L^2(X,\sigma) := \mathbb{C}$, see Section 5.2. This isomorphism produces natural operations in $L^2(\Gamma_X, \pi_\sigma)$ as images of the standard Fock space operators, see Section 5.3 below. In particular, we can consider the image of the annihilation operator from the Fock space as a natural version of a "gradient" operator in $L^2(\Gamma_X, \pi_\sigma)$. The differentiable structure in $L^2(\Gamma_X, \pi_\sigma)$ which appears in this way we consider as *external* because it is produced via transportation from the Fock space. Here we only would like to mention that the most important feature of this "gradient" (so-called Poissonian gradient) is that it produces (via a corresponding integration by parts formula) the orthogonal system of generalized Charlier polynomials on $L^2(\Gamma_X, \pi_\sigma)$, see Remark 5.3.11 - (5.28). In addition, we mention that in this setting as tangent space to each point $\gamma \in \Gamma_X$ we choose the same Hilbert space $L^2(X, \sigma)$.

As was shown in [AKR98a, Section 5] the intrinsic Dirichlet form of the measure π_σ can be represented also in terms of the external Dirichlet form $\mathcal{E}^P_{\pi_\sigma, H^X_\sigma}$ with coefficient operator H^X_σ (the Dirichlet operator associated with σ on X) which uses the external differentiable structure. This is a non trivial relation because the intrinsic Dirichlet form is a local form, however the extrinsic one is not. If we change the Poisson measure π_σ to a Gibbs measure μ on the configuration space Γ_X, which describes the equilibrium of an interacting particle systems, the corresponding intrinsic Dirichlet form can still be used for constructing the corresponding stochastic dynamics (cf. [AKR98b, Section 5]) or for constructing a quantum infinite particle Hamiltonian in models of quantum field theory, see [AKR97].

The aim of Section 5.5 is to show that even for the interacting case there is a transparent relation between the intrinsic Dirichlet form and the extrinsic one, see Theorem 5.5.3. The proof is based on the Nguyen-Zessin characterization of Gibbs measures (cf. [NZ79, Theorem 2] or Proposition 5.5.4 below) which is a consequence of the Mecke identity (cf. [Mec67, Satz 3.1] or (5.8) below), see Remark 5.5.5 below for more details.

As a consequence of the mentioned relation we prove the closability of the pre-Dirichlet form $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ on $L^2(\Gamma_X, \mu)$, where μ is a tempered grand canonical Gibbs measure, see Section 5.6 for this notion.

Another motivation for deriving Theorem 5.5.3 in Section 5.5 is to use this result for studying spectral properties of Hamiltonians of intrinsic Dirichlet forms associated with Gibbs measures.

5.1 Measures on configuration spaces

In this section we describe some facts about probability measures on configuration spaces which are necessary later on.

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by T_xX; and the tangent bundle endowed with its natural differentiable structure will be denoted by $TX = \cup_{x\in X} T_xX$. The Riemannian metric on X associates to each point $x \in X$ an inner product on T_xX which we denote by $\langle\cdot,\cdot\rangle_{T_xX}$. The associated norm will be denoted by $|\cdot|_{T_xX}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

5.1.1 The configuration space over a manifold

The *simple configuration space* $\Gamma := \Gamma_X$ over the manifold X is defined as the set of all locally finite subsets (simple configurations) in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}. \qquad (5.1)$$

Here (and below) $|A|$ denotes the cardinality of a set A. For any $Y \subset X$ we define

$$\Gamma_Y := \{\gamma \in \Gamma \,||\gamma \cap (X \backslash Y)| = 0\}.$$

We sometimes use the shorthand γ_Y for $\gamma \cap Y$, $Y \subset X$.

We can identify any $\gamma \in \Gamma_X$ with the corresponding sum of Dirac measures (i.e., non-negative integer-valued Radon measure), namely

$$\Gamma_X \ni \gamma \mapsto \sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_p(X) \subset \mathcal{M}(X), \qquad (5.2)$$

where $\sum_{x\in\emptyset} \varepsilon_x :=$ zero measure and $\mathcal{M}(X)$ (resp. $\mathcal{M}_p(X)$) denotes the set of all non-negative (resp. non-negative integer-valued) Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X such that all maps

$$\Gamma_X \ni \gamma \mapsto \langle \gamma, f\rangle := \int_X f(x) d\gamma(x) = \sum_{x\in\gamma} f(x),$$

are continuous. Here $f \in C_0(X)$ (the set of all real-valued continuous functions on X with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ-algebra.

For any $B \in \mathcal{B}(X)$ we define, as usual,

$$\Gamma \ni \gamma \mapsto N_B(\gamma) := |\gamma_B| \in \mathbb{N} \cup \{+\infty\}.$$

Then $\mathcal{B}(\Gamma) = \sigma(\{N_\Lambda | \Lambda \in \mathcal{O}_c(X)\})$. For any $A \in \mathcal{B}(X)$ we also define

$$\mathcal{B}_A(\Gamma) := \sigma(\{N_\Lambda | \Lambda \in \mathcal{B}_c(X), \Lambda \subset A\}).$$

For later use we recall the "localized" description of Γ_X. For any $n \in \mathbb{N}_0$ and any $Y \in \mathcal{B}(X)$ we define the *n-point configuration space* $\Gamma_Y^{(n)}$ as a subset of Γ_Y by

$$\Gamma_Y^{(n)} := \{\gamma \in \Gamma_Y | |\gamma| = n\},\ \Gamma_Y^{(0)} := \{\emptyset\},$$

and denote the corresponding σ-algebra by $\mathcal{B}(\Gamma_Y^{(n)})$.

There is a bijection

$$\tilde{Y}^n/\mathfrak{S}_n \to \Gamma_Y^{(n)},\ n \in \mathbb{N},\ Y \in \mathcal{B}(X), \tag{5.3}$$

where

$$\tilde{Y}^n := \{(x_1, \ldots, x_n) | x_i \in Y,\ x_i \neq x_j,\ \text{for } i \neq j\},$$

and $\mathfrak{S}_n$ denotes the permutation group over $\{1, \ldots, n\}$. Since this bijection is measurable in both directions the natural σ-algebra on $\tilde{Y}^n/\mathfrak{S}_n$ is isomorphic to $\mathcal{B}(\Gamma_Y^{(n)})$.

One can reconstruct Γ from the sets $\Gamma_\Lambda^{(n)}$ using the following scheme. First notice that we can write for any $\Lambda \in \mathcal{B}_c(X)$

$$\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}.$$

This space is equipped the σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ of disjoint union of the σ-algebras $\mathcal{B}(\Gamma_\Lambda^{(n)})$.

For any $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(X)$ with $\Lambda_1 \subset \Lambda_2$ there are natural maps

$$p_{\Lambda_2,\Lambda_1} : \Gamma_{\Lambda_2} \longrightarrow \Gamma_{\Lambda_1}, \qquad p_{\Lambda_1} : \Gamma \longrightarrow \Gamma_{\Lambda_1},$$

defined by $p_{\Lambda_2,\Lambda_1}(\gamma) := \gamma_{\Lambda_1},\ \gamma \in \Gamma_{\Lambda_2}$ (resp. $p_{\Lambda_1}(\gamma) = \gamma_{\Lambda_1},\ \gamma \in \Gamma$). It can be shown (cf. [Shi94]) that $(\Gamma, \mathcal{B}(\Gamma))$ coincides with the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_c(X)$, i.e., $\mathcal{B}(\Gamma)$ is the smallest σ-algebra on Γ such that all restriction mappings

$$\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda \in \Gamma_\Lambda,\ \Lambda \in \mathcal{B}_c(\Gamma),$$

are $\mathcal{B}(\Gamma)/\mathcal{B}(\Gamma_\Lambda)$-measurable.

$\mathcal{B}(\Gamma)$ is generated by the sets

$$C_{\Lambda,n} := \{\gamma \in \Gamma_X \,|\, |\gamma \cap \Lambda| = n\}, \tag{5.4}$$

where $\Lambda \in \mathcal{B}_c(X)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, see e.g., [GGV75] and [Shi94]. Note that for any $\Lambda \in \mathcal{B}(X)$ and all $n \in \mathbb{N}_0$ the set $C_{\Lambda,n}$ is, indeed, a Borel subset of Γ_X. Sets of the form (5.4) are called *cylinder* sets. That means $\mathcal{B}(\Gamma)$ is generated by the family of mappings $\{N_\Lambda,\ \Lambda \in \mathcal{B}_c(X)\}$.

5.1.2 Poisson measures

For the construction of a Poisson measure on Γ_X first we need to fix an intensity measure σ on the underlying manifold X. We take a density $\rho > 0$ m-a.s. such that $\rho^{1/2} \in H^{1,2}_{loc}(X)$ and put $d\sigma(x) = \rho(x)dm(x)$. Here $H^{1,2}_{loc}(X)$ denotes the local Sobolev space of order 1 in $L^2_{loc}(X,m)$. Then σ is a non-atomic Radon measure on X, in particular, $\sigma(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_c(X)$. Below we denote by $\mathcal{D} := C_0^\infty(X)$ the set of all C^∞-functions on X with compact support and by $\mathcal{D}'$ the dual space of $\mathcal{D}$.

For any $n \in \mathbb{N}$ we introduce the product measure $\sigma^{\otimes n}$ on $(X^n, \mathcal{B}(X^n))$. Of course $\sigma^{\otimes n}(X^n \backslash \tilde{X}^n) = 0$. Let $\Lambda \in \mathcal{B}_c(X)$ be fixed. Then the measure $\sigma^{\otimes n}$ can be considered as a finite measure on $\tilde{\Lambda}^n$. Let

$$\sigma_{\Lambda,n} := \sigma^{\otimes n} \circ (\mathrm{sym}^n_\Lambda)^{-1},$$

be the corresponding measure on $\Gamma^{(n)}_\Lambda$, where

$$\mathrm{sym}^n_\Lambda : \tilde{\Lambda}^n \to \Gamma^{(n)}_\Lambda,$$

given by

$$\mathrm{sym}^n_\Lambda((x_1, \ldots, x_n)) := \{x_1, \ldots, x_n\} \in \Gamma^{(n)}_\Lambda.$$

Then we consider the so-called *Lebesgue-Poisson measure* ν^Λ_σ on $\mathcal{B}(\Gamma_\Lambda)$, which coincides on each $\Gamma^{(n)}_\Lambda$ with the measure $\sigma_{\Lambda,n}$, as follows

$$\nu^\Lambda_\sigma := \sum_{n=0}^\infty \frac{1}{n!} \sigma_{\Lambda,n},$$

and $\sigma_{\Lambda,0}(\emptyset) := 1$. Considered as a measure on $\mathcal{B}(\Gamma_\Lambda)$ it is finite and $\nu_\sigma^\Lambda(\Gamma_\Lambda) = e^{\sigma(\Lambda)}$. Therefore, we can define a probability measure π_σ^Λ on Γ_Λ putting

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)}\nu_\sigma^\Lambda. \tag{5.5}$$

In order to obtain the existence of a unique probability measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\pi_\sigma^\Lambda = \pi_\sigma \circ p_\Lambda^{-1},\ \Lambda \in \mathcal{B}_c(X), \tag{5.6}$$

we notice that the family $\{\pi_\sigma^\Lambda | \Lambda \in \mathcal{B}_c(X)\}$ is consistent, i.e.,

$$\pi_\sigma^{\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1} = \pi_\sigma^{\Lambda_1},\ \Lambda_1, \Lambda_2 \in \mathcal{B}_c(X), \Lambda_1 \subset \Lambda_2,$$

and thus, by a version of Kolmogorov's theorem for the projective limit space Γ (cf. [Par67, Chap. V Theorem 5.1]) any such family determines uniquely a measure π_σ on $\mathcal{B}(\Gamma)$ such that $\pi_\sigma^\Lambda = \pi_\sigma \circ p_\Lambda^{-1}$. The measure π_σ is called *Poisson measure* with intensity measure σ.

Let us compute the Laplace transform of the measure π_σ. For a given $f \in C_0(X)$ we have supp$f \subset \Lambda$ for some $\Lambda \in \mathcal{B}_c(X)$. Then

$$\langle \gamma, f \rangle = \langle \gamma_\Lambda, f \rangle,\ \gamma \in \Gamma,$$

and

$$\int_\Gamma e^{\langle \gamma, f \rangle} d\pi_\sigma(\gamma) = \int_{\Gamma_\Lambda} e^{\langle \gamma, f \rangle} d\pi_\sigma^\Lambda(\gamma).$$

Next we use the infinite divisibility of the measure π_σ^Λ (cf. (5.5)) to write the right hand side of above equality as

$$\begin{aligned} & e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \exp(f(x_1) + \ldots + f(x_n)) d\sigma(x_1) \ldots d\sigma(x_n) \\ = \ & e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \left(\int_{\Lambda^n} e^{f(x)} d\sigma(x) \right)^n \\ = \ & \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x) \right). \end{aligned}$$

Thus, for all $f \in C_0(X)$ the following formula holds

$$l_{\pi_\sigma}(f) := \int_\Gamma e^{\langle \gamma, f \rangle} d\pi_\sigma(\gamma) = \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x) \right). \tag{5.7}$$

Remark 5.1.1 *1. Let us mention that (5.7) defines, via Minlos' theorem, the measure π_σ on a linear space $F(X)$ of generalized functions on X, see e.g., [GV68]. An additional analysis shows that the support of the measure π_σ consists of generalized functions of the form $\sum_{x\in\gamma}\varepsilon_x$, $\gamma \in \Gamma_X$, see e.g., [Oba87] and [Shi94], and then π_σ can be considered as a measure on Γ_X.*

2. By the same argument (5.7) holds for any $\mathcal{B}(X)$-measurable function f with compact support such that e^f is σ-integrable on supp$f$*. A simple limit-argument then implies that (5.7) holds for all f such that $e^f - 1 \in L^1(\sigma)$.*

The Poisson measure π_σ may be characterized using an integral equation which is known as Mecke identity, see e.g., [Mec67, Satz 3.1]

$$\int_\Gamma \left(\int_X h(\gamma, x) d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_X \int_\Gamma h(\gamma + \varepsilon_x, x) d\pi_\sigma(\gamma) d\sigma(x), \qquad (5.8)$$

where h is any non-negative, $\mathcal{B}(\Gamma)\times\mathcal{B}(X)$-measurable function. This equality will be used later in this chapter, see Section 5.3 below.

5.2 The Fock space isomorphism of Poisson space

In this section we prove the existence of an unitary isomorphism between the Fock space and the space of square integrable functions with respect to Poisson measure π_σ. We will produce this isomorphism using our general approach proposed in Chapter 4, see also [IK88], [Ito88], [KSS97], [KSSU98], and [NV95] for related results.

We already introduced in Example 3.1.9 (see page 53) a system of orthogonal polynomials on $L^2(S'(\mathbb{R}), \pi)$. Here, for practical reasons we use the triple

$$\mathcal{D} \subset L^2(\sigma) \subset \mathcal{D}', \;\; L^2(\sigma) := L^2(X, \sigma).$$

As in Example 3.1.9 we consider the following transformation on $\mathcal{D}$, $\alpha : \mathcal{D} \to \mathcal{D}$ defined by

$$\alpha(\varphi)(x) := \log(1 + \varphi(x)), \; -1 < \varphi \in \mathcal{D}, \; x \in X.$$

The transformation α satisfies the conditions from Subsection 4.1.1, i.e., $\alpha(0) = 0$ and α is invertible holomorphic in some neighborhood $\mathcal{U}_\alpha$ of zero. Using this transformation we introduce the normalized exponential $e^{\alpha}_{\pi_\sigma}(\cdot;\cdot)$ which is holomorphic on a neighborhood of zero $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha \subset \mathcal{D}$:

$$e^{\alpha}_{\pi_\sigma}(\varphi;\gamma) := \frac{\exp\langle\gamma, \alpha(\varphi)\rangle}{l_{\pi_\sigma}(\alpha(\varphi))} = \exp(\langle\gamma, \log(1+\varphi)\rangle - \langle\varphi\rangle_\sigma), \tag{5.9}$$

for any $\varphi \in \mathcal{U}'_\alpha$, $\gamma \in \Gamma$, where $\langle\varphi\rangle_\sigma := \int_{\mathbb{R}^d} \varphi(x)d\sigma(x)$.

We use the holomorphy of $e^{\alpha}_{\pi_\sigma}(\cdot;\gamma)$ on a neighborhood of zero to expand it in power series which, with Cauchy's inequality, polarization identity and kernel theorem, give us the following result

$$e^{\alpha}_{\pi_\sigma}(\varphi;\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle P^{\pi_\sigma,\alpha}_n(\gamma), \varphi^{\otimes n}\rangle,\ \varphi \in \mathcal{U}'_\alpha \subset \mathcal{U}_\alpha,\ \gamma \in \Gamma, \tag{5.10}$$

where $P^{\pi_\sigma,\alpha}_n : \Gamma \to \mathcal{D}'^{\hat{\otimes} n}$. $\{P^{\pi_\sigma,\alpha}_n(\cdot) =: C^{\sigma}_n(\cdot) | n \in \mathbb{N}_0\}$ is called the system of *generalized Charlier kernels* on Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$. From (5.10) it follows immediately that for any $\varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$ the function

$$\Gamma \ni \gamma \mapsto \langle C^{\sigma}_n(\gamma), \varphi^{(n)}\rangle,$$

is a polynomial of the order n on Γ. The system of functions

$$\{C^{\sigma}_n(\varphi^{(n)})(\gamma) := \langle C^{\sigma}_n(\gamma), \varphi^{(n)}\rangle,\ \forall\varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n},\ n \in \mathbb{N}_0\},$$

is called the system of *generalized Charlier polynomials* for the Poisson measure π_σ.

Proposition 5.2.1 *For any $\varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$ and $\psi^{(m)} \in \mathcal{D}^{\hat{\otimes} m}$ we have*

$$\int_\Gamma \langle C^{\sigma}_n(\gamma), \varphi^{(n)}\rangle\langle C^{\sigma}_m(\gamma), \psi^{(m)}\rangle d\pi_\sigma(\gamma) = \delta_{nm} n!(\varphi^{(n)}, \psi^{(n)})_{L^2(\sigma^{\otimes n})}.$$

Proof. Let $\varphi^{(n)}, \psi^{(m)}$ be given as in the proposition and such that $\varphi^{(n)} = \varphi^{\otimes n}$, $\psi^{(m)} = \psi^{\otimes m}$. Then for $z_1, z_2 \in \mathbb{C}$, and taking into account (5.7) and (5.9) we have

$$\int_\Gamma e^{\alpha}_{\pi_\sigma}(z_1\varphi;\gamma) e^{\alpha}_{\pi_\sigma}(z_2\psi;\gamma) d\pi_\sigma(\gamma)$$

$$
\begin{aligned}
&= \exp(-\langle z_1\varphi + z_2\psi\rangle_\sigma)\int_\Gamma \exp[\langle\gamma, \log((1+z_1\varphi)(1+z_2\psi))\rangle]d\pi_\sigma(\gamma) \\
&= \exp(-\langle z_1\varphi + z_2\psi\rangle_\sigma)\exp\left(\int_{\mathbb{R}^d}(\exp[\log((1+z_1\varphi)(1+z_2\psi))]-1)d\sigma\right) \\
&= \exp(z_1 z_2(\varphi,\psi)_{L^2(\sigma)}) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}z_1^n z_2^n(\varphi^{\otimes n},\psi^{\otimes n})_{L^2(\sigma^{\otimes n})}. \qquad (5.11)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\int_\Gamma e^\alpha_{\pi_\sigma}(z_1\varphi;\gamma)e^\alpha_{\pi_\sigma}(z_2\psi;\gamma)d\pi_\sigma(\gamma) \\
&= \sum_{n,m=0}^{\infty}\frac{z_1^n z_2^m}{n!m!}\int_\Gamma\langle C_n^\sigma(\gamma),\varphi^{\otimes n}\rangle\langle C_m^\sigma(\gamma),\psi^{\otimes m}\rangle d\pi_\sigma(\gamma). \qquad (5.12)
\end{aligned}
$$

Then a comparison of coefficients between (5.11) and (5.12), with polarization identity and linearity, gives the result. ■

Remark 5.2.2 *This proposition gives us the possibility to extend - in the* $L^2(\pi_\sigma) := L^2(\Gamma,\pi_\sigma)$ *sense - the class of* $\langle C_n^\sigma(\gamma),\varphi^{\otimes n}\rangle$*-functions to include kernels from the so-called n-particle Fock space over* $L^2(\sigma)$*.*

The Fock space (cf. Section 2.2) has the form

$$
\mathrm{Exp}L^2(\sigma) := \bigoplus_{n=0}^{\infty}\mathrm{Exp}_n L^2(\sigma),
$$

where $\mathrm{Exp}_n L^2(\sigma) := L^2(\sigma)^{\hat{\otimes} n}_{\mathbb{C}}$ and by definition $\mathrm{Exp}_0 L^2(\sigma) := \mathbb{C}$.

For any $F \in L^2(\pi_\sigma)$ there exists a sequence $(f^{(n)})_{n=0}^\infty \in \mathrm{Exp}L^2(\sigma)$ such that

$$
F(\gamma) = \sum_{n=0}^{\infty}\langle C_n^\sigma(\gamma), f^{(n)}\rangle, \qquad (5.13)
$$

and moreover

$$
\|F\|^2_{L^2(\pi_\sigma)} = \sum_{n=0}^{\infty} n!|f^{(n)}|^2_{L^2(\sigma^{\otimes n})}, \qquad (5.14)
$$

where the right hand side of (5.14) coincides with the square of the norm in $\mathrm{Exp}L^2(\sigma)$. And vice versa, any series of the form (5.13) with coefficients

$(f^{(n)})_{n=0}^{\infty} \in \mathrm{Exp}L^2(\sigma)$ gives a function from $L^2(\pi_\sigma)$. As a result we have the well-known isomorphism I_{π_σ} between $L^2(\pi_\sigma)$ and $\mathrm{Exp}L^2(\sigma)$.

Let us mention that $e^{\alpha}_{\pi_\sigma}(\psi;\cdot)$ is nothing as the coherent state in the Fock space picture. For any $\psi \in \mathcal{D}$, $\psi > -1$, we have

$$L^2(\pi_\sigma) \ni e^{\alpha}_{\pi_\sigma}(\psi;\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle C_n^\sigma(\cdot), \psi^{\otimes n}\rangle \mapsto \mathrm{Exp}\psi \in \mathrm{Exp}L^2\left(\sigma\right).$$

5.3 Extrinsic geometry on Poisson space

In this section we recall the extrinsic geometry on $L^2(\pi_\sigma)$ based on the isomorphism with the Fock space, see Section 5.2. Our approach is based on the general scheme from Chapter 4 but we should also mention [BLL95], [IK88], [KSSU98], [NV95], [Pri95] and references therein for related considerations.

5.3.1 Annihilation operator on Poisson space

Let us introduce a set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ which is dense in $L^2(\pi_\sigma)$.

Definition 5.3.1 *We define $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ as the set of all functions $F:\Gamma\to\mathbb{R}$ of the form*

$$F(\gamma) = f_F(\langle\gamma,\varphi_1\rangle,\ldots,\langle\gamma,\varphi_N\rangle),\ \gamma\in\Gamma, \tag{5.15}$$

where $\varphi_1,\ldots,\varphi_N \in \mathcal{D}$ and f_F is from $C_b^\infty(\mathbb{R}^N)$ (C^∞-functions on $\mathbb{R}^N$ with bounded derivatives). We will call a function $F:\Gamma\to\mathbb{R}$ cylindrical if F is $\mathcal{B}_\Lambda(\Gamma)$-measurable for some $\Lambda\in\mathcal{B}_c(X)$. These functions only depends on the value $\gamma\cap\Lambda$, $\gamma\in\Gamma$. Any $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is a cylindrical function in this sense. Elements from $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ are called smooth cylinder functions on Γ.

Let us define a "gradient" on functions $F:\Gamma\to\mathbb{R}$ which has specific properties on Poissonian spaces.

Definition 5.3.2 *We define the Poissonian gradient ∇^P as a mapping*

$$\nabla^P : \mathcal{F}C_b^\infty(\mathcal{D},\Gamma) \longrightarrow L^2(\pi_\sigma)\otimes L^2(\sigma),$$

given by

$$(\nabla^P F)(\gamma,x) := F(\gamma+\varepsilon_x) - F(\gamma),\ \gamma\in\Gamma,\ \ x\in X. \tag{5.16}$$

Remark 5.3.3 *1. Let us mention that the operation* $\Gamma \ni \gamma \mapsto \gamma + \varepsilon_x \in \Gamma$ *is well-defined because of the property*

$$\pi_\sigma\{\gamma \in \Gamma | x \in \gamma\} = 0, \ \forall x \in X.$$

The fact that $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \ni F \mapsto \nabla^P F \in L^2(\pi_\sigma) \otimes L^2(\sigma)$ *arises from the use of the Hilbert space* $L^2(\sigma)$ *as a tangent space at any point* $\gamma \in \Gamma$. *In Section 5.4 we will give another approach to construct geometry in* Γ, *there the tangent space to the configuration space* Γ *at the point* $\gamma \in \Gamma$ *will strongly depend on the point* γ *in contrast with what we suppose here, cf. Definition 5.4.2 below (see also Chapter 7 Section 7.2 for the corresponding results on compound Poisson space).*

2. *The Poissonian gradient appears from different points of view in many papers on Poissonian analysis, see e.g., [IK88], [NV95], [NV90], and references therein. In statistical physics of continuous systems it appears under algebraic approach, see e.g., [MM91, Chapter 2], [Rue69], and [Rue64]. Here our motivation relates to the gradient* G_α *(cf. Definition 4.2.2) which was computed for the case of Poisson measure* π *in Example 4.2.5.*

3. *The most important feature of the Poissonian gradient is that it produces (via a corresponding integration by parts) the orthogonal system of generalized Charlier polynomials, see Theorem 4.2.4 and Remark 5.3.11 - 2.*

4. *In Section 5.4 we will define another type of gradient on* Γ *(so-called intrinsic gradient) and in Section 5.5 we will give a relation between these two gradients through the associated Dirichlet forms (cf. Theorem 5.4.10 for the Poisson measure* π_σ *and Theorem 5.5.3 for the generalization to Gibbs measures). Form this point of view the relation between the Poissonian gradient and intrinsic gradient (see (5.44) and (5.52)) may be used to rewrite formulae making use of one gradient in therms of the other gradient.*

Next we give an explicit expression for the adjoint of the Poissonian gradient ∇^{P*}.

Proposition 5.3.4 *For any function* $F \in L^1(\pi_\sigma) \otimes L^1(\sigma)$ *we have* $F \in D(\nabla^{P*})$ *and the following equality holds*

$$(\nabla^{P*}F)(\gamma) = \int_X F(\gamma - \varepsilon_x, x)d\gamma(x) - \int_X F(\gamma, x)d\sigma(x),\ \gamma \in \Gamma, \qquad (5.17)$$

Proof. For $X = \mathbb{R}^d$ this proposition was proved in [KSSU98]. Let $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ be given. Then an application of (5.16) gives

$$\begin{aligned}(\nabla^P G, F)_{L^2(\pi_\sigma)\otimes L^2(\sigma)} &= \int_X \int_\Gamma G(\gamma + \varepsilon_x)F(\gamma, x)d\pi_\sigma(\gamma)d\sigma(x) \\ &\quad - \int_X \int_\Gamma G(\gamma)F(\gamma, x)d\pi_\sigma(\gamma)d\sigma(x). \qquad (5.18)\end{aligned}$$

Now we use the Mecke identity (5.8) on the right hand side of (5.18) to obtain

$$\int_\Gamma G(\gamma)\left[\int_X F(\gamma - \varepsilon_x, x)d\gamma(x) - \int_X F(\gamma, x)d\sigma(x)\right] d\pi_\sigma(\gamma),$$

which proves the proposition. ■

Now we are going to give an internal description of the annihilation operator on $L^2(\pi_\sigma)$.

The directional derivative is then defined as

$$\begin{aligned}(\nabla^P_\varphi f)(\gamma) &= ((\nabla^P f)(\gamma, \cdot), \varphi(\cdot))_{L^2(\sigma)} \\ &= \int_X (f(\gamma + \varepsilon_x) - f(\gamma))\varphi(x)d\sigma(x), \qquad (5.19)\end{aligned}$$

for any $\varphi \in \mathcal{D}$. Of course the operator

$$\nabla^P_\varphi : \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \longrightarrow L^2(\pi_\sigma),$$

is closable in $L^2(\pi_\sigma)$.

Proposition 5.3.5 *The closure of* ∇^P_φ *coincides with the image under* I_{π_σ} *of the annihilation operator* $a^-(\varphi)$ *in* $\mathrm{Exp}L^2(\sigma)$*, i.e.,* $I_{\pi_\sigma} a^-(\varphi) I_{\pi_\sigma}^{-1} = \nabla^P_\varphi$.

Proof. To prove this proposition it is enough to show this equality of operators in a total set in the core of the annihilation operator. Let $\psi \in \mathcal{U}'_\alpha$ be given, then having in mind (5.19) and (5.9) it follows that

$$\begin{aligned}(\nabla^P_\varphi e^\alpha_{\pi_\sigma}(\psi;\cdot))(\gamma) &= \int_X (e^\alpha_{\pi_\sigma}(\psi;\gamma+\varepsilon_x) - e^\alpha_{\pi_\sigma}(\psi;\gamma))\varphi(x)d\sigma(x) \\ &= e^\alpha_{\pi_\sigma}(\psi;\gamma)\int_X (\exp(\langle\varepsilon_x, \log(1+\psi)\rangle) - 1)\varphi(x)d\sigma(x) \\ &= (\psi,\varphi)_{L^2(\sigma)} e^\alpha_{\pi_\sigma}(\psi;\gamma). \qquad (5.20)\end{aligned}$$

On the other hand since $I_\sigma^{-1} e^\alpha_{\pi_\sigma}(\psi;\gamma) = \mathrm{Exp}\psi$ it follows that

$$a^-(\varphi)\mathrm{Exp}\psi = (\varphi,\psi)_{L^2(\sigma)}\mathrm{Exp}\psi.$$

Hence if we apply I_{π_σ} to this vector we just obtain the same result as (5.20) which had to be proven. ■

5.3.2 Creation operator on Poisson space

Proposition 5.3.6 *For any $\varphi \in \mathcal{D}$, $G \in \mathrm{Dom}(I_{\pi_\sigma} a^+(\varphi) I^{-1}_{\pi_\sigma})$, where $a^+(\varphi)$ is the creation operator in $\mathrm{Exp}L^2(\sigma)$, the following equality holds*

$$\begin{aligned}((\nabla^P_\varphi)^* G)(\gamma) &= \int_X G(\gamma - \varepsilon_x)\varphi(x)d\gamma(x) - G(\gamma)\int_X \varphi(x)d\sigma(x) \\ &= (G(\gamma - \varepsilon_\cdot), \varphi(\cdot))_{L^2(\gamma)} - G(\gamma)\langle\varphi\rangle_\sigma. \qquad (5.21)\end{aligned}$$

Remark 5.3.7 *In terms of chaos decomposition of $g \in \mathrm{Dom}((\nabla^P_\varphi)^*)$ the equality (5.21) was established in [NV95]. We give an independent proof of (5.21), which is based on the results on absolute continuity of Poisson measures, see e.g., [Sko57] and [Tak90].*

Proof. 1. First we give a version of the proof of (5.21) which uses the Mecke identity.

It follows from (5.19) that

$$\begin{aligned}(\nabla^P_\varphi F, G)_{L^2(\pi_\sigma)} &= \int_\Gamma ((\nabla^P F)(\gamma,\cdot), \varphi(\cdot))_{L^2(\sigma)} G(\gamma) d\pi_\sigma(\gamma) \\ &= ((\nabla^P F)(\cdot,\cdot), G(\cdot)\varphi(\cdot))_{L^2(\pi_\sigma)\otimes L^2(\sigma)}. \qquad (5.22)\end{aligned}$$

Whence using Proposition 5.3.4 we obtain

$$\begin{aligned}((\nabla^P_\varphi)^* G)(\gamma) &= ((\nabla^P)^* G\varphi)(\gamma) \\ &= \int_X G(\gamma - \varepsilon_x)\varphi(x)d\gamma(x) - G(\gamma)\int_X \varphi(x)d\sigma(x) \\ &= (G(\gamma - \varepsilon_\cdot), \varphi(\cdot))_{L^2(\gamma)} - G(\gamma)\langle\varphi\rangle_\sigma,\end{aligned}$$

which proves (5.21).
2. Alternatively, we give an independent prove of (5.21) based on absolute continuity of Poisson measure.

Let $\eta \in \mathcal{D}$ be such that $\eta(x) > -1$, $\forall x \in X$. Denote by σ_η the measure on X having density with respect to σ,

$$\frac{d\sigma_\eta}{d\sigma}(x) = 1 + \eta(x). \tag{5.23}$$

Lemma 5.3.8 *The Poisson measures π_σ and π_{σ_η} on $(\Gamma, \mathcal{B}(\Gamma))$ are mutually absolutely continuous and the Radon-Nikodym derivative $\frac{d\pi_{\sigma_\eta}}{d\pi_\sigma}(\gamma)$ coincides with the normalized exponential, i.e.,*

$$\frac{d\pi_{\sigma_\eta}}{d\pi_\sigma}(\gamma) = e^\alpha_{\pi_\sigma}(\eta;\gamma) = \exp(\langle\gamma, \log(1+\eta)\rangle - \langle\eta\rangle_\sigma).$$

Proof. Let $\eta \in \mathcal{D}$ be such that $\eta(x) > -1$, $\forall x \in X$. Then the Laplace transform of π_{σ_η}, given by (5.7) implies

$$\begin{aligned}
\int_\Gamma \exp(\langle\gamma,\varphi\rangle) d\pi_{\sigma_\eta}(\gamma) &= \exp\left(\int_X (e^{\varphi(x)} - 1)(1+\eta(x)) d\sigma(x)\right) \\
&= e^{-\langle\eta\rangle_\sigma} \exp\left(\int_X (e^{\varphi(x)+\log(1+\eta(x))} - 1) d\sigma(x)\right) \\
&= \int_\Gamma \exp(\langle\gamma,\varphi\rangle) \exp(\langle\gamma, \log(1+\eta)\rangle - \langle\eta\rangle_\sigma) d\pi_\sigma(\gamma).
\end{aligned}$$

■

In order to proof (5.21) it suffices to verify the equality

$$(\nabla^P_\varphi f, g)_{L^2(\pi_\sigma)} = \int_\Gamma f(\gamma)[(g(\gamma - \varepsilon.), \varphi(\cdot))_{L^2(\gamma)} - g(\gamma)\langle\varphi\rangle_\sigma] d\pi_\sigma(\gamma), \tag{5.24}$$

for $f(\gamma) = e^\alpha_{\pi_\sigma}(\psi;\gamma)$, $g(\gamma) = e^\alpha_{\pi_\sigma}(\eta;\gamma)$, ψ, η belong to a neighborhood of zero $\mathcal{U} \subset \mathcal{D}$, because the coherent states $\mathrm{Exp}\psi$, $\psi \in \mathcal{U}$ span a common core for the annihilation and creation operators.

Lemma 5.3.9 *For any $\varphi \in \mathcal{D}$ and for all ψ, η in a neighborhood of zero $\mathcal{U} \subset \mathcal{D}$, the following equality holds*

$$(\nabla^P_\varphi e^\alpha_{\pi_\sigma}(\psi;\cdot), e^\alpha_{\pi_\sigma}(\eta;\cdot))_{L^2(\pi_\sigma)} = (\psi,\varphi)_{L^2(\sigma)} \exp((\psi,\eta)_{L^2(\sigma)}). \tag{5.25}$$

Proof. Taking in account (5.20) we compute the left hand side of (5.25) to be

$$\begin{aligned}
&(\nabla^P_\varphi e^\alpha_{\pi_\sigma}(\psi;\cdot), e^\alpha_{\pi_\sigma}(\eta;\cdot))_{L^2(\pi_\sigma)} \\
=\ & (\psi,\varphi)_{L^2(\sigma)} \exp(-\langle\psi+\eta\rangle_\sigma) \int_\Gamma \exp[\langle\gamma, \log((1+\psi)(1+\eta))\rangle] d\pi_\sigma(\gamma) \\
=\ & (\psi,\varphi)_{L^2(\sigma)} \exp(-\langle\psi+\eta\rangle_\sigma) \exp\left(\int_X (\psi(x)+\eta(x)+\psi(x)\eta(x)) d\sigma(x)\right) \\
=\ & (\psi,\varphi)_{L^2(\sigma)} \exp((\psi,\eta)_{L^2(\sigma)}), \qquad (5.26)
\end{aligned}$$

which proves the statement of the lemma. ■

Further, the right hand side of (5.24) can be rewritten as follows

$$\begin{aligned}
&\int_\Gamma e^\alpha_{\pi_\sigma}(\psi;\gamma)\left[\sum_{x\in\gamma} e^{\langle\gamma-\varepsilon_x, \log(1+\eta)\rangle - \langle\eta\rangle_\sigma}\varphi(x) - e^\alpha_{\pi_\sigma}(\eta;\gamma)\langle\varphi\rangle_\sigma\right] d\pi_\sigma(\gamma) \\
=\ & \int_\Gamma e^\alpha_{\pi_\sigma}(\psi;\gamma) e^\alpha_{\pi_\sigma}(\eta;\gamma)\langle\tfrac{\varphi}{1+\eta}\rangle_\gamma d\pi_\sigma(\gamma) - \langle\varphi\rangle_\sigma \exp((\psi,\eta)_{L^2(\sigma)}). \qquad (5.27)
\end{aligned}$$

Let us state the following useful lemma.

Lemma 5.3.10 *1.* $\langle\psi\rangle_{\sigma_\eta} = \langle\psi\rangle_\sigma + (\psi,\eta)_{L^2(\sigma)}$, $\forall\psi,\eta\in\mathcal{D}$.

2. $e^\alpha_{\pi_{\sigma_\eta}}(\psi;\gamma) = \exp(-(\psi,\eta)_{L^2(\sigma)}) e^\alpha_{\pi_\sigma}(\psi;\gamma)$, $\forall\psi\in\mathcal{U}\subset\mathcal{D}$.

3. $\langle\gamma, \frac{\psi}{1+\eta}\rangle = \langle C_1^{\sigma_\eta}(\gamma), \frac{\psi}{1+\eta}\rangle + \langle\frac{\psi}{1+\eta}\rangle_{\sigma_\eta}$.

Proof. The non-trivial step is 3. Let us denote for simplicity $\frac{\psi}{1+\eta} =: \xi$

$$\begin{aligned}
\langle C_1^{\sigma_\eta}(\gamma), \xi\rangle &= \frac{d}{dt} e^\alpha_{\pi_{\sigma_\eta}}(t\xi;\gamma)|_{t=0} \\
&= \frac{d}{dt}\exp(\langle\gamma, \log(1+t\xi)\rangle - \langle t\xi\rangle_{\sigma_\eta})|_{t=0} \\
&= \frac{d}{dt}\sum_{x\in\gamma}\log(1+t\xi(x)) - \langle t\xi\rangle_{\sigma_\eta}|_{t=0} \\
&= \langle\gamma,\xi\rangle - \langle\xi\rangle_{\sigma_\eta}.
\end{aligned}$$

■

Now the rest of the proof follows from the previous lemma and (5.27), i.e.,

$$
\begin{aligned}
&\int_\Gamma e^\alpha_{\pi_\sigma}(\psi;\gamma)e^\alpha_{\pi_\sigma}(\eta;\gamma)\langle \tfrac{\varphi}{1+\eta}\rangle_\gamma d\pi_\sigma(\gamma) - \langle\varphi\rangle_\sigma \exp((\psi,\eta)_{L^2(\sigma)})\\
= \; & \exp((\psi,\eta)_{L^2(\sigma)})[\int_\Gamma e^\alpha_{\pi_\sigma}(\psi;\gamma)\langle \tfrac{\varphi}{1+\eta}\rangle_\gamma d\pi_{\sigma_\eta}(\gamma) - \langle\varphi\rangle_\sigma]\\
= \; & \exp((\psi,\eta)_{L^2(\sigma)})[\int_\Gamma e^\alpha_{\pi_\sigma}(\psi;\gamma)\langle C_1^{\sigma_\eta}(\gamma), \tfrac{\varphi}{1+\eta}\rangle d\pi_{\sigma_\eta}(\gamma)\\
& +\langle \tfrac{\varphi}{1+\eta}\rangle_{\sigma_\eta}\int_\Gamma e^\alpha_{\pi_{\sigma_\eta}}(\psi;\gamma)d\pi_{\sigma_\eta}(\gamma) - \langle\varphi\rangle_\sigma]\\
= \; & \exp((\psi,\eta)_{L^2(\sigma)})(\psi,\tfrac{\varphi}{1+\eta})_{L^2(\sigma_\eta)}\\
= \; & (\psi,\varphi)_{L^2(\sigma)}\exp((\psi,\eta)_{L^2(\sigma)}),
\end{aligned}
$$

which is the same as (5.26). This completes the proof. ■

Remark 5.3.11 *1. The pair $(\nabla^P_\varphi, \nabla^{P*}_\varphi)$, $\varphi \in \mathcal{D}$ verify the Canonical Commutation Relation (CCR), i.e.,*

(a) they are linear

$$a(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 a(\varphi_1) + \lambda_2 a(\varphi_2),$$

where $a(\varphi)$ represents either ∇^P_φ or ∇^{P*}_φ,

(b) $(\nabla^P_\varphi)^* = \nabla^{P*}_\varphi$,

(c) $[\nabla^P_\varphi, \nabla^P_\psi] = [\nabla^{P*}_\varphi, \nabla^{P*}_\psi] = 0$,

(d) $[\nabla^P_\varphi, \nabla^{P*}_\psi] = (\varphi,\psi)_{L^2(\sigma)}\mathbf{1}$, where $\mathbf{1}$ is the identity operator.

2. The operator $(\nabla^P_\varphi)^*$ plays the role of creation operator because of

$$((\nabla^P_{\varphi_1})^* \dots (\nabla^P_{\varphi_n})^* 1)(\gamma) = \langle C^\sigma_n(\gamma), \varphi_1 \otimes \dots \otimes \varphi_n\rangle. \tag{5.28}$$

5.4 Intrinsic geometry on Poisson space

We recall some results to be used below from [AKR98a], [AKR96b] to which we refer for the corresponding proofs and more details. In Chapter 6 we will generalize these result for the compound Poisson spaces, therefore here we present only the results and refer to [AKR98a, Sections 3-5] (see also lectures notes [Röc98]) for more details.

A homeomorphism $\psi : X \to X$ defines a transformation of Γ by

$$\Gamma \ni \gamma \mapsto \psi(\gamma) = \{\psi(x) | x \in \gamma\} = \sum_{x\in\gamma} \varepsilon_{\psi(x)}.$$

Any vector field $v \in V_0(X)$ (i.e., the set of all smooth vector fields on X with compact support) defines (via the exponential mapping) a one-parameter group ψ_t^v, $t \in \mathbb{R}$, of diffeomorphisms of X.

Definition 5.4.1 *For $F : \Gamma \to \mathbb{R}$ we define the directional derivative along the vector field v as (provided the right hand side exists)*

$$(\nabla_v^\Gamma F)(\gamma) := \frac{d}{dt} F(\psi_t^v(\gamma))|_{t=0}. \tag{5.29}$$

This definition applies to F in the set of smooth cylinder functions, (cf. Definition 5.3.1) of the form (5.15)

$$\begin{aligned}
(\nabla_v^\Gamma F)(\gamma) &= \sum_{i=1}^N \frac{\partial f_F}{\partial s_i}(\langle\gamma,\varphi_1\rangle,\ldots,\langle\gamma,\varphi_N\rangle) \int_X (\nabla_v^X \varphi_i)(x) d\gamma(x) \qquad (5.30)\\
&= \sum_{i=1}^N \frac{\partial f_F}{\partial s_i}(\langle\gamma,\varphi_1\rangle,\ldots,\langle\gamma,\varphi_N\rangle) \int_X \langle \nabla^X \varphi_i(x), v(x)\rangle_{T_xX} d\gamma(x)\\
&= \int_X \left\langle \sum_{i=1}^N \frac{\partial f_F}{\partial s_i}(\langle\gamma,\varphi_1\rangle,\ldots,\langle\gamma,\varphi_N\rangle)\nabla^X\varphi_i(x), v(x)\right\rangle_{T_xX} d\gamma(x)\\
&= ((\nabla^\Gamma F)(\gamma,\cdot), v(\cdot))_{L^2(X\to TX,\gamma)},
\end{aligned}$$

where

$$x \mapsto (\nabla_v^X \varphi)(x) = \langle \nabla^X \varphi(x), v(x)\rangle_{T_xX},$$

is the usual directional derivative on X along the vector field v and ∇^X denotes the gradient on X. $L^2(X \to TX, \gamma)$ denotes the space of γ-square integrable vector fields on X. It follows from the above computations that the *gradient* ∇^Γ on Γ is given by

$$(\nabla^\Gamma F)(\gamma, x) = \sum_{i=1}^N \frac{\partial f_F}{\partial s_i}(\langle\gamma,\varphi_1\rangle,\ldots,\langle\gamma,\varphi_N\rangle)\nabla^X\varphi_i(x),\ \gamma\in\Gamma,\ x\in X. \tag{5.31}$$

Hence, the expression of ∇_v^Γ on smooth cylinder functions given by (5.30) motivates the following definition.

Definition 5.4.2 *We define the tangent space $T_\gamma\Gamma$ to the configuration space Γ at the point $\gamma \in \Gamma$ as the Hilbert space $L^2(X \to TX, \gamma)$ of γ-square integrable vector fields on X, $V_\gamma : X \to TX$, equipped with the usual L^2-inner product*

$$\langle V_\gamma, W_\gamma \rangle_{T_\gamma\Gamma} := \langle V_\gamma, W_\gamma \rangle_{L^2(X \to TX, \gamma)} := \int_X \langle V_\gamma(x), W_\gamma(x) \rangle_{T_xX} d\gamma(x),$$

$V_\gamma, W_\gamma \in T_\gamma\Gamma$. The corresponding tangent bundle is

$$T\Gamma := \bigcup_{\gamma\in\Gamma} T_\gamma\Gamma.$$

Correspondingly, the finitely based vector fields on $(\Gamma, T\Gamma)$ can be defined as

$$\Gamma \ni \gamma \mapsto \sum_{i=1}^N F_i(\gamma) v_i \in C_0^\infty(X), \tag{5.32}$$

where $F_1, \ldots, F_N \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and $v_1, \ldots, v_N \in V_0(X)$. The collection of all such maps is denoted by $\mathcal{V}\mathcal{F}C_0^\infty(\mathcal{D}, \Gamma)$.

Remark 5.4.3

1. *We note that it follows from (5.31) that $\nabla^\Gamma F(\gamma) \in \mathcal{V}\mathcal{F}C_0^\infty(\mathcal{D}, \Gamma)$ for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, $\gamma \in \Gamma$.*

2. *Any $v \in V_0(X)$ can be considered as a "constant" vector field on Γ if we identify v with the corresponding class in $L^2(X \to TX, \gamma)$, i.e.,*

$$\Gamma \ni \gamma \mapsto V_\gamma = v \in T_\gamma\Gamma,$$

and

$$\langle v, v \rangle_{T_\gamma\Gamma} = \int_X |v(x)|^2_{T_xX} d\gamma(x).$$

Now we would like to compute the adjoint of ∇^Γ on $L^2(\pi_\sigma)$ which corresponds, of course, to an integration by parts with respect to π_σ. First we recall the integration by parts formula for the measure σ on the underlying manifold X.

The logarithmic derivative of the measure σ is given by the vector field

$$X \ni x \mapsto \beta^\sigma(x) := \nabla^X \log \rho(x) = \frac{\nabla^X \rho(x)}{\rho(x)} \in T_x X,$$

where $\beta^\sigma = 0$ on $\{\rho = 0\}$. Then the logarithmic derivative of σ along v is the function

$$X \ni x \mapsto \beta_v^\sigma(x) := \langle \beta^\sigma(x), v(x)\rangle_{T_x X} + \operatorname{div}^X v(x), \tag{5.33}$$

where $\operatorname{div}^X := \operatorname{div}_m^X$ denotes the divergence on X with respect to the volume element m. Analogously, we define $\operatorname{div}_\sigma^X$ as the divergence on X with respect to σ, i.e., $\operatorname{div}_\sigma^X$ is the dual operator on $L^2(\sigma)$ of ∇^X. For all $\varphi_1, \varphi_2 \in \mathcal{D}$ we have

$$\begin{aligned} &\int_X (\nabla_v^X \varphi_1)(x)\varphi_2(x) d\sigma(x) \\ = \quad & -\int_X \varphi_1(x)(\nabla_v^X \varphi_2)(x) d\sigma(x) - \int_X \varphi_1(x)\varphi_2(x)\beta_v^\sigma(x) d\sigma(x). \end{aligned} \tag{5.34}$$

Then, on the one hand we can rewrite (5.34) as an operator equality on the domain $\mathcal{D} \subset L^2(\sigma)$:

$$\nabla_v^{X*} = -\nabla_v^X - \beta_v^\sigma,$$

where the adjoint operator is considered with respect to $L^2(\sigma)$. Note that, obviously, $\beta_v^\sigma \in L^2(\sigma)$ for all $v \in V_0(X)$. On the other hand we have

$$\operatorname{div}_\sigma^X = \beta^\sigma. \tag{5.35}$$

Having the logarithmic derivative β_v^σ we introduce an analogous object for the Poisson measure π_σ.

Definition 5.4.4 *For any $v \in V_0(X)$ we define the logarithmic derivative of π_σ along v as the following function on Γ:*

$$\Gamma \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) := \langle \beta_v^\sigma, \gamma \rangle = \int_X [\langle \beta^\sigma(x), v(x)\rangle_{T_x X} + \operatorname{div}^X v(x)] d\gamma(x). \tag{5.36}$$

Theorem 5.4.5 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and any $v \in V_0(X)$ the following integration by parts formula for π_σ holds:*

$$\int_\Gamma (\nabla_v^\Gamma F)(\gamma) G(\gamma) d\pi_\sigma(\gamma) = \int_\Gamma F(\gamma)[-(\nabla_v^\Gamma G)(\gamma) - G(\gamma) B_v^{\pi_\sigma}(\gamma)] d\pi_\sigma(\gamma), \tag{5.37}$$

or $(\nabla_v^\Gamma)^* = -\nabla_v^\Gamma - B_v^{\pi_\sigma}$, as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ in $L^2(\pi_\sigma)$.

Definition 5.4.6 *For a measurable vector field $V : \Gamma \to T\Gamma$ the divergence $\mathrm{div}^{\Gamma}_{\pi_\sigma} V$ is defined via the duality relation for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by*

$$\int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma) = -\int_\Gamma F(\gamma)(\mathrm{div}^\Gamma_{\pi_\sigma} V)(\gamma) d\pi_\sigma(\gamma), \tag{5.38}$$

provided it exists (i.e., provided

$$F \mapsto \int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma),$$

is continuous on $L^2(\pi_\sigma)$).

Proposition 5.4.7 *Let V be a vector field such that $V \in \mathcal{V}\mathcal{F}C_0^\infty(\mathcal{D}, \Gamma)$ and has the form (5.32), then we have*

$$(\mathrm{div}^\Gamma_{\pi_\sigma} V)(\gamma) = \sum_{i=1}^N \langle (\nabla^\Gamma F_i)(\gamma), v_i\rangle_{T_\gamma\Gamma} + \sum_{i=1}^N F_i(\gamma) B^{\pi_\sigma}_{v_i}(\gamma). \tag{5.39}$$

For any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we introduce the *pre-Dirichlet form* which is generated by the intrinsic gradient ∇^Γ as

$$\mathcal{E}^\Gamma_{\pi_\sigma}(F, G) := \int_\Gamma \langle (\nabla^\Gamma F)(\gamma), (\nabla^\Gamma G)(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma). \tag{5.40}$$

We will also need the *classical pre-Dirichlet* form for the intensity measure σ which is given as

$$\mathcal{E}^X_\sigma(\varphi, \psi) := \int_X \langle \nabla^X \varphi(x), \nabla^X \psi(x)\rangle_{T_x X} d\sigma(x), \quad \text{for any } \varphi, \psi \in \mathcal{D}.$$

This form is associated with the *Dirichlet operator* H^X_σ which is given on $\mathcal{D}$ by

$$H^X_\sigma \varphi(x) := -\Delta^X \varphi(x) - \langle \beta^\sigma(x), \nabla^X \varphi(x)\rangle_{T_x X},$$

which satisfies

$$\mathcal{E}^X_\sigma(\varphi, \psi) = (H^X_\sigma \varphi, \psi)_{L^2(\sigma)}, \ \varphi, \psi \in \mathcal{D},$$

see e.g., [BK95] and [MR92].

The closure of this form on $L^2(\sigma)$ is defined by $(\mathcal{E}^X_\sigma, D(\mathcal{E}^X_\sigma))$. Note that $D(\mathcal{E}^X_\sigma)$ is nothing but the Sobolev space of order 1 in $L^2(\sigma)$ (sometimes also denoted by $H_0^{1,2}(X,\sigma)$). $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ generates a positive self-adjoint operator in $L^2(\sigma)$ (the so-called Friedrich's extension of H_σ^X, see e.g., [BKR97] and [RS75a]).

For this extension we preserve the previous notation H_σ^X and denote the domain by $D(H_\sigma^X)$.

For any $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, $(\nabla^\Gamma\nabla^\Gamma F)(\gamma,x,y) \in T_\gamma\Gamma \otimes T_\gamma\Gamma$ and we can define the Γ-*Laplacian* $(\Delta^\Gamma F)(\gamma) := \mathrm{Tr}(\nabla^\Gamma\nabla^\Gamma F)(\gamma) \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. We introduce a differential operator in $L^2(\pi_\sigma)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ by the formula

$$(H_{\pi_\sigma}^\Gamma F)(\gamma) = -\Delta^\Gamma F(\gamma) - \langle \mathrm{div}_\sigma^X (\nabla^\Gamma F)(\gamma,\cdot), \gamma\rangle. \tag{5.41}$$

Theorem 5.4.8 *The operator $H_{\pi_\sigma}^\Gamma$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_\sigma}^\Gamma$, i.e.,*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F,G) = (\!(H_{\pi_\sigma}^\Gamma F, G)\!)_{L^2(\pi_\sigma)}, \tag{5.42}$$

or $H_{\pi_\sigma}^\Gamma = -\mathrm{div}_{\pi_\sigma}^\Gamma \nabla^\Gamma$ on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. We call $H_{\pi_\sigma}^\Gamma$ the intrinsic Dirichlet operator of the measure π_σ.

There is an explicit relation between internal and external geometry, more precisely, between the corresponding Dirichlet forms associated with the gradients ∇^Γ and ∇^P. Actually, this relation holds for Poisson measures π_σ but this will serves as a motivation to produce an analogous result for Gibbs measures, cf. Section 5.5.

Recall from Section 5.2 the isomorphism I_{π_σ} between $L^2(\pi_\sigma)$ and the Fock space $\mathrm{Exp}L^2(\sigma)$ and the second quantization operator $d\mathrm{Exp}A$ on the Fock space introduced in Section 2.2. Thus, given a positive self-adjoint operator A in $L^2(\sigma)$ and the corresponding $d\mathrm{Exp}A$ on $\mathrm{Exp}L^2(\sigma)$ we denote by H_A^P the image of the operator $d\mathrm{Exp}A$ in the Poisson space $L^2(\pi_\sigma)$ under the isomorphism I_{π_σ}.

The following proposition gives explicit representation of the symmetric bilinear form associated to $H_{\pi_\sigma}^P$, see [AKR98a, Theorem 5.1].

Proposition 5.4.9 *Let $\mathcal{D} \subset D(A)$. Then the symmetric bilinear form corresponding to the operator H_A^P has the following form*

$$(\!(H_A^P F, G)\!)_{L^2(\pi_\sigma)} = \int_\Gamma (\nabla^P F(\gamma), A\nabla^P G(\gamma))_{L^2(\sigma)} d\pi_\sigma(\gamma),\ F,G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma). \tag{5.43}$$

The right hand side of (5.43) is called the "Poissonian pre-Dirichlet form" with coefficient operator A and is denoted by $\mathcal{E}_{\pi_\sigma,A}^P$.

Let us consider a special case of the second quantization operator $d\mathrm{Exp}A$, where the one-particle operator A coincides with the Dirichlet operator H_σ^X generated by the measure σ on X. Then we have the following theorem which relates the intrinsic Dirichlet operator $H_{\pi_\sigma}^\Gamma$ and the operator $H_{H_\sigma^X}^P$.

Theorem 5.4.10 $H_{\pi_\sigma}^\Gamma = H_{H_\sigma^X}^P$ *on* $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. *In particular, for all* $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned}
& \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma) \\
= & \int_\Gamma (\nabla^P F(\gamma), H_\sigma^X \nabla^P G(\gamma))_{L^2(\sigma)} d\pi_\sigma(\gamma) \\
= & \int_\Gamma \int_X \langle \nabla^X \nabla^P F(\gamma,x), \nabla^X \nabla^P G(\gamma,x)\rangle_{T_xX} d\sigma(x) d\pi_\sigma(\gamma). \quad (5.44)
\end{aligned}$$

5.5 Relation between intrinsic and extrinsic Dirichlet forms

Let us briefly recall the definition of grand canonical Gibbs measures on $(\Gamma, \mathcal{B}(\Gamma))$. We adopt the notation in [AKR98b] (see also [KRS98]), and refer the interested reader to the beautiful work by C. Preston, [Pre79], but also [Pre76], and [Geo79].

A function $\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ will be called a *potential* if and only if for all $\Lambda \in \mathcal{B}_c(X)$ we have $\Phi(\emptyset) = 0$, $\Phi = 1\!\!1_{\{N_X<\infty\}}\Phi$, and $\gamma \mapsto \Phi(\gamma_\Lambda)$ is $\mathcal{B}_\Lambda(\Gamma)$-measurable.

For $\Lambda \in \mathcal{B}_c(X)$ the *conditional energy* $E_\Lambda^\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E_\Lambda^\Phi(\gamma) := \begin{cases} \sum\limits_{\gamma' \subset \gamma, \gamma'(\Lambda)>0} \Phi(\gamma') & \text{if } \sum\limits_{\gamma' \subset \gamma, \gamma'(\Lambda)>0} |\Phi(\gamma')| < \infty, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.45)$$

where the sum of the empty set is defined to be zero.

Later on we will use conditional energies which satisfy an additional assumption, namely, the *stability condition*, i.e., there exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(X)$ and for all $\gamma \in \Gamma_\Lambda$

$$E_\Lambda^\Phi(\gamma) \geq -B|\gamma|.$$

Definition 5.5.1 *For any $\Lambda \in \mathcal{O}_c(X)$ and any $\gamma \in \Gamma$ define the measure $\Pi_\Lambda^\Phi(\gamma, \cdot)$ by*

$$\begin{aligned} \Pi_\Lambda^{\sigma,\Phi}(\gamma, \Delta) \;\; &:= \;\; 1\!\!1_{\{Z_\Lambda^{\sigma,\Phi}<\infty\}}(\gamma)[Z_\Lambda^{\sigma,\Phi}(\gamma)]^{-1} \int_\Gamma 1\!\!1_\Delta(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda) \\ &\quad \cdot \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] d\pi_\sigma(\gamma'),\ \Delta \in \mathcal{B}(\Gamma), \end{aligned} \tag{5.46}$$

where

$$Z_\Lambda^{\sigma,\Phi}(\gamma) := \int_\Gamma \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] d\pi_\sigma(\gamma'). \tag{5.47}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called grand canonical Gibbs measure with interaction potential Φ if for all $\Lambda \in \mathcal{O}_c(X)$

$$\mu \Pi_\Lambda^\Phi = \mu. \tag{5.48}$$

Let $\mathcal{G}_{gc}(\sigma, \Phi)$ denote the set of all such probability measures μ.

Remark 5.5.2

1. *It is known that $(\Pi_\Lambda^{\sigma,\Phi})_{\Lambda\in\mathcal{O}_c(X)}$ is a $(\mathcal{B}_{X\setminus\Lambda}(\Gamma))_{\Lambda\in\mathcal{O}_c(X)}$-specification in the sense of [Pre76, Section 6] or [Pre79].*

2. *For any $\gamma \in \Gamma$ the measure $\mu\Pi_\Lambda^\Phi$ in (5.48) is defined by*

$$(\mu \Pi_\Lambda^{\sigma,\Phi})(\Delta) := \int_\Gamma d\mu(\gamma) \Pi_\Lambda^{\sigma,\Phi}(\gamma, \Delta),\ \Delta \in \mathcal{B}(\Gamma) \tag{5.49}$$

and (5.48) are called Dobrushin-Landford-Ruelle (DLR) equations.

Here we will be interested only on the class of measures $\mathcal{G}_{gc}^1(\sigma, \Phi)$ consisting of all $\mu \in \mathcal{G}_{gc}(\sigma, \Phi)$ such that

$$\int_\Gamma \gamma(K) d\mu(\gamma) < \infty \text{ for all compact } K \subset X. \tag{5.50}$$

We define for any $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$ the pre-Dirichlet form $\mathcal{E}_\mu^\Gamma$ by

$$\mathcal{E}_\mu^\Gamma(F, G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{T_\gamma\Gamma} d\mu(\gamma),\ F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma). \tag{5.51}$$

After all our preparations we are now going to prove an analogue of (5.44) for $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$. We would like to emphasize that the corresponding formula (5.52) is not obtained from (5.44) by just replacing π_σ by $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$. The essential difference is, in addition, an extra factor involving the conditional energy E_Λ^Φ.

Theorem 5.5.3 *For any $\mu \in \mathcal{G}^1_{gc}(\sigma, \Phi)$, we have for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$*

$$\begin{aligned} \mathcal{E}^\Gamma_\mu(F,G) &= \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\mu(\gamma) \qquad (5.52)\\ &= \int_\Gamma\int_X \langle \nabla^X\nabla^P F(\gamma,x), \nabla^X\nabla^P G(\gamma,x)\rangle_{T_xX}\, e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x)d\mu(\gamma) \end{aligned}$$

Proof. Let us take any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ of the form (5.15). Then given $\gamma \in \Gamma$ and $x \in X$ (5.16) implies that

$$\begin{aligned} \nabla^X\nabla^P F(\gamma,x) &= \nabla^X F(\gamma+\varepsilon_x)\\ &= \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle\varphi_1,\gamma\rangle + \varphi_1(x), \ldots, \langle\varphi_N,\gamma\rangle + \varphi_N(x))\nabla^X\varphi_i(x). \end{aligned}$$

Let us define $\hat{F}_i(\gamma) := \frac{\partial g_F}{\partial s_i}(\langle\varphi_1,\gamma\rangle, \ldots, \langle\varphi_N,\gamma\rangle)$, $i = 1, \ldots, N$. Obviously, it is enough to prove the equality (5.52) for $F = G$. Thus, inserting the result of $\nabla^X\nabla^P F(\gamma,x)$ into the right hand side of (5.52) we obtain

$$\int_\Gamma\int_X \sum_{i,j=1}^N \langle\nabla^X\varphi_i(x), \nabla^X\varphi_j(x)\rangle_{T_xX} \hat{F}_i(\gamma+\varepsilon_x)\hat{F}_j(\gamma+\varepsilon_x) e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x)d\mu(\gamma). \qquad (5.53)$$

Then we need the following useful proposition which generalizes the Mecke identity to measures in $\mathcal{G}_{gc}(\sigma, \Phi)$, see [NZ79] and [MMW79].

Proposition 5.5.4 *Let $h : \Gamma \times X \to \mathbb{R}_+$ be $\mathcal{B}(\Gamma)\times\mathcal{B}(X)$-measurable, and let $\mu \in \mathcal{G}_{gc}(\sigma, \Phi)$. Then we have*

$$\int_\Gamma \left(\int_X h(\gamma,x)d\gamma(x)\right) d\mu(\gamma) = \int_X\int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\mu(\gamma)d\sigma(x). \qquad (5.54)$$

Using this proposition we transform (5.53) into

$$\int_\Gamma \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle\nabla^X\varphi_i(\cdot), \nabla^X\varphi_j(\cdot)\rangle_{TX}, \gamma\rangle d\mu(\gamma).$$

On the other hand using (5.31) we obtain

$$\langle\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T\Gamma} = \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle\nabla^X\varphi_i(\cdot), \nabla^X\varphi_j(\cdot)\rangle_{TX}, \gamma\rangle.$$

Therefore the equality on the dense $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is valid which proves the theorem. ■

Remark 5.5.5 *Let us give a heuristic proof of the Nguyen-Zessin characterization of Gibbs measures in (5.54) which really is a consequence of the Mecke identity (cf. (5.8)). Indeed, let us write (heuristically)*

$$d\mu(\gamma) = \frac{1}{Z^{\sigma,\Phi}} e^{-E^\Phi(\gamma)} d\pi_\sigma(\gamma).$$

Then the function $E^\Phi_{\{x\}}(\gamma+\varepsilon_x) = E^\Phi(\gamma+\varepsilon_x) - E^\Phi(\gamma)$ informally is the variation of the potential energy $E^\Phi(\gamma)$ when we add to the configuration γ an additional point $x \in X$. Using this representation we have

$$\begin{aligned} &\int_X \int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\mu(\gamma) d\sigma(x) \\ = \;& (Z^{\sigma,\Phi})^{-1} \int_X \int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\pi_\sigma(\gamma) d\sigma(x). \end{aligned}$$

Then we use the Mecke identity to transform the right hand side of the above equality into

$$\frac{1}{Z^{\sigma,\Phi}} \int_\Gamma \left(\int_X h(\gamma, x) e^{-E^\Phi(\gamma)} d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_\Gamma \left(\int_X h(\gamma, x) d\gamma(x) \right) d\mu(\gamma).$$

The rigorous proof in [NZ79] is obtained as a formalization of the heuristic computations above.

5.6 Closability of intrinsic Dirichlet forms

In this section we will prove the closability of the intrinsic Dirichlet form $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu) := L^2(\Gamma,\mu)$ for all $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$, using the integral representation (5.52) in Theorem 5.5.3. The closability of $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ over Γ is implied by the closability of an appropriate family of pre-Dirichlet forms over X. Let us describe this more precisely.

We define new intensity measures on X by $d\sigma_\gamma(x) := \rho_\gamma(x) dm(x)$, where

$$\rho_\gamma(x) := e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} \rho(x), \; x \in X, \gamma \in \Gamma \qquad (5.55)$$

It was shown in [AR90, Theorem 5.3] (in the case $X = \mathbb{R}^d$) that the components of the Dirichlet form $(\mathcal{E}^X_{\sigma_\gamma}, \mathcal{D}^{\sigma_\gamma})$ corresponding to the measure σ_γ are closable on $L^2(\mathbb{R}^d, \sigma_\gamma)$ if and only if σ_γ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and the Radon-Nikodym derivative satisfies some condition, see (5.56) below for details. This result allows us to prove the closability of $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ on $L^2(\mu)$. Let us first recall the above mentioned result.

Theorem 5.6.1 *(cf. [AR90, Theorem 5.3]) Let* ν *by a probability measure on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \in \mathbb{N}$ *and let* $\mathcal{D}^\nu$ *denote the* ν*-classes determined by* $\mathcal{D}$*. Then the forms* $(\mathcal{E}_{\nu,i}, \mathcal{D}^\nu)$ *defined by*

$$\mathcal{E}_{\nu,i}(u,v) := \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\nu,\ u, v \in \mathcal{D},$$

are well-defined and closable on $L^2(\mathbb{R}^d, \nu)$ *for* $1 \le i \le d$ *if and only if* ν *is absolutely continuous with respect to Lebesgue measure* λ^d *on* $\mathbb{R}^d$*, and the Radon-Nikodym derivative* $\rho = d\nu/d\lambda^d$ *satisfies the condition:*

$$\begin{aligned}&\text{for any } 1 \le i \le d \text{ and } \lambda^{d-1}\text{–a.e. } x \in \left\{ y \in \mathbb{R}^{d-1} | \int_{\mathbb{R}} \rho^{(i)}_y(s) d\lambda^1(s) > 0 \right\},\\ &\rho^{(i)}_x = 0\ \lambda^1\text{–a.e. on } \mathbb{R} \backslash R(\rho^{(i)}_x),\ \rho^{(i)}_x(s) := \rho(x_1, \ldots, x_{i-1}, s, x_i, \ldots, x_d),\\ &s \in \mathbb{R},\ \text{if } x = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1},\ \text{and where}\end{aligned} \tag{5.56}$$

$$R(\rho^{(i)}_x) := \left\{ t \in \mathbb{R} | \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho^{(i)}_x(s)} ds < \infty \text{ for some } \varepsilon > 0 \right\}. \tag{5.57}$$

There is an obvious generalization of Theorem 5.6.1 to the case where a Riemannian manifold X is replacing $\mathbb{R}^d$, to be formulated in terms of local charts. Since here we are only interested in the "if part" of Theorem 5.6.1, we now recall a slightly weaker sufficient condition for closability in the general case where X is a manifold as before.

Theorem 5.6.2 *Suppose* $\sigma_1 = \rho_1 m$*, where* $\rho_1 : X \to \mathbb{R}_+$ *is* $\mathcal{B}(X)$*-measurable such that*

$$\rho_1 = 0\ m\text{–a.e. on } X \backslash \left\{ x \in X | \int_{\Lambda_x} \frac{1}{\rho_1(x)} dm(x) < \infty \right\}, \tag{5.58}$$

where Λ_x is some open neighborhood of x. Then $(\mathcal{E}^X_{\sigma_1}, \mathcal{D}^{\sigma_1})$ defined by

$$\mathcal{E}^X_{\sigma_1}(u,v) := \int_X \langle \nabla^X u(x), \nabla^X v(x) \rangle_{T_xX} \, d\sigma_1(x); \; u, v \in \mathcal{D},$$

is closable on $L^2(\sigma_1)$.

The proof is a straightforward adaptation of the line of arguments in [MR92, Chap. II, Subsection 2a] (see also [ABR89, Theorem 4.2] for details). We emphasize that (5.58) e.g. always holds, if ρ_1 is lower semicontinuous, and that neither ν in Theorem 5.6.1 nor σ_1 in Theorem 5.6.2 is required to have full support, so e.g. ρ_1 is not necessarily strictly positive m-a.e. on X.

We are now ready to prove the closability of $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ on $L^2(\mu)$ under the above assumption.

Theorem 5.6.3 *Let $\mu \in \mathcal{G}^1_{gc}(\sigma, \Phi)$. Suppose that for μ-a.e. $\gamma \in \Gamma$ the function ρ_γ defined in (5.55) satisfies (5.58) (resp. (5.56) in case $X = \mathbb{R}^d$). Then the form $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ is closable on $L^2(\mu)$.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$ such that $F_n \to 0$, $n \to \infty$ in $L^2(\mu)$ and

$$\mathcal{E}^\Gamma_\mu(F_n - F_m, F_n - F_m) \underset{n,m\to\infty}{\longrightarrow} 0. \tag{5.59}$$

We have to show that

$$\mathcal{E}^\Gamma_\mu(F_{n_k}, F_{n_k}) \underset{k\to\infty}{\longrightarrow} 0 \tag{5.60}$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$. Let $(n_k)_{k\in\mathbb{N}}$ be a subsequence such that

$$\left(\int_\Gamma F^2_{n_k} d\mu\right)^{1/2} + \mathcal{E}^\Gamma_\mu(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

Then

$$\begin{aligned}
\infty &> \sum_{k=1}^\infty \mathcal{E}^\Gamma_\mu(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} \\
&\geq \sum_{k=1}^\infty \int_\Gamma \left(\int_X |\nabla^X \nabla^P (F_{n_{k+1}} - F_{n_k})(x,\gamma)|^2_{T_xX} \, e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x) \right)^{1/2} d\mu(\gamma) \\
&= \int_\Gamma \sum_{k=1}^\infty \left(\int_X |\nabla^X \nabla^P (F_{n_{k+1}} - F_{n_k})(x,\gamma)|^2_{T_xX} \, \rho_\gamma(x) dm(x) \right)^{1/2} d\mu(\gamma),
\end{aligned}$$

where we used Theorem 5.5.3 and (5.55). From the last expression we obtain that

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathcal{E}^X_{\sigma_\gamma}(u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k}, u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k})^{1/2} \qquad (5.61) \\ = & \sum_{k=1}^{\infty} \left(\int_X |\nabla^X \nabla^P (F_{n_{k+1}} - F_{n_k})(x,\gamma)|^2_{T_x X} \rho_\gamma(x) dm(x) \right)^{1/2} < \infty, \end{aligned}$$

for μ-a.e. $\gamma \in \Gamma$, where for $k \in \mathbb{N}$, $\gamma \in \Gamma$,

$$u^{(\gamma)}_{n_k}(x) := F_{n_k}(\gamma + \varepsilon_x) - F_{n_k}(\gamma),\ x \in X.$$

Note that $u^{(\gamma)}_{n_k} \in \mathcal{D}$. (5.61) implies that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}^X_{\sigma_\gamma}(u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}, u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}) \underset{k,l\to\infty}{\longrightarrow} 0. \qquad (5.62)$$

Let $\Lambda \subset \mathcal{O}_c(X)$.

Claim 1: For μ-a.e. $\gamma \in \Gamma$

$$\int_X (u^{(\gamma)}_{n_k}(x))^2 1\!\!1_\Lambda(x) d\sigma_\gamma(x) \underset{k\to\infty}{\longrightarrow} 0.$$

To prove Claim 1 we first note that for μ-a.e. $\gamma \in \Gamma$

$$\sigma_\gamma(\Lambda) < \infty,$$

as follows immediately from (5.54) (taking $h(\gamma, x) := 1\!\!1_\Lambda(x)$ for $x \in X$, $\gamma \in \Gamma$), since $\mu \in \mathcal{G}^1_{gc}(\sigma, \Phi)$. Therefore, for μ-a.e. $\gamma \in \Gamma$

$$\int_X F^2_{n_k}(\gamma) 1\!\!1_\Lambda(x) d\sigma_\gamma(x) = F^2_{n_k}(\gamma)\sigma_\gamma(\Lambda) \underset{k\to\infty}{\longrightarrow} 0. \qquad (5.63)$$

Furthermore, by (5.54)

$$\begin{aligned} & \int_\Gamma \int_X F^2_{n_k}(\gamma + \varepsilon_x) 1\!\!1_\Lambda(x) d\sigma_\gamma(x)(1 + \gamma(\Lambda))^{-1} d\mu(\gamma) \\ = & \int_\Gamma F^2_{n_k}(\gamma) \int_X \frac{1\!\!1_\Lambda(x)}{1 + \gamma(\Lambda) - 1\!\!1_\Lambda(x)} \gamma(dx) d\mu(\gamma) \\ \leq & \int_\Gamma F^2_{n_k}(\gamma) d\mu(\gamma) < \frac{1}{2^k}, \end{aligned}$$

because the integral with respect to γ is dominated by 1 for all $\gamma \in \Gamma$. Hence

$$\begin{aligned}\infty &> \sum_{k=1}^{\infty}\left(\int_\Gamma\int_X F_{n_k}^2(\gamma+\varepsilon_x)\mathbb{1}_\Lambda(x)d\sigma_\gamma(x)(1+\gamma(\Lambda))^{-1}d\mu(\gamma)\right)^{1/2}\\ &\geq \int_\Gamma \sum_{k=1}^{\infty}\left(\int_X F_{n_k}^2(\gamma+\varepsilon_x)\mathbb{1}_\Lambda(x)d\sigma_\gamma(x)\right)^{1/2}(1+\gamma(\Lambda))^{-1}d\mu(\gamma).\end{aligned}$$

Therefore, for μ-a.e. $\gamma \in \Gamma$

$$\int_X F_{n_k}^2(\gamma+\varepsilon_x)\mathbb{1}_\Lambda(x)d\sigma_\gamma(x) \underset{k\to\infty}{\longrightarrow} 0. \tag{5.64}$$

Then Claim 1 follows by (5.63) and (5.64).

Claim 2: For μ-a.e. $\gamma \in \Gamma$

$$|\nabla^X u_{n_k}^{(\gamma)}|_{TX} \underset{k\to\infty}{\longrightarrow} 0 \quad \sigma_\gamma\text{−a.e.}$$

To prove Claim 2 we first note that clearly (5.61) implies that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}^X_{\mathbb{1}_\Lambda\sigma_\gamma}(u_{n_k}^{(\gamma)}-u_{n_l}^{(\gamma)}, u_{n_k}^{(\gamma)}-u_{n_l}^{(\gamma)}) \underset{k,l\to\infty}{\longrightarrow} 0. \tag{5.65}$$

Hence we can apply Theorem 5.6.2 (resp. 5.6.1) to $\rho_1 := \mathbb{1}_\Lambda\rho_\gamma$ and conclude by Claim 1 and (5.65) that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}^X_{\mathbb{1}_\Lambda\sigma_\gamma}(u_{n_k}^{(\gamma)}, u_{n_k}^{(\gamma)}) \underset{k\to\infty}{\longrightarrow} 0,$$

hence by (5.61)

$$\mathbb{1}_\Lambda|\nabla^X u_{n_k}^{(\gamma)}|_{TX} \underset{k\to\infty}{\longrightarrow} 0 \quad \sigma_\gamma\text{−a.e.}$$

Since Λ was arbitrary, Claim 2 is proven.

From Claim 2 we now easily deduce (5.60) by (5.52) and Fatou's Lemma as follows:

$$\begin{aligned}\mathcal{E}^\Gamma_\mu(F_{n_k}, F_{n_k}) &\leq \int_\Gamma \liminf_{l\to\infty}\int_X |\nabla^X(u_{n_k}^{(\gamma)}-u_{n_l}^{(\gamma)})|^2_{TX}\, d\sigma_\gamma(x)d\mu(\gamma)\\ &\leq \liminf_{l\to\infty}\mathcal{E}^\Gamma_\mu(F_{n_k}-F_{n_l}, F_{n_k}-F_{n_l}),\end{aligned}$$

which by (5.59) can be made arbitrarily small for k large enough. ■

Remark 5.6.4 *The above method to prove closability of pre-Dirichlet forms on configuration spaces Γ_X extends immediately to the case where X is replaced by an infinite dimensional "manifold" such as the loop space (cf. [MR97]).*

Chapter 6

Analysis on compound Poisson spaces

The present chapter elaborates the L^2 structure for compound Poisson spaces. We note that for all of these compound Poisson processes the results of Chapter 4 immediately produce Gel'fand triples of test and generalized functions as well their characterizations and calculus.

The analysis on compound Poisson spaces can be done with the help of the analysis derived from Poisson spaces described in the previous chapter. That possibility is based on the existence of an unitary isomorphism U_Σ between compound Poisson spaces and Poisson spaces which allows us to transport the Fock structure from Poisson spaces to compound Poisson spaces, see Section 6.2 Proposition 6.2.4. The isomorphism U_Σ has been identified before by K. Itô, [Itô56] and A. Dermoune, [Der90]. We work out the details in Section 6.2.

The images of the annihilation and creation operators under the mentioned isomorphism are derived in Subsection 6.3.

In Section 6.4 we study in more detail the previous analysis in a particular case of compound Poisson measures, the so-called Gamma measure. Its Laplace transform is given by

$$l_{\mu_G^\sigma}(\varphi) := \exp(-\langle \log(1-\varphi)\rangle_\sigma),\ 1 > \varphi \in \mathcal{D}.$$

This measure can be seen as a special case of compound Poisson measure π_σ^τ for a specific choice of the measure τ, see Section 6.4, Remark 6.4.2 for details. From this point of view, of course, all structure may be implemented on Gamma space and it is possible to obtain the representations for the operators, see Remark 6.4.4. The question that still remains is to find intrinsic expressions for all these operators (creation, annihilation etc.) on Gamma space as for Poisson space.

The most intriguing feature of Gamma spaces we found is its Fock type structure. As in the Poisson case it is possible to define a transformation α on $\mathcal{D}$ (see (6.18) below) such that the normalized exponential $e^{\sigma}_{\mu^{\sigma}_{G}}(\varphi;\cdot)$ produces a complete system of orthogonal polynomials, the so-called system of generalized Laguerre polynomials. It leads to a Fock type realization of Gamma space as

$$L^2(\mu^{\sigma}_{G}) \simeq \bigoplus_{n=0}^{\infty} \mathrm{Exp}^{G}_{n} L^2(\sigma) =: \mathrm{Exp}^{G} L^2(\sigma),$$

where $\mathrm{Exp}^{G}_{n} L^2(\sigma) \subset \mathrm{Exp}_{n} L^2(\sigma)$ is a quasi-n-particle subspace of $\mathrm{Exp}^{G} L^2(\sigma)$. The point here is that the scalar product in $\mathrm{Exp}^{G}_{n} L^2(\sigma)$ (cf. Subsection 6.4.2, (6.25)) turns out to be different from the usual one given by $L^2(\sigma)^{\hat{\otimes} n}$. As a result the Fock space $\mathrm{Exp}^{G} L^2(\sigma)$ has a novel n-particle structure which is essentially different from traditional Fock picture.

6.1 Compound Poisson measures

This section is devoted to study the compound Poisson measures π^{τ}_{σ} on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$. Firstly we recall the Lévy canonical representation of all possible generalized white noise measures μ on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$, see [GV68], [Hid70], and [AW95]. These measures are defined by the characteristic functional of the form

$$C_{\mu}(\varphi) := \exp\left[ia(\varphi,1) - \frac{b^2|\varphi|^2}{2} + \int_{\mathbb{R}} (e^{is\varphi} - 1 - \frac{is\varphi}{1+s^2}, 1) d\beta(s)\right], \quad (6.1)$$

where $\varphi \in \mathcal{D}$, $a, b \in \mathbb{R}$, the measure β is such that $\beta(\{0\}) = 0$, $\int_{\mathbb{R}} s^2/(1+s^2) d\beta(s) < \infty$, $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and the norm in $L^2(\mathbb{R}^d)$, respectively. We take into account that such a measure is in general the convolution of a Gaussian and non-Gaussian measures. We will be interested in the non-Gaussian part of this class, i.e., in (6.1) $b = 0$. Furthermore, assume that

$$\int_{\mathbb{R}} s^2 d\beta(s) < \infty, \text{ and } \int_{-1}^{1} |s| d\beta(s) < \infty.$$

Then one can use the Kolmogorov canonical representation of $C_\mu(\varphi)$, as in e.g., [GV68]

$$C_\mu(\varphi) = \exp\left[ia(\varphi,1) + \int_{\mathbb{R}} (e^{is\varphi} - 1, 1)d\beta(s)\right],\ \varphi \in \mathcal{D}.$$

Let us define the compound Poisson measures on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$. Let τ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (finite or σ-finite) having all moments finite and such that $\tau(\{0\}) = 0$. In addition let σ be a non-atomic σ-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We denote

$$\psi_\tau(u) := \int_{\mathbb{R}} (e^{su} - 1)d\tau(s),\ u \in \mathbb{R}.$$

Definition 6.1.1 *A measure π_σ^τ on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$ is called a compound Poisson measure with Kolmogorov characteristic ψ_τ if its Laplace transform is given by, as e.g., [GGV75]*

$$\begin{aligned} l_{\pi_\sigma^\tau}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle \omega, \varphi\rangle) d\pi_\sigma^\tau(\omega) \\ &= \exp\left(\int_X \psi_\tau(\varphi(x))d\sigma(x)\right) \\ &= \exp\left(\int_X \int_{\mathbb{R}} (e^{s\varphi(x)} - 1)d\tau(s)d\sigma(x)\right),\ \varphi \in \mathcal{D}. \end{aligned} \tag{6.2}$$

Proposition 6.1.2 *1. Assume that τ satisfies the analyticity property:*

$$\exists C > 0 : \forall n \in \mathbb{N} \int_{\mathbb{R}} |s|^n d\tau(s) < C^n n!. \tag{6.3}$$

Then the Laplace transform of π_σ^τ is holomorphic at $0 \in \mathcal{D}_{\mathbb{C}}$.

2. Let $\tau(\mathbb{R}) < \infty$. Then

$$\pi_\sigma^\tau(\Omega) := \pi_\sigma^\tau\left(\left\{\sum_{x\in\gamma} s_x \varepsilon_x \in \mathcal{D}' | s_x \in \mathrm{supp}\tau,\ \gamma \in \Gamma\right\}\right) = 1,$$

where Γ is the configurations space defined in (5.1).

3. Let $\tau(\mathbb{R}) = \infty$. Then

$$\pi_\sigma^\tau(\Omega_\infty) := \pi_\sigma^\tau\left(\left\{\sum_{x\in\gamma_c} s_x\varepsilon_x \in \mathcal{D}' | s_x \in \operatorname{supp}\tau,\ \gamma_c \in \Gamma_c\right\}\right) = 1, \quad (6.4)$$

where Γ_c is the collection of all locally countable subets in X.

Proof. 1. By (6.3) the Kolmogorov characteristic ψ_τ is holomorphic on some neighborhood of $0 \in \mathbb{C}$. Then by (6.2) the Laplace transform $l_{\pi_\sigma^\tau}$ of π_σ^τ is holomorphic in some neighborhood of zero $\mathcal{U} \subset \mathcal{D}_\mathbb{C}$.
2, 3. At first assume $d = 1$. Then π_σ^τ corresponds to the distributional derivative of the compound Poisson process ξ_t and statements 2, 3 follow immediately from the properties of the paths of this process. Namely, almost every path ξ_t of compound Poisson process is right continuous step function with the jumps from suppτ. If τ is finite a measure, then any finite interval contains only finite number of the points of discontinuities of ξ_t (in this case ξ is called a generalized Poisson process). For infinite measure τ the set of discontinuities of ξ_t is locally countable, see e.g. [Tak67] and [Kin93].

For $d > 1$ the statements 2, 3 follow from the analogous results of the theory of Poisson measures, see e.g. [Kal74], [Kal83] and [KMM78]. ■

Remark 6.1.3 *Assume that τ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\{\xi_k, k \geq 1\}$ be a sequence of independent identically τ-distributed random variables and $N = \{N_t, t \geq 0\}$ be the standard Poisson process independent of $\{\xi_k, k \geq 1\}$. Then π_σ^τ is generated by the distributional derivative of the compound Poisson process*

$$\xi_t = \sum_{k=1}^{N_t} \xi_k.$$

Notice that we don't consider here measures μ corresponding to the distributional derivatives of doubly stochastic Poisson processes and fields.

6.2 Isomorphism between compound Poisson and Poisson spaces

Let us define a measure $\hat{\sigma}$ on $(\hat{X}, \mathcal{B}(\hat{X}))$ (here $\hat{X} := X \times \mathbb{R}$) as the product measure of the measures τ and σ, i.e.,

$$d\hat{\sigma}(\hat{x}) := d\tau(s)d\sigma(x),\ \hat{x} = (x, s) \in X \times \mathbb{R}.$$

Define the Poisson measure $\pi_{\hat{\sigma}}$ with intensity measure $\hat{\sigma}$ on $(\Gamma_{\hat{X}}, \mathcal{B}(\Gamma_{\hat{X}}))$ via its Laplace transform

$$\begin{aligned} l_{\pi_{\hat{\sigma}}}(\hat{\varphi}) &= \int_{\Gamma_{\hat{X}}} \exp(\langle \hat{\gamma}, \hat{\varphi} \rangle) d\pi_{\hat{\sigma}}(\hat{\gamma}) \\ &= \exp\left(\int_{\hat{X}} (e^{\hat{\varphi}(\hat{x})} - 1) d\hat{\sigma}(\hat{x}) \right), \ \hat{\varphi} \in \mathcal{D}(\hat{X}). \end{aligned} \tag{6.5}$$

The intensity measure $\hat{\sigma}$ has the following property: for any $x \in X$, $\Delta \in \mathcal{B}(\mathbb{R})$ such that $\tau(\Delta) < \infty$ we have

$$\hat{\sigma}(\Delta \times \{x\}) = \tau(\Delta)\sigma(\{x\}) = 0.$$

This property yields that $\pi_{\hat{\sigma}}$ is concentrated on a smaller set than $\Gamma_{\hat{X}}$. Namely, let us define $\hat{\Gamma} \subset \Gamma_{\hat{X}}$ as follows

$$\hat{\Gamma} := \left\{ \hat{\gamma} \in \Gamma_{\hat{X}} | \hat{\gamma} = \sum_{\hat{x}_i \in \hat{\gamma}} \varepsilon_{\hat{x}_i}, \ \hat{x}_i = (x_i, s_i) \in X \times \mathbb{R}, \, x_i \neq x_j, \ i \neq j \right\}. \tag{6.6}$$

Proposition 6.2.1 *The measure $\pi_{\hat{\sigma}}$ is concentrated on the set $\hat{\Gamma} \in \mathcal{B}(\Gamma_{\hat{X}})$.*

Proof. One can deduce this result from the theory of point processes, see e.g., [KMM78, Chap. 1], [Kal83], and [Kin93]. ■

Remark 6.2.2 *Let $d = 1$. The measure $\pi_{\hat{\sigma}}$ corresponds to the distributional derivative of the independently marked Poisson process with the intensity measure σ and space of marks $(\mathbb{R}, \tau)$ (for more details on marked processes see, e.g., [Kin93] and [BL95]). For $d > 1$ there exists analogous connection between $\pi_{\widehat{\sigma}}$ and independently marked Poisson fields with the same intensity and marking.*

It follows from (6.3) that the Laplace transform $l_{\pi_{\hat{\sigma}}}$ is well defined for $\hat{\varphi}(s, x) = p(s)\varphi(x)$ where $p(s) = \sum_{k=0}^{m} p_k s^k$ $(p_0 \neq 0)$ is a polynomial and $\varphi \in \mathcal{D}$ (cf. [LRS97]). Let us put $\hat{\varphi}(s, x) = s\varphi(x)$, $\varphi \in \mathcal{D}$ in (6.5). Then by (5.7) we obtain

$$l_{\pi_{\sigma}^{\tau}}(\varphi) = l_{\pi_{\hat{\sigma}}}(s\varphi), \ \varphi \in \mathcal{D}.$$

Then it follows that the compound Poisson measure π_σ^τ is the image of $\pi_{\hat{\sigma}}$ under the transformation $\Sigma : \hat{\Gamma} \to \Sigma\hat{\Gamma} = \Omega \subset \mathcal{D}'$ given by

$$\hat{\Gamma} \ni \hat{\gamma} \mapsto (\Sigma\hat{\gamma})(\cdot) = \Sigma \left(\sum_{\hat{x_i} \in \hat{\gamma}} \varepsilon_{\hat{x_i}} \right)(\cdot) := \sum_{(s_i, x_i) \in \hat{\gamma}} s_i \varepsilon_{x_i}(\cdot) \in \Omega \subset \mathcal{D}', \tag{6.7}$$

i.e., $\forall B \in \mathcal{B}(\mathcal{D}')$

$$\pi_\sigma^\tau(B) = \pi_\sigma^\tau(B \cap \Omega) = \pi_{\hat{\sigma}}(\Sigma^{-1}(B \cap \Omega)),$$

where $\Sigma^{-1}\Delta$ is the pre-image of the set Δ.

The latter equality may be rewritten in the following form

$$\int_{\mathcal{D}'} 1\!\!1_B(\omega) d\pi_\sigma^\tau(\omega) = \int_\Omega 1\!\!1_B(\omega) d\pi_\sigma^\tau(\omega) = \int_{\hat{\Gamma}} 1\!\!1_B(\Sigma\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}),$$

which is analogous to the well known change of variable formula for the Lebesgue integral. Namely, for any $h \in L^1(\mathcal{D}', \pi_\sigma^\tau) = L^1(\Omega, \pi_\sigma^\tau)$ the function $h \circ \Sigma \in L^1(\hat{\Gamma}, \pi_{\hat{\sigma}})$ and

$$\int_\Omega h(\omega) d\pi_\sigma^\tau(\omega) = \int_{\hat{\Gamma}} h(\Sigma\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}). \tag{6.8}$$

Remark 6.2.3 *It is worth noting that there exists on Ω an inverse map $\Sigma^{-1} : \Omega \to \hat{\Gamma}$. And we obtain that $\pi_{\hat{\sigma}}$ on $\hat{\Gamma}$ is the image of π_σ^τ on Ω under the map Σ^{-1}, i.e., $\forall \hat{C} \in \mathcal{B}(\hat{\Gamma})$, $\pi_{\hat{\sigma}}(\hat{C}) = \pi_\sigma^\tau(\Sigma\hat{C})$ or after rewriting*

$$\int_{\hat{\Gamma}} 1\!\!1_{\hat{C}}(\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}) = \int_\Omega 1\!\!1_{\Sigma\hat{C}}(\omega) d\pi_\sigma^\tau(\omega) = \int_\Omega 1\!\!1_{\hat{C}}(\Sigma^{-1}\omega) d\pi_\sigma^\tau(\omega).$$

As before we easily can write the corresponding change of variables formula, namely for any $\hat{f} \in L^1(\hat{\Gamma}, \pi_{\hat{\sigma}})$ the function $\hat{f} \circ \Sigma^{-1} \in L^1(\Omega, \pi_\sigma^\tau)$ and

$$\int_{\hat{\Gamma}} \hat{f}(\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}) = \int_\Omega \hat{f}(\Sigma^{-1}\omega) d\pi_\sigma^\tau(\omega). \tag{6.9}$$

Now we construct a unitary isomorphism U_Σ between the Poisson space $L^2(\pi_{\hat{\sigma}}) := L^2(\hat{\Gamma}, \pi_{\hat{\sigma}})$ and the compound Poisson space $L^2(\pi_\sigma^\tau) := L^2(\Omega, \pi_\sigma^\tau)$. Namely,

$$L^2(\Omega, \pi_\sigma^\tau) \ni h \mapsto U_\Sigma h := h \circ \Sigma \in L^2(\hat{\Gamma}, \pi_{\hat{\sigma}})$$

and

$$L^2(\hat{\Gamma}, \pi_{\hat{\sigma}}) \ni \hat{f} \mapsto U_\Sigma^{-1}\hat{f} = \hat{f} \circ \Sigma^{-1} \in L^2(\Omega, \pi_\sigma^\tau). \tag{6.10}$$

The isometry of U_Σ and U_Σ^{-1} follows from (6.8) and (6.9), respectively

As a result we have established the following proposition.

Proposition 6.2.4 *The map U_Σ is a unitary isomorphism between the Poisson space $L^2(\pi_{\hat\sigma})$ and the compound Poisson space $L^2(\pi_\sigma^\tau)$.*

Remark 6.2.5 *In the space $L^2(\pi_{\widehat\sigma})$ we have a basis of generalized Charlier polynomials, annihilation and creation operators etc. Now we can use the unitary isomorphism U_Σ in order to transport the Fock structure from $L^2(\pi_{\widehat\sigma})$ to $L^2(\pi_\sigma^\tau)$.*

6.3 Annihilation and creation operators on compound Poisson space

In this section we will use the isomorphism from the last section in order to transport the Fock structure from $L^2(\pi_{\hat\sigma})$ to the compound Poisson space $L^2(\pi_\sigma^\tau)$.

Let $\triangledown_{\hat\varphi}^P$, $(\triangledown_{\hat\varphi}^P)^*$, $\hat\varphi \in \mathcal{D}(\hat X)$ be the annihilation and creation operators on Poisson space $L^2(\pi_{\hat\sigma})$. Their images under U_Σ

$$U_\Sigma^{-1}\,\triangledown_{\hat\varphi}^P\,U_\Sigma,\;\; U_\Sigma^{-1}(\triangledown_{\hat\varphi}^P)^* U_\Sigma \tag{6.11}$$

play the role of annihilation and creation operators in compound Poisson space $L^2(\pi_\sigma^\tau)$. Let us calculate the actions of (6.11).

The set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$, (dense in $L^2(\pi_\sigma^\tau)$) consists of all functions of the form

$$\begin{aligned} h(\omega) &= H(\langle\omega,\varphi_1\rangle,\ldots,\langle\omega,\varphi_N\rangle) \\ &= H(\langle\Sigma^{-1}\omega, s\varphi_1\rangle,\ldots,\langle\Sigma^{-1}\omega, s\varphi_N\rangle), \end{aligned} \tag{6.12}$$

where (generating directions) $\varphi_1,\ldots,\varphi_N \in \mathcal{D}$ and H (generating function for h) is from $C_b^\infty(\mathbb{R}^N)$. Whence it follows that

$$\mathcal{F}C_b^\infty(\mathcal{D},\Omega) = U_\Sigma^{-1}\mathcal{F}C_b^\infty(\mathcal{D}(\hat X),\hat\Gamma).$$

By (5.19) for any $\hat f \in \mathcal{F}C_b^\infty(\mathcal{D}(\hat X),\hat\Gamma)$ we have

$$(\nabla_{\hat\varphi}^P \hat f)(\hat\gamma) = \int_{\hat X}[\hat f(\hat\gamma+\varepsilon_{\hat x}) - \hat f(\hat\gamma)]\hat\varphi(\hat x)d\hat\sigma(\hat x). \tag{6.13}$$

Proposition 6.3.1 *For any $h \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ the operator $U_\Sigma^{-1} \bigtriangledown_{\hat\varphi}^P U_\Sigma$ has the following form*

$$(U_\Sigma^{-1} \bigtriangledown_{\hat\varphi}^P U_\Sigma h)(\omega) = \int_X \int_{\mathbb{R}} [h(\omega + s\varepsilon_x) - h(\omega)]\hat\varphi(s,x)d\tau(s)d\sigma(x).$$

Proof. Let $h \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ be given and denote $U_\Sigma h = h \circ \Sigma =: \hat h$ and $\Sigma^{-1}w =: \hat\gamma$. Taking into account (6.11) and (6.12) we obtain

$$\begin{aligned}(U_\Sigma^{-1} \bigtriangledown_{\hat\varphi}^P U_\Sigma h)(\omega) &= (\bigtriangledown_{\hat\varphi}^P \hat h)(\hat\gamma) \\ &= \int_{\hat X} [\hat h(\hat\gamma + \varepsilon_{\hat x}) - \hat h(\hat\gamma)]\hat\varphi(\hat x)d\hat\sigma(\hat x). \end{aligned} \tag{6.14}$$

Now we use the definition of $\hat h$, the additivity of the map Σ and the obvious equality $\Sigma\varepsilon_{\hat x} = s\varepsilon_x$ for $\hat x = (s,x)$. With this (6.14) turns out to be

$$\int_{\hat X} [h(\Sigma(\hat\gamma + \varepsilon_{\hat x})) - h(\Sigma\hat\gamma)\hat\varphi(\hat x)]d\hat\sigma(\hat x) = \int_{\hat X} [h(\omega + s\varepsilon_x) - h(\omega)]\hat\varphi(\hat x)d\hat\sigma(\hat x). \tag{6.15}$$

The result of the proposition follows then by definition of $\hat\sigma$. ■

Putting $\hat\varphi\,(\hat x) = \phi\,(s)\,\varphi\,(x)$ in (6.15) we obtain

$$\begin{aligned} &(U_\Sigma^{-1} \bigtriangledown_{\phi\varphi}^P U_\Sigma h)(\omega) \\ = &\int_X \left(\int_{\mathbb{R}} [h(\omega + s\varepsilon_x) - h(\omega)]\phi(s)d\tau(s) \right) \varphi(x)d\sigma(x). \end{aligned}$$

Let us note that by (6.1) we can admit not only bounded functions $\phi\,(s)$ but also polynomials. For finite τ and $\phi \equiv 1$ we have the following formula for the *annihilation operator* $\bigtriangledown_\varphi^{CP}$ in compound Poisson space $L^2(\Omega, \pi_\sigma^\tau)$:

$$\begin{aligned}(\bigtriangledown_\varphi^{CP} h)(\omega) &:= (U_\Sigma^{-1} \bigtriangledown_\varphi^P U_\Sigma h)(\omega) \\ &= \int_X \left(\int_{\mathbb{R}} [h(\omega + s\varepsilon_x) - h(\omega)]d\tau(s) \right) \varphi(x)d\sigma(x). \end{aligned} \tag{6.16}$$

Example 6.3.2

1. *Let $\tau = \varepsilon_1$, then $\pi_\sigma^\tau = \pi_\sigma$ and, of course, (6.16) coincides with (5.19).*

2. *Let $\tau = \frac{1}{2}(\varepsilon_{-1} + \varepsilon_1)$ (for $d = 1$ π_σ^τ is generated by the so called telegraph process) then the annihilation operator $\bigtriangledown_\varphi^{CP}$ has the form*

$$(\bigtriangledown_\varphi^{CP} h)(\omega) = \int_{\mathbb{R}} \left[\frac{1}{2}h(\omega + \varepsilon_x) + \frac{1}{2}h(\omega - \varepsilon_x) - h(\omega) \right] \varphi(x)d\sigma(x).$$

Example 6.3.3 *Let $h \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ be given by*

$$\begin{aligned} h(\omega) &= \exp\left(\langle\omega,\log(1+\eta)\rangle - \langle\eta\rangle_\sigma \int_{\mathbb{R}} s d\tau(s)\right) \\ &= \exp(\langle\omega,\log(1+\eta)\rangle - \langle\eta\rangle_\sigma m_1(\tau)), \end{aligned}$$

for $\mathcal{D} \ni \eta > -1$. Then the annihilation operator $\bigtriangledown_\varphi^{CP}$ applied to h can be computed to be

$$\begin{aligned} (\bigtriangledown_\varphi^{CP} h)(\omega) &= (\bigtriangledown_\varphi^{P} \hat{h})(\hat{\gamma}) \\ &= \int_{\hat{X}} [\hat{h}(\hat{\gamma}+\varepsilon_{(s,x)}) - \hat{h}(\hat{\gamma})]\varphi(x) d\hat{\sigma}(s,x) \\ &= \int_{\hat{X}} [h(\omega + s\varepsilon_x) - h(\omega)]\varphi(x) d\hat{\sigma}(s,x) \\ &= h(\omega)\int_{\hat{X}} [(1+\eta(x))^s - 1]\varphi(x) d\hat{\sigma}(s,x) \\ &= \langle((1+\eta)^{\cdot} - 1),\varphi\rangle_{\hat{\sigma}} h(\omega). \end{aligned}$$

Example 6.3.4 *Let $h \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ be given by*

$$h(\omega) = \exp(-\langle\phi\rangle_\tau\langle\varphi\rangle_{\hat{\sigma}}) \prod_{x\in\gamma}(1+\phi(s_x)\varphi(x)),\ \omega = \sum_{x\in\gamma} s_x\varepsilon_x \in \Omega.$$

It is clear that $(U_\Sigma^{-1} e^\alpha_{\pi_{\hat{\sigma}}}(\hat{\varphi};\cdot))(\omega) = h(\omega)$ $(\hat{\varphi} := \phi\varphi)$ as it can be easily seen from the definitions of U_Σ^{-1} and $e_{\pi_{\hat{\sigma}}}(\hat{\varphi};\cdot)$ given in (6.10) and (5.9), respectively. On the other hand from (5.20) we have

$$(\nabla_{\hat{\psi}}^P e^\alpha_{\pi_{\hat{\sigma}}}(\hat{\varphi};\cdot))(\hat{\gamma}) = (\hat{\varphi},\hat{\psi})_{L^2(\hat{\sigma})} e^\alpha_{\pi_{\hat{\sigma}}}(\hat{\varphi};\hat{\gamma}),$$

and therefore $(U_\Sigma^{-1} \bigtriangledown_{\hat{\psi}}^P U_\Sigma h)(\omega) = (\hat{\varphi},\hat{\psi})_{L^2(\hat{\sigma})} h(\omega)$ which says that

$$(\bigtriangledown_\psi^{CP} h)(\omega) = (\varphi,\psi)_{L^2(\sigma)} h(\omega).$$

Now we proceed to compute an expression for the creation operator on compound Poisson space.

Proposition 6.3.5 *Let $g \in L^2(\Omega,\pi_\sigma^\tau)$ and $\hat{\varphi} \in \mathcal{D}(\hat{X})$ be given such that $U_\Sigma g \in \mathrm{Dom}(I_{\hat{\sigma}} a^+(\hat{\varphi}) I_{\hat{\sigma}}^{-1})$. Then the operator $U_\Sigma^{-1}(\bigtriangledown_{\hat{\varphi}}^P)^* U_\Sigma$ has the following representation*

$$(U_\Sigma^{-1}(\bigtriangledown_{\hat{\varphi}}^P)^* U_\Sigma g)(\omega) = \int_{\hat{X}} g(\omega - s\varepsilon_x)\hat{\varphi}(s,x) d\hat{\gamma}(s,x) - g(\omega)\langle\hat{\varphi}\rangle_{\hat{\sigma}}.$$

Proof. We know from (5.21) that for any $\hat{g} \in \mathrm{Dom}(I_{\hat{\sigma}} a^+(\hat{\varphi}) I_{\hat{\sigma}}^{-1})$ the creation operator $(\bigtriangledown_{\hat{\varphi}}^P)^*$ on Poisson space $L^2(\hat{\Gamma}, \pi_{\hat{\sigma}})$ has the form

$$((\bigtriangledown_{\hat{\varphi}}^P)^* \hat{g})(\hat{\gamma}) = \int_{\hat{X}} \hat{g}(\hat{\gamma} - \varepsilon_{\hat{x}}) \hat{\varphi}(\hat{x}) d\hat{\gamma}(\hat{x}) - \hat{g}(\hat{\gamma}) \langle \hat{\varphi} \rangle_{\hat{\sigma}}.$$

On the other hand,

$$\begin{aligned} (U_\Sigma^{-1} (\bigtriangledown_{\hat{\varphi}}^P)^* U_\Sigma g)(\omega) &= ((\bigtriangledown_{\hat{\varphi}}^P)^* \hat{g})(\hat{\gamma}) \\ &= \int_{\hat{X}} \hat{g}(\hat{\gamma} - \varepsilon_{\hat{x}}) \hat{\varphi}(\hat{x}) d\hat{\gamma}(\hat{x}) - \hat{g}(\hat{\gamma}) \langle \hat{\varphi} \rangle_{\hat{\sigma}} \\ &= \int_{\hat{X}} g(\omega - s\varepsilon_x) \hat{\varphi}(s,x) d\hat{\gamma}(s,x) - g(\omega) \langle \hat{\varphi} \rangle_{\hat{\sigma}}, \end{aligned}$$

which proves the result of the proposition. ■

As before if we choose $\hat{\varphi} = 1\varphi$, in the case when τ is finite, then we have the following form for the *creation operator* $(\bigtriangledown_\varphi^{CP})^*$ in compound Poisson space $L^2(\Omega, \pi_\sigma^\tau)$:

$$\begin{aligned} ((\bigtriangledown_\varphi^{CP})^* g)(\omega) &:= (U_\Sigma^{-1} (\bigtriangledown_\varphi^P)^* U_\Sigma g)(\omega) \\ &= \int_{\hat{X}} g(\omega - s\varepsilon_x) \varphi(x) d\hat{\gamma}(s,x) - g(\omega) \tau(\mathbb{R}) \langle \varphi \rangle_\sigma. \end{aligned}$$

Remark 6.3.6 *The generalized Charlier polynomials in $L^2(\pi_{\hat{\sigma}})$, according to (5.28), have the following representation*

$$((\nabla_{\hat{\varphi}}^P)^{*n} 1)(\hat{\gamma}) = \langle C_n^{\hat{\sigma}}(\hat{\gamma}), \hat{\varphi}^{\otimes n} \rangle.$$

Their images under U_Σ^{-1} have the following form

$$\begin{aligned} (U_\Sigma^{-1} \langle C_n^{\hat{\sigma}}(\cdot), \hat{\varphi}^{\otimes n} \rangle)(\omega) &= \langle C_n^{\hat{\sigma}}(\Sigma^{-1}\omega), \hat{\varphi}^{\otimes n} \rangle \\ &= (U_\Sigma^{-1} (\nabla_{\hat{\varphi}}^P)^{*n} U_\Sigma 1)(\omega). \end{aligned}$$

In particular for finite measure τ and $\hat{\varphi} = \varphi$ we obtain

$$((\nabla_\varphi^{CP})^{*n} 1)(\omega) = \langle C_n^{\hat{\sigma}}(\Sigma^{-1}\omega), \varphi^{\otimes n} \rangle.$$

6.4 The special case of Gamma measure

6.4.1 Definition and properties

In this section we consider the classical (real) Schwartz triple

$$\mathcal{D}(\mathbb{R}^d) =: \mathcal{D} \subset L^2(\mathbb{R}^d) \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^d).$$

Definition 6.4.1 *We call Gamma noise the measure μ_G^σ on the measure space $(\mathcal{D}',\mathcal{B}(\mathcal{D}'))$ determined via its Laplace transform*

$$\begin{aligned} l_{\mu_G^\sigma}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle \omega, \varphi \rangle) d\mu_G^\sigma(\omega) \\ &= \exp(-\langle \log(1-\varphi) \rangle_\sigma), \ 1 > \varphi \in \mathcal{D}. \end{aligned}$$

Remark 6.4.2 *In order to apply Minlos' theorem we note that μ_G^σ is a special case of π_σ^τ for the choice of τ as follows*

$$\tau(\Delta) = \int_{\Delta \cap]0,\infty[} \frac{e^{-s}}{s} ds, \ \Delta \in \mathcal{B}(\mathbb{R}). \tag{6.17}$$

Whence by Minlos' theorem μ_G^σ is well-defined, of course $l_{\mu_G^\sigma}$ is an analytic function.

Remark 6.4.3 *Let us explain the term "Gamma noise". If $d = 1$ and $\sigma = m$, then for any $t > 0$ the value of the Laplace transform*

$$l_{\mu_G^m}(\lambda 1\!\!1_{[0,t]}) = \exp[-t \log(1-\lambda)], \ \lambda < 1,$$

coincides with the Laplace transform $l_{\xi(t)}(\lambda)$ of a random variable $\xi(t)$ having two-side Gamma distribution, i.e., the density of the distribution function has the form

$$p_t(x) = \frac{1}{2} \frac{|x|^{t-1} e^{-|x|}}{\Gamma(t)}, \ t > 0,$$

where $\Gamma(t)$ is the Gamma function. The process $\{\xi(t), t > 0; \xi(0) := 0\}$ is known as Gamma process, see e.g., [Tak67, Section 19]. Thus the triple $(\mathcal{D}', \mathcal{B}(\mathcal{D}'), \mu_G^m)$ is a direct representation of the generalized stochastic process $\{\dot{\xi}(t), t \geq 0\}$ (detailed information on generalized stochastic process can be *found in [GV68]) which is a distributional derivative of the Gamma process $\{\xi(t), t \geq 0\}$. In other words, the image of μ_G^m under the transformation*

$$\mathcal{D}' \ni \omega \longmapsto G_t(\omega) := \langle \omega, 1\!\!1_{[0,t]} \rangle \in \mathbb{R},\ t \in \mathbb{R}_+$$

coincides with the two-sided Gamma distribution, i.e.,

$$(\mu_G^m \circ G_t^{-1})(\Delta) = \int_\Delta p_t(x) dm(x),\ \Delta \in \mathcal{B}(\mathbb{R}).$$

So the term "Gamma noise" is natural for μ_G^m.

Remark 6.4.4 *As μ_G^σ is a special case of compound Poisson measure one can obtain the representations of generalized Charlier polynomials, annihilation and creation operators etc. along the lines of Subsection 6.3. It is worth noting that $\tau(\mathbb{R}) = \infty$ nevertheless one can set in (6.13) $\widehat{\varphi}(\widehat{x}) = 1\varphi(x)$ and obtain the representation (6.14)*

$$(\nabla_\varphi^G h)(\omega) = \int_{\mathbb{R}^d} \int_0^\infty (h(\omega + s\varepsilon_x) - h(\omega)) \frac{e^{-s}}{s} ds \varphi(x) d\sigma(x)$$

for the annihilation operator in Gamma space. Indeed, by (6.12)

$$\begin{aligned} h(\omega + s\varepsilon_x) - h(\omega) \quad &= \quad H((\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle) + s(\varphi_1(x), \ldots, \varphi_N(x))) \\ &\quad - H(\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle),\ H \in C_b^\infty(\mathbb{R}^N) \end{aligned}$$

whence by Lagrange theorem it follows that $|h(\omega + s\varepsilon_x) - h(\omega)| \leq Cs$. Therefore the integral over $[0, \infty)$ converges and the right hand side of above equality is well-defined.

Remark 6.4.5 *Let us assume that $d = 1$ and $\sigma = m$. Then μ_G^m corresponds to a distributional derivative of the Gamma process $\xi = \{\xi(t), t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ (see Remark 6.4.3). The Gamma process is Lévy one, such that $\mathbb{E}[\xi(t)] = t$ and $\mathbb{E}[(\xi(t) - t)^2] = t$, $t \geq 0$. Thus the centered Gamma process $\{\xi(t) - t, t \geq 0\}$ is a normal martingale and one can define the n-multiple stochastic integrals $I_n(f^{(n)}, \xi)$, $f^{(n)} \in \mathrm{Exp}_n L^2(m)$ with respect to ξ and the space of chaos decomposable random variables from $L^2(\Omega, \mathcal{F}, P)$:*

$$\mathfrak{C}(\xi) = \left\{ \sum_{n=0}^\infty I_n(f^{(n)}, \xi) \,|\, f^{(n)} \in \mathrm{Exp}_n L^2(m), \sum_{n=0}^\infty n! |f^{(n)}|^2 < \infty \right\},$$

(for more details see [Mey93]).

It follows from the results of [Der90] that ξ does not possess CRP, i.e., $\mathfrak{C}(\xi)$ is a proper subspace of $L^2(\Omega, \mathcal{F}_\xi, P) =: L^2(\xi)$, where $\mathcal{F}_\xi$ denotes a σ-algebra which is generated by the collection $\{\xi(t), t \geq 0\}$.

6.4.2 Chaos decomposition of gamma space

Let us now consider a function $\alpha : \mathcal{D} \longrightarrow \mathcal{D}$ defined by

$$\alpha(\varphi)(x) := \frac{\varphi(x)}{\varphi(x)-1},\ \varphi \in \mathcal{D},\ x \in \mathbb{R}^d. \tag{6.18}$$

We stress that α is a holomorphic function on a neighborhood of zero $\mathcal{U}_\alpha \subset \mathcal{D}$, in other words $\alpha \in \mathrm{Hol}_0(\mathcal{D}, \mathcal{D})$.

Because of the holomorphy of $l_{\mu_G^\sigma}$ and $l_{\mu_G^\sigma}(0) = 1$, there exists a neighborhood of zero $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha$ such that the normalized exponential $e^\alpha_{\mu_G^\sigma}(\varphi;\omega)$ is holomorphic for any $\varphi \in \mathcal{U}'_\alpha$ and $\omega \in \mathcal{D}'$. Then

$$\begin{aligned} e^\alpha_{\mu_G^\sigma}(\varphi;\omega) &:= \frac{\exp(\langle \omega, \alpha(\varphi)\rangle)}{l_{\mu_G^\sigma}(\alpha(\varphi))} \\ &= \exp\left(\left\langle \omega, \frac{\varphi}{\varphi-1}\right\rangle - \langle \log(1-\varphi)\rangle_\sigma\right),\ \varphi \in \mathcal{U}'_\alpha. \end{aligned} \tag{6.19}$$

We use the holomorphy of $\varphi \mapsto e^\alpha_{\mu_G^\sigma}(\varphi;\omega)$ to expand it in a power series which, with Cauchy's inequality, polarization identity and kernel theorem, give us

$$e^\alpha_{\mu_G^\sigma}(\varphi;\omega) = \sum_{n=0}^\infty \frac{1}{n!}\langle P_n^{\mu_G^\sigma,\alpha}(\omega), \varphi^{\otimes n}\rangle,\ \varphi \in \mathcal{U}'_\alpha,\ \omega \in \mathcal{D}', \tag{6.20}$$

where $P_n^{\mu_G^\sigma,\alpha} : \mathcal{D}' \to \mathcal{D}'^{\widehat{\otimes} n}$. $\{P_n^{\mu_G^\sigma,\alpha}(\cdot) =: L_n^\sigma(\cdot) | n \in \mathbb{N}_0\}$ is called the system of *generalized Laguerre kernels* on Gamma space $(\mathcal{D}', \mathcal{B}(\mathcal{D}'), \mu_G^\sigma)$. From (6.20) it follows immediately that for any $\varphi^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$ the function

$$\mathcal{D}' \ni \omega \mapsto \langle L_n^\sigma(\omega), \varphi^{(n)}\rangle,$$

is a polynomial of the order n on $\mathcal{D}'$. The system of functions

$$\{L_n^\sigma(\varphi^{(n)})(\omega) := \langle L_n^\sigma(\omega), \varphi^{(n)}\rangle,\ \forall \varphi^{(n)} \in \mathcal{D}^{\widehat{\otimes} n},\ n \in \mathbb{N}_0\},$$

is called the system of *generalized Laguerre polynomials* for the Gamma measure μ_G^σ. In one-dimensional case this system coincides with the system of even continuations of Laguerre polynomials $\{L_n^{(t-1)}, n \in \mathbb{N}_0\}$ which are orthogonal with respect to the density $p_t(x)$, see e.g. [Rai71]. Namely in the notation of Remark 6.4.3 the following equality holds

$$\langle L_n^m(\omega), \mathbb{1}_{[0,t]}^{\otimes n}\rangle = L_n^{(t-1)}(G_t(\omega)).$$

Now we proceed establishing the following result. Let $\varphi, \psi \in \mathcal{U}'_\alpha$ be given, then using (6.19) it follows that for any $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned}
&\int_{\mathcal{D}'} e^\alpha_{\mu^\sigma_G}(\lambda_1\varphi;\omega)e^\alpha_{\mu^\sigma_G}(\lambda_2\psi;\omega)d\mu^\sigma_G(\omega)\\
=\ & e^{\langle -\log(1-\lambda_1\varphi)-\log(1-\lambda_2\psi)\rangle_\sigma}\int_{\mathcal{D}'}\exp(\langle\omega, \tfrac{\lambda_1\varphi}{\lambda_1\varphi-1}+\tfrac{\lambda_2\psi}{\lambda_2\psi-1}\rangle)d\mu^\sigma_G(\omega)\\
=\ & \exp\left(\langle -\log(1-\lambda_1\varphi)-\log(1-\lambda_2\psi)\rangle_\sigma - \langle\log(1-\tfrac{\lambda_1\varphi}{\lambda_1\varphi-1}-\tfrac{\lambda_2\psi}{\lambda_2\psi-1})\rangle_\sigma\right)\\
=\ & \exp(-\langle\log(1-\lambda_1\varphi\lambda_2\psi)\rangle_\sigma)\\
=\ & l_{\mu^\sigma_G}(\lambda_1\lambda_2\varphi\psi). \qquad (6.21)
\end{aligned}$$

Since $l_{\mu^\sigma_G} \in \mathcal{M}_a(\mathcal{D}')$, then (6.21) turns out to be an analytic function on λ_1 and λ_2. Hence

$$l_{\mu^\sigma_G}(\lambda_1\lambda_2\varphi\psi) = \sum_{n=0}^{\infty}\frac{1}{n!}(\lambda_1\lambda_2)^n(\varphi^{\otimes n}, \psi^{\otimes n})_{\mathrm{Exp}^G_n L^2(\sigma)}, \qquad (6.22)$$

where the coefficients $(\varphi^{\otimes n}, \psi^{\otimes n})_{\mathrm{Exp}^G_n L^2(\sigma)}$ are given by

$$(\varphi^{\otimes n}, \psi^{\otimes n})_{\mathrm{Exp}^G_n L^2(\sigma)} = \frac{d^n}{dt^n}\exp(-\langle\log(1-t\varphi\psi)\rangle_\sigma))|_{t=0}$$

and $\mathrm{Exp}^G_n L^2(\sigma)$ stands for a quasi-n-particle subspace of $\mathrm{Exp}^G L^2(\sigma)$ defined by (6.26) below.

By using the formula, see e.g., [Bou50] and [GR81],

$$\begin{aligned}
&\frac{d^n}{dt^n}e^{f(t)}\\
=\ & \sum_{\substack{i_1+2i_2+\cdots+ki_k=n\\ i_1,i_2,\ldots,i_k\in\mathbb{N}_0}}\frac{n!}{i_1!\ldots i_k!}\left(\frac{f^{(1)}(t)}{1!}\right)^{i_1}\left(\frac{f^{(2)}(t)}{2!}\right)^{i_2}\cdots\left(\frac{f^{(k)}(t)}{k!}\right)^{i_k}e^{f(t)}
\end{aligned}$$

it follows that

$$\begin{aligned}
&(\varphi^{\otimes n}, \psi^{\otimes n})_{\mathrm{Exp}^G_n L^2(\sigma)}\\
=\ & \sum_{\substack{i_1+2i_2+\cdots+ki_k=n\\ i_1,i_2,\ldots,i_k\in\mathbb{N}_0}}\frac{n!}{i_1!i_2!\ldots i_k!}\frac{1}{2^{i_2}\ldots k^{i_k}}
\end{aligned}$$

$$\cdot \left(\int_{\mathbb{R}^d} \varphi(x)\psi(x)d\sigma(x)\right)^{i_1} \left(\int_{\mathbb{R}^d} \varphi^2(x)\psi^2(x)d\sigma(x)\right)^{i_2}$$
$$\cdots \left(\int_{\mathbb{R}^d} \varphi^k(x)\psi^k(x)d\sigma(x)\right)^{i_k}. \tag{6.23}$$

On the other hand

$$\begin{aligned} & \int_{\mathcal{D}'} e^{\alpha}_{\mu^{\sigma}_G}(\lambda_1\varphi;\omega)e^{\alpha}_{\mu^{\sigma}_G}(\lambda_2\psi;\omega)d\mu^{\sigma}_G(\omega) \\ = & \sum_{n,m=0}^{\infty} \frac{\lambda_1^n\lambda_2^m}{n!m!}\int_{\mathcal{D}'} \langle L^{\sigma}_n(\omega),\varphi^{\otimes n}\rangle\langle L^{\sigma}_m(\omega),\psi^{\otimes m}\rangle d\mu^{\sigma}_G(\omega). \end{aligned} \tag{6.24}$$

Then a comparison of coefficients between (6.22) and (6.24) gives us

$$\int_{\mathcal{D}'} \langle L^{\sigma}_n(\omega),\varphi^{\otimes n}\rangle\langle L^{\sigma}_m(\omega),\psi^{\otimes m}\rangle d\mu^{\sigma}_G(\omega) = \delta_{nm}n!(\varphi^{\otimes n},\psi^{\otimes n})_{\mathrm{Exp}^G_n L^2(\sigma)},$$

which shows the orthogonality property of the system $\{L^{\sigma}_n(\cdot)|n\in\mathbb{N}_0\}$.

Since $(\cdot,\cdot)_{\mathrm{Exp}^G_n L^2(\sigma)}$ is n-linear we can extend it by polarization, linearity and continuity to general smooth kernels $\varphi^{(n)}$, $\psi^{(n)} \in \mathrm{Exp}^G_n L^2(\sigma)$. To this end we proceed as follows.

First we consider a partition of the numbers $I_n := \{1,2,\ldots,n\}$ in

$$I_n = \bigcup_{\alpha} I_{\alpha} =: \mathcal{I}^{(n)}.$$

Then for each such partition $\mathcal{I}^{(n)}$, we define i_k by

$$i_k := \#\{I_{\alpha}||I_{\alpha}|=k\},\ 1\le k\le n.$$

Finally we define the contraction of the kernel $\varphi^{(n)}$ with respect to $\mathcal{I}^{(n)}$ as

$$\varphi^{(n)}_{\mathcal{I}^{(n)}}(x_1,x_2,\ldots,x_n) := \varphi^{(n)}(x_{i_1},x_{i_2},\ldots,x_{i_k}),$$

where $x_{i_m} = (x_m,x_m,\ldots,x_m)$ (m-times), $1\le m\le k$.

Hence the inner product is given by

$$\begin{aligned} & (\varphi^{(n)},\psi^{(n)})_{\mathrm{Exp}^G_n L^2(\sigma)} \\ = & \sum_{\mathcal{I}^{(n)}} n!\left(\prod_{k=1}^{n}\frac{1}{i_k!k^{i_k}}\right)\left(n!\prod_{k=1}^{n}\frac{1}{(k!)^{i_k}\,i_k!}\right)^{-1} \end{aligned}$$

$$\cdot \int_{\mathbb{R}^{dn}} \varphi^{(n)}_{\mathcal{I}^{(n)}}(x)_1^n \psi^{(n)}_{\mathcal{I}^{(n)}}(x)_1^n d\sigma^{\otimes n}(x)_1^n$$

$$= \sum_{\mathcal{I}^{(n)}} \prod_{k=1}^{n} ((k-1)!)^{i_k} \int_{\mathbb{R}^{dn}} \varphi^{(n)}_{\mathcal{I}^{(n)}}(x)_1^n \psi^{(n)}_{\mathcal{I}^{(n)}}(x)_1^n d\sigma^{\otimes n}(x)_1^n, \qquad (6.25)$$

where $(x)_1^n := (x_1, \dots x_n)$ and the sum extends over all possible partition $\mathcal{I}^{(n)}$ of I_n.

Hence we have established the proposition.

Proposition 6.4.6 *Let $\varphi, \psi \in \mathcal{D}$ be given. Then the system of generalized Laguerre polynomials verifies the following orthogonality property*

$$\int_{\mathcal{D}'} \langle L_n^\sigma(\omega), \varphi^{(n)} \rangle \langle L_m^\sigma(\omega), \psi^{(m)} \rangle d\mu_G^\sigma(\omega) = \delta_{nm} n! (\varphi^{(n)}, \psi^{(n)})_{\mathrm{Exp}_n^G L^2(\sigma)},$$

where $(\varphi^{(n)}, \psi^{(n)})_{\mathrm{Exp}_n^G L^2(\sigma)}$ is defined by (6.25) above.

As a consequence of the last proposition we have established the following isomorphism

$$I_{\mu_G^\sigma} : L^2(\mu_G^\sigma) \longrightarrow \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n^G L^2(\sigma) =: \mathrm{Exp}^G L^2(\sigma). \qquad (6.26)$$

Therefore for any $F \in L^2(\mu_G^\sigma)$ there is a sequence $(f^{(n)})_{n=0}^\infty \in \mathrm{Exp}^G L^2(\sigma)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} \langle L_n^\sigma(\omega), f^{(n)} \rangle,$$

moreover

$$\|F\|_{L^2(\mu_G^\sigma)} = \sum_{n=0}^{\infty} n! |f^{(n)}|^2_{\mathrm{Exp}_n^G L^2(\sigma)}.$$

Remark 6.4.7 *Hence we see that the Gamma noise does not produce the standard Fock type isomorphism since the inner product $(\cdot, \cdot)_{\mathrm{Exp}_n^G L^2(\sigma)}$ do not coincide with the inner product in the n-particle subspace, $L^2(\sigma)^{\hat{\otimes} n}$.*

Remark 6.4.8 *The orthogonal polynomials of independent-increment processes (in particular, Gamma-process) were constructed in [KS76]. It is worth noting that these polynomials of Gamma process differ from generalized Laguerre polynomials.*

Chapter 7

Differential Geometry on compound Poisson space

Starting with the work of Gelfand et al. [GGV75], many researchers consider representations on compound Poisson spaces Ω, see also [Ism96]. Hence it is natural to ask about geometry and analysis on these spaces. In the language of Chapter 5 this corresponds to the internal geometry on Ω. In this chapter we develop in detail the internal geometry on compound configuration spaces which generalizes the results from Chapter 5, Section 5.4.

On the other hand, in statistical physics of continuous systems compound Poisson measures (or more general marked Poisson measures) and their Gibbsian perturbation are used for the description of many concrete models, see e.g., [AGL78], [GZ93], [GH96], and [MM91].

The geometry is constructed via a "lifting procedure" and is completely determined by the Riemannian structure on X (cf. Subsection 7.2.3). In particular, we obtain the corresponding intrinsic gradient ∇^{Ω_X}, divergence $\mathrm{div}^{\Omega_X}_{\pi^\tau_\sigma}$, and Laplace-Beltrami operator $\Delta^{\Omega_X} = \mathrm{div}^{\Omega_X}_{\pi^\tau_\sigma} \nabla^{\Omega_X}$. For details we refer to the main body of this chapter. Here we only mention that the "tangent space" $T_\omega\Omega_X$ to Ω_X at the point $\omega \in \Omega_X$ is given by

$$T_\omega\Omega_X := L^2(X \to TX; \omega),\ \omega \in \Omega_X,$$

i.e., the space of ω-square integrable vector fields on X. Since each $T_\omega\Omega_X$ is thus a Hilbert space (endowed with the corresponding L^2-inner product $\langle\cdot,\cdot\rangle_{T_\omega\Omega_X}$ coming from the measure ω) Ω_X obtains a Riemannian-type structure which is non-trivial (i.e., varies with ω) even when $X = \mathbb{R}^d$, see Subsection 7.2.1 for details.

Let us stress that the "test" functions $\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$ (resp. "test" vector fields $\mathcal{V}\mathcal{F}C_0^\infty(\mathcal{D},\Omega_X)$) we consider as domains for our gradient ∇^{Ω_X} (resp. $\mathrm{div}_{\pi_\sigma^\tau}^{\Omega_X}$) above are of cylinder type, i.e., $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$ if and only if

$$\omega\mapsto F(\omega)=g_F(\langle\omega,\varphi_1\rangle,\dots,\langle\omega,\varphi_N\rangle),$$

for some $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N\in\mathcal{D}:=C_0^\infty(X)$, $g_F\in C_b^\infty(\mathbb{R}^N)$ (and V correspondingly, cf. (7.5)). Hence so far the analysis on Ω_X is basically finite dimensional. However, one can do generic infinite dimensional analysis on Ω_X by introducing the first order Sobolev space $H_0^{1,2}(\Omega_X,\pi_\sigma^\tau)$ by closing the corresponding Dirichlet form

$$\mathcal{E}_{\pi_\sigma^\tau}^{\Omega_X}(F,G)=\int_{\Omega_X}\langle(\nabla^{\Omega_X}F)(\omega),(\nabla^{\Omega_X}G)(\omega)\rangle_{T_\omega\Omega_X}d\pi_\sigma^\tau(\omega),$$

on $L^2(\Omega_X,\pi_\sigma^\tau)$, i.e., a function $F\in H_0^{1,2}(\Omega_X,\pi_\sigma^\tau)$ is together with its gradient $\nabla^{\Omega_X}F$ obtained as a limit in $L^2(\Omega,\pi_\sigma^\tau)$ of a sequence $F_n\in\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$, resp. $\nabla^{\Omega_X}F_n$, $n\in\mathbb{N}$. Thus such F really depends on infinitely many points in X (cf. Subsections 7.4.1, and 7.4.2).

The diffusion process determined by this geometry is identified in the following way. We regard every compound configuration $\omega\in\Omega_X$ as depending on two variables, namely $\omega=(\gamma_\omega,m_\omega)$ (or more general marked configuration) and this allowed us to obtain the following embedding

$$L^2(\Gamma_X,\pi_\sigma)\hookrightarrow L^2(\Omega_X,\pi_\sigma^\tau).$$

As a result we may apply operators acting on $L^2(\Gamma_X,\pi_\sigma)$, e.g., ∇^Γ, $\nabla^{\Gamma*}$ to the space $L^2(\Omega_X,\pi_\sigma^\tau)$ acting on part of the variables, see e.g., [BK95]. It turns out that the following equality holds

$$(\nabla^\Omega F)(\omega)=(\nabla^\Gamma F)((\gamma_\omega,m_\omega)),\ \omega=(\gamma_\omega,m_\omega)\in\Omega_X,$$

and from this relation it is not hard to obtain relations between the Dirichlet operators as well as between the correspondings semigroups. Therefore the process associated to our Dirichlet form is nothing but the process $X_t^{\gamma_\omega}$, $t\geq 0$, together with marks, i.e.,

$$\Xi_t=(X_t^{\gamma_\omega},m_\omega),\ t\geq 0,$$

where $X_t^{\gamma_\omega}$ is just the equilibrium process on Γ_X, see [AKR98a, Section 6] for details. We describe this procedure in details in Section 7.5.

On the basis of the results described above we provide a corresponding representation of the associated Lie algebra of compactly supported vector fields. We also exhibit explicit formulas for the corresponding generators, see Section 7.3.

Finally in Section 7.6 we prove in detail the existence of a marked Poisson measure over the marked Poisson space Ω_X^M, where M is a complete separable metric space with a probability measure. Hence all the results obtained in this chapter extend with obvious changes to marked Poisson spaces.

7.1 The group of diffeomorphisms and compound Poisson measures

Let us denote the group of all diffeomorphisms on X by $\mathrm{Diff}(X)$ and by $\mathrm{Diff}_0(X)$ the subgroup of all diffeomorphisms $\phi : X \to X$ with compact support, i.e., which are equal to the identity outside of a compact set (depending on ϕ).

For any $f \in C_0(X)$ we have a continuous functional

$$\Omega \ni \omega \mapsto \langle \omega, f \rangle = \int_X f(x) d\omega(x) = \sum_{x \in \gamma_\omega} s_x f(x),$$

and given $\phi \in \mathrm{Diff}_0(X)$ we have

$$\begin{aligned} \langle \phi^* \omega, f \rangle &= \int_X f(x) d\omega(\phi^{-1}(x)) \\ &= \sum_{x \in \gamma_\omega} s_x f \circ \phi(x) \\ &= \langle \omega, f \circ \phi \rangle. \end{aligned}$$

Any $\phi \in \mathrm{Diff}_0(X)$ defines (pointwise) a transformation of any subset of X and, consequently, the diffeomorphism ϕ has the following "lifting" from X to Ω:

$$\Omega \ni \omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_x \mapsto \phi^* \omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_{\phi(x)} \in \Omega,$$

because for any $f \in C_0(X)$

$$\int_X f(x) d(\phi^* \omega)(x) = \int_X f(\phi(x)) d\omega(x)$$

$$
\begin{aligned}
&= \sum_{x\in\gamma_\omega} s_x f(\phi(x)) \\
&= \int_X f(y) \sum_{x\in\gamma_\omega} s_x \varepsilon_{\phi(x)}(dy).
\end{aligned}
$$

This mapping is obviously measurable and we can define the image $\phi^*\pi_\sigma^\tau$ of the measure π_σ^τ under ϕ as usually by $\phi^*\pi_\sigma^\tau = \pi_\sigma^\tau \circ \phi^{-1}$, i.e.,

$$
(\phi^*\pi_\sigma^\tau)(A) = \pi_\sigma^\tau(\phi^{-1}(A)), \; A \in \mathcal{B}(\Omega).
$$

The following proposition shows that this transformation is nothing but a change of the intensity measure σ, and τ is preserved.

Proposition 7.1.1 *For any $\phi \in \mathrm{Diff}_0(X)$ we have*

$$
\phi^*\pi_\sigma^\tau = \pi_{\phi^*\sigma}^\tau.
$$

Proof. Due to the characterization of the measures it is enough to compute the Laplace transform of the measure $\phi^*\pi_\sigma^\tau$, to show the property.

Let $f \in C_0(X)$ be given. Then the Laplace transform of $\phi^*\pi_\sigma^\tau$ is given by

$$
\begin{aligned}
\int_\Omega \exp(\langle \omega, f\rangle) d(\phi^*\pi_\sigma^\tau)(\omega) &= \int_\Omega \exp(\langle \omega, f\rangle) d\pi_\sigma^\tau(\phi^{-1}(\omega)) \\
&= \int_\Omega \exp(\langle \omega, f\circ\phi\rangle) d\pi_\sigma^\tau(\omega) \\
&= \exp\left(\int_X \int_0^\infty (e^{sf\circ\phi(x)} - 1) d\tau(s) d\sigma(x)\right) \\
&= \int_\Omega \exp(\langle \omega, f\rangle) d\pi_{\phi^*\sigma}^\tau(\omega),
\end{aligned}
$$

which is just the Laplace transform of the measure $\pi_{\phi^*\sigma}^\tau$. ■

For any $\phi \in \mathrm{Diff}_0(X)$ we introduce the Radon-Nikodym density of σ as

$$
\begin{cases}
p_\phi^\sigma(x) \;:= \dfrac{d(\phi^*\sigma)}{d\sigma}(x) = \dfrac{\rho(\phi^{-1}(x))}{\rho(x)} \dfrac{dm(\phi^{-1}(x))}{dm(x)} = \dfrac{\rho(\phi^{-1}(x))}{\rho(x)} J_m^\phi(x), \\
\qquad\qquad \text{if } x \in \{0<\rho<\infty\} \cap \{0 < \rho\circ\phi^{-1} < \infty\}; \\
p_\phi^\sigma(x) \;:= 1, \text{ otherwise},
\end{cases}, \tag{7.1}
$$

where J_m^ϕ is the Jacobian determinant of ϕ (with respect to the Riemannian volume m), see e.g., [Boo75]. Note that $p_\phi^\sigma(x) \equiv 1$ outside a compact.

The next proposition is a consequence of the Proposition 6.2.4, the Skorokhod theorem on absolute continuity of Poisson measures, see e.g., [Sko57], [Tak90], and also [Shi94]. It shows that π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$.

Proposition 7.1.2 *The compound Poisson measure π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ and for any $\phi \in \mathrm{Diff}_0(X)$ we have $p_\phi^{\pi_\sigma^\tau} = p_\phi^{\pi_{\lambda_\tau \sigma}}$, where $\lambda_\tau = \tau(\mathbb{R})$, i.e.,*

$$p_\phi^{\pi_\sigma^\tau}(\omega) = \frac{d(\phi^* \pi_\sigma^\tau)}{d\pi_\sigma^\tau}(\omega) = \prod_{x \in \gamma_\omega} p_\phi^\sigma(x) \exp\left(\lambda_\tau \int_X (1 - p_\phi^\sigma(x)) d\sigma(x)\right).$$

Proof. Given $\phi \in \mathrm{Diff}_0(X)$ then $\hat{\phi} := \phi \otimes \mathrm{id} \in \mathrm{Diff}(\hat{X})$. Hence having in mind the isomorphism described in Section 6.2 the Radon-Nikodym density of π_σ^τ with respect to the group $\mathrm{Diff}_0(X)$ is given by

$$\begin{aligned}
& p_\phi^{\pi_\sigma^\tau}(\omega) = U_\Sigma^{-1} p_{\phi \otimes \mathrm{id}}^{\pi_{\hat{\sigma}}}(\omega) \\
= & \prod_{\hat{x} \in \hat{\gamma}_\omega} \frac{d\hat{\sigma} \circ (\phi \otimes \mathrm{id})^{-1}}{d\hat{\sigma}}(\hat{x}) \exp\left(\int_{X \times \mathbb{R}_+} \left(1 - \frac{d\hat{\sigma} \circ (\phi \otimes \mathrm{id})^{-1}}{d\hat{\sigma}}(\hat{x})\right) d\hat{\sigma}(\hat{x})\right) \\
= & \prod_{\hat{x} \in \hat{\gamma}_\omega} p_\phi^\sigma(x) \exp\left(\lambda_\tau \int_X (1 - p_\phi^\sigma(x)) d\sigma(x)\right) \\
= & \prod_{x \in \gamma_\omega} p_\phi^\sigma(x) \exp\left(\lambda_\tau \int_X (1 - p_\phi^\sigma(x)) d\sigma(x)\right) \\
= & p_\phi^{\pi_{\lambda_\tau \sigma}}(\gamma_\omega),
\end{aligned}$$

where we have used [AKR98a, Proposition 2.2]. ■

Remark 7.1.3 *We would like to stress that the above Radon-Nikodym density for the measure π_σ^τ is valid only for finite measure τ, but a more careful analysis shows that it is possible to include infinite measure τ (this implies that the support of π_σ^τ is Ω_∞, cf. Proposition 6.1.2 - 3), and the results state below essentially do not change.*

7.2 Intrinsic geometry on compound Poisson spaces

The underlying differentiable structure on X has a natural lifting to the configuration space Ω. As a result there appear in Ω objects such as the gradient, the tangent space etc. Below we describe the corresponding constructions in details.

7.2.1 The tangent bundle of Ω

Let us recall that $V(X)$ is the set of all C^∞-vector fields on X (i.e., smooth sections of TX). We will use a subset $V_0(X) \subset V(X)$ consisting of all vector fields with compact support. $V_0(X)$ can be considered as an infinite dimensional Lie algebra which corresponds to the group $\mathrm{Diff}_0(X)$ in the following sense: for any $v \in V_0(X)$ we can construct the flow of this vector field as a collection of mappings $\phi_t^v : X \to X$, $t \in \mathbb{R}$ obtained by integrating the vector field.

More precisely, for any $x \in X$ the curve

$$\mathbb{R} \ni t \longmapsto \phi_t^v(x) \in X,$$

is defined as the solution to the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\phi_t^v(x) = v(\phi_t^v(x)) \\ \\ \phi_0^v(x) = x \end{cases}.$$

That no explosion is possible and ϕ_t^v is well-defined for each $t \in \mathbb{R}$, is a consequence of $v \in V_0(X)$ (the latter implies that v is a complete vector field). The mappings $\{\phi_t^v, t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in the group $\mathrm{Diff}_0(X)$ (see e.g., [Boo75]), that is,

$$1)\ \forall t \in \mathbb{R}\ \phi_t^v \in \mathrm{Diff}_0(X)$$

$$2)\ \forall t, s \in \mathbb{R}\ \phi_t^v \circ \phi_s^v = \phi_{t+s}^v.$$

Let us fix $v \in V_0(X)$. Having the group ϕ_t^v, $t \in \mathbb{R}$, we can consider for any $\omega \in \Omega$ the curve

$$\mathbb{R} \ni t \longmapsto \phi_t^v(w) \in \Omega.$$

Definition 7.2.1 *For a function $F : \Omega \to \mathbb{R}$ we define the directional derivative along the vector field $v \in V_0(X)$ as*

$$(\nabla_v^\Omega F)(\omega) := \frac{d}{dt} F(\phi_t^{v*}\omega)|_{t=0},$$

provided the right hand side exists.

We note that $\nabla_v^\Omega F$ is closely related to the concept of the Lie derivative corresponding to a special class of vector fields on Ω, see below.

Let us introduce a special class of smooth functions on Ω which play an important role in our considerations below. We introduce $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ as the set of all functions $F : \Omega \to \mathbb{R}$ of the form

$$F(\omega) = g_F(\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle), \ \omega \in \Omega, \tag{7.2}$$

where (generating directions) $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and $g_F(s_1, \ldots, s_N)$ (generating function for F) is from $C_b^\infty(\mathbb{R}^N)$.

For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ of the form (7.2) and given $v \in V_0(X)$ we have

$$\begin{aligned} F(\phi_t^{v*}\omega) &= g_F(\phi_t^{v*}\omega, \varphi_1 \rangle, \ldots, \langle \phi_t^{v*}\omega, \varphi_N \rangle) \\ &= g_F(\langle \omega, \varphi_1 \circ \phi_t^v \rangle, \ldots, \langle \omega, \varphi_N \circ \phi_t^v \rangle) \end{aligned}$$

and, therefore, an application of Definition 7.2.1 gives

$$\begin{aligned} (\nabla_v^\Omega F)(\omega) &= \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle) \langle \omega, \nabla_v^X \varphi_i \rangle & (7.3) \\ &= \int_X \left\langle \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle) \nabla^X \varphi_i(x), v(x) \right\rangle_{T_x X} d\omega(x) \\ &= \langle \nabla^\Omega F(\omega, \cdot), v(\cdot) \rangle_{L^2(X \to TX, \omega)}, \end{aligned}$$

where $\nabla_v^X \varphi$ is the directional (or Lie) derivative of $\varphi : X \to \mathbb{R}$ along the vector field $v \in V_0(X)$, i.e.,

$$(\nabla_v^X \varphi)(x) = \langle \nabla^X \varphi(x), v(x) \rangle_{T_x X},$$

and ∇^X denotes the gradient on X.

The expression of ∇_v^Ω on smooth cylinder functions given by (7.3) motivates the following definition.

Definition 7.2.2 *We introduce the tangent space $T_\omega\Omega$ to the configuration space Ω at the point $\omega \in \Omega$ as the Hilbert space of measurable ω-square - integrable vector fields, $V_\omega : X \to TX$, with the scalar product*

$$\left\langle V_\omega^1, W_\omega \right\rangle_{T_\omega\Omega} = \int_X \langle V_\omega^1(x), V_\omega^2(x)\rangle_{T_xX} d\omega(x), \tag{7.4}$$

$V_\omega^1, W_\omega \in T_\omega\Omega$. The corresponding tangent bundle is

$$T\Omega := \bigcup_{\omega\in\Omega} T_\omega\Omega.$$

Correspondingly, the finitely based vector fields on $(\Omega, T\Omega)$ can be defined as

$$\Omega \ni \omega \mapsto \sum_{i=1}^{N} F_i(\omega)v_i \in C_0^\infty(X), \tag{7.5}$$

where $F_1, \ldots, F_N \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ and $v_1, \ldots, v_N \in V_0(X)$. We denote the collection of all such maps by $\mathcal{VF}C_0^\infty(\mathcal{D}, \Omega)$.

Let us stress that any $v \in V_0(X)$ can be considered as a "constant" vector field on Ω such that

$$\Omega \ni \omega \longmapsto V_\omega(\cdot) = v(\cdot) \in T_\omega\Omega,$$

$$\langle v, v\rangle_{T_\omega\Omega} = \int_X |v(x)|^2_{T_xX} d\omega(x) < \infty.$$

Usually in Riemannian geometry, having the directional derivative and a Hilbert space as the tangent space we can introduce the gradient.

Definition 7.2.3 *We define the intrinsic gradient of a function $F : \Omega \to \mathbb{R}$ as the mapping*

$$\Omega \ni \omega \longmapsto (\nabla^\Omega F)(\omega) \in T_\omega\Omega,$$

such that for any $v \in V_0(X)$

$$(\nabla_v^\Omega F)(\omega) = \langle(\nabla^\Omega F)(\omega), v\rangle_{T_\omega\Omega}. \tag{7.6}$$

Note that (7.6), in particular, implies that $\nabla_v^\Omega F$ is the directional derivative along the "constant" vector field v on Ω. Furthermore, by (7.3) for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ of the form (7.2) the gradient is given by

$$(\nabla^\Omega F)(\omega; x) = \sum_{i=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle\omega, \varphi_1\rangle, \ldots, \langle\omega, \varphi_N\rangle)\nabla^X \varphi_i(x),\ \omega \in \Omega, x \in X. \tag{7.7}$$

7.2.2 Integration by parts on compound Poisson space

Let the compound configuration space Ω be equipped with the compound Poisson measure π_σ^τ (cf. Section 6.1). The set $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ is a dense subset in the space $L^2(\Omega,\mathcal{B}(\Omega),\pi_\sigma^\tau) =: L^2(\pi_\sigma^\tau)$. For any vector field $v \in V_0(X)$ we have a differential operator in $L^2(\pi_\sigma^\tau)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ given by

$$\mathcal{F}C_b^\infty(\mathcal{D},\Omega) \ni F \longmapsto \nabla_v^\Omega F \in L^2(\pi_\sigma^\tau).$$

Our aim now is to compute the adjoint operator $\nabla_v^{\Omega *}$ in $L^2(\pi_\sigma^\tau)$. It corresponds, of course, to an integration by parts formula with respect to the measure π_σ^τ.

Definition 7.2.4 *For any $v \in V_0(X)$ we define the logarithmic derivative of the compound Poisson measure π_σ^τ along v as the following function on Ω:*

$$\Omega \ni \omega \mapsto B_v^{\pi_\sigma^\tau}(\omega) := \langle \gamma_\omega, \beta_v^\sigma \rangle = \int_X [\langle \beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x)] d\gamma_\omega(x). \tag{7.8}$$

A motivation for this definition is given by the following integration by parts formula.

Theorem 7.2.5 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ and any $v \in V_0(X)$ we have*

$$\int_\Omega (\nabla_v^\Omega F)(\omega) G(\omega) d\pi_\sigma^\tau(\omega) = \int_\Omega F(\omega)[-(\nabla_v^\Omega G)(\omega) - G(\omega) B_v^{\pi_\sigma^\tau}(\omega)] d\pi_\sigma^\tau(\omega), \tag{7.9}$$

or

$$\nabla_v^{\Omega *} = -\nabla_v^\Omega - B_v^{\pi_\sigma^\tau}, \tag{7.10}$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ in $L^2(\pi_\sigma^\tau)$.

Proof. Due to Proposition 7.1.1 we have that

$$\int_\Omega F(\phi_t^v(\omega)) G(\omega) d\pi_\sigma^\tau(\omega) = \int_\Omega F(\omega) G(\phi_{-t}^v \omega) d\pi_{\phi_t^{v*}\sigma}^\tau(\omega).$$

Differentiating this equation with respect to t and interchanging $\frac{d}{dt}$ with the integrals, by Definition 7.2.1 the left hand side becomes (7.9). To see that the right hand side also coincides with (7.9) we note that

$$\frac{d}{dt} G(\phi_{-t}^v(\omega))|_{t=0} = -(\nabla_v^\Omega G)(\omega),$$

and (by Proposition 7.1.2)

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{d\pi^\tau_{\phi_t^{v*}\sigma}}{d\pi^\tau_\sigma}(\omega)\right]\Bigg|_{t=0}\\
=\ &\frac{d}{dt}\left[\prod_{x\in\gamma_\omega}\frac{\rho(\phi_t^{v*}(x))}{\rho(x)}J_m^{\phi_t^v}(x)\right]\Bigg|_{t=0}\\
&+\frac{d}{dt}\left[\exp\left\{\lambda_\tau\int_X\left(1-\frac{\rho(\phi_t^{v*}(x))}{\rho(x)}J_m^{\phi_t^v}(x)\right)d\sigma(x)\right\}\right]\Bigg|_{t=0}.
\end{aligned}
$$

Using (5.33) and the formula $\frac{d}{dt}[J_m^{\phi_t^v}(x)]|_{t=0} = -\mathrm{div}^X v(x)$, the latter expressions becomes equal to

$$
\begin{aligned}
&- \sum_{x\in\gamma_\omega}[\langle\beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x)]\\
&+ \lambda_\tau\int_X[\langle\beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x)]d\sigma(x)\\
&= -\sum_{x\in\gamma_\omega}\beta_v^\sigma(x) + \lambda_\tau\int_X\beta_v^\sigma(x)d\sigma(x) = -B_v^{\pi_\sigma^\tau}(\omega),
\end{aligned}
$$

where we have used the equality

$$\int_X\beta_v^\sigma(x)d\sigma(x) = -\int_X(\nabla_v^{X*}1)(x)d\sigma(x) = 0.$$

This completes the proof. ■

Definition 7.2.6 *For a vector field*

$$V : \Omega \ni \omega \longmapsto V_\omega \in T_\omega\Omega$$

the intrinsic divergence $\mathrm{div}^\Omega_{\pi^\tau_\sigma}V$ *is defined via the duality relation*

$$\int_\Omega\langle V_\omega, (\nabla^\Omega F)(\omega)\rangle_{T_\omega\Omega}d\pi^\tau_\sigma(\omega) = -\int_\Omega F(\omega)(\mathrm{div}^\Omega_{\pi^\tau_\sigma}V)(\omega)d\pi^\tau_\sigma(\omega), \qquad (7.11)$$

for all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$*, provided it exists (i.e., provided*

$$F \longmapsto \int_\Omega\langle V_\omega, (\nabla^\Omega F)(\omega)\rangle_{T_\omega\Omega}d\pi^\tau_\sigma(\omega),$$

is continuous on $L^2(\pi^\tau_\sigma)$*).*

The existence of the divergence, of course, requires some smoothness of the vector field. A class of smooth vector fields on Ω for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 7.2.7 *For any vector field $V_\omega \in \mathcal{VFC}_b^\infty(\mathcal{D},\Omega)$ of the form*

$$V_\omega(x) = \sum_{j=1}^{N} G_j(\omega) v_j(x),\ \omega \in \Omega,\ x \in X, \tag{7.12}$$

we have

$$\begin{aligned}(\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) &= \sum_{j=1}^{N} (\nabla^\Omega_{v_j} G_j)(\omega) + \sum_{j=1}^{N} B^{\pi^\tau_\sigma}_{v_j}(\omega) G_j(\omega) \\ &= \sum_{j=1}^{N} \langle (\nabla^\Omega G_j)(\omega), v_j \rangle_{T_\omega\Omega} + \sum_{j=1}^{N} \langle \gamma, \beta^\sigma_{v_j} \rangle G_j(\omega). \end{aligned} \tag{7.13}$$

Proof. Due to the linearity of ∇^Ω it is sufficient to consider the case $N = 1$, i.e., $V_\omega(x) = G(\omega)v(x)$. Then for all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ and the Definition 7.2.6 we have

$$\begin{aligned}\int_\Omega (\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) &= -\int_\Omega \langle V_\omega, (\nabla^\Omega F)(\omega) \rangle_{T_\omega\Omega} d\pi^\tau_\sigma(\omega) \\ &= -\int_\Omega G(\omega) \langle v, (\nabla^\Omega F)(\omega) \rangle_{T_\omega\Omega} d\pi^\tau_\sigma(\omega) \\ &= -\int_\Omega G(\omega) (\nabla^\Omega_v F)(\omega) d\pi^\tau_\sigma(\omega) \\ &= -\int_\Omega (\nabla^{\Omega *}_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\ &= \int_\Omega (\nabla^\Omega_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\ &\quad + \int_\Omega B^{\pi^\tau_\sigma}_v(\omega) G(\omega) F(\omega) d\pi^\tau_\sigma(\omega), \end{aligned}$$

where we have used (7.10). Hence

$$\begin{aligned}(\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) &= (\nabla^\Omega_v G)(\omega) + B^{\pi^\tau_\sigma}_v(\omega) G(\omega) \\ &= \langle (\nabla^\Omega G)(\omega), v \rangle_{T_\omega\Omega} + \langle \gamma, \beta^\sigma_v \rangle G(\omega). \end{aligned}$$

■

In the next subsection we give an equivalent description via a "lifting rule" of the above differential objects on Ω.

7.2.3 A lifting of the geometry

In the consideration above we have constructed some objects related to the differential geometry of the space Ω. Now we present an interpretation of all the above formulas via a simple "lifting rule".

Any function $\varphi \in \mathcal{D}$ generates a (cylinder) function on Ω by the formula

$$L_\varphi(\omega) := \langle \omega, \varphi \rangle, \ \omega \in \Omega. \tag{7.14}$$

We will call L_φ the lifting of φ. As before any vector field $v \in V_0(X)$ can be considered as a vector field on Ω (the lifting of v) which we denote by L_v, see Definition 7.2.2. For $v, w \in V_0(X)$ formula (7.4) can be written as

$$\langle L_v, L_w \rangle_{T_\omega \Omega} = L_{\langle v,w \rangle_{TX}}(\omega), \tag{7.15}$$

i.e., the scalar product of lifting vector fields is computed as the lifting of the scalar product $\langle v(x), w(x) \rangle_{T_x X} = \varphi(x)$. This rule can be used as a definition of the tangent space $T_\omega \Omega$.

Formula (7.3) has now the following interpretation:

$$(\nabla_v^\Omega L_\varphi)(\omega) = L_{\nabla_v^X \varphi}(\omega), \ \omega \in \Omega,$$

and the gradient of L_φ is nothing but the lifting of the corresponding underlying gradient:

$$(\nabla^\Omega L_\varphi)(\omega) = L_{\nabla^X \varphi}(\omega).$$

As follows from (7.8) the logarithmic derivative $B_v^{\pi_\sigma^\tau} : \Omega \to \mathbb{R}$ is obtained via the same lifting procedure of the corresponding logarithmic derivative $\beta_v^\sigma : X \to \mathbb{R}$, namely,

$$B_v^{\pi_\sigma^\tau}(\omega) = L_{\beta_v^\sigma}(\gamma_\omega).$$

Or, equivalently, one has for the divergence of a lifted vector field:

$$\mathrm{div}_{\pi_\sigma^\tau}^\Omega (L_v) = L_{\mathrm{div}_\sigma^X v}.$$

7.3 Representations of the Lie algebra of vector fields

Using the property of quasi-invariance of the compound Poisson measure π_σ^τ we can define a unitary representation of the diffeomorphism group $\mathrm{Diff}_0(X)$

in the space $L^2(\pi_\sigma^\tau)$, see [GGV75]. Namely, for $\phi \in$Diff$_0(X)$ we define a unitary operator

$$(V_{\pi_\sigma^\tau}(\phi)F)(\omega) := F(\phi(\omega))\sqrt{\frac{d\pi_\sigma^\tau(\phi(\omega))}{d\pi_\sigma^\tau(\omega)}},\ F \in L^2(\pi_\sigma^\tau).$$

Then we have

$$V_{\pi_\sigma^\tau}(\phi_1)V_{\pi_\sigma^\tau}(\phi_2) = V_{\pi_\sigma^\tau}(\phi_1 \circ \phi_2),\ \phi_1, \phi_2 \in \mathrm{Diff}_0(X).$$

As in Subsection 7.2.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms $\phi_t^v, t \in \mathbb{R}$. It generates a one-parameter unitary group

$$V_{\pi_\sigma^\tau}(\phi_t^v) := \exp[itJ_{\pi_\sigma^\tau}(v)],\ t \in \mathbb{R}, \tag{7.16}$$

where $J_{\pi_\sigma^\tau}(v)$ denotes the self-adjoint generator of this group.

Proposition 7.3.1 *For any $v \in V_0(X)$ the following operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ holds:*

$$J_{\pi_\sigma^\tau}(v) = \frac{1}{i}\nabla_v^\Omega + \frac{1}{2i}B_v^{\pi_\sigma^\tau}. \tag{7.17}$$

Proof. Let $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ be given. Then differentiating the left hand side of (7.16) at $t = 0$ we get

$$\begin{aligned} \frac{d}{dt}(V_{\pi_\sigma^\tau}(\phi_t^v)F)(\omega)|_{t=0} &= \frac{d}{dt}F(\phi_t^v(\omega))|_{t=0} + F(\omega)\frac{1}{2}\frac{d}{dt}\left.\frac{d\pi_\sigma^\tau(\phi_t^v(\omega))}{d\pi_\sigma^\tau(\omega)}\right|_{t=0} \\ &= (\nabla_v^\Omega F)(\omega) + \frac{1}{2}F(\omega)B_v^{\pi_\sigma^\tau}(\omega), \end{aligned}$$

where we have used the form of the operator $V_{\pi_\sigma^\tau}(\phi_t^v)$, the definition of the directional derivative ∇_v^Ω and Theorem 7.2.5. On the other hand the same procedure on the right hand side of (7.16) produce $i(J_{\pi_\sigma^\tau}(v)F)(\omega)$. Hence the result of the proposition follows. ■

Remark 7.3.2 *More generally, one can study a family of self-adjoint operators $J(v)$, $v \in V_0(X)$, in a Hilbert space $\mathcal{H}$ which gives a representation of the Lie algebra $V_0(X)$ in the sense of the following commutation relation:*

$$[J(v_1), J(v_2)] = -iJ([v_1, v_2]) \tag{7.18}$$

(on a dense domain in $\mathcal{H}$), where $[v_1, v_2] = \langle v_1, \nabla v_2\rangle_{TX} - \langle v_2, \nabla v_1\rangle_{TX}$ is the Lie-bracket of the vector fields $v_1, v_2 \in V_0(X)$. In the case discussed, this relation is a direct consequence of (7.17). Thus, we have constructed a compound Poisson space representation of the Lie algebra $V_0(X)$.

Let us define, in addition, a unitary representation of the additive group $\mathcal{D}$ given by the formula

$$(U_{\pi_\sigma^\tau}(f)F)(\omega) := \exp(i\langle\omega, f\rangle)F(\omega),\ F \in L^2(\pi_\sigma^\tau),\ \omega \in \Omega,$$

for any $f \in \mathcal{D}$. As usual, the semi-direct product $\mathcal{G} := \mathcal{D} \wedge \mathrm{Diff}_0(X)$ of the groups $\mathcal{D}$ and $\mathrm{Diff}_0(X)$ is defined as the set of pairs (f, ϕ) with multiplication operation

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1),$$

see e.g., [GGV75]. Let us introduce for any element $(f, \phi) \in \mathcal{G}$ the following operator on $L^2(\pi_\sigma^\tau)$:

$$W_{\pi_\sigma^\tau}(f, \phi) := U_{\pi_\sigma^\tau}(f)V_{\pi_\sigma^\tau}(\phi).$$

These operators are unitary and form a representation of the group $\mathcal{G}$. If we introduce multiplication operators $\rho_{\pi_\sigma^\tau}(f)$, $f \in \mathcal{D}$, as self-adjoint operators on $L^2(\pi_\sigma^\tau)$ which are defined for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ by the formula

$$(\rho_{\pi_\sigma^\tau}(f)F)(\omega) := \langle\omega, f\rangle F(\omega),\ \omega \in \Omega,$$

then $U_{\pi_\sigma^\tau}(f) = \exp[i\rho_{\pi_\sigma^\tau}(f)]$ and the form of the multiplication in $\mathcal{G}$ implies

$$[\rho_{\pi_\sigma^\tau}(f), J_{\pi_\sigma^\tau}(v)] = i\rho_{\pi_\sigma^\tau}(\nabla_v^X f)$$

(on a dense domain in $L^2(\pi_\sigma^\tau)$) for all $f \in \mathcal{D}$, $v \in \mathrm{Diff}_0(X)$. We also have the relation $[\rho_{\pi_\sigma^\tau}(f_1), \rho_{\pi_\sigma^\tau}(f_2)] = 0$. The family of operators $J_{\pi_\sigma^\tau}(v), \rho_{\pi_\sigma^\tau}(f)$, $v \in V_0(X)$, $f \in \mathcal{D}$, thus forms a compound Poisson representation of an infinite-dimensional Lie algebra. In the particular case when $\tau = \varepsilon_1$ this representation is known as Lie algebra of currents in non relativistic quantum field theory, e.g., [GGPS74].

7.4 Dirichlet forms on compound Poisson space

7.4.1 Definition of the intrinsic Dirichlet form

We start by introducing some useful spaces of cylinder functions on Ω in addition to $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$. By $\mathcal{FP}(\mathcal{D},\Omega)$ we denote the set of all cylinder functions of the form (7.2) in which the generating function g_F is a polynomial on $\mathbb{R}^N$, i.e., $g_F \in \mathcal{P}(\mathbb{R}^N)$. Analogously, we define $\mathcal{F}C_p^\infty(\mathcal{D},\Omega)$, where now $g_F \in C_p^\infty(\mathbb{R}^N)$ (the set of all C^∞-functions f on $\mathbb{R}^N$ such that f and all its partial derivatives of any order are polynomially bounded).

We have obviously

$$\begin{aligned} \mathcal{F}C_b^\infty(\mathcal{D},\Omega) &\subset \mathcal{F}C_p^\infty(\mathcal{D},\Omega), \\ \mathcal{FP}(\mathcal{D},\Omega) &\subset \mathcal{F}C_p^\infty(\mathcal{D},\Omega), \end{aligned}$$

and these spaces are algebras with respect to the usual operations. The existence of the Laplace transform $l_{\pi_\sigma^\tau}(f)$, $f \in \mathcal{D}$, implies $\mathcal{F}C_p^\infty(\mathcal{D},\Omega) \subset L^2(\pi_\sigma^\tau)$.

Note that after the embedding $\Omega \hookrightarrow \mathcal{D}'$ (see Subsection 6.1) and a natural extension to $\mathcal{D}'$ the space $\mathcal{FP}(\mathcal{D},\mathcal{D}')$ is nothing but the well-known space of cylinder polynomials on $\mathcal{D}'$, see [BK95, Chapter 2].

Definition 7.4.1 *For $F, G \in \mathcal{F}C_p^\infty(\mathcal{D},\Omega)$ we introduce a pre-Dirichlet form as*

$$\mathcal{E}_{\pi_\sigma^\tau}^\Omega(F,G) = \int_\Omega \langle (\nabla^\Omega F)(\omega), (\nabla^\Omega G)(\omega)\rangle_{T_\omega\Omega} d\pi_\sigma^\tau(\omega). \tag{7.19}$$

Note that for $F, G \in \mathcal{F}C_p^\infty(\mathcal{D},\Omega)$ formula (7.7) is still valid and therefore

$$\left\langle \nabla^\Omega F, \nabla^\Omega G \right\rangle_{T\Omega} \in \mathcal{F}C_p^\infty(\mathcal{D},\Omega),$$

such that (7.19) is well-defined.

We will call $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$ the intrinsic pre-Dirichlet form corresponding to the compound Poisson measure π_σ^τ on Ω. The name "intrinsic" means that $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$ is associated with the geometry of Ω generated by the original Riemannian structure of X, in particular, by the intrinsic gradient ∇^Ω. In the next subsection we shall prove the closability of $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$.

7.4.2 Intrinsic Dirichlet operators

Let us introduce a differential operator $H^{\Omega}_{\pi^{\tau}_{\sigma}}$ on the domain $\mathcal{F}C^{\infty}_{b}(\mathcal{D},\Omega)$ which is given on any $F \in \mathcal{F}C^{\infty}_{b}(\mathcal{D},\Omega)$ of the form

$$F(\omega) = g_F(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle),\ \omega \in \Omega, g_F \in C^{\infty}_{b}(\mathbb{R}^N), \varphi_1, \ldots, \varphi_N \in \mathcal{D}, \tag{7.20}$$

by the formula

$$\begin{aligned}
&(H^{\Omega}_{\pi^{\tau}_{\sigma}} F)(\omega) \\
:=\ & \sum_{i,j=1}^{N} \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle) \int_X \langle \nabla^X \varphi_i(x), \nabla^X \varphi_j(x)\rangle_{T_xX} d\omega(x) \\
& - \sum_{i=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle) \int_X \Delta^X \varphi_i(x) d\omega(x) \qquad (7.21) \\
& - \sum_{i=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle) \int_X \langle \nabla^X \varphi_i(x), \beta^{\sigma}(x)\rangle_{T_xX} d\omega(x),
\end{aligned}$$

where Δ^X denotes the Laplace-Beltrami operator on X. In this formula all expressions are from $\mathcal{F}C^{\infty}_{b}(\mathcal{D},\Omega)$ or have the form $\langle \omega, \psi\rangle$, $\omega \in \Omega$, $\psi \in \mathcal{D}$, except for the functions $\langle \omega, h_i\rangle$, $\omega \in \Omega$, with

$$h_i(x) = \langle (\nabla^X \varphi_i)(x), \beta^{\sigma}(x)\rangle_{T_xX},\ x \in X, i = 1, \ldots, N.$$

To clarify the situation with these functions note that due to the assumption on σ we have $\rho^{1/2} \in H^{1,2}_{loc}(X)$ which gives $h_i \in L^1(\sigma)$ and these functions have compact supports. Therefore $h_i \in L^1(\sigma)$, $j = 1, \ldots, N$. On the other hand we know that a function $\langle \omega, f\rangle$, $\omega \in \Omega$, is from $L^2(\pi^{\tau}_{\sigma})$ if $f \in L^1(\sigma) \cap L^2(\sigma)$. The latter follows from the formula for the second moment of the measure π^{τ}_{σ}, namely

$$\int_{\Omega} \langle \omega, f\rangle^2 d\pi^{\tau}_{\sigma}(\omega) = m_2(\tau) \int_X f^2(x) d\sigma(x) + (m_1(\tau))^2 \left(\int_X f(x) d\sigma(x) \right)^2, \tag{7.22}$$

where $m_1(\tau)$ and $m_2(\tau)$ are the first and second moment of the measure τ on $\mathbb{R}$, respectively. Equation (7.22) is a direct consequence of (6.2). As a result the right hand side of (7.21) is well-defined. To show that the operator $H^{\Omega}_{\pi^{\tau}_{\sigma}}$ is well-defined we still have to show that its definition does not depend on the representation of F in (7.20) which will be done below.

Remark 7.4.2 *In the applications to the study of unitary representations of the group* $\mathrm{Diff}_0(X)$ *given by compound Poisson measures, there is usually an additional assumption on the smoothness of the density* $\rho := d\sigma/dm$*, namely* $\rho \in C^\infty(X)$, $\rho(x) > 0$, $x \in X$*, see e.g. [GGV75]. In this case it is obvious that the operator* $H^\Omega_{\pi^\tau_\sigma}$ *preserves the spaces* $\mathcal{F}C^\infty_p(\mathcal{D}, \Omega)$ *and* $\mathcal{F}\mathcal{P}(\mathcal{D}, \Omega)$.

Let us also consider the classical pre-Dirichlet form corresponding to the measure σ on X :

$$\mathcal{E}^X_\sigma(\varphi, \psi) := \int_X \langle \nabla^X \varphi(x), \nabla^X \psi(x)\rangle_{T_x X} d\sigma(x), \tag{7.23}$$

where $\varphi, \psi \in \mathcal{D}$. This form is associated with the Dirichlet operator H^X_σ which is given on $\mathcal{D}$ by

$$(H^X_\sigma \varphi)(x) := -\Delta^X \varphi(x) - \langle \beta^\sigma(x), \nabla^X \varphi(x)\rangle_{T_x X}, \tag{7.24}$$

and which satisfies

$$\mathcal{E}^X_\sigma(\varphi, \psi) = (H^X_\sigma \varphi, \psi)_{L^2(\sigma)}, \ \varphi, \psi \in \mathcal{D}.$$

Using the underlying Dirichlet operator H^X_σ we obtain the representation

$$\begin{aligned} &(H^\Omega_{\pi^\tau_\sigma} F)(\omega) \\ = \ &\sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle)\langle \nabla^X \varphi_i, \nabla^X \varphi_j\rangle_{T_\omega \Omega} \\ &+ \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle)\langle \omega, H^X_\sigma \varphi_i\rangle. \end{aligned}$$

Let us define for any $\omega \in \Omega$, $x, y \in X$

$$\begin{aligned} &(\nabla^\Omega \nabla^\Omega F)(\omega, x, y) \\ := \ &\sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle)\nabla^X \varphi_i(x) \otimes \nabla^X \varphi_j(y) \in T_\omega \Omega \otimes T_\omega \Omega. \end{aligned}$$

Then

$$\begin{aligned} \triangle^\Omega F(\omega) \ &:= \ \mathrm{Tr}(\nabla^\Omega \nabla^\Omega F)(\omega) \\ &= \ \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle)\langle \nabla^X \varphi_i, \nabla^X \varphi_j\rangle_{T_\omega \Omega}. \end{aligned}$$

Hence the operator $H^\Omega_{\pi^\tau_\sigma}$ can be written as

$$(H^\Omega_{\pi^\tau_\sigma}F)(\omega) = -(\Delta^\Omega F)(\omega) - \langle \omega, \mathrm{div}^X_\sigma(\nabla^\Omega F)(\omega;\cdot)\rangle.$$

The following theorem implies that both $H^\Omega_{\pi^\tau_\sigma}$ and Δ^Ω are well-defined as linear operators on $\mathcal{F}C^\infty_b(\mathcal{D},\Omega)$, i.e., independently of the representation of F on (7.20).

Theorem 7.4.3 *The operator $H^\Omega_{\pi^\tau_\sigma}$ is associated with the intrinsic Dirichlet form $\mathcal{E}^\Omega_{\pi^\tau_\sigma}$, i.e., for all $F, G \in \mathcal{F}C^\infty_b(\mathcal{D},\Omega)$ we have*

$$\mathcal{E}^\Omega_{\pi^\tau_\sigma}(F,G) = (H^\Omega_{\pi^\tau_\sigma}F, G)_{L^2(\pi^\tau_\sigma)},$$

or

$$H^\Omega_{\pi^\tau_\sigma} = -\mathrm{div}^\Omega_{\pi^\tau_\sigma}\nabla^\Omega \text{ on } \mathcal{F}C^\infty_b(\mathcal{D},\Omega).$$

We call $H^\Omega_{\pi^\tau_\sigma}$ the intrinsic Dirichlet operator of the measure π^τ_σ.

Proof. For any $F \in \mathcal{F}C^\infty_b(\mathcal{D},\Omega)$ of the form (7.20) we have

$$(\nabla^\Omega F)(\omega;x) = \sum_{j=1}^N \frac{\partial g_F}{\partial s_i}(\langle\omega,\varphi_1\rangle,\ldots,\langle\omega,\varphi_N\rangle)\nabla^X\varphi_i(x).$$

By (7.13) we conclude that

$$\mathrm{div}^\Omega_{\pi^\tau_\sigma}(\nabla^\Omega F) = -H^\Omega_{\pi^\tau_\sigma}F$$

which by (7.11) for $F, G \in \mathcal{F}C^\infty_b(\mathcal{D},\Omega)$ gives

$$\begin{aligned}(H^\Omega_{\pi^\tau_\sigma}F, G)_{L^2(\pi^\tau_\sigma)} &= -\int_\Omega \mathrm{div}^\Omega_{\pi^\tau_\sigma}(\nabla^\Omega F)(\omega)G(\omega)d\pi^\tau_\sigma(\omega)\\ &= \int_\Omega \big\langle(\nabla^\Omega F)(\omega),(\nabla^\Omega G)(\omega)\big\rangle_{T_\omega\Omega}d\pi^\tau_\sigma(\omega).\end{aligned}$$

■

Corollary 7.4.4 *The Dirichlet form $(\mathcal{E}^\Omega_{\pi^\tau_\sigma}, \mathcal{F}C^\infty_b(\mathcal{D},\Omega))$ is closable on the space $L^2(\pi^\tau_\sigma)$. Its closure $(\mathcal{E}^\Omega_{\pi^\tau_\sigma}, D(\mathcal{E}^\Omega_{\pi^\tau_\sigma}))$ is associated with a positive definite self-adjoint operator, the Friedrichs extension of $H^\Omega_{\pi^\tau_\sigma}$ which we also denote by $H^\Omega_{\pi^\tau_\sigma}$ (and its domain by $D(H^\Omega_{\pi^\tau_\sigma})$).*

Clearly, ∇^Ω also extends to $D(\mathcal{E}^\Omega_{\pi^\tau_\sigma})$. We denote this extension again by ∇^Ω.

Corollary 7.4.5 *Let*

$$F(\omega) = g_F(\langle\omega,\varphi_1\rangle,\ldots,\langle\omega,\varphi_N\rangle), \qquad \omega\in\Omega, g_F\in C_b^\infty(\mathbb{R}^N), \\ \varphi_1,\ldots,\varphi_N\in D(\mathcal{E}^\Omega_\sigma).$$

Then $F\in D(\mathcal{E}^\Omega_{\pi^\tau_\sigma})$ *and*

$$(\nabla^\Omega F)(\cdot) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle\cdot,\varphi_1\rangle,\ldots,\langle\cdot,\varphi_N\rangle)\nabla^X\varphi_i.$$

Proof. By approximation this is an immediate consequence of (7.7) and the fact that for all $1\le i\le N$

$$(m_1(\tau))^{-1}\int_\Omega \langle\omega,|\nabla^X\varphi_i|^2_{TX}\rangle d\pi^\tau_\sigma(\omega) = \mathcal{E}^X_\sigma(\varphi_i,\varphi_i).$$

■

Remark 7.4.6 *Of course the domain* $D(\mathcal{E}^\Omega_{\pi^\tau_\sigma})$ *of the Dirichlet form* $\mathcal{E}^\Omega_{\pi^\tau_\sigma}$ *is nothing but the Sobolev space* $H_0^{1,2}(\Omega,\pi^\tau_\sigma)$ *on* Ω *of order 1 in* $L^2(\Omega,\pi^\tau_\sigma)$*.*

7.5 Identification of the diffusion process on compound Poisson space

In this section we will prove the existence of a diffusion process corresponding to our Dirichlet form $(\mathcal{E}^\Omega_{\pi^\tau_\sigma},\mathcal{D}(\mathcal{E}^\Omega_{\pi^\tau_\sigma}))$. For a general theory of processes corresponding to Dirichlet forms we refer to [MR92, Chapter IV], see also [Fuk80].

After all our preparation and taking into account the general description of compound Poisson space given in Section 7.6 we will see that this process is nothing but a direct consequence ("lifting") of the corresponding process on the space of simple configurations Γ_X, see [AKR98a, Section 6] for a detailed description. Let us clarify this in more detail. After Section 7.6 there is an obvious identification between compound configurations $\omega\in\Omega$

and a marked configurations $(\gamma_\omega, m_\omega) \in \Omega_X^{\mathbb{R}}$ which gives the possibility to obtain an embedding from $L^2(\Gamma_X, \pi_\sigma)$ into $L^2(\Omega_X, \pi_\sigma^\tau)$, i.e.,

$$L^2(\Omega_X, \pi_\sigma^\tau) \ni F(\omega) = G(\gamma_\omega) \in L^2(\Gamma_X, \pi_\sigma).$$

Hence all operators acting in $L^2(\Gamma_X, \pi_\sigma)$, e.g., ∇^Γ, $\nabla^{\Gamma *}$, $H_{\pi_\sigma}^\Gamma$ etc. are applicable on $L^2(\Omega_X, \pi_\sigma^\tau)$ with respect to part of the variables. Moreover we have the following relation

$$(\nabla^\Omega F)(\omega) = (\nabla^\Gamma F)((\gamma_\omega, m_\omega)),\ \omega = (\gamma_\omega, m_\omega) \in \Omega,$$

for the intrinsic gradient, from which everything else follows, see below.

Let us consider a probability measure τ on $\mathbb{R}$ (or more general a probability measure on the space of marks M, cf. Section 7.6). In what follows we always identify any compound configuration $\omega \in \Omega$ (or in general marked configuration) with $(\gamma_\omega, m_\omega)$, i.e.,

$$\Omega \ni \omega \rightsquigarrow (\gamma_\omega, m_\omega).$$

From Subsection 7.1 we have for any diffeomorphism $\phi \in \mathrm{Diff}_0(X)$ its action on $(\gamma_\omega, m_\omega)$ is given by

$$\phi(\gamma_\omega, m_\omega) = (\phi(\gamma_\omega), m_\omega).$$

It follows from Proposition 7.1.2 and the assumption on τ that the Radon-Nikodym density of π_σ^τ and π_σ with respect to the group $\mathrm{Diff}_0(X)$ are equal, i.e., $p_\phi^{\pi_\sigma^\tau}(\omega) = p_\phi^{\pi_\sigma}(\gamma_\omega)$, where γ_ω corresponds to ω.

Let us compute the action of the gradient ∇^Γ on cylinder functions $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$. To this end let $v \in V_0(X)$ be a vector field on X with compact support and ϕ_t^v the corresponding flow. Then by definition we have

$$(\nabla_v^\Gamma F)((\gamma_\omega, m_\omega)) := \frac{d}{dt} F((\phi_t^v(\gamma_\omega), m_\omega))|_{t=0}.$$

On cylinder functions of the form

$$F((\gamma_\omega, m_\omega)) = g_F(\langle(\gamma_\omega, m_\omega), \varphi_1\rangle, \ldots, \langle(\gamma_\omega, m_\omega), \varphi_N\rangle),$$

where $\omega = (\gamma_\omega, m_\omega) \in \Omega$, $g_F \in C_b^\infty(\mathbb{R}^N)$, and $\varphi_1, \ldots \varphi_N \in \mathcal{D}$, the above definition gives

$$\sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle(\gamma_\omega, m_\omega), \varphi_1\rangle, \ldots, \langle(\gamma_\omega, m_\omega), \varphi_N\rangle)\langle(\gamma_\omega, m_\omega), \nabla_v^X \varphi_i\rangle.$$

Therefore the following equality on $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ (dense in $L^2(\pi_\sigma^\tau)$) of the directional derivatives holds (cf. (5.31) and (7.7))

$$(\nabla_v^\Omega F)(\omega) = (\nabla_v^\Gamma F)((\gamma_\omega, m_\omega)),$$

which implies the equality between the intrinsic gradients, i.e.,

$$(\nabla^\Omega F)(\omega) = (\nabla^\Gamma F)((\gamma_\omega, m_\omega)).$$

From these considerations on the intrinsic gradient we get a relation between the Dirichlet forms, namely

$$\begin{aligned}
\mathcal{E}_{\pi_\sigma^\tau}^\Omega(F,G) &= \int_\Omega \langle (\nabla^\Omega F)(\omega), (\nabla^\Omega G)(\omega)\rangle_{T_\omega\Omega} d\pi_\sigma^\tau(\omega) \\
&= \int_\Omega \langle (\nabla^\Gamma F)((\gamma_\omega, m_\omega)), (\nabla^\Gamma G)((\gamma_\omega, m_\omega))\rangle_{T_{\gamma_\omega}\Gamma} d\pi_\sigma^\tau(\omega) \\
&= \mathcal{E}_{\pi_\sigma^\tau}^\Gamma(F,G).
\end{aligned}$$

On the other hand the above relation between Dirichlet forms allowed us to derive easily the following relation for the intrinsic Dirichlet operators

$$(H_{\pi_\sigma^\tau}^\Omega F)(\omega) = (H_{\pi_\sigma,\gamma_\omega}^\Gamma F)(\gamma_\omega, m_\omega),\ F \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega),$$

where $H_{\pi_\sigma,\gamma_\omega}^\Gamma$ acts with respect to the variable γ_ω. Of course the corresponding semigroups (whose generators are $H_{\pi_\sigma^\tau}^\Omega$ and $H_{\pi_\sigma,\gamma_\omega}^\Gamma$) are related by

$$e^{-tH_{\pi_\sigma^\tau}^\Omega} = e^{-tH_{\pi_\sigma,\gamma_\omega}^\Gamma} \otimes \mathbf{1}_{m_\cdot},\ t > 0.$$

From this it follows that the process, Ξ_t, $t \geq 0$, on compound Poisson space (or in marked Poisson space) is nothing but the equilibrium process $X_t^{\gamma_\omega}$ (distorted Brownian motion on Ω) together with marks of the corresponding configuration, i.e.,

$$\Xi_t = \{X_t^{\gamma_\omega}, m_\omega\}.$$

For more detailed description of properties of the process $X_t^{\gamma_\omega}$ we refer to [AKR98a, Section 6].

7.6 Marked Poisson measures

In this section we present some general results on marked Poisson measures which generalizes the compound Poisson measures introduced in Section 6.1 as well as the construction of measures on configuration spaces from Section 5.1. For more detailed information on marked Poisson processes we refer to [BL95, Chapter 6], [Kin93, Chapter 5], [MM91], and references therein.

7.6.1 Definitions and measurable structure

In this subsection we describe the space of marked configurations as well as its associated measurable structure. Before we recall the definition of the configuration space over the Cartesian space $X \times M$ between a Riemannian manifold X and a complete separable metric space M.

The configuration space $\Gamma_{X\times M}$ over the Cartesian product $X \times M$ is defined as the set of all locally finite subsets (configurations) in $X \times M$:

$$\Gamma_{X\times M} := \{\hat{\gamma} \subset X \times M \,|\, |\hat{\gamma} \cap K| < \infty \text{ for any compact } K \subset X \times M\}.$$

Let us now introduce the space of marked configurations which will plays the same role as Γ_X played for Poisson measure but now for the marked Poisson measure, see Subsection 7.6.3 below. It is defined as

$$\Omega_X^M := \{\omega = (\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_X, m_\omega \in M^\omega\}.$$

Here M^ω stands for the set of all maps $\gamma_\omega \ni x \mapsto m_x \in M$. We may also write the marked configuration space Ω_X^M as a subspace of $\Gamma_{X\times M}$ as follows

$$\Omega_X^M := \{\omega = \{(x, m_x)\} \subset \Gamma_{X\times M} | \{x\} = \gamma_\omega \in \Gamma_X, m_x \in M\}.$$

For any $\Lambda \subset X$ we define in an analogous way the set Ω_Λ^M, i.e.,

$$\Omega_\Lambda^M := \{\omega = (\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_\Lambda, m_\omega \in M^\omega\},$$

and

$$\Omega_\Lambda^M := \{\omega = \{(x, m_x)\} \subset \Gamma_{\Lambda\times M} | \{x\} = \gamma_\omega \in \Gamma_\Lambda, m_x \in M\}.$$

In order to describe the σ-algebra $\mathcal{B}(\Omega_X^M)$ we proceed as follows. Let $\Lambda \in \mathcal{O}_c(X)$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be given. We define an equivalent relation $\sim$ on $(\Lambda \times M)^n$ setting

$$((x_1, m_{x_1}), (x_2, m_{x_2}), \ldots, (x_n, m_{x_n})) \sim ((y_1, m_{y_1}), (y_2, m_{y_2}), \ldots, (y_n, m_{y_n}))$$

iff there exists a permutation $\pi \in \mathfrak{S}_n$ (the group of permutations of n elements) such that

$$(x_i, m_{x_i}) = (y_{\pi(i)}, m_{y_{\pi(i)}}),\ \forall i = 1, \ldots, n.$$

Hence we obtain the quotient space $(\Lambda \times M)^n/\mathfrak{S}_n$ by means of $\sim$. Then we introduce the subset of $(\Lambda \times M)^n/\mathfrak{S}_n$, $\Omega_\Lambda^M(n)$, defined as follows

$$\Omega_\Lambda^M(n) := \{((x_1, m_{x_1}), \ldots, (x_1, m_{x_1})) | x_i \in \Lambda, x_i \neq x_j,\ i \neq j, m_{x_i} \in M\},$$

or equivalently

$$\Omega_\Lambda^M(n) := \{(\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_\Lambda^{(n)}, m_\omega \in M^\omega\},\ \Omega_\Lambda^M(0) := \{\emptyset\}$$

The space $\Omega_\Lambda^M(n)$ is endowed with the relative metric from $(\Lambda \times M)^n / \mathfrak{S}_n$, i.e.,

$$\delta([x],[y]) = \inf_{x' \in [x], y' \in [y]} d^n(x', y'),$$

where d^n is the metric defined on $(\Lambda \times M)^n$ driven from the original metrics on X and M. Therefore $\Omega_\Lambda^M(n)$ becomes a metrizable topological space.

It is obvious that

$$\Omega_\Lambda^M = \bigsqcup_{n=0}^{\infty} \Omega_\Lambda^M(n).$$

This space can be equipped with the topology of disjoint union of topological spaces, namely, the strongest topology on Ω_Λ^M such that all the embeddings

$$i_n : \Omega_\Lambda^M(n) \hookrightarrow \Omega_\Lambda^M,\ n \in \mathbb{N}_0,$$

are continuous. $\mathcal{B}(\Omega_\Lambda^M)$ stands for the corresponding Borel σ-algebra.

For any $\Lambda \in \mathcal{O}_c(X)$ there are natural restriction maps

$$p_\Lambda : \Omega_X^M \to \Omega_\Lambda^M,$$

defined by

$$p_\Lambda(\gamma_\omega, m_\omega) := (\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda}) \in \Omega_\Lambda^M,\ (\gamma_\omega, m_\omega) \in \Omega_X^M. \tag{7.25}$$

The topology on Ω_X^M is defined as the weakest topology making all the mappings p_Λ continuous. The associated Borel σ-algebra is denoted by $\mathcal{B}(\Omega_X^M)$.

7.6.2 The projective limit

Finally we want to show that Ω_X^M coincides with the projective limit of the family of topological spaces $\{\Omega_\Lambda^M | \Lambda \in \mathcal{O}_c(X)\}$. First we recall the definition of projective limit of topological spaces, see e.g., [BD68, Chapter 3] and [Sch71, Chapter 2].

Definition 7.6.1 *Let $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ be given with $\Lambda_1 \subset \Lambda_2$. There are natural maps*

$$p_{\Lambda_2,\Lambda_1} : \Omega_{\Lambda_2}^M \to \Omega_{\Lambda_1}^M$$

defined by

$$p_{\Lambda_2,\Lambda_1}(\gamma_\omega, m_\omega) := (\gamma_\omega \cap_{\Lambda_1}, m_{\omega|\gamma_\omega\cap\Lambda_1}) \in \Omega^M_{\Lambda_1},\ (\gamma_\omega, m_\omega) \in \Omega^M_{\Lambda_2}.$$

The projective limit of the family $\{\Omega^M_\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ *denoted by*

$$\operatorname*{prlim}_{\Lambda\in\mathcal{O}_c(X)} \Omega^M_\Lambda,$$

is a topological space Ω *and a family of continuous projections*

$$P_\Lambda : \Omega \to \Omega^M_\Lambda,\ \Lambda \in \mathcal{O}_c(X),$$

such that the following two conditions are satisfied:

1. *If* $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ *with* $\Lambda_1 \subset \Lambda_2$*, then*

$$P_{\Lambda_1} = p_{\Lambda_2,\Lambda_1} \circ P_{\Lambda_2}.$$

2. *If* Ω' *is a topological space and*

$$P'_\Lambda : \Omega' \to \Omega^M_\Lambda,\ \Lambda \in \mathcal{O}_c(X),$$

a family of continuous projections which fulfills condition 1 above, then there exists a unique continuous map $u : \Omega' \to \Omega$ *such that* $P'_\Lambda = P_\Lambda \circ u$*, for all* $\Lambda \in \mathcal{O}_c(X)$*.*

Remark 7.6.2 *The projective limit of the family* $\{\Omega^M_\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ *exists and is unique in the following sense: let* Ω *and* Ω' *be projective limits, then there exists a map* $u : \Omega \to \Omega'$ *such that* u *and* u^{-1} *are continuous, see e.g., [Par67, Chap. 4].*

Theorem 7.6.3 *The space of marked configurations* Ω^M_X *is the projective limit of the family* $\{\Omega^M_\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ *together with the family of projections* $\{p_\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ *(defined in (7.25)) and*

$$\mathcal{B}(\Omega^M_X) = \sigma(p_\Lambda^{-1}(\mathcal{B}(\Omega^M_\Lambda)); \Lambda \in \mathcal{O}_c(X)).$$

In other words there exists a bicontinuous bijective mapping between Ω^M_X *and the projective limit. This will be denoted by* $\Omega^M_X \simeq \operatorname{pr}\lim_{\Lambda\in\mathcal{O}_c(X)} \Omega^M_\Lambda$*.*

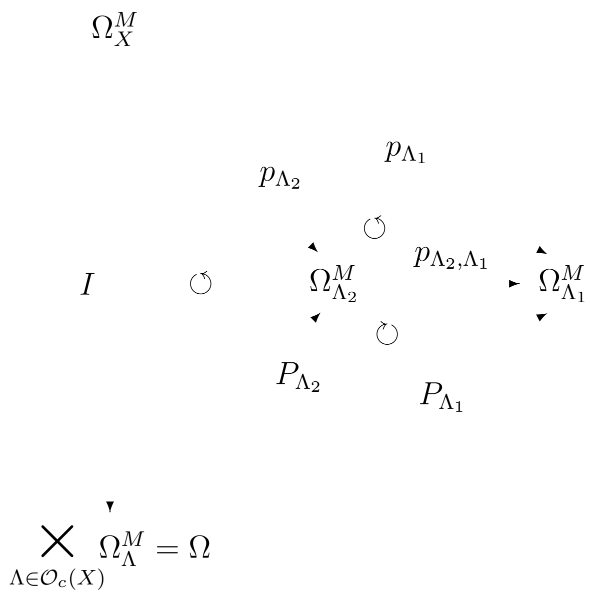

Figure 7.1: Diagram used in constructing projective limit.

Proof. We will use always in the proof the convention that $\Lambda, \Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ with $\Lambda_1 \subset \Lambda_2$. First we verify condition 1 of Definition 7.6.1. This can easily be done as follows

$$\begin{aligned} p_{\Lambda_2,\Lambda_1} \circ p_{\Lambda_2}(\gamma_\omega, m_\omega) &= p_{\Lambda_2,\Lambda_1}(\gamma_\omega \cap \Lambda_2, m_{\omega|\gamma_\omega\cap\Lambda_2}) \\ &= (\gamma \cap \Lambda_1, m_{\omega|\gamma_\omega\cap\Lambda_1}) \\ &= p_{\Lambda_1}(\gamma_\omega, m_\omega), \end{aligned}$$

for $(\gamma_\omega, m_\omega) \in \Omega_X^M$ which is the desired result.

Let us now construct a version of the projective limit of the family $\{\Omega_\Lambda^M, \Lambda \in \mathcal{O}_c(X)\}$, see e.g., [Par67]. As Ω we take

$$\Omega := \left\{ \omega \in \bigtimes_{\Lambda\in\mathcal{O}_c(X)} \Omega_\Lambda^M | p_{\Lambda_2,\Lambda_1}((\omega)_{\Lambda_2}) = (\omega)_{\Lambda_1} \right\},$$

where $(\omega)_\Lambda$ denotes the Λ-component of ω. We define the projections P_Λ by $P_\Lambda(\omega) := (\omega)_\Lambda$ for any $\omega \in \Omega$ and define the σ-algebra as $\mathcal{B}(\Omega) := \sigma(\{P_\Lambda | \Lambda \in \mathcal{O}_c(X)\})$, see diagram in Figure 7.1.

Now we define a (bijective) mapping $I : \Omega_X^M \to \Omega$ by

$$(I(\gamma_\omega, m_\omega))_\Lambda := p_\Lambda(\gamma_\omega, m_\omega) \in \Omega_\Lambda^M,\ (\gamma_\omega, m_\omega) \in \Omega_X^M, \Lambda \in \mathcal{O}_c(X).$$

We first show that I is well defined, this means that

$$p_{\Lambda_2,\Lambda_1}((I(\gamma_\omega, m_\omega))_{\Lambda_2}) = (\gamma_\omega \cap \Lambda_1, m_{\omega|\gamma_\omega\cap\Lambda_1}).$$

Indeed we have

$$\begin{aligned} p_{\Lambda_2,\Lambda_1}((I(\gamma_\omega, m_\omega))_{\Lambda_2}) &= p_{\Lambda_2,\Lambda_1} \circ p_{\Lambda_2}(\gamma_\omega, m_\omega) \\ &= p_{\Lambda_1}(\gamma_\omega, m_\omega) \\ &= (\gamma_\omega \cap \Lambda_1, m_{\omega|\gamma_\omega\cap\Lambda_1}) \end{aligned}$$

which proves that I is well defined. Let us prove in addition that I is a bijective mapping between Ω_X^M and Ω.
Injectivity. Let $(\gamma_\omega, m_\omega)$, $(\gamma_{\omega'}, m_{\omega'}) \in \Omega_X^M$ such that $I(\gamma_\omega, m_\omega) = I(\gamma_{\omega'}, m_{\omega'})$. That means by definition of I that $(\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega\cap\Lambda}) = (\gamma_{\omega'} \cap \Lambda, m_{\omega'|\gamma_{\omega'}\cap\Lambda})$ for all $\Lambda \in \mathcal{O}_c(X)$. Since the manifold X can be written as countable union of sets from $\mathcal{O}_c(X)$, i.e.,

$$X = \bigcup_{n\in\mathbb{N}_0} \Lambda_n,\ \Lambda_n \in \mathcal{O}_c(X), n \in \mathbb{N}_0$$

this implies that $(\gamma_\omega, m_\omega) = (\gamma_{\omega'}, m_{\omega'})$ and therefore the injectivity of I is proved.
Surjectivity. Let $\omega = ((\omega)_\Lambda)_{\Lambda\in\mathcal{O}_c(X)} = ((\gamma_\omega^*, m_\omega^*)_\Lambda)_{\Lambda\in\mathcal{O}_c(X)} \in \Omega$ be given and take a family of pairewise disjoint subsets from $\mathcal{O}_c(X)$, $\{\Lambda_n, n \in \mathbb{N}\}$ such that

$$X = \bigsqcup_{n\in\mathbb{N}} \Lambda_n;$$

moreover we may assume that for any $\Lambda \in \mathcal{O}_c(X)$ $\exists m \in \mathbb{N}$ such that

$$\Lambda \subset \bigsqcup_{n=1}^{m} \Lambda_n =: \Lambda_n^m. \tag{7.26}$$

Let us define an element (γ, m_γ) from Ω_X^M as follows:

$$\gamma_\omega := \bigcup_{n\in\mathbb{N}} (\gamma_\omega^* \cap \Lambda_n)$$

and

$$m_\omega : \bigcup_{n\in\mathbb{N}} (\gamma_\omega^* \cap \Lambda_n) \ni x \mapsto m_x \in M,\ x \in \gamma_\omega^* \cap \Lambda_n, n_0 \in \mathbb{N}.$$

First we note that the assumption (7.26) gives

$$\gamma_\omega \cap \Lambda = \bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda), \ \Lambda \in \mathcal{O}_c(X).$$

Secondly we must prove that $(I(\gamma_\omega, m_\omega))_\Lambda = (\omega)_\Lambda$ for any $\Lambda \in \mathcal{O}_c(X)$. From the definition of I and p_Λ we have

$$(I(\gamma_\omega, m_\omega))_\Lambda := p_\Lambda(\gamma_\omega, m_\omega) := (\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda})$$

and from the above representation for $\gamma_\omega \cap \Lambda$ we obtain

$$\begin{aligned}
(\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda}) &= \left(\bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda), m_{\omega|\gamma_\omega \cap \Lambda} \right) \\
&= \bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda, m_{\omega|\gamma_\omega^* \cap \Lambda_n \cap \Lambda}) \\
&= \bigcup_{n=1}^{m} p_{\Lambda_n \cap \Lambda}(\gamma_\omega^*, m_\omega) = \bigcup_{n=1}^{m} p_\Lambda(\gamma_\omega^* \cap \Lambda_n, m_{\omega|\gamma_\omega^* \cap \Lambda_N}) \\
&= p_\Lambda(\gamma_\omega^* \cap \Lambda_n^m, m_{\omega|\gamma_\omega^* \cap \Lambda_n^m}) = (\gamma_\omega^* \cap \Lambda, m_{\omega|\gamma_\omega^* \cap \Lambda}) \\
&= (\omega)_\Lambda
\end{aligned}$$

which proves the surjectivity of I.

Taking into account condition 1 of Definition 7.6.1 it follows that I is continuous. Hence only remained to proof that $\mathcal{B}(\Omega_X^M)$ and $\mathcal{B}(\Omega)$ coincide. This is an immediate consequence of the definition of the σ-algebras and the continuity of I. ■

7.6.3 Marked Poisson measure

The underlying manifold X is endowed with a non-atomic Radon measure σ, (cf. Section 5.1). Let a probability measure τ be given on the space M. The space $\hat{X} := X \times M$ is endowed with the product measure between σ and τ denoted by $\hat{\sigma}$, i.e., $\hat{\sigma} := \sigma \otimes \tau$.

Remark 7.6.4 *For σ-additive measure τ we may also allowed that $\tau(M) = \infty$ by an monotone argument.*

The measure $\hat{\sigma}^{\otimes n}$ can be considered as a finite measure on $(\Lambda \times M)^n$ for any $\Lambda \in \mathcal{O}_c(X)$ which induces on $\Omega_\Lambda^M(n)$ the measure

$$\hat{\sigma}_{\Lambda,n} = \frac{1}{n!}\hat{\sigma}^{\otimes n},\ n \geq 0,\ \hat{\sigma}_{\Lambda,0}(\emptyset) := 1.$$

Then we consider a measure $\lambda_{\hat{\sigma}}^\Lambda$ on Ω_Λ^M which coincides on each $\Omega_\Lambda^M(n)$ with the measure $\hat{\sigma}_{\Lambda,n}$ as follows

$$\lambda_{\hat{\sigma}}^\Lambda := \sum_{n=0}^{\infty} \hat{\sigma}_{\Lambda,n} = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\sigma}^{\otimes n}.$$

The measure $\lambda_{\hat{\sigma}}^\Lambda$ is finite on Ω_Λ^M and $\lambda_{\hat{\sigma}}^\Lambda(\Omega_\Lambda^M) = e^{\sigma(\Lambda)}$, therefore we define a probability measure $\mu_{\hat{\sigma}}^\Lambda$ on Ω_Λ^M setting

$$\mu_{\hat{\sigma}}^\Lambda := e^{-\sigma(\Lambda)}\lambda_{\hat{\sigma}}^\Lambda. \tag{7.27}$$

The measure $\mu_{\hat{\sigma}}^\Lambda$ has the following property

$$\mu_{\hat{\sigma}}^\Lambda(\Omega_\Lambda^M(n)) = \frac{1}{n!}\sigma^{\otimes n}(\Lambda)e^{-\sigma(\Lambda)},$$

which gives the probability of the occurrence of exactly n points of the marked Poisson process (with arbitrary values of marks) inside the volume Λ.

In order to obtain the existence of a unique probability measure $\mu_{\hat{\sigma}}$ on $\mathcal{B}(\Omega_X^M)$ such that

$$\mu_{\hat{\sigma}}^\Lambda = p_\Lambda^* \mu_{\hat{\sigma}},\ \Lambda \in \mathcal{O}_c(X),$$

on should check the consistency property of the family $\{\mu_{\hat{\sigma}}^\Lambda | \Lambda \in \mathcal{O}_c(X)\}$. In other words one should verify the following equality of measures, see diagram in Figure 7.2.

$$\mu_{\hat{\sigma}}^{\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1} = \mu_{\hat{\sigma}}^{\Lambda_1},\ \Lambda_1, \Lambda_2 \in \mathcal{O}_c(X), \Lambda_1 \subset \Lambda_2. \tag{7.28}$$

It is known, see e.g., [GGV75], that the σ-algebra $\mathcal{B}(\Omega_\Lambda^M)$ coincides with the σ-algebra generated by the cylinder sets from Ω_Λ^M, $C_{B,n}^\Lambda$, $B \in \mathcal{O}_c(\Lambda)$, $n \in \mathbb{N}_0$. Here $C_{B,n}^\Lambda$ has the following representation

$$C_{B,n}^\Lambda := \{\omega = (\gamma_\omega, m_\omega) \in \Omega_\Lambda^M || \gamma_\omega \cap B| = n\}.$$

Hence for a given $B \in \mathcal{O}_c(\Lambda_1)$, $n \in \mathbb{N}_0$ the pre-image under $p_{\Lambda_2,\Lambda_1}$of the cylinder set $C_{B,n}^{\Lambda_1}$ from $\Omega_{\Lambda_1}^M$ is a cylinder set from $\Omega_{\Lambda_2}^M$, i.e.,

$$p_{\Lambda_2,\Lambda_1}^{-1}(C_{B,n}^{\Lambda_1}) = \{\omega = (\gamma_\omega, m_\omega) \in \Omega_{\Lambda_2}^M || \gamma_\omega \cap B| = n\} = C_{B,n}^{\Lambda_2}.$$

$$(\Omega_X^M, \mu_{\hat\sigma})$$

$$p_{\Lambda_2} \qquad \circlearrowleft \qquad p_{\Lambda_1}$$

$$(\Omega_{\Lambda_2}^M, \mu_{\hat\sigma}^{\Lambda_2}) \xrightarrow{p_{\Lambda_2,\Lambda_1}} (\Omega_{\Lambda_1}^M, \mu_{\hat\sigma}^{\Lambda_1})$$

Figure 7.2: Diagram used to prove the consistency property of the family $\{\mu_{\hat\sigma}^{\Lambda} | \Lambda \in \mathcal{O}_c(X)\}$.

On the other hand it is well known, see e.g., [AKR98a] and [Shi94] that

$$\mu_{\hat\sigma}^{\Lambda_2}(C_{B,n}^{\Lambda_2}) = \frac{1}{n!}\sigma^{\otimes n}(B)e^{-\sigma(B)},$$

which is the same as $\mu_{\hat\sigma}^{\Lambda_1}(C_{B,n}^{\Lambda_1})$. Therefore the consistency property (7.28) is proved.

It is possible to compute in closed form the Laplace transform of the measure $\mu_{\hat\sigma}$. Let f be a continuous function on $\hat X$ such that the supp$f \subset \Lambda \times M$ for some $\Lambda \in \mathcal{O}_c(X)$. Let $\omega = (\gamma_\omega, m_\omega)$ be an element of Ω_X^M and define the pairing between f and ω by

$$\langle f, \omega \rangle := \sum_{x \in \gamma_\omega} f(x, m_x).$$

Then we have

$$\int_{\Omega_X^M} e^{\langle f,\omega\rangle} d\mu_{\hat\sigma}(\omega) = \int_{\Omega_\Lambda^M} e^{\langle f,\omega\rangle} d\mu_{\hat\sigma} \circ p_\Lambda^{-1}(\omega) = \int_{\Omega_\Lambda^M} e^{\langle f,\omega\rangle} d\mu_{\hat\sigma}^{\Lambda}(\omega).$$

Using (7.27) the last integral is equal to

$$\begin{aligned} & e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Lambda\times M)^n} \exp\left(\sum_{k=0}^{n} f(x_k, m_{x_k})\right) d\hat\sigma(x_1, m_{x_1}) \dots d\hat\sigma(x_n, m_{x_n}) \\ = \; & e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{\Lambda\times M} e^{f(x,m_x)} d\hat\sigma(x, m_x)\right)^n \\ = \; & \exp\left(\int_{\hat X} (e^{f(x,m_x)} - 1) d\hat\sigma(x, m_x)\right). \end{aligned}$$

That is, for any f in the above conditions the following formula holds:

$$l_{\mu_{\hat{\sigma}}}(f) = \int_{\Omega_X^M} e^{\langle f,\omega\rangle} d\mu_{\hat{\sigma}}(\omega) = \exp\left(\int_{\hat{X}} (e^{f(x,m_x)} - 1) d\hat{\sigma}(x, m_x)\right).$$

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