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MULTIPLE INTERSECTION LOCAL TIMES IN TERMS OF WHITE NOISE

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We show how multiple intersections of Brownian motion can be expressed in terms of generalized white noise functionals. We also calculate the kernels of their chaos expansions and discuss their L^2 properties.

1. Introduction

The intersections of the Wiener process have been under study (at least) since 1940.[10] In 1957 Dvoretzky *et al.*[3] proved that, with probability one, Brownian paths in three-dimensional space have no triple points; in 1978 Wolpert,[15] studied some intersection properties of independent Wiener processes in the plane (e.g., the dimension of the set of intersections). He also constructed a functional that measured the extent to which the trajectories of k processes intersect. In Ref. 12 Rosen gave a generalization of Varadhan's formula for $n = 2$, presenting a simple prescription for "renormalizing" the local time for n-fold intersections of planar Brownian motion. He also presented a new proof for the joint continuity of the renormalized local time. In the same year (1986) Le Gall[9] approximated the intersections of Brownian motions by the intersections of "Wiener sausages" and used this approximation to prove Taylor's conjecture related to the Hausdorff measure of the set of multiple points of planar Brownian motions. In 1988, Dynkin,[4] made a study of a class of random fields associated with multiple points of a random walk in the plane. Shieh[13] used Hida's theory[6] of generalized Brownian functionals to establish a stochastic integral formula concerning the multiple intersection local times of planar Brownian motion. Within the same framework de Faria *et al.*[2] calculated the chaos — or multiple Wiener integral — expansion for a regularized form of the local time of simple self intersections of d-dimensional Brownian motion. H. Watanabe[14] has shown that as d increases, successive truncations (omission of lowest order chaos) of the expansion are sufficient to ensure that the truncated local time is a generalized

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functional of Brownian motion. In Ref. 2 one also finds its limiting form as the regularization is removed. The effectiveness of these truncations comes from the fact that the kernel functions of increasing order are less and less singular in the L^1 sense. On the other hand, if one studies the weak limit of the suitably regularized local time, as in Refs. 1 and 16, one finds[1] that (in the L^2 sense) all kernels are equally singular and contribute to the limit.

Here we study the corresponding expression for intersections of higher order k. For arbitrary dimension d and order of intersections k we determine the number $r = r(k,d)$ of truncations sufficient to obtain a generalized functional of Brownian motion and calculate the kernels of the chaos expansion. We further show the L^2 property for all kernels in dimension one and two; multiplicative renormalization in higher dimensions will be studied elsewhere.[5]

2. Notation and Background from White Noise Analysis

We consider a d-tuple of independent Gaussian white noises $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^d)$, correspondingly, d-tuples of Schwartz test functions $\mathbf{f} = (f_1, \ldots, f_d) \in S(R, R^d)$, and use the following notation:

$$\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{N}^d, \qquad n = \sum_{i=1}^{d} n_i, \qquad \mathbf{n}! = \prod_{i=1}^{d} n_i!\,,$$

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^{d} \int_{\mathbf{R}} dt f_i^2(t)\,,$$

$$\langle F_{\mathbf{n}}, \mathbf{f}^{\otimes \mathbf{n}} \rangle = \int_{\mathbf{R}^n} d^n t F_{\mathbf{n}}(t_1, \ldots, t_n) \cdot \bigotimes_{i=1}^{d} f_i^{\otimes n_i}(t_1, \ldots, t_n)$$

and similarly for $\langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle$, where

$$: \boldsymbol{\omega}^{\otimes \mathbf{n}} := \bigotimes_{i=1}^{d} : \omega_i^{\otimes n_i} : .$$

For elements of the Hilbert space (L^2) of square integrable functions F we have the chaos expansion

$$F(\omega) = \sum_{\mathbf{n}=0}^{\infty} \langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle \tag{1}$$

with kernel functions $F_{\mathbf{n}}$ in Fock space.

Generalized functions are obtained via a Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*\,,$$

where (S) is constructed by considering only rapidly converging sequences of smooth kernel functions, for details see Ref. 6 or 7.

The generalized functionals in $(S)^*$ are conveniently characterized by their action on "Wick exponentials"

$$: \exp(\langle \boldsymbol{\omega}, \mathbf{f} \rangle) :\equiv \sum_{\mathbf{n}=0}^{+\infty} \frac{1}{\mathbf{n}!} \langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, \mathbf{f}^{\otimes \mathbf{n}} \rangle \, .$$

For this we make

Definition 2.1. The transformation defined for all test functions $\mathbf{f} \in S(R, R^d)$ via the bilinear dual product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $(S)^* \times (S)$ by

$$(S\Phi)(\mathbf{f}) \equiv \langle\!\langle \Phi, : \exp(\langle \cdot, \mathbf{f} \rangle) : \rangle\!\rangle$$

is called the S-transform of Φ.

The multilinear expansion of $S(\Phi)$

$$(\mathrm{S}\Phi)(\mathbf{f}) = \sum_{\mathbf{n}=0}^{\infty} \langle \varphi_{\mathbf{n}}, \mathbf{f}^{\otimes \mathbf{n}} \rangle$$

extends the chaos expansion to $\Phi \in (S)^*$, with distribution valued kernels $\varphi_{\mathbf{n}}$, such that

$$\langle\!\langle \Phi, F \rangle\!\rangle = \sum_{\mathbf{n}=0}^{\infty} \mathbf{n}! \langle \varphi_{\mathbf{n}}, F_{\mathbf{n}} \rangle. \tag{2}$$

The S-transforms of generalized functionals are "U-functionals" in the sense of the following:

Definition 2.2. A mapping G: $S(R, R^d) \to C$ is called a "U-functional" if

(a) $G(\lambda \mathbf{f}_1 + \mathbf{f}_2)$ has an entire extension to $\lambda \in C$ for any pair $\mathbf{f}_i \in S(R, R^d)$ of test functions, and
(b) $|G(z\mathbf{f}_i)| \leq C_1 \exp(C_2 |z^2 B(\mathbf{f}_i)|)$ with $C_i > 0$ and B a continuous quadratic form on $S(R, R^d)$.

This in fact allows for a complete characterization of generalized functionals $\Phi \in (S)^*$ admitting the formal decomposition

$$\Phi(\omega) = \sum_{\mathbf{n}=0}^{+\infty} \langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, \varphi_{\mathbf{n}} \rangle$$

with symmetric generalized function kernels $F_{\mathbf{n}}$, extending (1) in the sense of (2). The following is useful to control the space $(S)^*$ of generalized functions.

Theorem 2.1. *The following are equivalent*:

(a) *G is a U-functional.*
(b) *G is the S-transform of a generalized functional $\Phi \in (S)^*$.*

For a proof in the framework of general Gaussian spaces see Ref. 7 where one also finds the following corollaries which we shall use below. The first one concerns the convergence of sequences:

Corollary 2.1. *Let (Ω, B, m) be a measure space, and Φ_λ a mapping defined on Ω with values in $(S)^*$. We assume that the S-transform of Φ_λ*

(a) *is an m-measurable function of λ for any test function vector $\mathbf{f} \in S(\mathbf{R}, \mathbf{R}^d)$;*
(b) *obeys a U-functional estimate*

$$|(S\Phi_\lambda)(z\mathbf{f})| \leq C_1(\lambda) \exp(C_2(\lambda)|z^2 B(\mathbf{f})|)$$

for some fixed continuous quadratic form B and for $C_1 \in L^1(m), C_2 \in L^\infty(m)$.

Then Φ_λ is Bochner-integrable

$$\int_\Omega dm(\lambda)\Phi_\lambda \in (S)^*$$

and

$$S\left(\int_\Omega dm(\lambda)\Phi_\lambda\right)(\mathbf{f}) = \int_\Omega dm(\lambda)(S(\Phi_\lambda)(\mathbf{f}))\,.$$

3. Truncated Self-Intersection Local Times

Let B: $S' * [0, +\infty) \to \mathbf{R}^d$ be the d-dimensional Brownian motion defined as $B_k(\omega, t) = \langle \omega_k, \mathbf{1}_{[0,t)} \rangle$, $k = 1, \ldots, d$. We introduce a renormalization of the well-studied Hida distribution $\delta(B(t_2) - B(t_1))$ through subtraction of the lowest terms in its chaos expansion in analogy to the procedure proposed in Ref. 14. This is most conveniently done using its S-transform which is

$$S(\mathbf{f})\colon \delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \to \left(\frac{1}{\sqrt{2\pi|t_2 - t_1|}}\right)^d \exp\left(-\frac{\sum_{i=1}^d (\int_{t_1}^{t_2} f_i(x)dx)^2}{2|t_2 - t_1|}\right).$$

Using the shorthand notation

$$\exp_r(x) \equiv \sum_{n=r}^\infty \frac{x^n}{n!}$$

we set

Definition 3.1.

$$\delta_r(\mathbf{B}(t_2) - \mathbf{B}(t_1))(\mathbf{f}) \equiv S^{-1}\left(\frac{1}{(\sqrt{2\pi(t_2 - t_1)})^d} \exp_r\left(-\frac{\sum_{i=1}^d (\int_{t_1}^{t_2} f_i(x)dx)^2}{2|t_2 - t_1|}\right)\right).$$

Proposition 3.1. *Let $r > d/2 - 1$, and $\Delta_m = \{0 < t_1 < \cdots < t_m < 1\}$. Then the (truncated) m-tuple point local time is a Hida distribution*

$$L_m \equiv \int_{\Delta_m} \delta_r(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \cdots \delta_r(\mathbf{B}(t_m) - \mathbf{B}(t_{m-1}))d^m t \in (S)^*\,.$$

Remark 3.1. In particular, one sees that for dimensions $d = 2$ and 3 it is sufficient to "center" the δ-functions by subtracting their expectations (choosing $r = 1$).

Proof. The S-transform of the integrand is

$$D(t_1, \ldots, t_m) = \prod_{k=1}^{m-1} \left[\left(\frac{1}{2\pi(t_{k+1} - t_k)} \right)^{d/2} \exp_r \left(- \frac{\sum_{i=1}^{d} (\int_{t_k}^{t_{k+1}} f_i(x) dx)^2}{2(t_{k+1} - t_k)} \right) \right] . \quad (3)$$

We use the fact that

$$\exp_r(x) \leq |\exp_r(x)| \leq |x|^r e^{|x|}$$

to obtain

$$|D(t_1, \ldots, t_m)| \leq a \prod_{k=1}^{m-1} (t_{k+1} - t_k)^{r-d/2} \exp \left(b \sum_{k=1}^{m-1} \sup_{1 \leq i \leq d} \left(\sup_{t_k \leq x \leq t_{k+1}} |f_i(x)| \right)^2 \right) \quad (4)$$

for suitable constants a, b. This now allows application of the Bochner integration theorem, Corollary 2.1. Since D is measurable, it is sufficient to note that in the above estimate

(a) the exponent may be bounded by the square of a continuous norm on $\mathbf{f}$,
(b) the coefficient of the exponential in (3) is integrable:

$$\int_{\Delta_m} d^m t \prod_{k=1}^{m-1} (t_{k+1} - t_k)^{r-d/2} = \frac{(\Gamma(1 + r - d/2))^{m-1}}{\Gamma(m + 1 + (m-1)(r - d/2))} ,$$

$$\text{for } r - \frac{d}{2} + 1 > 0 . \qquad \square$$

4. The Chaos Expansion

In this section our goal will be to determine the kernel functions F of the chaos expansion

$$L_m(\omega) = \sum_{\mathbf{n}} \langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle .$$

We restrict ourselves to the study of triple points ($m = 3$) to limit the notational effort. It will turn out that all these kernel functions can be given in terms of certain functions of just four variables which we shall introduce now to prepare the main result below.

Definition 4.1. Let $d = 1, 2, 3, \ldots$ and p, s be non-negative integers larger than $d/2 - 1$. For $x_1 < x_2 < x_3 < x_4$ we define

$$g_{ps}(x_1, x_2, x_3, x_4) \equiv \int_{x_2}^{x_3} (t - x_1)^{1-p-d/2} (x_4 - t)^{1-s-d/2} dt .$$

Remark 4.1. We collect a certain number of properties of these functions for later use in the Appendix.

Furthermore, we shall use the notation of the following:

Definition 4.2. Let $x = (x_1, \dots, x_n) \in R^n$ and σ any permutation such that

$$x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} .$$

Then for $p \leq n$ we use the notation

$$(x)_p \equiv x_{\sigma(p)} .$$

In particular, $(x)_1$ is the smallest of the x_i, and $(x)_n$ the largest.

Using (3) we have

$$(SL_3)(\mathbf{f}) = \int_{\Delta_3} d^3t S[\delta_r(\mathbf{B}(t_2) - \mathbf{B}(t_1))](\mathbf{f}) S[\delta_r(\mathbf{B}(t_3) - \mathbf{B}(t_2))](\mathbf{f})$$

$$= \sum_{\substack{\mathbf{p}:\ p \geqslant r \\ \mathbf{s}:\ s \geqslant r}} \frac{(2\pi)^{-d}}{(-2)^{p+s}\mathbf{p}!\mathbf{s}!} \int_{\Delta_3} d^3t \Bigg[(t_2 - t_1)^{-\frac{d}{2}-p} (t_3 - t_2)^{-\frac{d}{2}-s}$$

$$\times \prod_{i=1}^{d} \left(\left| \int_{t_1}^{t_2} f_i(u^i) du^i \right|^{2p_i} \left| \int_{t_2}^{t_3} f_i(v^i) dv^i \right|^{2s_i} \right) \Bigg] .$$

To identify the kernel functions φ we need to bring the terms $T_{\mathbf{p},\mathbf{s}}$ of this sum into the form

$$T_{\mathbf{p},\mathbf{s}} = \langle \varphi_{2\mathbf{p},2\mathbf{s}}, \mathbf{f}^{\otimes 2\mathbf{n}} \rangle . \tag{5}$$

For $p_i \neq 0$ and $s_i \neq 0$ we have

$$\left| \int_{t_1}^{t_2} f_i(u) du \right|^{2p_i} \left| \int_{t_2}^{t_3} f_i(u) du \right|^{2s_i}$$

$$= \int \mathbf{1}_{[t_1,t_2]^{2p_i}}(u^i) f_i^{\otimes 2p_i}(u^i) d^{2p_i} u^i \cdot \int \mathbf{1}_{[t_2,t_3]^{2s_i}}(v^i) f_i^{\otimes 2s_i}(v^i) d^{2s_i} v^i .$$

Next we want to symmetrize this integral with respect to all the integration variables: $2p_i$ of them are restricted to the first interval, and $2s_i$ to the second. Using symmetry we first change the integration to ordered p_i- and s_i-tuples and obtain

$$(2p_i)!(2s_i)! \int \mathbf{1}_A(u^i) f_i^{\otimes 2p_i}(u^i) d^{2p_i} u^i \cdot \int \mathbf{1}_B(v^i) f_i^{\otimes 2s_i}(v^i) d^{2s_i} v^i ,$$

where

$$A = \{u^i \colon t_1 \leq u_1^i \leq \cdots \leq u_{2p_i}^i \leq t_2\}$$

and

$$B = \{v^i \colon t_2 \leq v_1^i \leq \cdots \leq v_{2s_i}^i \leq t_3\} .$$

In other words the integral runs over all ordered $2p_i + 2s_i$-tuples $w^i = (u^i, v^i)$ with the following restrictions. Denoting the smallest of the w^i by $(w^i)_1$, the $2p_i$th by $(w^i)_{2p_i}$, the next larger by $(w^i)_{2p_i+1}$ and the largest by $(w^i)_{2n_i}$ we can rewrite the above integral as one over ordered $2p_i + 2s_i$-tuples

$$(2p_i)!(2s_i)! \int \mathbf{1}_C(w^i) f_i^{\otimes 2n_i}(w^i) d^{2n_i} w^i \,,$$

where $n_i \equiv p_i + s_i$ and

$$C = \{w^i \colon t_1 \le (w^i)_1 \le (w^i)_{2p_i} \le t_2 \le (w^i)_{2p_i+1} \le (w^i)_{2n_i} \le t_3\} \,.$$

Note that the indicator function so defined is invariant under perturbations, hence it extends to a symmetric function on arbitrary $2p_i + 2s_i$-tuples, and we can rewrite the integral as

$$\frac{(2p_i)!(2s_i)!}{(2n_i)!} \int_{R^{2n_i}} \mathbf{1}_C(w^i) f_i^{\otimes 2p_i + 2s_i}(w^i) d^{2n_i} w^i \,.$$

To obtain $T_{\mathbf{p},\mathbf{s}}$ we must take the product over the $i = 1, \dots, d$, and perform the t_i-integrations:

$$\begin{aligned} T_{\mathbf{p},\mathbf{s}} &= \frac{(2\pi)^{-d}}{(-2)^{p+s}\mathbf{p}!\mathbf{s}!} \int_{\Delta_3} d^3t (t_2 - t_1)^{-\frac{d}{2}-p} (t_3 - t_2)^{-\frac{d}{2}-s} \\ &\quad \cdot \prod_{i=1}^{d} \frac{(2p_i)!(2s_i)!}{(2n_i)!} \int_{R^{2n_i}} \mathbf{1}_C(w^i) f_i^{\otimes 2n_i}(w^i) d^{2n_i} w^i \\ &= \frac{(2\pi)^{-d}(2\mathbf{p})!(2\mathbf{s})!}{(-2)^n \mathbf{p}!\mathbf{s}!(2\mathbf{n})!} \int_{\Delta_3} d^3t (t_2 - t_1)^{-\frac{d}{2}-p} (t_3 - t_2)^{-\frac{d}{2}-s} \\ &\quad \cdot \int_{R^{2n}} \prod_{i=1}^{d} (\mathbf{1}_C(w^i) f_i^{\otimes 2(p_i+s_i)}(w^i)) d^{2n} w \,. \end{aligned}$$

The product of the indicator functions is the indicator function of the set

$$\begin{aligned} D = \{(w^1, \dots, w^d) \colon t_1 &\le (w^i)_1 \le (w^i)_{2p_i} \le t_2 \\ &\le (w^i)_{2p_i+1} \le (w^i)_{2n_i} \le t_3, \forall\, i = 1, \dots, d\} \,. \end{aligned}$$

This implies for $w = (w^1, \dots, w^d)$ the conditions

$$\begin{aligned} & x_1(w) \equiv \min_i (w^i)_1 \ge 0, \\ & x_2(w) \equiv \max_i (w^i)_{2p_i} \le x_3(w) \equiv \min_i (w^i)_{2p_i+1} \,, \\ & x_4(w) \equiv \max_i (w^i)_{2n_i} \le 1 \end{aligned}$$

which we shall denote by

$$w \in E_{\mathbf{p},\mathbf{s}} \subseteq [0,1]^{2p+2s}$$

and for any such w we have the following range for the t_i-integrations

$$0 \le t_1 \le x_1(w)\,,$$
$$x_2(w) \le t_2 \le x_3(w)\,,$$
$$x_4(w) \le t_3 \le 1\,.$$

Hence for positive p and s

$$T_{\mathbf{p},\mathbf{s}} = \frac{(2\pi)^{-d}(2\mathbf{p})!(2\mathbf{s})!}{(-2)^n\mathbf{p}!\mathbf{s}!(2\mathbf{n})!}\int_{E_{\mathbf{p},\mathbf{s}}} d^{2n}w \prod_{i=1}^{d}(f_i^{\otimes 2n_i}(w^i))\int_0^{x_1(w)} dt_1 \int_{x_2(w)}^{x_3(w)} dt_2 \int_{x_4(w)}^{1} dt_3$$
$$\times (t_2-t_1)^{-d/2-p}(t_3-t_2)^{-d/2-s}\,.$$

For $d=1$ and p, s equal to zero

$$T_{0,0} = \frac{1}{2\pi}\int_{\Delta_3} d^3t \frac{1}{\sqrt{t_3-t_2}\sqrt{t_2-t_1}} = \frac{1}{4}$$

while

$$T_{p,0} = \frac{(2\pi)^{-1}}{(-2)^p p!}\int_{E_{p,0}} d^{2p}w(f^{\otimes 2p}(w))\int_0^{x_1(w)} dt_1 \int_{x_2(w)}^{1} dt_2$$
$$\times \int_{t_2}^{1} dt_3 (t_2-t_1)^{-1/2-p}(t_3-t_2)^{-1/2}$$
$$= \frac{(2\pi)^{-1}}{(-2)^p p!}\int_{E_{p,0}} d^{2p}w(f^{\otimes 2p}(w))\frac{4}{2p-1}\int_{x_2}^{1} dt_2(1-t_2)^{1/2}$$
$$\times ((t_2-x_1)^{1/2-p}-(t_2-0)^{1/2-p})$$

and

$$T_{0,s} = \frac{(2\pi)^{-1}}{(-2)^s s!}\int_{E_{0,s}} d^{2s}w(f^{\otimes 2s}(w))\int_0^{1} dt_1 \int_{t_1}^{x_3(w)} dt_2$$
$$\times \int_{x_4}^{1} dt_3 (t_2-t_1)^{-1/2}(t_3-t_2)^{-1/2-s}$$
$$= \frac{(2\pi)^{-1}}{(-2)^s s!}\frac{4}{2s-1}\int_0^{x_3} dt_2\sqrt{t_2}((x_4-t_2)^{1/2-s}-(1-t_2)^{1/2-s})\,.$$

Note that for $d=1$ (6) is trivial

$$E_{p,s} = [0,1]^{2n}$$

since in this case

$$x_1(w) = (w)_1\,,$$
$$x_2(w) = (w)_{2p} \le x_3(w) = (w)_{2p+1}\,,$$
$$x_4(w) = (w)_{2n}\,.$$

Now we use the functions defined in Definition 4.1 to calculate (s, p positive)

$$\int_0^{x_1(w)} dt_1 \int_{x_2(w)}^{x_3(w)} dt_2 \int_{x_4(w)}^{1} dt_3 (t_2 - t_1)^{-d/2-p} (t_3 - t_2)^{-d/2-s}$$

$$= \frac{1}{(1-p-d/2)(1-s-d/2)} (g_{ps}(x_1, x_2, x_3, x_4) - g_{ps}(x_1, x_2, x_3, 1)$$

$$-g_{ps}(0, x_2, x_3, x_4) + g_{ps}(0, x_2, x_3, 1)) \,.$$

In conclusion we have, by comparing with Eq. (5)

$$\varphi_{2\mathbf{p},2\mathbf{s}}(w) = \frac{(2\pi)^{-d}(2\mathbf{p})!(2\mathbf{s})!}{(-2)^n \mathbf{p}!\mathbf{s}!(2\mathbf{n})!} \frac{1}{(1-p-d/2)(1-s-d/2)} \mathbf{1}_{E_{\mathbf{p},\mathbf{s}}}(w)$$

$$\cdot (g_{ps}(x_1, x_2, x_3, x_4) - g_{ps}(x_1, x_2, x_3, 1) - g_{ps}(0, x_2, x_3, x_4)$$

$$+ g_{ps}(0, x_2, x_3, 1)) \,.$$

For $d = 1, s = 0$ and $p \geq 1$ the last term is accordingly

$$(g_{p0}(0, x_2, 1, 1) - g_{p0}(x_1, x_2, 1, 1))$$

and for $p = 0$, $s \geq 1$

$$(g_{0s}(0, 0, x_3, 1) - g_{0s}(0, 0, x_3, x_4))$$

except for $d = 1$ and p or s equal to zero, where

$$\varphi_{0,0} = \frac{1}{2\pi} \int_{\Delta_3} d^3 t \frac{1}{\sqrt{t_3 - t_2}\sqrt{t_2 - t_1}} = \frac{1}{4} \,.$$

We have proved the following:

Theorem 4.1. *The kernel functions* $\varphi_{2\mathbf{n}}$ *of*

$$L_3(\omega) = \sum_{\substack{\mathbf{n}=(p_1+s_1,\ldots,p_d+s_d) \\ \mathbf{p}:p\geq r, \mathbf{s}:s\geq r}} \langle :\omega^{\otimes 2\mathbf{n}}:, \varphi_{2\mathbf{n}} \rangle$$

are given by

$$\varphi_{2\mathbf{n}}(w) = \sum_{2\mathbf{p}+2\mathbf{s}=2\mathbf{n}} \varphi_{2\mathbf{p},2\mathbf{s}}(w)$$

with

$$\varphi_{2\mathbf{p},2\mathbf{s}}(w) = \frac{(2\pi)^{-d}(2\mathbf{p})!(2\mathbf{s})!}{(-2)^n \mathbf{p}!\mathbf{s}!(2\mathbf{n})!} \frac{1}{(1-p-d/2)(1-s-d/2)} \mathbf{1}_{E_{\mathbf{p},\mathbf{s}}}(w)$$

$$\cdot (g_{ps}(x_1, x_2, x_3, x_4) - g_{ps}(x_1, x_2, x_3, 1) - g_{ps}(0, x_2, x_3, x_4)$$

$$+ g_{ps}(0, x_2, x_3, 1)) \,,$$

where the functions g are given by (4) and $x_i = x_i(w)$ and the sets E by (6), except for $d = 1$ and p or s equal to zero, where

$$\varphi_{0,0} = \frac{1}{4}\,,$$

$$\varphi_{p,0}(w) = \frac{2}{\pi(-2)^p p!(1-2p)}\mathbf{1}_{[0,1]^{2n}}(w)(g_{p0}(0, x_2, 1, 1) - g_{p0}(x_1, x_2, 1, 1))$$

and

$$\varphi_{0,s}(w) = \frac{2}{\pi(-2)^s s!(1-2s)}\mathbf{1}_{[0,1]^{2n}}(w)(g_{0s}(0, 0, x_3, 1) - g_{0s}(0, 0, x_3, x_4))\,.$$

5. L^2-Norms

Proposition 5.1. *For $d = 1, 2$ the kernels $\varphi_{2\mathbf{n}}$ are in $\hat{L}^2(\mathbf{R}^{2n})$.*

Proof. By Remark A.2 in the Appendix it is sufficient to show that

$$\|g_{ps}(x_1, x_2, x_3, x_4)\|^2_{L^2([0,1]^{2(p+s)})} < \infty\,.$$

For $s, p \neq 0$ the L.H.S. is equal to

$$\int_{\Delta_4}\left(\int_{x_1}^{x_2} d^{2(p-1)}u\right)\left(\int_{x_3}^{x_4} d^{2(s-1)}v\right) g^2_{ps}(x_1, x_2, x_3, x_4)d^4x$$

$$= \int_{\Delta_4}(x_2 - x_1)^{2(p-1)}(x_4 - x_3)^{2(s-1)} g^2_{ps}(x_1, x_2, x_3, x_4)d^4x$$

and the estimate in (A.1) is sufficient to show that the L^2-norms are finite. □

The remaining cases are covered by Remark A.3.

Appendix A

Here are some properties and estimates used in the main text to control the functions

$$g_{ps}(x_1, x_2, x_3, x_4) \equiv \int_{x_2}^{x_3}(t - x_1)^{1-p-d/2}(x_4 - t)^{1-s-d/2}dt\,,$$

where $d = 1, 2, 3, \ldots$ and p, s are non-negative integers larger than $d/2 - 1$.

The integral defining g may be evaluated in terms of hypergeometric functions but the following estimate seems to be more useful.

Remark A.1. For p, $s \geq 1$ there exist the bounds

$$0 \leq g_{ps}(x_1, x_2, x_3, x_4) \leq (x_2 - x_1)^{1-p}(x_4 - x_3)^{1-s}$$

$$\cdot \begin{cases} \pi & \text{if } d = 1\,, \\ \dfrac{1}{x_4 - x_1}\log\dfrac{x_4 - x_2}{x_4 - x_3}\dfrac{x_3 - x_1}{x_2 - x_1} & \text{if } d = 2\,. \end{cases} \qquad \text{(A.1)}$$

Remark A.2. g is monotonic increasing in x_1 except for $d = 1$, $p = 0$, and decreasing in x_4 except for $d = 1$, $s = 0$.

Remark A.3. For $p = 0$ (and hence $d = 1$) and $s \geq 1$

$$0 \leq g_{0s}(0, 0, x_3, x_4) \leq 2(x_4 - x_3)^{1-s} x_3^{1/2} x_4^{1/2}\,.$$

Similarly, for $s = 0$ (and hence $d = 1$) and $p \geq 1$

$$0 \leq g_{p0}(x_1, x_2, 1, 1) \leq 2(x_2 - x_1)^{1-p}(1 - x_2)^{1/2}(1 - x_1)^{1/2}\,.$$

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