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Ergodicity of canonical Gibbs measures with respect to the diffeomorphism group

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For general potentials we prove that every canonical Gibbs measure on configurations over a manifold X is quasi-invariant w.r.t. the group of diffeomorphisms on X. We show that this quasi-invariance property also characterizes the class of canonical Gibbs measures. From this we conclude that the extremal canonical Gibbs measures are just the ergodic ones w.r.t. the diffeomorphism group. Thus we provide a whole class of different irreducible representations.

1 Introduction

The group $\mathrm{Diff}_0(X)$ of diffeomorphisms with compact support over a manifold X has been frequently studied as an example of an infinite dimensional group, see e.g. [18], [24] and references therein. In addition, its study is motivated by problems of mathematical physics, see e.g. [7], [12], [14], and for an overview [4]. Namely, the unitary representations of this group can be considered as models of non-relativistic quantum field theory. Since there is little hope to describe all unitary and irreducible representations of the diffeomorphism groups, usually one confines oneself to regular representations in the sense of Mackey originating from a quasi-invariant measure μ. Unfortunately, examples of this kind of measures are rare, see e.g. [18].

Starting from the works of [16], [20], [12], and [9], configuration spaces are considered as underlying measurable spaces. On X the quasi-invariant measures are just the volume forms (e.g. Riemannian volume or Lebesgue measure on $\mathbb{R}^d$). Let σ be such a volume form. The space of n-point configurations $\Gamma_X^{(n)} := \{\eta \subset X \mid |\eta| = n\}$ is a kind of homogeneous space of $\mathrm{Diff}_0(X)$ for the action $\phi(\eta) := \{\phi(x) \mid x \in \eta\}$. Also in this case the volume forms are the quasi-invariant measures. All of them are equivalent to each other and in particular equivalent to the symmetrization of $\sigma^{\otimes n}$. In applications to physics the points of a set η are sometimes interpreted as the positions of n indistinguishable particles. Strictly speaking, we work with the so-called simple configurations, i.e. we assume that two particles cannot have the same position. For our purpose this is reasonable, because a $\mathrm{Diff}_0(X)$-ergodic measure μ on the space of configurations with coinciding points has either $\mu(\Gamma_X) = 1$ or 0.

For the space of infinite configurations

$$\Gamma_X := \{\gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}$$

the situation is richer. Under the natural action $\phi(\gamma) := \{\phi(x) \mid x \in \gamma\}$ of $\mathrm{Diff}_0(X)$ on Γ_X the subspaces of n-point configurations are orbits of the group, which can be treated separately. Thus we focus on measures concentrated on infinite configurations. Well studied is the case of Poisson measures $\pi_{z\sigma}$, see [9], [17], and [30]. Note that these measures are ergodic although they are not supported by a single orbit of the diffeomorphism group. For different $z > 0$ the representations corresponding to $\pi_{z\sigma}$ are inequivalent.

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Already in [9] it was noted that Gibbs measures form a much wider class of examples and (under an abstract condition on the regularity of the underlying specification) quasi-invariance was proven. In [31] the case of finite range potentials was handled. Technical difficulties arise only if one drops this restriction. Therefore, in Subsection 3.3 we prove quasi-invariance for concrete conditions on the potential and the class of admissible Gibbs measures.

In [9] and [18] the authors showed that a regular representation of $\mathrm{Diff}_0(X)$ is irreducible iff the corresponding measure is ergodic. In Theorem 4.3 we proof that the extremal elements of the convex set of Gibbs measures are just the $\mathrm{Diff}_0(X)$-ergodic ones. The proof is essentially based on the characterization of Gibbs measures by their Radon-Nikodym derivatives, cf. Theorem 3.10. The ergodic decomposition of Gibbs measures discussed in [31] (cf. the second definition of ergodicity on the bottom of page 623) is actually the decomposition of Gibbs measures into extremal ones in the sense of Dynkin-Föllmer-Martin boundary, see e.g. [26]. For the group of translations on a Hilbert space similar results holds for Gibbs measures for lattice systems, for a detailed consideration and references see e.g. [1].

Although one cannot expect neither that the quasi-invariant measures have full support on a single orbit, nor uniqueness (w.r.t. equivalence of measures) and one is lacking information about the finite dimensional distributions or characteristics of the measure; nevertheless the proofs of this paper are rather short and simple due to the applied techniques. Besides the conceptional background of configuration space analysis, the following two techniques seem to play an important role.

First, the idea to derive ergodicity of extremal Gibbs measures using their characterization by Radon-Nikodym derivatives seems to be new. The difficult direction is to show that extreme Gibbs measures are ergodic. Typically, one tries to prove that every a.s. invariant function is a.s. equal to a tail-field measurable function. This point is not considered in [31]. We cannot generalize this approach to invariant measures. The technique of Lemma 2.7 in [1] seems to be based on the linear structure of the group. To prove this lemma the characterization of Gibbs measures is not used, whereas we show that ergodicity is a direct consequence of this characterization. Notwithstanding several considerations from the study of invariant measures can be preserved, cf. Section 4.

Secondly, the concept of specification, i.e. characterization by conditional expectations, appears to be suitable. Be aware that typically neither the conditional expectations nor the "characterization" by Radon-Nikodym derivatives determine the measure uniquely. Specifications are constructed from potentials $V : \bigsqcup_{n\in\mathbb{N}} \Gamma_X^{(n)} \to \mathbb{R}$. For simplicity, we consider in the main body of the paper only pair potentials and the generalization to general potentials is postponed to Section 5. The quasi-invariance of a measure μ on Γ_X with admissible Radon-Nikodym derivatives is equivalent to the quasi-invariance of its conditional probabilities. For technical reasons this holds only for a countable subgroup. Nevertheless, one can construct a subgroup which is large enough to still characterize the conditional probabilities, as measures on $\Gamma_X^{(n)}$, by their Radon-Nikodym derivatives, cf. Section 3. As Gibbs measures are defined via their conditional probabilities this yields the characterization. Usually, one works in mathematical statistical mechanics with the set of grand canonical Gibbs measures $\mathcal{G}_{\mathrm{gc}}(z, V)$ for an activity $z > 0$. It turns out to be useful to work with a filtration of σ-algebras leading to canonical Gibbs measures $\mathcal{G}_{\mathrm{c}}(V)$, cf. Subsection 2.2. In this case the conditional probabilities are supported on $\Gamma_X^{(n)}$, whereas for the grand canonical ensemble they are supported on $\bigsqcup_{n\in\mathbb{N}_0} \Gamma_X^{(n)}$ which is not a single orbit of $\mathrm{Diff}_0(X)$. Hence the conditional probability measures are not determined uniquely up to equivalence by their Radon-Nikodym derivatives. Both concepts were already used in the context of characterization of Gibbs measures by an integration by parts formula, cf. [3]. This is the infinitesimal version of the characterization by Radon-Nikodym derivatives, however it is not equivalent even for quite natural differentiable potentials, cf. for a discussion of this fact in the one dimensional case Remark 3.6 (ii) in [1]. These concepts seem to be new in the area of infinite dimensional group theory. One should expect that in general extremal canonical Gibbs measures are extremal grand canonical Gibbs measures:

$$\mathrm{ext}(\mathcal{G}_{\mathrm{c}}(V)) \;=\; \{\delta_\emptyset\} \cup \bigcup_{z>0} \mathrm{ext}(\mathcal{G}_{\mathrm{gc}}(z, V))\,.$$

This is the so-called equivalence of ensembles, see [10] for the case $X = \mathbb{R}^d$ and [27] for an abstract consideration. Note that $\mathcal{G}_{\mathrm{gc}}(z, 0) = \{\pi_{z\sigma}\}$. In general $\mathrm{ext}(\mathcal{G}_{\mathrm{gc}}(z, V))$ can have more than one element; in statistical mechanics this effect is called phase transition. Existence of Gibbs measures are well studied, see for example

[29] for pair potentials and $X = \mathbb{R}^d$; see also e.g. [22] for partial results for general spaces X. The representations corresponding to measures from $\mathrm{ext}(\mathcal{G}_c(V))$ are mutually inequivalent.

Clearly, conditions on the class of potentials under consideration are necessary. First of all the potentials should have no hard core, as otherwise the Gibbs measures are not even quasi-invariant. For finite range potentials, the Radon-Nikodym derivatives and the densities of the conditional probabilities are obviously well defined . An a priori information about the measure is needed to handle also non finite range potentials; for example a support property or a bound for the first correlation function. The potential should be lower and upper regular. These technical details are collected in Subsection 3.2. The concepts of harmonic analysis on configuration spaces, namely the K-transform of A. Lenard, essentially simplify these considerations. The concrete assumptions on the potential are weaker than what is usually assumed. Standard techniques of statistical mechanics, see e.g. [29], would work as well, however in the case of general potentials they would be rather intricate and for marked systems too restrictive if applied naively. Using the proposed technique these generalizations are direct. This is pointed out in Section 5.

2 Preliminaries

2.1 Configuration spaces

Let X be a connected, oriented Riemannian l-dimensional C^∞-manifold with a volume element m and metric d. For simplicity, we consider X to be a geodesically complete and non-compact space. We denote by $\mathcal{O}(X)$ the family of all open subsets of X, and by $\mathcal{B}(X)$ the corresponding σ-algebra on X. $\mathcal{O}_c(X)$, $\big($respectively $\mathcal{B}_c(X)\big)$ denotes the system of all sets in $\mathcal{O}(X)$ (respectively $\mathcal{B}(X)$) which are bounded (and hence have compact closure). Define the space $\Gamma_Y^{(n)}$ of n-point configurations in $Y \in \mathcal{B}(X)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ by

$$\Gamma_Y^{(n)} := \{\eta \subset Y \mid |\eta| = n\}, \quad \Gamma_Y^{(0)} := \{\emptyset\}, \tag{2.1}$$

where $|\eta|$ denotes the cardinality of the set η. For each $\Lambda \in \mathcal{B}_c(X)$ define the mapping N_Λ by $N_\Lambda : \Gamma_X^{(n)} \to \mathbb{N}_0$, $\eta \mapsto |\eta \cap \Lambda|$. For short we denote by $\eta_\Lambda := \eta \cap \Lambda$. The space of all n-point configurations $\Gamma_X^{(n)}$ is an $n \cdot \dim(X)$-dimensional C^∞-manifold, because it is equal to $\widetilde{X^n}/S_n$. Here $\widetilde{X^n} := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$ and S_n is the symmetric group over $\{1, \ldots, n\}$. A basis of the topology on $\Gamma_X^{(n)}$ is given by sets of the form

$$U_1 \hat{\times} \ldots \hat{\times} U_n := \big\{\eta \in \Gamma_X^{(n)} \mid N_{U_1}(\eta) = 1, \ldots, N_{U_n}(\eta) = 1\big\},$$

where $U_i \in \mathcal{O}_c(X)$, $U_i \cap U_j = \emptyset$. As for each $\{x_1, \ldots, x_n\} \in U_1 \hat{\times} \ldots \hat{\times} U_n$ there exists a unique i_k with $x_{i_k} \in U_k$ one can construct a chart of $\Gamma_X^{(n)}$ using n-charts (U_i, h_i) of X by

$$h_1 \hat{\times} \cdots \hat{\times} h_n (\{x_1, \ldots, x_n\}) := \big(h_1(x_{i_1}), \ldots, h_n(x_{i_n})\big). \tag{2.2}$$

The family of all open sets on $\Gamma_X^{(n)}$ we denote by $\mathcal{O}\big(\Gamma_X^{(n)}\big)$. The corresponding Borel σ-algebra $\mathcal{B}\big(\Gamma_X^{(n)}\big)$ is also equal to $\sigma(N_\Lambda \mid \Lambda \in \mathcal{B}_c(X))$. The space of finite configurations is $\Gamma_0 := \Gamma_{0,X}$ and $\Gamma_{0,X} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_X^{(n)}$ equipped with the topology $\mathcal{O}(\Gamma_{0,Y})$ of disjoint union.

The configuration space $\Gamma := \Gamma_X$ over X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma := \{\gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}. \tag{2.3}$$

The space Γ equipped with the vague topology $\mathcal{O}(\Gamma)$ is Polish, see e.g. [19]. The corresponding Borel σ-algebra is $\mathcal{B}(\Gamma) = \sigma(N_\Lambda \mid \Lambda \in \mathcal{O}_c(X))$. The configuration space Γ is the projective limit of the spaces $\Gamma_\Lambda := \{\gamma \in \Gamma \mid \gamma_{\Lambda^c} = \emptyset\}$, $\Lambda^c = X \backslash \Lambda$ $\Lambda \in \mathcal{O}_c(X)$ and projections $p_\Lambda : \Gamma \to \Gamma_\Lambda,\ \gamma \mapsto \gamma_\Lambda$. Notice that as measurable spaces $\Gamma_\Lambda = \Gamma_{0,\Lambda} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_\Lambda^{(n)}$. Furthermore, for any $\Lambda \in \mathcal{O}_c(X)$ we define the following filtration of σ-algebras on Γ

$$\mathcal{B}_\Lambda(\Gamma) := \sigma\big(\big\{N_{\Lambda'} \mid \Lambda' \in \mathcal{B}_c(X) \text{ with } \Lambda' \subset \Lambda\big\}\big). \tag{2.4}$$

The σ-algebras $\mathcal{B}_\Lambda(\Gamma)$ and $\mathcal{B}(\Gamma_\Lambda)$ are σ-isomorphic. Denote by $L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$ the set of all $\mathcal{B}_\Lambda(\Gamma)$-measurable functions on Γ and by $L^0(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ the space of $\mathcal{B}(\Gamma_\Lambda)$-measurable functions on Γ_Λ. Recall that $F : \Gamma \mapsto \mathbb{R}$ is $\mathcal{B}_\Lambda(\Gamma)$-measurable iff $F\restriction_{\Gamma_\Lambda} \in L^0(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ and we have the following connection $F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda)$. For more details concerning configuration space analysis and its applications see e.g. [2], [3] and [22] or [21] and references therein.

2.2 Canonical Gibbs measures

Let $\rho \in L^1_{\mathrm{loc}}(X, m)$ be a m-a.s. strictly positive (in general non-integrable) function and define $\sigma := \rho m$. Interesting in this context is the case $\sigma(X) = \infty$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ has full measure on $\widetilde{X^n}$ and its symmetrization σ_n is a measure on $\Gamma_0^{(n)}$. The Lebesgue-Poisson measure on Γ_0 is $\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma_n$. The Poisson measure $\pi_{z\sigma}$ is defined as the projective limit of the probability measures $\pi^\Lambda_{z\sigma} := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}$ on Γ_Λ, $\Lambda \in \mathcal{O}_c(X)$. We now describe a bigger class of probability measures on the configuration space, the so-called canonical Gibbs measures, cf. [11] and [27]. A measurable symmetric function $V : \widetilde{X^2} \to \mathbb{R}$ is called a pair potential. The energy functional $E : \Gamma_0 \to \mathbb{R}$ is defined by $E(\eta) := \sum_{\{x,y\} \subset \eta} V(x, y)$, with $E(\eta) := 0$ for $|\eta| \leq 1$. Let $\eta \in \Gamma_0$ and $\gamma \in \Gamma$ be given, then the interaction energy between η and γ is defined as

$$W(\eta, \gamma) := \begin{cases} \sum\limits_{x \in \eta,\, y \in \gamma} V(x, y), & \text{if} \quad \sum\limits_{x \in \eta,\, y \in \gamma} |V(x, y)| < \infty\,, \\ +\infty\,, & \text{otherwise}\,. \end{cases} \tag{2.5}$$

For any $\Lambda \in \mathcal{B}_c(X)$ the conditional energy $E_\Lambda : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is given by $E_\Lambda(\gamma) := E(\gamma_\Lambda) + W(\gamma_\Lambda, \gamma_{\Lambda^c})$.

Definition 2.1 The *canonical specification* Π^c_Λ, $\Lambda \in \mathcal{B}_c(X)$ is defined for any $\gamma \in \Gamma$ and $F \in \mathcal{B}(\Gamma)$ by (cf. [27])

$$\Pi^c_\Lambda(F, \gamma) := \frac{\mathbf{1}_{\{0 < Z_\Lambda < \infty\}}(\gamma)}{Z_\Lambda(\gamma)} \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} \mathbf{1}_F(\eta \cup \gamma_{\Lambda^c}) e^{-E(\eta) - W(\eta, \gamma_{\Lambda^c})}\, \sigma_{|\gamma_\Lambda|}(d\eta) \tag{2.6}$$

and $Z_\Lambda(\gamma) := \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} e^{-E(\eta) - W(\eta, \gamma_{\Lambda^c})}\, \sigma_{|\gamma_\Lambda|}(d\eta)$. A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a canonical Gibbs measure iff $\mu \Pi^c_\Lambda = \mu$, for all $\Lambda \in \mathcal{B}_c(X)$ (the analogue of the (DLR)-equations for canonical Gibbs measures). We denote by $\mathcal{G}_c(V)$ the set of all such probability measures μ.

It has been shown in [27] that, in fact, $\left(\Pi^c_\Lambda\right)_{\Lambda \in \mathcal{B}_c(X)}$ is a $(\mathcal{F}_{\Lambda^c})_{\Lambda \in \mathcal{B}_c(X)}$-specification in the sense of [8], where

$$\mathcal{F}_{\Lambda^c} := \mathcal{F}_{\Lambda^c}(\Gamma) := \mathcal{B}_{X \setminus \Lambda}(\Gamma) \vee \sigma\big(N_\Lambda^{-1}(\{n\}) \mid n \in \mathbb{N}_0\big)\,, \tag{2.7}$$

where N_Λ is considered as a function on Γ. (Our definition differs slightly from [27], but for stable potentials fulfilling the conditions introduced in Subsection 3.2 they coincide). For all $z > 0$ the Poisson measure $\pi_{z\sigma}$ is the unique element in $\mathcal{G}_{gc}(z, 0)$ and it is also in $\mathcal{G}_c(0)$ $\Big($for $\sigma(X) < \infty$ furthermore $\mathbf{1}_{\Gamma_X^{(n)}} \sigma^{(n)} \in \mathcal{G}_c(0)\Big)$. The grand canonical Gibbs measures $\mathcal{G}_{gc}(z, V)$ are given by the $(\mathcal{B}_\Lambda(\Gamma))_{\Lambda \in \mathcal{B}_c(X)}$-specification

$$\Pi^{gc}_\Lambda(F, \gamma) := \frac{\mathbf{1}_{\{\Xi^z_\Lambda < \infty\}}(\gamma)}{\Xi^z_\Lambda(\gamma)} \int_{\Gamma_\Lambda} \mathbf{1}_F(\gamma_{X \setminus \Lambda} \cup \eta) e^{-E_\Lambda(\gamma_{X \setminus \Lambda} \cup \eta)}\, \pi^\Lambda_{z\sigma}(d\eta)\,. \tag{2.8}$$

Often one cannot work with the class of all Gibbs measures, but one has to restrict oneself to a subclass, for example the subclass defined by Assumption 3.6 below. Frequently, one assumes an a priori information about the support. Measures with this property we will call tempered in the following. In general one expects that the extremal canonical Gibbs measures are just extremal grand canonical Gibbs measure for a suitable value of z, under an abstract condition this was proven in [27]. This fact is called the equivalence of canonical and grand canonical ensemble. For the case $X = \mathbb{R}^d$, $\sigma = m$, and a continuous, finite range potential V, such that there exists a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+_0$ with $V(x, y) \geq \psi(|x - y|)$ and $\lim_{r \to 0} \psi(r) r^d = \infty$ H.-O. Georgii showed in [10] the equivalence of ensembles for Gibbs measures tempered in the following sense:

$$\mu\left(\left\{\gamma \in \Gamma \;\middle|\; \limsup_{k \to \infty} \frac{N_{\Delta_k}(\gamma)}{\sigma(\Delta_k)} < \infty\right\}\right) = 1\,,$$

where Δ_k are the cubes centered at 0 of side length $2k$, $k \in \mathbb{N}$. The Gibbs measures tempered in this sense are a face of the class of all Gibbs measures. Note that in the case $\sigma(X) < \infty$, $V = 0$ the equivalence of ensembles does not hold as the extremal canonical Gibbs measures $\mathbf{1}_{\Gamma_X^{(n)}}\sigma^{(n)} \notin \mathcal{G}_{gc}(z,0)$. For more details see e.g. Section 4.1 in [11].

Let μ be a probability measure on the Polish space $(\Gamma, \mathcal{B}(\Gamma))$. Using [25, Theorem V. 8.1] there exists for any $\Lambda \in \mathcal{B}_c(X)$ a probability kernel $\mu_\Lambda : \mathcal{B}(\Gamma_\Lambda) \times \Gamma \to \mathbb{R}^+$ such that for any $F \in L^0(\Gamma, \mathcal{B}(\Gamma))$ which is either positive or integrable

$$\mathbb{E}_\mu(F|\mathcal{F}_\Lambda^c)(\gamma) \;=\; \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F(\eta \cup \gamma_{X\setminus\Lambda})\, \mu_\Lambda(d\eta, \gamma)\,,\ \mu\text{-a.s.} \tag{2.9}$$

and $\mu_\Lambda(\{\eta \in \Gamma_\Lambda \mid N_\Lambda(\eta) = N_\Lambda(\gamma)\}, \gamma) = 1$ μ-a.s. Moreover, for all $F \in \mathcal{B}(\Gamma)$ the function $\mu_\Lambda(F, \cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable.

Corollary 2.2 *Let μ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. Then for any $\Lambda \in \mathcal{B}_c(X)$ and any positive $F \in L^0(\Gamma, \mathcal{B}(\Gamma))$*

$$\int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F(\eta \cup \gamma_{\Lambda^c})\, \mu_\Lambda(d\eta, \gamma) \;=\; \int_\Gamma F(\eta)\, \Pi_\Lambda^c(d\eta, \gamma)\,,\ \mu\text{-a.s.}$$

iff $\mu \in \mathcal{G}_c(\sigma, V)$.

2.3 K-transform

Let $G : \Gamma_0 \to \mathbb{R}$ be a function such that $\operatorname{supp}(G) \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$ for some $\Lambda \in \mathcal{O}_c(X)$, $N \in \mathbb{N}$. Then we define $KG : \Gamma \to \mathbb{R}$ by $(KG)(\gamma) := \sum_{\xi \Subset \gamma} G(\xi)$, where the sum is extended over all finite subconfigurations ξ from γ, in symbols $\xi \Subset \gamma$. The K-transform appears from different points of view in statistical mechanics and also probability theory, see e.g. [5], [23], [6] and [21] for details. A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ has finite local moments, in symbols $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, iff for all $\Lambda \in \mathcal{O}_c(X)$, $n \in \mathbb{N}_0$ we have $\int_\Gamma |\gamma_\Lambda|^n\, \mu(d\gamma) < \infty$. For $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ the correlation measure ρ_μ corresponding to μ is uniquely defined by

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(\eta) \;=\; \int_\Gamma (KG)(\gamma)\mu(\gamma)\,, \tag{2.10}$$

$G : \Gamma_0 \to \mathbb{R}_0^+$. Since Γ_0 is the disjoint union of the family of measurable spaces $\big(\Gamma_{0,X}^{(n)}\big)_{n\in\mathbb{N}_0}$, ρ_μ can be expressed through its components $\rho_\mu^{(n)}$, $n \in \mathbb{N}_0$. For a canonical (grand canonical) Gibbs measure the σ-finite measures $\rho_\mu^{(n)}$ are absolutely continuous w.r.t. σ_n and their Radon-Nikodym derivatives are the well-known correlation functions, cf. e.g. [28].

3 Characterization by Radon-Nikodym derivatives

3.1 Diffeomorphism group and characterization on finite configuration spaces

Let us denote the group of all diffeomorphisms $\phi : X \to X$ which are equal to identity outside of a compact set by $\mathrm{Diff}_0(X)$. The corresponding Lie algebra is $\mathrm{Vect}_0(X)$, the set of all vector fields $v : X \to TX$ with compact support. For technical reasons, cf. the proof of Theorem 3.10, we have to work with a countable subgroup $\mathrm{Diff}_{\mathrm{small}}(X)$, which still locally characterizes measures by their quasi-invariance. More precisely, for any open connected set $O \in \mathcal{O}_c(X)$ and any measure $\widetilde{\sigma}$ on O which is quasi-invariant for all $\phi \in \mathrm{Diff}_{\mathrm{small}}(O)$ with the same Radon-Nikodym derivatives as σ, there exists $k > 0$ such that $\widetilde{\sigma} = k\sigma$. Here we denote by $\mathrm{Diff}_{\mathrm{small}}(O)$ the subgroup of all $\phi \in \mathrm{Diff}_{\mathrm{small}}(X)$ with support in O. Such a subgroup may be constructed in the following way, see e.g. [31] and [9]:

As the topology of the manifold X has a countable basis we may reduce ourselves w.l.o.g. to $B_1 := \{x \in X \mid |x| < 1\}$. For every $n \in \mathbb{N}$ consider $\chi_n \in C^\infty(\mathbb{R}^l)$ such that $\chi_n\!\restriction B_{1-\frac{1}{n}} = 1$ and $\chi_n\!\restriction B_1^c = 0$. Then for every unit vector $e_i \in \mathbb{R}^l$, $i = 1, \ldots, l$ we can construct the vector field

$$v_{i,n}(x) \;:=\; \chi_n(x) e_i\,.$$

For each of the aforementioned $v_{i,n}$ consider the corresponding flow $\phi_t^{v_{i,n}}$, $t \in \mathbb{R}$. As elements of $\mathrm{Diff}_{\mathrm{small}}(X)$ we consider all finite combinations of the flows $\phi_t^{v_{i,n}}$ for $n \in \mathbb{N}$, $i = 1, \ldots, l$ and $t \in \mathbb{Q}$. If $\widetilde{\sigma}$ is $\mathrm{Diff}_{\mathrm{small}}(X)$ quasi-invariant, then $\widetilde{\sigma}$ is quasi-invariant under translation given by te_i, $t \in \mathbb{Q}$, $i = 1, \ldots, l$ and $\frac{1}{\rho}\,\widetilde{\sigma}$ is even invariant. Adjusting the classical proof for the characterization of Lebesgue measure by translation invariance we see that there exists a constant $k \geq 0$ s.t. $\widetilde{\sigma}(dx) = k\rho(x)\, m(dx)$.

Following the same line of arguments we see the following: let $O_j \in \mathcal{O}_c(X)$ be connected, $j = 1, \ldots, n$, μ a measure on $\times_{j=1}^n O_j$ and $r : \times_{j=1}^n O_j \to \mathbb{R}_0^+$ a μ-a.s. strictly positive function. If μ is $\times_{j=1}^n \mathrm{Diff}_{\mathrm{small}}(O_j)$-quasi-invariant with Radon-Nikodym derivative

$$\frac{d(\phi^*\mu)}{d\mu}(x) \;=\; \frac{r(\phi^{-1}(x))}{r(x)}\, J\phi^{-1}(x)\,,$$

then μ has the form $\mu(dx) = kr(x)\, m^{\otimes n}(dx)$, where k is a positive constant and $J\phi$ is the Jacobian determinant w.r.t. m.

The next lemma is not a trivial corollary of the previous results. We consider an embedding of $\mathrm{Diff}(\Lambda)$ into $\mathrm{Diff}_0\big(\Gamma_\Lambda^{(n)}\big)$ in the following way

$$\{x_1, \ldots, x_n\} \;\longmapsto\; \{\phi(x_1), \ldots, \phi(x_n)\}\,,$$

instead of the full group $\mathrm{Diff}_0\big(\Gamma_\Lambda^{(n)}\big)$, which has elements of the following form

$$\{x_1, \ldots, x_n\} \;\longmapsto\; \{\phi_1(x_1, \ldots, x_n), \ldots, \phi_n(x_1, \ldots, x_n)\}$$

or $\times_{i=1}^n \mathrm{Diff}_{\mathrm{small}}(\Lambda)$, which has elements of the form

$$\{x_1, \ldots, x_n\} \;\longmapsto\; \{\phi_1(x_1), \ldots, \phi_n(x_n)\}\,.$$

In order to prove that this smaller group already characterizes quasi-invariant measures, we essentially use the fact that configurations cannot contain particles with the same position, see e.g. [9].

Lemma 3.1 *Let μ be a probability measure on $\big(\Gamma_\Lambda^{(n)}, \mathcal{B}\big(\Gamma_\Lambda^{(n)}\big)\big)$, $\Lambda \in \mathcal{O}_c(X)$ connected, $n \in \mathbb{N}$. Let $r : \Gamma_\Lambda^{(n)} \to \mathbb{R}^+$ be a measurable mapping which is μ-a.s. strictly positive. If μ is quasi-invariant w.r.t.* $\mathrm{Diff}_{\mathrm{small}}(X)$ *with Radon-Nikodym derivatives given by*

$$\frac{d(\phi^*\mu)}{d\mu}(\eta) \;=\; \frac{r(\phi^{-1}(\eta))}{r(\eta)} \prod_{x\in\eta} J\phi^{-1}(x)\,, \quad \textit{for all} \quad \eta \in \Gamma_\Lambda^{(n)}\,, \tag{3.1}$$

then $\mu(d\eta) = kr(\eta)\lambda_m|_{\Gamma_\Lambda^{(n)}}(d\eta)$, for some constant $k > 0$.

P r o o f. Using charts of the manifold $\Gamma_\Lambda^{(n)}$ (cf. Subsection 2.1) of the form $(U_1 \hat{\times} \ldots \hat{\times} U_n, h_1 \hat{\times} \ldots \hat{\times} h_n)$ where $(U_i, h_i)_{i=1}^n$ are charts of X, we may reduce our considerations to open sets $O := O_1 \times \ldots \times O_n \subset \mathbb{R}^{nl}$ with $O_i \cap O_j = \emptyset$, $i \neq j$. We keep the same notation for the objects transported from $\Gamma_\Lambda^{(n)}$ to O. For any measurable positive function F on O and for all diffeomorphisms $\phi_i \in \mathrm{Diff}_{\mathrm{small}}(O_i)$ we define $\phi := \phi_1 \circ \ldots \circ \phi_n \in \mathrm{Diff}_{\mathrm{small}}(O)$. As the O_i, $i = 1, \ldots, n$ are mutually disjoint, for $(x_1, \ldots, x_n) \in O_1 \times \ldots \times O_n$ we have

$$(\phi(x_1), \ldots, \phi(x_n)) \;=\; (\phi_1(x_1), \ldots, \phi_n(x_n))\,.$$

Therefore, using (3.1) we obtain quasi-invariance of μ on O w.r.t. $\times_{i=1}^n \mathrm{Diff}_{\mathrm{small}}(O_i)$. Thus, according to the results discussed before, there exists a constant k such that $\mu = kr\, m$ on O. As $\Gamma_\Lambda^{(n)}$ is connected we obtain the claimed result. □

3.2 Conditions on the interactions

All assumptions concerning the potential V are collected in this subsection. They enter in the rest of the work only via Corollary 3.8.

Proposition 3.2 *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $G : X \to \mathbb{R}^+ \cup \{\infty\}$ be a $\mathcal{B}(X)$ -measurable function. If*

$$\int_X (G(x) \wedge 1)\, \rho^{(1)}_\mu(dx) \;<\; \infty \tag{3.2}$$

and

$$\rho^{(1)}_\mu\left(\{x \in X \mid G(x) = \infty\}\right) \;=\; 0\,,$$

then the series $\sum_{x\in\gamma} G(x)$ is μ-a.s. convergent.

P r o o f. Denote by A the following subset of X

$$A \;:=\; \{x \in X \mid G(x) \le 1\}\,.$$

Taking into account that $(K\mathbf{1}_A G)(\gamma) = \sum_{x\in\gamma_A} G(x)$, we obtain that

$$\mathbb{E}_\mu\left(\sum_{x\in\gamma_A} G(x)\right) \;=\; \int_A G(x)\, \rho^{(1)}_\mu(dx) \;<\; \infty\,.$$

This implies that $\sum_{x\in\gamma_A} G(x)$ is μ-a.s. convergent. On the other hand, the sum $\sum_{x\in\gamma_{A^c}} G(x)$ contains only finite many summands, because

$$\mathbb{E}_\mu(N_{A^c}) \;=\; \int_{A^c} \rho^{(1)}_\mu(dx) \;\le\; \int_X (G(x) \wedge 1)\, \rho^{(1)}_\mu(dx) \;<\; \infty\,.$$

Furthermore, μ-a.s. all these summands are finite

$$\begin{aligned}\mu\left(\{\gamma \in \Gamma \mid \text{exists } x \in \gamma \text{ s.t. } G(x) = \infty\}\right) \;&\le\; \int_\Gamma \sum_{x\in\gamma} \mathbf{1}_{G^{-1}(\{\infty\})}(x)\, \mu(d\gamma)\\ &=\; \rho^{(1)}_\mu\left(\{x \in X \mid G(x) = \infty\}\right) \;=\; 0\,.\end{aligned}$$

□

Assumption 3.3 *Bounded below*: There exists a $B \ge 0$ such that $V(x, y) > -2B$ for all $x, y \in X$.

Assumption 3.4 *No hard core*: For each $\delta > 0$ we have

$$\sup_{\substack{x,y\in X\\ d(x,y)>\delta}} V(x, y) \;<\; \infty\,.$$

Assumption 3.5 *Regularity*: For all $\Lambda \in \mathcal{O}_c(X)$ we have

$$\int_X \left(\sup_{x\in\Lambda} |V(x, y)| \wedge 1\right) \sigma(dy) \;<\; \infty\,.$$

Assumption 3.6 $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and the first correlation measure $\rho^{(1)}_\mu$ corresponding to μ is absolutely continuous w.r.t. σ and we have

$$\frac{d\rho^{(1)}_\mu}{d\sigma}(x) \;\le\; C_1 \quad \text{for some} \quad C_1 \;>\; 0\,.$$

Remark 3.7 If μ is a measure on $\Gamma_{\mathbb{R}^d}$ then all assumptions are fulfilled if the conditions of [29] and Assumption 3.4 holds, i.e. μ is tempered in the sense of D. Ruelle and there exist $R > 0$, and positive bounded decreasing functions $\psi_1 : (0, R] \to \mathbb{R}$, $\psi_2 : [R, \infty) \to \mathbb{R}$ with $\int_0^R \psi_1(r) r^{d-1}\, dr = \infty$ and $\int_R^\infty \psi_2(r) r^{d-1}\, dr < \infty$ such that $V(x, y) \geq \psi_1(|x - y|)$ for $|x - y| \leq R$ and $|V(x, y)| \leq \psi_2(|x - y|)$ for $|x - y| \geq R$. In particular, V is then superstable, lower and upper regular. For finite range potentials Assumption 3.5 is trivially fulfilled and Assumption 3.6 is not anymore necessary to obtain Corollary 3.8. Assumption 3.6 might be replaced by a support condition for μ. For hard core potentials the Gibbs measure is not even quasi-invariant w.r.t. $\mathrm{Diff}_{\mathrm{small}}(X)$. Essential supremum w.r.t. σ_2 would be sufficient in Assumption 3.4. V, however, is typically continuous for $x \neq y$.

Corollary 3.8 *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be a measure fulfilling Assumption 3.6, and V a potential satisfying Assumptions 3.3–3.5. Let $\Lambda \in \mathcal{O}_c(X)$ with $\sigma(\partial\Lambda) = 0$. Then for μ-a.a. $\gamma \in \Gamma$*

$$\sum_{x \in \gamma_{\Lambda^c}} \sup_{y \in \Lambda} |V(x, y)| \; < \; \infty \,,$$

and $0 < Z_\Lambda(\gamma) < \infty$. Moreover, for all $x \in \Lambda$ the sum

$$W(\{x\}, \gamma_{\Lambda^c}) \; = \; \sum_{y \in \gamma_{\Lambda^c}} V(x, y)$$

is absolutely convergent.

Proof. Apply Proposition 3.2 for $G(y) := \mathbf{1}_{\Lambda^c}(y) \sup_{x \in \Lambda} |V(x, y)|$. Note that

$$\int_X (G(y) \wedge 1)\, \rho^{(1)}_\mu(dy) \; \leq \; C_1 \int_X \left(\sup_{x \in \Lambda} |V(x, y)| \wedge 1 \right) \sigma(dy)$$

and $G(y) < \infty$ if $y \notin \partial\Lambda$. Hence, for μ-a.a. γ there exists a constant $C_\Lambda(\gamma)$ such that

$$\sum_{x \in \gamma_{\Lambda^c}} \sup_{y \in \Lambda} |V(x, y)| \; \leq \; C_\Lambda(\gamma) \,.$$

Let $\eta \in \Gamma_\Lambda$ be given, then we have $W(\eta, \gamma_{\Lambda^c}) \leq C_\Lambda(\gamma)\, |\eta|$ and therefore

$$Z_\Lambda(\gamma) \; = \; \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} e^{-E(\eta) - W(\eta, \gamma_{\Lambda^c})}\, \pi^\Lambda_\sigma(d\eta) \; \geq \; \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} e^{-E(\eta) - C_\Lambda(\gamma)|\eta|}\, \pi^\Lambda_\sigma(d\eta) \; > \; 0 \,.$$

Analogously, we see that $Z_\Lambda(\gamma) < \infty$ μ-a.s. □

3.3 Characterization results

First, the Radon-Nikodym derivatives for a canonical Gibbs measure w.r.t. the diffeomorphism group are derived.

Theorem 3.9 *Let $\mu \in \mathcal{G}_c(V) \cap \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ fulfilling Assumption 3.6 for a potential V fulfilling the Assumptions 3.3–3.5. Then μ is $\mathrm{Diff}_0(X)$-quasi-invariant and*

$$\frac{d(\phi^*\mu)}{d\mu}(\gamma) \; = \; \exp\big(- E_{\mathrm{rel}}\big(\phi^{-1}(\gamma), \gamma\big)\big)\, \frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma) \,, \tag{3.3}$$

where $\phi \in \mathrm{Diff}_0(X)$ and

$$E_{\mathrm{rel}}(\phi^{-1}(\gamma), \gamma) \; := \; \sum_{\{x,y\} \in \gamma} \big(V\big(\phi^{-1}(x), \phi^{-1}(y)\big) - V(x, y) \big) \,,$$

i.e. E_{rel} is the relative energy (in analogy to the lattice case, see e.g. [15]), and where

$$\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma) \; := \; \prod_{x \in \gamma} \frac{\rho(\phi^{-1}(x))}{\rho(x)}\, J\phi^{-1}(x) \,.$$

P r o o f. Let $\phi \in \mathrm{Diff}_0(X)$ be given and $F : \Gamma \to \mathbb{R}_0^+$ a $\mathcal{B}_\Lambda(\Gamma)$-measurable function for a $\Lambda \in \mathcal{O}_c(X)$ with $\sigma(\partial\Lambda) = 0$ and w.l.o.g. $\mathrm{supp}\,\phi \subset \Lambda$. Then by Definition 2.1 we have

$$\int_\Gamma F(\phi(\gamma))\,\mu(d\gamma) \;=\; \int_\Gamma \frac{\mathbf{1}_{\{0<Z_\Lambda<\infty\}}(\gamma)}{Z_\Lambda(\gamma)} \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F(\phi(\eta)) e^{-E(\eta)-W(\eta,\gamma_{\Lambda^c})}\,\sigma_{|\gamma_\Lambda|}(d\eta)\mu(\gamma)\,. \quad (3.4)$$

We note that

$$\frac{d(\phi^*\sigma_{|\gamma_\Lambda|})}{d\sigma_{|\gamma_\Lambda|}}(\eta) \;=\; \frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)$$

and

$$\begin{aligned} E_{\mathrm{rel}}\big(\phi^{-1}(\eta\cup\gamma_{\Lambda^c}),\eta\cup\gamma_{\Lambda^c}\big) \;&=\; E\big(\phi^{-1}(\eta)\big) + W\big(\phi^{-1}(\eta),\gamma_{\Lambda^c}\big) - E(\eta) - W(\eta,\gamma_{\Lambda_c}) \\ &=\; E_\Lambda(\phi^{-1}(\gamma)) - E_\Lambda(\gamma)\,, \end{aligned}$$

since all sums are absolutely convergent according to Corollary 3.8 for μ-a.a. $\gamma \in \Gamma$. Therefore, applying the usual Radon-Nikodym theorem to the manifold $\Gamma_\Lambda^{(|\gamma_\Lambda|)}$ on the right-hand side of (3.4)

$$\int_\Gamma \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F(\eta) e^{-E_{\mathrm{rel}}(\phi^{-1}(\eta\cup\gamma_{\Lambda^c}),\eta\cup\gamma_{\Lambda^c})}\,\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)\,\Pi_\Lambda^c(d\eta,\gamma)\,\mu(d\gamma)\,.$$

The result now follows by Definition 2.1. □

We proceed to show that (3.3) already characterizes canonical Gibbs measures.

Theorem 3.10 *Let be given a measure* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *fulfilling Assumption* 3.6 *and a potential* V *fulfilling Assumptions* 3.3–3.5. *If for all* $\phi \in \mathrm{Diff}_{\mathrm{small}}(X)$ *we have*

$$\frac{d(\phi^*\mu)}{d\mu}(\gamma) \;=\; \exp\big(-E_{\mathrm{rel}}\big(\phi^{-1}(\gamma),\gamma\big)\big)\,\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma)\,, \quad (3.5)$$

then μ *is a canonical Gibbs measure, i.e.,* $\mu \in \mathcal{G}_c(V)$.

P r o o f. $E_{\mathrm{rel}}\big(\phi^{-1}(\gamma),\gamma\big) = \sum_{\{x,y\}\in\gamma} V\big(\phi^{-1}(x),\phi^{-1}(y)\big) - V(x,y)$ is well defined and the series is μ-a.s. absolutely convergent according to Corollary 3.8. Let ϕ be a diffeomorphism from $\mathrm{Diff}_{\mathrm{small}}(X)$ and choose $\Lambda \in \mathcal{O}_c(X)$ connected with $\sigma(\partial\Lambda) = 0$ such that $\mathrm{supp}\,\phi \subset \Lambda$. Take a $F = F_1 \cdot F_2$ where $F_1 \in L^0(\Gamma,\mathcal{B}_\Lambda(\Gamma))$ and $F_2 \in L^0(\Gamma,\mathcal{B}_{\Lambda^c}(\Gamma))$. If we denote by μ_Λ the conditional probability measure of μ w.r.t. $\mathcal{F}_{\Lambda^c}$, we then have to show that μ_Λ is equal to (2.6) μ-a.s. Hence, using the definition of conditional probability we can write

$$\int_\Gamma F(\phi(\gamma))\,\mu(d\gamma) \;=\; \int_\Gamma \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F_2(\gamma)F_1(\phi(\eta))\,\mu_\Lambda(d\eta,\gamma)\,\mu(d\gamma)\,.$$

On the other hand we have

$$\begin{aligned} \int_\Gamma F(\phi(\gamma))\,\mu(d\gamma) \;&=\; \int_\Gamma F_2(\gamma) \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F_1(\eta) e^{-E_{\mathrm{rel}}(\phi^{-1}(\eta)\cup\gamma_{\Lambda^c},\eta\cup\gamma_{\Lambda^c})} \\ &\qquad\qquad \cdot\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)\,\mu_\Lambda(d\eta,\gamma)\,\mu(d\gamma)\,. \end{aligned}$$

Because of the countability of $\mathrm{Diff}_{\mathrm{small}}(X)$ for μ-a.a. γ the following holds for all $F_3 \in L^0(\Gamma,\mathcal{B}(\Gamma_\Lambda))$

$$\begin{aligned} \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F_3(\phi(\eta))\,\mu_\Lambda(d\eta,\gamma) \;&=\; \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} F_3(\eta) e^{-E_{\mathrm{rel}}(\phi^{-1}(\eta)\cup\gamma_{\Lambda^c},\eta\cup\gamma_{\Lambda^c})} \\ &\qquad\qquad \cdot\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)\,\mu_\Lambda(d\eta,\gamma)\,. \end{aligned}$$

Now we apply Lemma 3.1 for $r_\gamma(\eta) := e^{-E_\Lambda(\eta\cup\gamma_{\Lambda^c})}\prod_{x\in\eta}\rho(x)$. According to Corollary 3.8 for μ-a.a. γ we have $r_\gamma > 0$ and

$$\frac{r_\gamma\big(\phi^{-1}(\eta)\big)}{r_\gamma(\eta)}\prod_{x\in\eta} J\phi^{-1}(x) \;=\; e^{-E_{\mathrm{rel}}(\phi^{-1}(\eta)\cup\gamma_{\Lambda^c},\eta\cup\gamma_{\Lambda^c})}\,\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)\,.$$

Thus, the measure μ_Λ is of the form

$$\mu_\Lambda(d\eta,\gamma) \;=\; k\mathbf{1}_{\{N_\Lambda=N_\Lambda(\gamma)\}}(\eta)\,e^{-E_\Lambda(\eta\cup\gamma_{\Lambda^c})}\,\sigma_{|\gamma_\Lambda|}(d\eta)\,.$$

Since μ_Λ is a probability measure on Γ_Λ we have $k = (Z_\Lambda(\gamma))^{-1}$ and $0 < Z_\Lambda(\gamma) < \infty$ (cf. Corollary 3.8). Thus $\mu \in \mathcal{G}_c(V)$ by Definition 2.1. □

4 Ergodicity

A measure μ on Γ is called $\mathrm{Diff}_0(X)$-ergodic if the μ-a.s. constant functions are the only bounded measurable functions $F : \Gamma \to \mathbb{R}^+$ which have the property $F\circ\phi = F$ μ-a.s. for all $\phi \in \mathrm{Diff}_0(X)$. A measure μ from the convex set $\mathcal{G}_c(V)$ is called extreme if for all $\mu_1,\mu_2 \in \mathcal{G}_c(V)$ and $0 \le \alpha \le 1$ with $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, it follows that $\mu = \mu_1 = \mu_2$. The tail field σ-algebra $\mathcal{F}_\infty(\Gamma)$ is defined by

$$\mathcal{F}_\infty(\Gamma) \;:=\; \bigcap_{\Lambda\in\mathcal{B}_c(X)} \mathcal{F}_{\Lambda^c}(\Gamma)\,.$$

The following results from [26, Th. 2.1, Lemma 2.4] are used in this section.

Lemma 4.1 *Let $\mu,\mu' \in \mathcal{G}_c(V)$, and let $F \in L^0(\Gamma,\mathcal{B}(\Gamma))$ such that F is positive and $\int_\Gamma F(\gamma)\,\mu(d\gamma) = 1$.*

(i) *μ is extreme iff μ is trivial on $\mathcal{F}_\infty(\Gamma)$, i.e., $\mu(B)$ is either 0 or 1 for each $B \in \mathcal{F}_\infty(\Gamma)$.*

(ii) *$F\mu \in \mathcal{G}_c(V)$ iff $\mathbb{E}_\mu[F|\mathcal{F}_\infty(\Gamma)] = F$ μ-a.s.*

(iii) *If $\mu \ne \mu'$ then $\mu \perp \mu'$, i.e. there exists a $B \in \mathcal{F}_\infty(\Gamma)$ with $\mu(B) = 1$ and $\mu'(B) = 0$.*

We call a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ admissible if μ fulfills the Assumption 3.6. The set of all admissible measures is convex and the set of all admissible canonical Gibbs measures $\mathcal{G}_{c,a}(V)$ is a face of $\mathcal{G}_c(V)$, in symbols

$$\mathrm{ext}\,(\mathcal{G}_{c,a}(V)) \;=\; \mathrm{ext}\,(\mathcal{G}_c(V)) \cap \mathcal{G}_{c,a}(V)\,.$$

The following lemma contains the part of the proof specific for the relation between Gibbs measures and the diffeomorphism group which is based on the characterization theorem via Radon-Nikodym derivatives. The main result follows then by general considerations for Gibbs measures, cf. e.g. [26].

Lemma 4.2 *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)\cap\mathcal{G}_c(V)$ fulfilling Assumption 3.6 for a potential V fulfilling Assumptions 3.3–3.5. Let $F : \Gamma \to \mathbb{R}^+$ be a measurable bounded function with $\int_\Gamma F(\gamma)\,\mu(d\gamma) = 1$ such that $F\circ\phi = F$ μ-a.s. for all $\phi \in \mathrm{Diff}_{\mathrm{small}}(X)$. Then $\nu := F\mu$ is also a canonical Gibbs measure.*

P r o o f. Let $G : \Gamma \to \mathbb{R}^+$ be another measurable bounded function, then it follows that

$$\int_\Gamma G(\phi(\gamma))\,\nu(d\gamma) \;=\; \int_\Gamma G(\gamma)F(\phi^{-1}(\gamma))\,d(\phi^*\mu)(\gamma) \;=\; \int_\Gamma G(\gamma)\,\frac{d(\phi^*\mu)}{d\mu}(\gamma)\,\nu(d\gamma)\,.$$

Therefore ν-a.s. we have $\frac{d(\phi^*\mu)}{d\mu}(\gamma) = \frac{d(\phi^*\nu)}{d\nu}(\gamma)$. Furthermore, for any measurable $H : \Gamma_0 \to \mathbb{R}^+$ and for $C := \sup_{\gamma\in\Gamma}|F(\gamma)|$ we have

$$\int_{\Gamma_0} H(\eta)\,\rho_\nu(d\eta) \;=\; \int_\Gamma (KH)(\gamma)F(\gamma)\,\mu(d\gamma) \;\le\; C\int_\Gamma (KH)(\gamma)\,\mu(d\gamma)$$

which implies that $\rho_\nu(d\eta) \le C\,\rho_\mu(d\eta)$. Hence with μ also ν fulfills the assumptions of Theorem 3.10 (characterization theorem via Radon-Nikodym derivatives) and we deduce that ν is a canonical Gibbs measure. □

We are now prepared to prove the main result of this section.

Theorem 4.3 *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \cap \mathcal{G}_c(V)$ fulfilling Assumptions* 3.6 *for a potential V fulfilling Assumptions* 3.3–3.5. *μ is extreme iff it is* $\mathrm{Diff}_{\mathrm{small}}(X)$*-ergodic.*

Proof. Assume that μ is an extreme measure. Let $F : \Gamma \to \mathbb{R}^+$ be a measurable bounded function such that $F \circ \phi = F$ μ-a.s. for all $\phi \in \mathrm{Diff}_{\mathrm{small}}(X)$. W.l.o.g. we may assume that $\int_\Gamma F(\gamma)\,\mu(d\gamma) = 1$. According to Lemma 4.2 we have $F\mu \in \mathcal{G}_c(V)$ and applying Lemma 4.1 (ii) we obtain

$$\mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) \;=\; F\,,\; \mu\text{-a.s.}\,.$$

According to Lemma 4.1 (i) the measure μ is trivial on $\mathcal{F}_\infty(\Gamma)$, this implies that F is constant μ-a.s. Hence μ is $\mathrm{Diff}_{\mathrm{small}}(X)$-ergodic.

Conversely, assume that μ is $\mathrm{Diff}_{\mathrm{small}}(X)$ -ergodic and there exist $\mu_1, \mu_2 \in \mathcal{G}_c(V)$ such that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. Thus $\mu_1 \ll \mu$ and there exists a measurable function $F : \Gamma \to \mathbb{R}^+$ with $\int_\Gamma F(\gamma)\,\mu(d\gamma) = 1$ such that $\mu_1 = F\mu$. It follows from Lemma 4.1 (ii) that $\mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F$ μ -a.s. and, hence for any $\phi \in \mathrm{Diff}_{\mathrm{small}}(X)$ we obtain

$$F \circ \phi \;=\; \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) \circ \phi \;=\; \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) \;=\; F\,.$$

In the second equality we use the fact that $\mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma))$ is $\mathcal{F}_{\Lambda^c}(\Gamma)$-measurable for a certain $\Lambda \in \mathcal{B}_c(X)$ such that $\mathrm{supp}\,\phi \subset \Lambda$. Since μ is $\mathrm{Diff}_{\mathrm{small}}(X)$-ergodic it implies that F is constant μ-a.s. Therefore $\mu_1 = \mu_2 = \mu$ and this proves that μ is extreme. □

As a direct consequence of the previous theorem and the general Proposition 2.4 in [26] one obtains the following result. This result completes the considerations in [31] as the identification of the decomposition in extremal Gibbs measures and $\mathrm{Diff}_0(X)$-ergodic measures is shown.

Corollary 4.4 *Let V be a potential fulfilling Assumptions* 3.3–3.5. *Then there exists a measurable structure on* $\mathrm{ext}(\mathcal{G}_{c,a}(V))$ *such that $\mu \in \mathcal{G}_{c,a}(V)$ if and only if there exists a unique probability measure P on* $\mathrm{ext}(\mathcal{G}_{c,a}(V))$ *s.t. for all bounded measurable functions F on Γ*

$$\int_\Gamma F(\gamma)\,\mu(d\gamma) \;=\; \int_{\mathrm{ext}(\mathcal{G}_{c,a}(V))} \int_\Gamma F(\gamma)\,\mu_{ex}(d\gamma)\,P(d\mu_{ex})\,.$$

Each $\mu_{ex} \in \mathrm{ext}(\mathcal{G}_{c,a}(V))$ is $\mathrm{Diff}_0(X)$*-ergodic.*

Furthermore, for each $\mu_{ex} \in \mathrm{ext}\,(\mathcal{G}_{c,a}(V))$ exist a boundary condition $\gamma \in \Gamma$ and a sequence of volumes $(\Lambda_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)$ with $\Lambda_n \uparrow X$ such that for all $\Lambda \in \mathcal{B}_c(X)$ and all $B \in \mathcal{B}_\Lambda(\Gamma)$

$$\lim_{n\to\infty} \Pi^c_{\Lambda_n}(B,\gamma) \;=\; \mu_{ex}(B)\,.$$

We are now ready to state the result concerning the irreducibility of the unitary representation V_μ of the group $\mathrm{Diff}_0(X)$ associated with $\mu \in \mathcal{G}_c(V)$. The proof is a consequence of the results in this section and Theorem 1, §3 in [9] or Corollary 28.1 in Chapter 5 of [18].

Theorem 4.5 *Let $\mu \in \mathcal{G}_c(V) \cap \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be an admissible canonical Gibbs measure fulfilling Assumption* 3.6 *for a potential V fulfilling Assumptions* 3.3–3.5. *Then the unitary representation*

$$(V_\mu(\phi)F)(\gamma) \;:=\; \sqrt{\frac{d\phi^*\mu}{d\mu}(\gamma)}\,F\big(\phi^{-1}(\gamma)\big)\,, \quad F \in L^2(\Gamma,\mu)\,,\;\; \phi \in \mathrm{Diff}_0(X) \tag{4.1}$$

is irreducible iff μ is extreme.

If $\mu_1, \mu_2 \in \mathrm{ext}\,(\mathcal{G}_c(V)) \cap \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ with $\mu_1 \neq \mu_2$ then V_{μ_1} is inequivalent to V_{μ_2}.

Remark 4.6 Let V_1, V_2 be two different potentials fulfilling Assumptions 3.3–3.5. For $\mu_i \in \mathrm{ext}(\mathcal{G}_c(V_i)) \cap \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ it does not hold in general that $\mu_1 \perp \mu_2$. The difference of the potentials must be "singular" enough, for example translation invariance may be assumed. In general this is a non trivial question.

5 Conclusions and generalizations

Under weak assumptions on the potential we have proved that the canonical Gibbs measures are characterized by their Radon-Nikodym derivatives. To do this we used conditional expectations to reduce ourselves to finite configurations. The absence of coinciding points allowed us to reduce further to quasi-invariant measures on open subsets of $\mathbb{R}^l$. Surely, this result holds not only for the whole group $\mathrm{Diff}_0(X)$, but also for reasonable subgroups. A related topic are the marked systems: in addition to the manifold structure, the one particle space has the structure of a fiber bundle, for simplicity take another manifold S and consider $X \times S$ as the one particle space. X here still describes the positions of the particles and S an internal degree of freedom like momentum, spin, charge, dipole-moment, type of particle, quantumness, etc. Typically in applications the intensity measure on S is finite and hence the thermodynamical limit is trivial in the direction of S. The difficulty is to get sufficient conditions on the interaction general enough for applications; uniform bounds for the influence of the marks are too restrictive. The ideas of Subsection 3.2 may be generalized to this case. Gibbs measures in the marked situation have full measure on a subspace of $\Gamma_{X\times S}$, the marked configuration space, i.e. the space of all configuration $\widehat{\gamma} \in \Gamma_{X\times S}$ such that for all $(x, s), (y, t) \in \widehat{\gamma}$ with $(x, s) \neq (y, t)$ it holds $x \neq y$. This formalizes the aforementioned interpretation of S as an internal degree of freedom. It is natural to consider a subgroup of diffeomorphism which respect the marked structure, these are all diffeomorphisms $\widehat{\phi} \in \mathrm{Diff}_0(X \times S)$ of the form $\widehat{\phi}(x, s) := (\phi(x), \psi(x, s))$. Furthermore, we may assume that $\psi(x, \cdot)$ is from the structure group of our bundle. More concretely, assume for example that S is a Lie group and $\psi(x, s) = \widetilde{\psi}(x) \cdot s$, $\widetilde{\psi} : X \to S$. This is not only of general mathematical interest, but also motivated from the view point of applications for example in quantum field theory. In [13] we show, following the same line of proof as in this paper, that it is essential to have the characterization by Radon-Nikodym derivatives for measures on the one particle space $X \times S$ to derive the characterization result for canonical Gibbs measures. A direct generalization of Theorem 4.3 implies that also in this case the extremal canonical Gibbs measures are ergodic w.r.t. the considered subgroup of diffeomorphisms. However, in general the corresponding representations on the corresponding L^2-spaces will be not any longer irreducible. The analysis of this situation will be part of future investigations.

Another line of generalization is to consider more general interactions than pair potentials, more explicitly consider functions $V : \Gamma_0 \to \mathbb{R}$ and define the corresponding conditional energy for $\eta \in \Gamma_\Lambda$, $\gamma \in \Gamma$ by

$$E_\Lambda\big(\eta \cup \gamma_{\Lambda^c}\big) := \sum_{\substack{\eta' \subset \eta \\ \eta' \neq \emptyset}} \sum_{\xi \Subset \gamma_{\Lambda^c}} V\big(\eta' \cup \xi\big)\,, \tag{5.1}$$

if the series is absolutely convergent and by $+\infty$ otherwise. As in the case of pair potentials the main technical difficulty is to give concrete conditions on the measure and potential such that uniformly the convergence of the series (5.1) may be controlled.

Assumption 5.1 Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be given and ρ_μ the corresponding correlation measure. Assume that ρ_μ is absolutely continuous w.r.t. λ_σ. Let $V : \Gamma_0 \to \mathbb{R}$ be a continuous function. Assume that for all $\Lambda \in \mathcal{O}_c(X)$ with $\sigma(\partial\Lambda) = 0$ and all $n \in \mathbb{N}$, $\delta = 1/m$, $m \in \mathbb{N}$

$$\int_{\Gamma_{\Lambda^c}} \left(\sup_{\eta \in \Gamma^{(n)}_{\delta,\Lambda}} \big|V\big(\eta \cup \xi\big)\big| \wedge 1 \right) \rho_\mu(d\xi) \;<\; \infty\,, \tag{5.2}$$

where

$$\Gamma^{(n)}_{\delta,\Lambda} := \left\{ \eta \in \Gamma^{(n)}_\Lambda \;\middle|\; d(x, y) > \delta \text{ for all } \{x, y\} \subset \eta \right\}.$$

Under this assumption the techniques of Subsection 3.2 show that for each Λ as in the conditions of that assumption ($n \in \mathbb{N}$, δ) and for μ-a.a. $\gamma \in \Gamma$ the series in (5.1) is uniformly convergent for all $\eta \in \Gamma^{(n)}_\Lambda$.

Theorem 5.2 *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $V : \Gamma_0 \to \mathbb{R}$ be given which fulfills Assumption 5.1. Then $\mu \in \mathcal{G}_{\mathrm{c}}(V)$ iff μ is quasi-invariant w.r.t. $\mathrm{Diff}_0(X)$ with Radon-Nikodym derivatives (3.3), where in this case the relative energy $E_{\mathrm{rel}}(\phi(\gamma), \gamma) := E_\Lambda(\phi(\gamma)) - E_\Lambda(\gamma)$ for a $\Lambda \in \mathcal{O}_c(X)$ with $\sigma(\partial\Lambda) = 0$ and $\mathrm{supp}\,\phi \subset \Lambda$. The measure μ is ergodic w.r.t. $\mathrm{Diff}_0(X)$ iff it is an extreme element of $\mathcal{M}^1_{\mathrm{fm}}(\Gamma) \cap \mathcal{G}_{\mathrm{c}}(V)$ and then the corresponding representation V_μ is irreducible.*

A sufficient condition for (5.2) is for example the following: (we consider for simplicity $X = \mathbb{R}^d$, $\sigma = m$) Let $\psi : \mathbb{R}^+ \to [0,1]$ be a decreasing function with $\int_0^\infty \psi(r) r^{d-1}\, dr < \infty$. Assume that

$$|V(\eta)| \wedge 1 \;\le\; \psi\left(\max_{\{x,y\}\subset\eta} |x-y|\right)^{|\eta|} .$$

and ρ_μ fulfills the Ruelle bound.

A natural question is if there exist quasi-invariant measures on Γ which are not Gibbs measures in the above general sense. Their conditional probability measures are absolutely continuous anyhow; the difficulty is to show regularity for the corresponding densities under weak assumptions on the Radon-Nikodym derivatives. This will be the subject for a forthcoming paper.

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