Representation of diffeomorphisms on compound Poisson space

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Abstract

In this talk we exhibit a new approach to analysis and geometry for a class of infinite dimensional manifolds, namely, compound configuration spaces as a natural generalization of the work [AKR98a]. This framework allowed us to obtain a representation of the Lie-algebra of compactly supported vector fields on X on compound Poisson space.

1 Introduction

Started with the work of Gelfand et al. [GGV75], many researchers consider representations on compound Poisson space, see also [Ism96]. Hence it is natural to ask about geometry and analysis on this space. On the other hand in statistical physics of continuous systems compound Poisson measures and their Gibbsian perturbation are used for description of many concrete models, see e.g. [AGL78].

In constructing analysis and geometry in the space of simple configurations Γ_X over a manifold X, i.e.,

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\},$$

an important tool is the action of the group of diffeomorphism $\mathrm{Diff}_0(X)$ on X which are equal to the identity outside a compact on the configuration space Γ_X (cf. [AKR98a]).

In this talk we present a natural extension of the results obtained in [AKR98a] to the case of compound configuration space Ω_X over a Riemannian manifold X, i.e., the space of $\mathbb{R}_+$-valued measures on X of the form

$$\Omega_X = \{\omega = \sum_{x\in\gamma_\omega} s_x \varepsilon_x \in \mathcal{D}' | s_x \in \operatorname{supp}\tau,\ \gamma_\omega \in \Gamma_X\},$$

where τ is a finite measure on $\mathbb{R}_+$. This geometry is constructed via a "lifting procedure" and is completely determined by the Riemannian structure on X.

The problem of analysis and geometry on infinite dimensional spaces is highly connected with the lacking of a good notion of "volume element" which is due to the fact that there is no Lebesgue measure on infinite dimensional linear spaces. In Subsection 3.2 we prove that the compound Poisson measure π_σ^τ on Ω_X for which ∇^{Ω_X} and $\operatorname{div}_{\pi_\sigma^\tau}^{\Omega_X}$ become dual operators on $L^2(\Omega_X, \pi_\sigma^\tau)$ (w.r.t. $\langle\cdot,\cdot\rangle_{T\Omega_X}$) is the right "volume element" corresponding to our differential geometry on Ω_X.

Let us stress that the "test" functions $\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$ (resp. "test" vector fields $\mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$) we consider as domains for our gradient ∇^{Ω_X} (resp. $\operatorname{div}_{\pi_\sigma^\tau}^{\Omega_X}$) above are of cylinder type, i.e., $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$ if and only if

$$\omega \mapsto F(\omega) = g_F(\langle\omega,\varphi_1\rangle,\ldots,\langle\omega,\varphi_N\rangle) \tag{1}$$

for some $N \in \mathbb{N}$, $\varphi_1,\ldots,\varphi_N \in \mathcal{D} := C_0^\infty(X)$, $g_F \in C_b^\infty(\mathbb{R}^N)$, and $V \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$ if and only if

$$\Omega_X \ni \omega \mapsto V(\omega) = \sum_{i=1}^N G_i(\omega) v_i, \tag{2}$$

$N \in \mathbb{N}$, $G_i \in \mathcal{F}C_b^\infty(\mathcal{D},\Omega_X)$, $v_i \in V_0(X)$, $i = 1,\ldots,N$. Here $C_0^\infty(X)$, $V_0(X)$ denote the set of all smooth functions, resp. vector fields on X with compact support, $C_b^\infty(\mathbb{R}^N)$ the set of all functions on $\mathbb{R}^N$ with all derivatives of any order bounded, and for all $\varphi \in C_0^\infty(X)$

$$\langle\omega,\varphi\rangle := \int_X \varphi(x)\,d\omega(x) = \sum_{x\in\gamma_\omega} s_x\varphi(x).$$

We would like to emphasize the contents of Subsection 3.3. It is well-known, see [GGV75, Sect. 6] that there is a canonical unitary representation on compound Poisson space, i.e., $L^2(\Omega_X, \pi_\sigma^\tau)$, of the group of diffeomorphisms

$\mathrm{Diff}_0(X)$. On the basis of our results described above, we provide a corresponding representation of the associated Lie algebra of compactly supported vector fields. We also exhibit explicit formulas for the corresponding generators.

Last we would like to mention that most of the results obtained extend in a natural way (along the lines of the work [AKR98b] and [AGL78]) to the case where compound Poisson measures are replaced by Gibbs measures of Ruelle type.

2 Measures on configuration spaces

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by T_xX; and the tangent bundle endowed with its natural differentiable structure will be denoted by $TX = \cup_{x\in X} T_xX$. The Riemannian metric on X associates to each point $x \in X$ an inner product on T_xX which we denote by $\langle \cdot, \cdot \rangle_{T_xX}$. The associated norm will be denoted by $|\cdot|_{T_xX}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

We can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_p(X) \subset \mathcal{M}(X),$$

where $\sum_{x\in\emptyset} \varepsilon_x :=$ zero measure, and $\mathcal{M}(X)$ (resp. $\mathcal{M}_p(X)$) denotes the set of all positive (resp. positive integer-valued) Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology. Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ-algebra.

2.1 Poisson measures

For the construction of a Poisson measure on Γ_X first we need to fix an intensity measure σ on the underlying manifold X. We take a density $\rho > 0$ m-a.s. such that $\rho^{1/2} \in L^1_{loc}(X)$ and put $d\sigma(x) = \rho(x)dm(x)$. Then σ is a non-atomic Radon measure on X, in particular, $\sigma(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_c(X)$.

There are different ways to define the Poisson measure π_σ with intensity σ on Γ_X, see e.g. [AKR98a] and [GV68]. Here we characterize π_σ by its Laplace transform.

Definition 2.1 *The Laplace transform of π_σ is given for $f \in C_0(X)$ by*

$$\begin{aligned} l_{\pi_\sigma}(f) &= \int_{\Gamma_X} \exp(\langle\gamma, f\rangle) d\pi_\sigma(\gamma) \\ &= \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x)\right). \end{aligned} \tag{3}$$

2.2 Compound Poisson measures

Let τ be a non-negative finite measure on $\mathbb{R}_+ :=]0, \infty[$ having all moments finite and satisfying the analyticity property

$$\exists C > 0 : \forall n \in \mathbb{N}_0 \quad \int_0^\infty |s|^n d\tau(s) < C^n n!. \tag{4}$$

We denote $\mathcal{D} := C_0^\infty(X)$ (the set of C^∞-functions on X with compact support) equipped with the usual topology, see e.g. [Aub82].

Definition 2.2 *A measure π_σ^τ on $\mathcal{D}'$ is called a compound Poisson measure if its Laplace transform is given for $\varphi \in \mathcal{D}$ by*

$$\begin{aligned} l_{\pi_\sigma^\tau}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle\omega, \varphi\rangle) d\pi_\sigma^\tau(\omega) \\ &= \exp\left(\int_X \int_0^\infty (e^{s\varphi(x)} - 1) d\tau(s) d\sigma(x)\right), \end{aligned} \tag{5}$$

see e.g. [GGV75].

The measure π_σ^τ has the following properties.

Proposition 2.3 *1. π_σ^τ has an analytic Laplace transform.*

2. π_σ^τ is supported on $\Omega := \Omega_X$, the space of compound configurations, i.e.,

$$\Omega = \{\omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_x \in \mathcal{D}' | s_x \in \operatorname{supp} \tau,\ \gamma_\omega \in \Gamma_X\},$$

in other words $\pi_\sigma^\tau(\Omega) = 1$.

3. If $\text{supp}\tau = \{1\}$*, i.e.,* $d\tau(s) = \varepsilon_1(ds)$*, then* $\pi_\sigma^\tau = \pi_\sigma$*.*

For the proof of the above proposition we refer to [KSSU98], [Oba84]. Actually the compound Poisson measure is a special case of the marked Poisson measure where instead $\omega = \sum_{x\in\gamma_\omega} s_x\varepsilon_x$ we have $\omega = (\gamma_\omega, m_\omega)$, $\gamma_\omega \in \Gamma_X$ and m_ω is a mapping from γ_ω to $\mathbb{R}_+$, see [KSS98] for more details.

2.3 The isomorphism between Poisson and compound Poisson spaces

Let us define a measure $\hat{\sigma}$ on $(X \times \mathbb{R}_+, \mathcal{B}(X \times \mathbb{R}_+))$ as the product measure of the measures τ and σ, i.e.,

$$d\hat{\sigma}(\hat{x}) := d\tau(s)d\sigma(x),\ \hat{x} = (x,s) \in X \times \mathbb{R}_+.$$

We define the Poisson measure $\pi_{\hat{\sigma}}$ with intensity measure $\hat{\sigma}$ on $(\Gamma_{X\times\mathbb{R}_+}, \mathcal{B}(\Gamma_{X\times\mathbb{R}_+}))$ via its Laplace transform

$$\begin{aligned} l_{\pi_{\hat{\sigma}}}(\hat{\varphi}) &= \int_{\hat{\Gamma}} \exp(\langle\hat{\gamma},\hat{\varphi}\rangle) d\pi_{\hat{\sigma}}(\hat{\gamma}) \\ &= \exp\left(\int_{X\times\mathbb{R}_+} (e^{\hat{\varphi}(\hat{x})} - 1) d\hat{\sigma}(\hat{x})\right),\ \hat{\varphi} \in \mathcal{D}(X \times \mathbb{R}_+). \end{aligned} \tag{6}$$

We notice that indeed $\pi_{\hat{\sigma}}$ is concentrated in a smaller space denoted by $\hat{\Gamma}$ of the following form

$$\hat{\Gamma} = \left\{\hat{\gamma} = \sum_{\hat{x}_i\in\hat{\gamma}} \varepsilon_{\hat{x}_i},\ \hat{x}_i = (x_i, s_i) \in X \times \mathbb{R}_+,\ x_i \neq x_j,\ i \neq j\right\}.$$

This follows from the construction of Poisson measure and the fact that σ is a non-atomic measure on X. It follows from (4) that the Laplace transform $l_{\pi_{\hat{\sigma}}}$ is well defined for $\hat{\varphi}(s,x) = p(s)\varphi(x)$ where $p(s) = \sum_{k=0}^m p_k s^k$ $(p_0 \neq 0)$ is a polynomial and $\varphi \in \mathcal{D}$ (cf. [LRS97]). Let us put $\hat{\varphi}(s,x) = s\varphi(x)$, $\varphi \in \mathcal{D}$ in (6). Then by (3) we obtain

$$l_{\pi_\sigma^\tau}(\varphi) = l_{\pi_{\hat{\sigma}}}(s\varphi),\ \varphi \in \mathcal{D}.$$

Then it follows that the compound Poisson measure π_σ^τ is the image of $\pi_{\hat{\sigma}}$ under the transformation $\Sigma : \hat{\Gamma} \to \Sigma\hat{\Gamma} = \Omega \subset \mathcal{D}'$ given by

$$\hat{\Gamma} \ni \hat{\gamma} \mapsto (\Sigma\hat{\gamma})(\cdot) = \Sigma\left(\sum_{\hat{x}_i\in\hat{\gamma}} \varepsilon_{\hat{x}_i}\right)(\cdot) := \sum_{(s_i,x_i)\in\hat{\gamma}} s_i\varepsilon_{x_i}(\cdot) \in \Omega \subset \mathcal{D}', \tag{7}$$

i.e., $\forall B \in \mathcal{B}(\mathcal{D}')$

$$\pi_\sigma^\tau(B) = \pi_\sigma^\tau(B \cap \Omega) = \pi_{\hat\sigma}(\Sigma^{-1}(B \cap \Omega)),$$

where $\Sigma^{-1}\Delta$ is the pre-image of the set Δ.

Then for any $h \in L^1(\mathcal{D}', \pi_\sigma^\tau) = L^1(\Omega, \pi_\sigma^\tau)$ the function $h \circ \Sigma \in L^1(\hat\Gamma, \pi_{\hat\sigma})$ we have

$$\int_\Omega h(\omega) d\pi_\sigma^\tau(\omega) = \int_{\hat\Gamma} h(\Sigma\hat\gamma) d\pi_{\hat\sigma}(\hat\gamma). \tag{8}$$

It is worth noting that there exists on Ω an inverse map $\Sigma^{-1} : \Omega \to \hat\Gamma$. And we obtain that $\pi_{\hat\sigma}$ on $\hat\Gamma$ is the image of π_σ^τ on Ω under the map Σ^{-1}, i.e., $\forall \hat C \in \mathcal{B}(\hat\Gamma)$, $\pi_{\hat\sigma}(\hat C) = \pi_\sigma^\tau(\Sigma \hat C)$. Thus for any $\hat f \in L^1(\hat\Gamma, \pi_{\hat\sigma})$ the function $\hat f \circ \Sigma^{-1} \in L^1(\Omega, \pi_\sigma^\tau)$ we obtain

$$\int_{\hat\Gamma} \hat f(\hat\gamma) d\pi_{\hat\sigma}(\hat\gamma) = \int_\Omega \hat f(\Sigma^{-1}\omega) d\pi_\sigma^\tau(\omega). \tag{9}$$

Now we construct a unitary isomorphism U_Σ between the Poisson space $L^2(\pi_{\hat\sigma}) := L^2(\hat\Gamma, \pi_{\hat\sigma})$ and the compound Poisson space $L^2(\pi_\sigma^\tau) := L^2(\Omega, \pi_\sigma^\tau)$. Namely,

$$L^2(\Omega, \pi_\sigma^\tau) \ni h \mapsto U_\Sigma h := h \circ \Sigma \in L^2(\hat\Gamma, \pi_{\hat\sigma})$$

and

$$L^2(\hat\Gamma, \pi_{\hat\sigma}) \ni \hat f \mapsto U_\Sigma^{-1} \hat f = \hat f \circ \Sigma^{-1} \in L^2(\Omega, \pi_\sigma^\tau).$$

The isometry of U_Σ and U_Σ^{-1} follows from (8) and (9), respectively

As a result we have established the following proposition.

Proposition 2.4 *The map U_Σ is a unitary isomorphism between the Poisson space $L^2(\pi_{\hat\sigma})$ and the compound Poisson space $L^2(\pi_\sigma^\tau)$.*

2.4 The group of diffeomorphisms and compound Poisson measures

Let us denote the group of all diffeomorphisms on X by $\mathrm{Diff}(X)$ and by $\mathrm{Diff}_0(X)$ the subgroup of all diffeomorphisms $\phi : X \to X$ with compact support, i.e., which are equal to the identity outside of a compact set (depending of ϕ). For any $f \in C_0(X)$ we have a continuous functional

$$\Omega \ni \omega \mapsto \langle \omega, f \rangle = \int_X f(x) d\omega(x) = \sum_{x \in \gamma_\omega} s_x f(x)$$

and given $\phi \in \mathrm{Diff}_0(X)$ we have $\langle \phi^*\omega, f\rangle = \langle \omega, f\circ\phi\rangle$.

Any $\phi \in \mathrm{Diff}_0(X)$ defines (pointwisely) a transformation of any subset of X and, consequently, the diffeomorphism ϕ has the following "lifting" from X to Ω :

$$\Omega \ni \omega = \sum_{x\in\gamma_\omega} s_x\varepsilon_x \mapsto \phi^*\omega = \sum_{x\in\gamma_\omega} s_x\varepsilon_{\phi(x)} \in \Omega.$$

This mapping is obviously measurable and we can define the image measure of π_σ^τ under ϕ as usually by $\phi^*\pi_\sigma^\tau = \pi_\sigma^\tau \circ \phi^{-1}$, i.e., $(\phi^*\pi_\sigma^\tau)(A) = \pi_\sigma^\tau(\phi^{-1}(A)),\ A \in \mathcal{B}(\Omega)$.

The following proposition shows that this transformation is nothing but a change of the intensity measure σ and τ is preserved.

Proposition 2.5 *For any $\phi \in \mathrm{Diff}_0(X)$ we have $\phi^*\pi_\sigma^\tau = \pi_{\phi^*\sigma}^\tau$.*

Proof. Due to the characterization of the measures it is enough to compute the Laplace transform of the measure $\phi^*\pi_\sigma^\tau$ to show the property.

Let $f \in C_0(X)$ be given. Then the Laplace transform of $\phi^*\pi_\sigma^\tau$ is given by

$$\begin{aligned}
\int_\Omega \exp(\langle\omega, f\rangle) d(\phi^*\pi_\sigma^\tau)(\omega) &= \int_\Omega \exp(\langle\omega, f\rangle) d\pi_\sigma^\tau(\phi^{-1}(\omega)) \\
&= \int_\Omega \exp(\langle\omega, f\circ\phi\rangle) d\pi_\sigma^\tau(\omega) \\
&= \exp\left(\int_X \int_0^\infty (e^{sf(x)} - 1) d\tau(s) d(\phi^*\sigma)(x)\right) \\
&= \int_\Omega \exp(\langle\omega, f\rangle) d\pi_{\phi^*\sigma}^\tau(\omega)
\end{aligned}$$

which is just the Laplace transform of the measure $\pi_{\phi^*\sigma}^\tau$. ■

For any $\phi \in \mathrm{Diff}_0(X)$ we introduce the Radon-Nikodym density of σ as

$$\begin{cases}
p_\phi^\sigma(x) := \frac{d(\phi^*\sigma)}{d\sigma}(x) = \frac{\rho(\phi^{-1}(x))}{\rho(x)} \frac{dm(\phi^{-1}(x))}{dm(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} J_m(\phi)(x), \\
\qquad \text{if } x \in \{0 < \rho < \infty\} \cap \{0 < \rho\circ\phi^{-1} < \infty\}; \\
p_\phi^\sigma(x) := 1, \text{ otherwise},
\end{cases} \tag{10}$$

where $J_m(\phi)$ is the Jacobian determinant of ϕ (with respect to the Riemannian volume m), see e.g. [Boo75]. Note that $p_\phi^\sigma(x) \equiv 1$ outside a compact.

The next proposition is a consequence of the Proposition 2.4, Skorokhod's theorem on absolute continuity of Poisson measures, see e.g. [Sko57], [Tak90] and also [Shi94]. It shows that π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$.

Proposition 2.6 *The compound Poisson measure π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ and for any $\phi\in\mathrm{Diff}_0(X)$ we have $p_\phi^{\pi_\sigma^\tau}=p_\phi^{\pi_{\lambda_\tau\sigma}}$, where $\lambda_\tau=\tau(\mathbb{R}_+)$, i.e.,*

$$p_\phi^{\pi_\sigma^\tau}(\omega)=\frac{d(\phi^*\pi_\sigma^\tau)}{d\pi_\sigma^\tau}(\omega)=\prod_{x\in\gamma_\omega}p_\phi^\sigma(x)\exp\left(\lambda_\tau\int_X(1-p_\phi^\sigma(x))d\sigma(x)\right).$$

Proof. Given $\phi\in\mathrm{Diff}_0(X)$ then $\hat{\phi}:=\phi\otimes\mathrm{id}\in\mathrm{Diff}(X\times\mathbb{R}_+)$. Hence having in mind the isomorphism described in Subsection 2.3 the Radon-Nikodym density of π_σ^τ with respect to the group $\mathrm{Diff}_0(X)$ is given by

$$\begin{aligned}
&p_\phi^{\pi_\sigma^\tau}(\omega)=U_\Sigma^{-1}p_{\phi\otimes\mathrm{id}}^{\pi_{\hat\sigma}}(\omega)\\
=&\prod_{\hat{x}\in\hat{\gamma}_\omega}\frac{d\hat\sigma\circ(\phi\otimes\mathrm{id})^{-1}}{d\hat\sigma}(\hat{x})\exp\left(\int_{X\times\mathbb{R}_+}\left(1-\frac{d\hat\sigma\circ(\phi\otimes\mathrm{id})^{-1}}{d\hat\sigma}(\hat{x})\right)d\hat\sigma(\hat{x})\right)\\
=&\prod_{\hat{x}\in\hat{\gamma}_\omega}p_\phi^\sigma(x)\exp\left(\lambda_\tau\int_X(1-p_\phi^\sigma(x))d\sigma(x)\right)\\
=&\ p_\phi^{\pi_{\lambda_\tau\sigma}}(\gamma_\omega),
\end{aligned}$$

where we have used [AKR98a, Prop. 2.2]. ■

3 Differential geometry on compound Poisson space

The underlying differentiable structure on X has a natural lifting to the configuration space Ω. As a result appear in Ω objects such as the gradient, the tangent space etc. Below we describe the corresponding constructions in details.

3.1 The tangent bundle of Ω

Let $V(X)$ be the set of all C^∞-vector fields on X (i.e., smooth sections of TX). We will use a subset $V_0(X)\subset V(X)$ consisting of all vector fields with compact support. $V_0(X)$ can be considered as an infinite dimensional Lie algebra which corresponds to the group $\mathrm{Diff}_0(X)$ in the following sense: for any $v \in V_0(X)$ we can construct the flow of this vector field as a collection of mappings $\phi_t^v : X \to X$, $t \in \mathbb{R}$ obtained by integrating the vector field.

Let us fix $v \in V_0(X)$. Having the group ϕ_t^v, $t \in \mathbb{R}$, we can consider for any $\omega \in \Omega$ the curve $\mathbb{R} \ni t \longmapsto \phi_t^v(w) \in \Omega$.

Definition 3.1 *For a function $F : \Omega \to \mathbb{R}$ we define the directional derivative along the vector field $v \in V_0(X)$ as*

$$(\nabla_v^\Omega F)(\omega) := \frac{d}{dt} F(\phi_t^{v*}\omega)|_{t=0},$$

provided the right hand side exists.

A class of functions where ∇_v^Ω can be computed explicitly is $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$. Namely let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ be of the form (1) and $v \in V_0(X)$, then we have

$$\begin{aligned} F(\phi_t^{v*}\omega) &= g_F(\langle \phi_t^{v*}\omega, \varphi_1 \rangle, \ldots, \langle \phi_t^{v*}\omega, \varphi_N \rangle) \\ &= g_F(\langle \omega, \varphi_1 \circ \phi_t^v \rangle, \ldots, \langle \omega, \varphi_N \circ \phi_t^v \rangle) \end{aligned}$$

and, therefore, an application of Definition 3.1 gives

$$(\nabla_v^\Omega F)(\omega) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle) \langle \omega, \nabla_v^X \varphi_i \rangle, \qquad (11)$$

where $\nabla_v^X \varphi$ is the directional (or Lie) derivative of $\varphi : X \to \mathbb{R}$ along the vector field $v \in V_0(X)$, i.e.,

$$(\nabla_v^X \varphi)(x) = \langle \nabla^X \varphi(x), v(x) \rangle_{T_x X},$$

where ∇^X denotes the gradient on X.

The expression of ∇_v^Ω on smooth cylinder functions given by (11) motivates the following definition.

Definition 3.2 *We introduce the tangent space $T_\omega \Omega$ to the configuration space Ω at the point $\omega \in \Omega$ as the Hilbert space of measurable ω-square-integrable sections (measurable vector fields) $V_\omega : X \to TX$ with the scalar product*

$$\langle V_\omega^1, V_\omega^2 \rangle_{T_\omega \Omega} = \int_X \langle V_\omega^1(x), V_\omega^2(x) \rangle_{T_x X} \, d\omega(x) \qquad (12)$$

$V^1_\omega, V^2_\omega \in T_\omega\Omega$. *The corresponding tangent bundle is*

$$T\Omega = \bigcup_{\omega\in\Omega} T_\omega\Omega.$$

Any $v \in V_0(X)$ *can be considered as a "constant" vector field on* Ω *such that* $\Omega \ni \omega \mapsto V_\omega(\cdot) = v(\cdot) \in T_\omega\Omega$ *and* $\langle v, v\rangle_{T_\omega\Omega} = \int_X |v(x)|^2_{T_xX} d\omega(x)$.

Usually in Riemannian geometry, having the directional derivative and a Hilbert space as the tangent space we can introduce the gradient.

Definition 3.3 *We define the intrinsic gradient of a function* $F : \Omega \to \mathbb{R}$ *as the mapping* $\Omega \ni \omega \mapsto (\nabla^\Omega F)(\omega) \in T_\omega\Omega$ *such that for any* $v \in V_0(X)$

$$(\nabla^\Omega_v F)(\omega) = \left\langle (\nabla^\Omega F)(\omega), v \right\rangle_{T_\omega\Omega}. \tag{13}$$

Note that (13), in particular, implies that $\nabla^\Omega_v F$ is the directional derivative along the "constant" vector field v on Ω. Furthermore, by (11) for any $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$ of the form (1) gives

$$(\nabla^\Omega F)(\omega; x) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1\rangle, \ldots, \langle \omega, \varphi_N\rangle)\nabla^X \varphi_i(x),\ \omega \in \Omega, x \in X. \tag{14}$$

3.2 Integration by parts and divergence on compound Poisson space

Let the compound configuration space Ω be equipped with the compound Poisson measure π^τ_σ (cf. Subsection 2.2). The set $\mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$ is a dense subset in $L^2(\Omega, \mathcal{B}(\Omega), \pi^\tau_\sigma) =: L^2(\pi^\tau_\sigma)$. For any vector field $v \in V_0(X)$ we have a differential operator in $L^2(\pi^\tau_\sigma)$ on the domain $\mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$ given by

$$\mathcal{F}C^\infty_b(\mathcal{D}, \Omega) \ni F \longmapsto \nabla^\Omega_v F \in L^2(\pi^\tau_\sigma).$$

Our aim is to compute the adjoint operator $\nabla^{\Omega *}_v$ in $L^2(\pi^\tau_\sigma)$. It corresponds, of course, to an integration by parts formula w.r.t. the measure π^τ_σ.

To this end we recall first of all the integration by parts formula for the measure σ. The logarithmic derivative of σ is given by the vector field

$$X \ni x \longmapsto \beta^\sigma(x) := \frac{\nabla^X \rho(x)}{\rho(x)} \in T_xX.$$

(where as usual $\beta^\sigma := 0$ on $\{\rho = 0\}$). For all $\varphi_1, \varphi_2 \in \mathcal{D}$ we have

$$\begin{aligned}
&\int_X (\nabla_v^X \varphi_1)(x)\varphi_2(x)d\sigma(x) \\
= &-\int_X \varphi_1(x)(\nabla_v^X \varphi_2)(x)d\sigma(x) - \int_X \varphi_1(x)\varphi_2(x)\beta_v^\sigma(x)d\sigma(x), \quad (15)
\end{aligned}$$

where

$$\beta_v^\sigma(x) := \langle \beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x) \quad (16)$$

is the so-called logarithmic derivative of the measure σ along the vector field v and $\mathrm{div}^X :=\mathrm{div}_m^X$ is the divergence on X with respect to m. Analogously, we define div_σ^X as the divergence on X with respect to σ, i.e., div_σ^X is the dual operator on $L^2(X, \sigma) =: L^2(\sigma)$ of ∇^X. Then on the one hand we can rewrite (15) as an operator equality on the domain $\mathcal{D} \subset L^2(\sigma)$:

$$\nabla_v^{X*} = -\nabla_v^X - \beta_v^\sigma,$$

where the adjoint operator is considered with respect to $L^2(\sigma)$. Note that, obviously, $\beta_v^\sigma \in L^2(\sigma)$ for all $v \in V_0(X)$. On the other hand we have

$$\mathrm{div}_\sigma^X = \beta^\sigma. \quad (17)$$

Having the logarithmic derivative β_v^σ we introduce an analogous object for the compound Poisson measure.

Definition 3.4 *For any $v \in V_0(X)$ we define the logarithmic derivative of the compound Poisson measure π_σ^τ along v as the following function on Ω :*

$$\Omega \ni \omega \mapsto B_v^{\pi_\sigma^\tau}(\omega) := \langle \gamma_\omega, \beta_v^\sigma\rangle = \int_X [\langle \beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x)]d\gamma_\omega(x). \quad (18)$$

A motivation for this definition is given by the following integration by parts formula.

Theorem 3.5 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ and any $v \in V_0(X)$ we have*

$$\begin{aligned}
&\int_\Omega (\nabla_v^\Omega F)(\omega)G(\omega)d\pi_\sigma^\tau(\omega) \\
= &-\int_\Omega F(\omega)(\nabla_v^\Omega G)(\omega)d\pi_\sigma^\tau(\omega) - \int_\Omega F(\omega)G(\omega)B_v^{\pi_\sigma^\tau}(\omega)d\pi_\sigma^\tau(\omega), \quad (19)
\end{aligned}$$

or

$$\nabla_v^{\Omega *} = -\nabla_v^{\Omega} - B_v^{\pi_\sigma^\tau} \tag{20}$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Omega)$ *in* $L^2(\pi_\sigma^\tau)$.

Proof. Due to Proposition 2.5 we have that

$$\int_\Omega F(\phi_t^v(\omega))G(\omega)d\pi_\sigma^\tau(\omega) = \int_\Omega F(\omega)G(\phi_{-t}^v\omega)d\pi_{\phi_t^{v*}\sigma}^\tau(\omega).$$

Differentiating this equation with respect to t and interchanging $\frac{d}{dt}$ with the integrals, by Definition 3.1 the left hand side becomes (19). To see that the right hand side also coincides with (19) we note that

$$\frac{d}{dt}G(\phi_{-t}^v(\omega))|_{t=0} = -(\nabla_v^\Omega G)(\omega)$$

and (by Proposition 2.6)

$$\begin{aligned}
& \frac{d}{dt}\left[\frac{d\pi_{\phi_t^{v*}\sigma}^\tau}{d\pi_\sigma^\tau}(\omega)\right]\Bigg|_{t=0} \\
= & \frac{d}{dt}\left[\prod_{x\in\gamma_\omega}\frac{\rho(\phi_t^{v*}(x))}{\rho(x)}J_m(\phi_t^v)(x)\right]\Bigg|_{t=0} \\
& +\frac{d}{dt}\left[\exp\left\{\lambda_\tau\int_X\left(1-\frac{\rho(\phi_t^{v*}(x))}{\rho(x)}J_m(\phi_t^v)(x)\right)d\sigma(x)\right\}\right]\Bigg|_{t=0}.
\end{aligned}$$

Using (16) and the formula $\frac{d}{dt}[J_m(\phi_t^v)(x)]|_{t=0} = -\mathrm{div}^X v(x)$, the latter expressions becomes equal to

$$\begin{aligned}
- & \sum_{x\in\gamma_\omega}[\langle\beta^\sigma(x),v(x)\rangle_{T_xX}+\mathrm{div}^X v(x)] \\
+ & \lambda_\tau\int_X[\langle\beta^\sigma(x),v(x)\rangle_{T_xX}+\mathrm{div}^X v(x)]d\sigma(x) \\
= & -\sum_{x\in\gamma_\omega}\beta_v^\sigma(x)+\lambda_\tau\int_X\beta_v^\sigma(x)d\sigma(x) = -B_v^{\pi_\sigma^\tau}(\omega),
\end{aligned}$$

where we have used the equality

$$\int_X\beta_v^\sigma(x)d\sigma(x) = -\int_X(\nabla_v^{X*}1)(x)d\sigma(x) = 0.$$

This completes the proof. ■

Definition 3.6 *For a vector field $V : \Omega \ni \omega \mapsto V_\omega \in T_\omega\Omega$ the intrinsic divergence $\mathrm{div}^\Omega_{\pi^\tau_\sigma} V$ is defined via the duality relation*

$$\int_\Omega \left\langle V_\omega, (\nabla^\Omega F)(\omega)\right\rangle_{T_\omega\Omega} d\pi^\tau_\sigma(\omega) = -\int_\Omega F(\omega)(\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) d\pi^\tau_\sigma(\omega) \tag{21}$$

for all $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$, provided it exists (i.e., provided

$$F \longmapsto \int_\Omega \left\langle V_\omega, (\nabla^\Omega F)(\omega)\right\rangle_{T_\omega\Omega} d\pi^\tau_\sigma(\omega)$$

is continuous on $L^2(\pi^\tau_\sigma)$).

The existence of the divergence, of course, requires some smoothness of the vector field. A class of smooth vector fields on Ω for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.7 *Let $V \in \mathcal{VF}C^\infty_b(\mathcal{D}, \Omega)$ be a vector field of the form (2), then we have*

$$(\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) = \sum_{j=1}^N (\nabla^\Omega_{v_j} G_j)(\omega) + \sum_{j=1}^N B^{\pi^\tau_\sigma}_{v_j}(\omega) G_j(\omega). \tag{22}$$

Proof. Due to the linearity of ∇^Ω it is sufficient to consider the case $N = 1$, i.e., $V_\omega(x) = G(\omega)v(x)$. Then for all $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$ and the Definition 3.6 we have

$$\begin{aligned}
\int_\Omega (\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) &= -\int_\Omega G(\omega) \left\langle v, (\nabla^\Omega F)(\omega)\right\rangle_{T_\omega\Omega} d\pi^\tau_\sigma(\omega) \\
&= -\int_\Omega (\nabla^{\Omega*}_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\
&= \int_\Omega (\nabla^\Omega_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\
&\quad + \int_\Omega B^{\pi^\tau_\sigma}_v(\omega) G(\omega) F(\omega) d\pi^\tau_\sigma(\omega),
\end{aligned}$$

where we have used (20). ■

Let us stress that all the above formulas can be obtained via a "lifting rule".

1. (lifting of functions): any function $\varphi \in \mathcal{D}$ generates a (cylinder) function on Ω (lifting of φ) by the formula

$$L_\varphi(\omega) := \langle \omega, \varphi \rangle, \ \omega \in \Omega. \tag{23}$$

2. (lifting of vector fields): let L_v be the lifting of the vector field $v \in V_0(X)$ on Ω, (cf. Definition 3.2). Then for any $v, w \in V_0(X)$ formula (12) can be written as

$$\langle L_v, L_w \rangle_{T_\omega \Omega} = L_{\langle v,w \rangle_{TX}}(\omega), \tag{24}$$

   i.e., the scalar product of lifting vector fields is computed as the lifting of the scalar product $\langle v(x), w(x) \rangle_{T_x X} = \varphi(x)$. This rule can be used as a definition of the tangent space $T_\omega \Omega$.

3. (lifting of gradient): the gradient in (11) has now the following interpretation:

$$(\nabla_v^\Omega L_\varphi)(\omega) = L_{\nabla_v^X \varphi}(\omega), \ \omega \in \Omega,$$

   and the gradient of L_φ is nothing but the lifting of the corresponding underlying gradient:

$$(\nabla^\Omega L_\varphi)(\omega) = L_{\nabla^X \varphi}(\omega).$$

4. (lifting of logarithmic derivative): it follows from (18) that the logarithmic derivative $B_v^{\pi_\sigma^\tau} : \Omega \to \mathbb{R}$ is obtained via the same lifting procedure of the corresponding logarithmic derivative $\beta_v^\sigma : X \to \mathbb{R}$, namely,

$$B_v^{\pi_\sigma^\tau}(\omega) = L_{\beta_v^\sigma}(\gamma_\omega).$$

   Or, equivalently, one has for the divergence of a lifted vector field:

$$\mathrm{div}_{\pi_\sigma^\tau}^\Omega (L_v) = L_{\mathrm{div}_\sigma^X v}.$$

3.3 Representations of the Lie algebra of vector fields

Using the property of quasi-invariance of the compound Poisson measure π_σ^τ we can define a unitary representation of the diffeomorphism group $\mathrm{Diff}_0(X)$

in the space $L^2(\pi_\sigma^\tau)$, see [GGV75]. Namely, for $\phi \in$Diff$_0(X)$ we define a unitary operator

$$(V_{\pi_\sigma^\tau}(\phi)F)(\omega) := F(\phi(\omega))\sqrt{\frac{d\pi_\sigma^\tau(\phi(\omega))}{d\pi_\sigma^\tau(\omega)}},\ F \in L^2(\pi_\sigma^\tau).$$

Then we have

$$V_{\pi_\sigma^\tau}(\phi_1)V_{\pi_\sigma^\tau}(\phi_2) = V_{\pi_\sigma^\tau}(\phi_1 \circ \phi_2),\ \phi_1, \phi_2 \in \mathrm{Diff}_0(X).$$

as in Subsection 3.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms $\phi_t^v, t \in \mathbb{R}$. It generates a one-parameter unitary group

$$V_{\pi_\sigma^\tau}(\phi_t^v) := \exp[itJ_{\pi_\sigma^\tau}(v)],\ t \in \mathbb{R}, \tag{25}$$

where $J_{\pi_\sigma^\tau}(v)$ denotes the self-adjoint generator of this group.

Proposition 3.8 *For any $v \in V_0(X)$ the following operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ holds:*

$$J_{\pi_\sigma^\tau}(v) = \frac{1}{i}\nabla_v^\Omega + \frac{1}{2i}B_v^{\pi_\sigma^\tau}. \tag{26}$$

Proof. Let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ be given. Then differentiating the left hand side of (25) at $t = 0$ we get

$$\begin{aligned}\frac{d}{dt}(V_{\pi_\sigma^\tau}(\phi_t^v)F)(\omega)|_{t=0} &= \frac{d}{dt}F(\phi_t^v(\omega))|_{t=0} + F(\omega)\frac{1}{2}\frac{d}{dt}\left.\frac{d\pi_\sigma^\tau(\phi_t^v(\omega))}{d\pi_\sigma^\tau(\omega)}\right|_{t=0} \\ &= (\nabla_v^\Omega F)(\omega) + \frac{1}{2}F(\omega)B_v^{\pi_\sigma^\tau}(\omega),\end{aligned}$$

where we have used the form of the operator $V_{\pi_\sigma^\tau}(\phi_t^v)$, the definition of the directional derivative ∇_v^Ω and Theorem 3.5. On the other hand the same procedure on the right hand side of (25) produce $i(J_{\pi_\sigma^\tau}(v)F)(\omega)$. Hence the result of the proposition follows. ■

Remark 3.9 *More generally, one can study a family of self-adjoint operators $J(v)$, $v \in V_0(X)$, in a Hilbert space $\mathcal{H}$ which gives a representation of the Lie algebra $V_0(X)$ in the sense of the following commutation relation:*

$$[J(v_1), J(v_2)] = -iJ([v_1, v_2]) \tag{27}$$

(on a dense domain in $\mathcal{H}$), where $[v_1, v_2] = \langle v_1, \nabla v_2\rangle_{TX} - \langle v_2, \nabla v_1\rangle_{TX}$ is the Lie-bracket of the vector fields $v_1, v_2 \in V_0(X)$. In the case discussed, this relation is a direct consequence of (26). Thus, we have constructed a compound Poisson space representation of the Lie algebra $V_0(X)$.

Let us define, in addition, a unitary representation of the additive group $\mathcal{D}$ given by the formula

$$(U_{\pi_\sigma^\tau}(f)F)(\omega) := \exp(i\langle\omega, f\rangle)F(\omega),\ F \in L^2(\pi_\sigma^\tau),\ \omega \in \Omega,$$

for any $f \in \mathcal{D}$. As usual, the semi-direct product $\mathcal{G} := \mathcal{D} \wedge \mathrm{Diff}_0(X)$ of the groups $\mathcal{D}$ and $\mathrm{Diff}_0(X)$ is defined as the set of pairs (f, ϕ) with multiplication operation

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1),$$

see e.g. [GGV75]. Let us introduce for any element $(f, \phi) \in \mathcal{G}$ the following operator on $L^2(\pi_\sigma^\tau)$:

$$W_{\pi_\sigma^\tau}(f, \phi) := U_{\pi_\sigma^\tau}(f)V_{\pi_\sigma^\tau}(\phi).$$

These operators are unitary and form a representation of the group $\mathcal{G}$. If we introduce multiplication operators $\rho_{\pi_\sigma^\tau}(f)$, $f \in \mathcal{D}$, as self-adjoint operators on $L^2(\pi_\sigma^\tau)$ which are defined for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ by the formula

$$(\rho_{\pi_\sigma^\tau}(f)F)(\omega) := \langle\omega, f\rangle F(\omega),\ \omega \in \Omega,$$

then $U_{\pi_\sigma^\tau}(f) = \exp[i\rho_{\pi_\sigma^\tau}(f)]$ and the form of the multiplication in $\mathcal{G}$ implies

$$[\rho_{\pi_\sigma^\tau}(f), J_{\pi_\sigma^\tau}(v)] = i\rho_{\pi_\sigma^\tau}(\nabla_v^X f)$$

(on a dense domain in $L^2(\pi_\sigma^\tau)$) for all $f \in \mathcal{D}$, $v \in \mathrm{Diff}_0(X)$. We also have the relation $[\rho_{\pi_\sigma^\tau}(f_1), \rho_{\pi_\sigma^\tau}(f_2)] = 0$. The family of operators $J_{\pi_\sigma^\tau}(v), \rho_{\pi_\sigma^\tau}(f)$, $v \in V_0(X)$, $f \in \mathcal{D}$, thus forms a compound Poisson representation of an infinite-dimensional Lie algebra. The particular case when $\tau = \varepsilon_1$ this representation is known as Lie algebra of currents in non relativistic Quantum field theory, e.g. [GGPS74].

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