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ON RELATIONS BETWEEN *A PRIORI* BOUNDS FOR MEASURES ON CONFIGURATION SPACES

YURI KONDRATIEV* and TOBIAS KUNA†

Fakultät für Mathematik and Research Center BiBos,
Universität Bielefeld, 33615 Bielefeld, Germany
**kondrat@mathematik.uni-bielefeld.de*
†*tkuna@mathematik.uni-bielefeld.de*

OLEKSANDR KUTOVIY

Fakultät für Mathematik, Universität Bielefeld,
33615 Bielefeld, Germany
kutoviy@mathematik.uni-bielefeld.de

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Some *a priori* bounds for measures on configuration spaces are considered. We establish relations between them and consequences for corresponding measures (such as support properties etc.). Applications to Gibbs measures are discussed.

Keywords: Configuration space; correlation functions; Gibbs state.

1. Introduction

The space of configurations Γ_X over a Riemannian manifold X consists of all locally finite subsets of X. Such spaces play an important role in the topology, the theory of point processes, the mathematical physics and several other areas of the mathematics and its applications. As objects of infinite dimensional analysis, configuration spaces form a class of infinite dimensional manifolds which are not in the well-known categories of Banach or Fréchet manifolds. Nevertheless, they can be equipped with a natural differentiable structure (coming from the underlying manifold X) with quite rich analytic and geometrical properties, see Refs. 1 and 2.

The measure theory on configuration spaces has several specific aspects comparing with the well-developed one in the case of linear spaces. Namely, in the linear case we have useful relations between such characteristics of measure as moments,

*Also at Institute of Mathematics, Kyiv, Ukraine.

the Laplace transform, support and integrability properties for some classes of functions on linear spaces, see e.g. Ref. 4 for a review and related historical comments and references. These characteristics need to be modified properly in configuration space analysis. Important instructive ideas in this area are coming from the theory of stochastic processes and statistical physics. In these applications measures on configuration spaces correspond to point processes and states of continuous systems respectively and in both areas we already have many deep results concerning properties of particular classes of such measures.

The point of view developed in the present paper is motivated mainly by results of classical statistical mechanics of continuous systems. In particular, in pioneering works of Dobrushin[6] and Ruelle[24] dedicated to the study of equilibrium states (*Gibbs measures*) in the case of pair potentials several properties of these measures related with analysis of their characteristics were discovered. Namely, the first characteristic of configuration space measures is the system of correlation functions (that is the system of reduced moments or coincidence densities in the point process theory). Correlation functions can be considered as an analog of the moments of measures in the linear space analysis. In the case of superstable pair potentials they satisfy the so-called Ruelle bound (RB)[24] which is very useful in applications. Another important bound obtained in the same paper is related to the density of finite volume projections of Gibbs measures (Ruelle probability bound (RPB)) which also became a standard technical tool in the equilibrium statistical physics. In particular, (RPB) gives information about the support of Gibbs measures. Dobrushin[6] proved exponential integrability w.r.t. Gibbs measures of some local functions on configuration spaces (Dobrushin exponential bound (DEB)) which also gives useful information about these measures.

In the present paper we consider measures on configuration spaces which satisfy (some generalizations of) the mentioned bounds. We have shown that these bounds, in fact, are related among each other and do not need to be restricted to the class of Gibbs measures. This is important, in particular, in applications to non-equilibrium problems. More precisely, in the study of the dynamics (e.g. Hamiltonian or stochastic) of continuous systems we need, typically, to restrict the class of initial states assuming one or another kind of *a priori* bounds on them. Actually, the necessity to transport the description of the time evolution from the traditional classical mechanics point of view (in terms of particle trajectories) to the evolution of states is a specific point in the rigorous statistical physics of continuous systems. We refer the reader to the excellent discussion of this concept in the review by Dobrushin, Sinai and Suhov.[7] In concrete examples we can see that the possibility to construct the time evolution of an initial state depends on the level of the deviation from the equilibrium state (i.e. on the information about "how non-equilibrium is the initial state").

Moreover, even in the case when the initial state is a Gibbs measure, the time evolution usually does not preserve the Gibbs property (at least, in the class of Gibbs measures with interactions of a finite order). But we can expect that the time evolution can be realized in a class on configuration space measures with certain *a priori* bounds. This hope is supported, in particular, by recent results on the stochastic dynamics of infinite particle systems.[15] One of the aims of this paper is to clarify which kinds of *a priori* bounds can be reasonable, in principle, for measures in the configuration space analysis and how modifications of these bounds are reflected in the properties of the measures (e.g., support properties etc.).

Note that even in the case of Gibbs measures with pair potentials, modifications of classical bounds are useful. For example, a generalization of the Ruelle bound for correlation functions, which we discussed in this paper, was already used essentially in Ref. 2 for the construction of equilibrium gradient stochastic dynamics of continuous systems with pair singular interactions. An additional motivation for the analysis developed in this paper is related with an important class of so-called fermion and boson measures, see e.g. Ref. 21 and references therein. Such measures are defined via explicitly given correlation functions and do not admit clear Gibbs type descriptions. The only way to study the properties of such measures is to use the bounds on correlation functions and their consequences.

2. General Facts and Notations

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. By $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all open and Borel sets, respectively. $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ denote the system of all sets in $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, respectively, which are bounded. The space of n-point configuration is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \{\eta \subset \mathbb{R}^d \,|\, |\eta| = n\}\,, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,,$$

where $|A|$ denotes the cardinality of the set A. Analogously the space $\Gamma_{0,\Lambda}^{(n)}$ is defined for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, which we denote for short by $\Gamma_\Lambda^{(n)}$.

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping $N_\Lambda : \Gamma_0^{(n)} \to \mathbb{N}_0$; $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For short we write $\eta_\Lambda := \eta \cap \Lambda$. As a set, $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n | x_k \neq x_l \quad \text{if } k \neq l\}\,,$$

i.e. $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group over $\{1, \ldots, n\}$. Hence $\Gamma_0^{(n)}$ inherits the structure of an $n \cdot d$-dimensional manifold. Applying this we can introduce a topology $\mathcal{O}(\Gamma_0^{(n)})$ on $\Gamma_0^{(n)}$. The corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0^{(n)})$ coincides with $\sigma(N_\Lambda | \Lambda \in \mathcal{B}_c(\mathbb{R}^d))$. The space of finite configurations $\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$ is equipped with the topology of disjoint union $\mathcal{O}(\Gamma_0)$. A set $B \in \mathcal{B}(\Gamma_0)$ (the corresponding Borel σ-algebra) is called bounded if there exists a $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and an $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$.

The configuration space

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, |\gamma \cap \Lambda| < \infty\,, \quad \text{for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\}$$

is equipped with the vague topology. The Borel σ-algebra $\mathcal{B}(\Gamma)$ is equal to the smallest σ-algebra for which all the mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$ are measurable, i.e.

$$\mathcal{B}(\Gamma) = \sigma(N_\Lambda | \Lambda \in \mathcal{B}_c(\mathbb{R}^d))$$

and filtration on Γ given by

$$\mathcal{B}_\Lambda(\Gamma) := \sigma(N_{\Lambda'} | \Lambda' \in \mathcal{B}_c(\mathbb{R}^d),\ \Lambda' \subset \Lambda) .$$

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a projection $p_\Lambda : \Gamma \to \Gamma_\Lambda$; $p_\Lambda(\gamma) := \gamma_\Lambda$ and w.r.t. this projections Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$. The following classes of function are used in the following: $L^0(\Gamma_0)$ is the set of all measurable functions on Γ_0, $L^0_{\mathrm{ls}}(\Gamma_0)$ is the set of functions which have additionally a local support, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ such that $G \upharpoonright_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$. $L^0_{\mathrm{bs}}(\Gamma_0)$ denotes the measurable functions with bounded support, $B(\Gamma_0)$ the set of bounded measurable functions. On Γ we consider the set of a cylinder functions $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable function $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. These functions are characterized by the following relation: $F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda)$.

Next we would like to describe some facts from Harmonic analysis on configuration space based on Refs. 11 and 13.

The following mapping between functions on Γ_0, e.g. $L^0_{\mathrm{ls}}(\Gamma_0)$, and functions on Γ, e.g. $\mathcal{F}L^0(\Gamma)$, plays a key role in our further considerations:

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi) , \quad \gamma \in \Gamma ,$$

where $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, see e.g. Refs. 18 and 19. The summation in the latter expression is extended over all finite subconfigurations of γ, in symbols $\xi \Subset \gamma$. K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi) , \quad \eta \in \Gamma_0 . \tag{1}$$

It is easy to see that for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $F \in \mathcal{F}L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$

$$K^{-1}F(\eta) = 1\!\!1_{\Gamma_\Lambda} (\eta) K^{-1}F(\eta) , \quad \forall\, \eta \in \Gamma_0 . \tag{2}$$

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ which have finite local moments of all orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$. A measure ρ on Γ_0 is called locally finite iff $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$, the set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e. for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\, (K^*\mu)(d\eta) .$$

$\rho_\mu := K^*\mu$ we call the correlation measure corresponding to μ.

As shown in Ref. 11 for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta)\,, \tag{3}$$

is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\mu(d\gamma)\,. \tag{4}$$

Fix a non-atomic and locally finite measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$ and hence on $\Gamma_0^{(n)}$. The measure on $\Gamma_0^{(n)}$ we denote by $\sigma^{(n)}$.

The *Lebesgue–Poisson measure* $\lambda_{z\sigma}$ on Γ_0 is defined as $\lambda_{z\sigma} := \sum_{n=0}^\infty \frac{z^n}{n!}\sigma^{(n)}$. Here $z > 0$ is the so-called activity parameter. The restriction of $\lambda_{z\sigma}$ to Γ_Λ will also be denoted by $\lambda_{z\sigma}$.

The *Poisson measure* $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi^\Lambda_{z\sigma}\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$, where $\pi^\Lambda_{z\sigma}$ is the measure on Γ_Λ defined by $\pi^\Lambda_{z\sigma} := e^{-z\sigma(\Lambda)}\lambda_{z\sigma}$.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous w.r.t. $\pi_{z\sigma}$ iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi^\Lambda_{z\sigma} = \pi_{z\sigma} \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_\Lambda(\mathbb{R}^d)$. In this case $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t $\lambda_{z\sigma}$. We denote by $k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_\sigma}(\eta)$, $\eta \in \Gamma_0$.

The functions

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}^+$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\})\,, & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0\,, & \text{otherwise} \end{cases} \tag{5}$$

are well-known correlation functions of statistical physics, see e.g. Refs. 24 and 25.

3. *A Priori* Bounds

Let σ be Lebesgue measure and $\|x\| = \max_k |x_k|$, $x \in \mathbb{R}^d$. For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, let

$$l_\Lambda = \sup_{x,y \in \Lambda} \|x - y\|$$

and $|\Lambda|$ denote the Lebesgue measure of Λ. The symbol $|\cdot|$ may therefore represent cardinality or Lebesgue measure, but the meaning will always be clear from the context.

Let $V:\ \Gamma_0^{(2)} \to \mathbb{R}$ be a pair potential.

Definition 3.1. A potential V is called stable (see Ref. 25) iff there exists a constant $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any configuration $\gamma \in \Gamma_\Lambda$ holds

$$\sum_{\{x,y\} \subset \gamma} V(x,y) \geq -B|\gamma|\,. \tag{6}$$

In the following we assume that all potentials under consideration are stable.

Consider $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ locally absolutely continuous w.r.t. $\pi_{z\sigma}$ and three types of bounds on it.

We will say that a measure μ satisfies the generalized Ruelle bound with potential V if the following holds:

- $(\mathrm{GRB})_V$: The correlation function $k_\mu(\eta)$ satisfies the inequality

$$k_\mu(\eta) \le C^{|\eta|} \exp\left[- \sum_{\{x,y\}\subset\eta} V(x,y) \right], \quad \eta \in \Gamma_0\,, \tag{7}$$

  with some $C > 0$.

We will say that a measure μ satisfies the Ruelle's probability bound if the following holds:

- (RPB): For any $g > 0$ there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ and $N \in \mathbb{N}_0$

$$\mu(\{\gamma \,|\, |\gamma_\Lambda| \ge N\}) \le \exp\left\{ -\alpha \frac{N^2}{l_\Lambda^d} + \delta l_\Lambda^d \right\}. \tag{8}$$

We will say that a measure μ satisfies the Dobrushin's exponential bound of type $\lambda > 0$ and order $p > 0$ if the following holds:

- $(\mathrm{DEB})_{(\lambda,\ p)}$: For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ there exists a constant $C_\Lambda > 0$ such that

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p} \mu(d\gamma) < C_\Lambda\,. \tag{9}$$

Remark 3.1. Obviously, for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ with $l_\Lambda = 0$ the bound (9) holds automatically. Therefore, in the sequel we will consider $(\mathrm{DEB})_{(\lambda,\ p)}$ only for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda > 0$.

Definition 3.2. A potential V is called *superstable* in the sense of Ginibre (see Refs. 9 and 20) iff for any $g > 0$ there exist $A > 0$ and $B \ge 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ and any configuration $\gamma \in \Gamma_\Lambda$ holds

$$\sum_{\{x,y\}\subset\gamma} V(x,y) \ge A\frac{|\gamma|^2}{l_\Lambda^d} - B|\gamma|\,. \tag{10}$$

In the sequel, we will sometimes write α_g, δ_g, A_g, B_g, instead of α, δ, A, B, to emphasize that these constants depend on g.

Theorem 3.1. (a) *For any* $\lambda > 0$ *and* $p \in (0,1]$

$$(\mathrm{GRB})_V \Rightarrow (\mathrm{DEB})_{(\lambda,\ p)}\,.$$

(b) *Let* V *be superstable in the sense of Ginibre. Then*

(i) $$(\mathrm{GRB})_V \Rightarrow (\mathrm{RPB})\,,$$

(ii) *for any* $\lambda > 0$ *and* $p \in (1, 2)$

$$(\mathrm{GRB})_V \Rightarrow (\mathrm{DEB})_{(\lambda,\ p)},$$

(iii) *for any* $\lambda > 0$ *and* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $0 < l_\Lambda^d \le A_{l_\Lambda}\lambda^{-1}$

$$(\mathrm{GRB})_V \Rightarrow (\mathrm{DEB})_{(\lambda,\ 2)}.$$

(c) *For any* $\lambda > 0$ *and* $p \in (0, 2)$

$$(\mathrm{RPB}) \Rightarrow (\mathrm{DEB})_{(\lambda,\ p)}.$$

(d) *For any* $\lambda > 0$ $(\mathrm{DEB})_{(\lambda,\ 2)}$ *with* $C_\Lambda \le e^{\delta l_\Lambda^d}, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\delta > 0$ *implies* (RPB).

Proof. (a) Using (1), stability of V and according to the bound on the correlation functions we have

$$\int_{\Gamma_\Lambda} |K^{-1}[e^{\lambda|\eta|^p}]|\rho_\mu(d\eta)$$

$$= \int_{\Gamma_\Lambda} \left| \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} e^{\lambda|\xi|^p} \right| \rho_\mu(d\eta)$$

$$\le \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi\subset\{x_1,\ldots,x_n\}} e^{\lambda|\xi|^p} C^n e^{-\sum_{\{x,y\}\subset\{x_1,\ldots,x_n\}} V(x,y)} dx_1\cdots dx_n$$

$$\le \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi\subset\{x_1,\ldots,x_n\}} e^{\lambda|\xi|} C^n e^{Bn} dx_1\cdots dx_n = \exp\left\{2zC|\Lambda|e^{\lambda+B}\right\}.$$

Due to (2) and (4) we conclude that

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p}\mu(d\gamma) = \int_{\Gamma_\Lambda} K^{-1}[e^{\lambda|\eta|^p}]\rho_\mu(d\eta) \le \exp\{2zC|\Lambda|e^{\lambda+B}\}.$$

(b) Now suppose that V is superstable in the sense of Ginibre.

(i) Define $S_\Lambda := \{\gamma \in \Gamma \,|\, |\gamma_\Lambda| \ge N\}$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Let $g > 0$ be any given variable. Then, using (1) for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ we have

$$\int_{\Gamma_\Lambda} |K^{-1}[1\!\!1_{S_\Lambda}(\eta)]|\rho_\mu(d\eta)$$

$$= \int_{\Gamma_\Lambda} \left| \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|}\, 1\!\!1_{S_\Lambda}(\xi) \right| \rho_\mu(d\eta) = \int_{\Gamma_\Lambda} \left| 1\!\!1_{S_\Lambda}(\eta) \sum_{\xi\subset\eta, |\xi|\ge N} (-1)^{|\eta\setminus\xi|} \right| \rho_\mu(d\eta).$$

According to the bound on the correlation functions and the superstability for given g, the latter expression can be estimated by

$$\int_{\Gamma_\Lambda} 1\!\!1_{S_\Lambda}(\eta) C^{|\eta|} \sum_{\xi\subset\eta, |\xi|\ge N} e^{-\sum_{\{x,y\}\subset\eta} V(x,y)} \lambda_{z\sigma}(d\eta)$$

$$\le \int_{\Gamma_\Lambda} 1\!\!1_{S_\Lambda}(\eta)(2C)^{|\eta|} e^{-A|\eta|^2 l_\Lambda^{-d} + B|\eta|} \lambda_{z\sigma}(d\eta) \le \exp\left\{-A\frac{N^2}{l_\Lambda^d} + 2zCe^B l_\Lambda^d\right\}.$$

In the last inequality we have used the fact that integration actually extends only over all $\eta \in \Gamma_\Lambda : |\eta| \geq N$.

Finally, (2) and (4) give us

$$\mu(\{\gamma |\, |\gamma_\Lambda| \geq N\}) = \int_{\Gamma_\Lambda} K^{-1}[\mathbb{1}_{S_\Lambda}(\eta)]\rho_\mu(d\eta) \leq \exp\left\{-A\frac{N^2}{l_\Lambda^d} + 2zCe^B l_\Lambda\right\}.$$

(ii) Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d), l_\Lambda > 0$ be arbitrary and fixed. Using (1) we have

$$\int_{\Gamma_\Lambda} |K^{-1}[e^{\lambda|\eta|^p}]|\rho_\mu(d\eta) \leq \int_{\Gamma_\Lambda} \sum_{\xi\subset\eta} \exp\{\lambda|\xi|^p\}\rho_\mu(d\eta).$$

The estimation for the correlation functions and the superstability of V for $g = l_\Lambda$ imply the following bound for the latter integral

$$\int_{\Gamma_\Lambda} \sum_{\xi\subset\eta} e^{\lambda|\xi|^p - Al_\Lambda^{-d}|\eta|^2 + B|\eta|} C^{|\eta|}\lambda_{z\sigma}(d\eta)$$

$$\leq \int_{\Gamma_\Lambda} (2C)^{|\eta|} e^{B|\eta|} e^{\lambda|\eta|^p - Al_\Lambda^{-d}|\eta|^2}\lambda_{z\sigma}(d\eta) \leq e^{2zC|\Lambda|e^B + C^\star_\Lambda}, \tag{11}$$

where $C^\star_\Lambda > 0$ is some constant such that

$$\lambda|\eta|^p - \frac{A}{l_\Lambda^d}|\eta|^2 \leq C^\star_\Lambda.$$

Therefore, using (2) and (4) we have

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p}\mu(d\gamma) = \int_{\Gamma_\Lambda} K^{-1}[e^{\lambda|\eta|^p}]\rho_\mu(d\eta) \leq e^{2zC|\Lambda|e^B + C^\star_\Lambda}.$$

(iii) Doing the same as in (ii) for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $0 < l_\Lambda \leq A_{l_\Lambda}\lambda^{-1}$ we obtain

$$\int_{\Gamma_\Lambda} |K^{-1}[e^{\lambda|\eta|^2}]|\rho_\mu(d\eta)$$

$$\leq \sum_{n=0}^\infty \frac{(zC)^n}{n!}\int_{\Lambda^n} \sum_{\xi\subset\{x_1,\ldots,x_n\}} e^{\lambda|\xi|^2 - A_{l_\Lambda}l_\Lambda^{-d}n^2 + B_{l_\Lambda}n} dx_1\cdots dx_n. \tag{12}$$

Since $l_\Lambda^d \leq A_{l_\Lambda}\lambda^{-1}$, (12) is bounded by

$$\sum_{n=0}^\infty \frac{(zC)^n}{n!}\int_{\Lambda^n} \sum_{\xi\subset\{x_1\cdots x_n\}} e^{\lambda[|\xi|^2 - n^2] + B_{l_\Lambda}n} dx_1\cdots dx_n \leq e^{2zC|\Lambda|e^{B_{l_\Lambda}}}.$$

The statement is now a direct consequence of (2) and (4).

(c) To prove this part of the theorem we need the following lemma which follows directly from the definition of distribution function for a random variable.

Lemma 3.1. *For any measurable $\xi : \Gamma \to \mathbb{R}_+$ and differentiable $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$ we have*

$$\int_\Gamma f\circ\xi(\gamma)\mu(d\gamma) = \int_0^\infty f'(x)\mu(\{\gamma \in \Gamma | \xi(\gamma) > x\})dx.$$

Using Lemma 3.1 for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda > 0$ we have

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p}\mu(d\gamma) = \int_0^\infty \mu\left(\left\{\gamma\middle||\gamma_\Lambda| > \frac{(\ln y)^{\frac{1}{p}}}{\lambda^{\frac{1}{p}}}\right\}\right) dy\,. \tag{13}$$

Due to (RPB) for $g = l_\Lambda$ we bound (13) by

$$\int_0^{\exp[(2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}]} 1dy + \int_{\exp[(2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}]}^\infty \exp\left\{-\alpha\frac{(\ln y)^{2/p}}{\lambda^{2/p}l_\Lambda^d} + \delta l_\Lambda^d\right\} dy\,.$$

A direct computation gives the following estimation for the latter expression:

$$2\exp[\delta l_\Lambda^d + (2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}]\,.$$

(d) Let $(\mathrm{DEB})_{(\lambda,\ 2)}$ holds for some $\lambda > 0$ with $C_\Lambda \le e^{\delta l_\Lambda^d}$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\delta > 0$ and $g > 0$ be arbitrary and given.

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ consider a function $g_\Lambda(x) = e^{\alpha l_\Lambda^{-d}x^2}$, $x \ge 0$, $0 \le \alpha \le \lambda g^d$. This function is increasing and $\int_\Gamma g_\Lambda(|\gamma_\Lambda|)\mu(d\gamma) \le C_\Lambda$ (it follows from $(\mathrm{DEB})_{(\lambda,\ 2)}$ and inequality $l_\Lambda \ge g$).

Thus, the generalized Chebyshev inequality shows that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$:

$$\mu(\{\gamma|\,|\gamma_\Lambda| \ge N\}) \le \frac{\int_\Gamma g_\Lambda(|\gamma_\Lambda|)\mu(d\gamma)}{e^{\alpha N^2 l_\Lambda^{-d}}} \le C_\Lambda e^{-\alpha N^2 l_\Lambda^{-d}} \le e^{-\alpha N^2 l_\Lambda^{-d} + \delta l_\Lambda^d}\,. \qquad \square$$

For each $i \in \mathbb{Z}^d$, let

$$Q_i = \{r \in \mathbb{R}^d | i_k - 1/2 < r_k \le i_k + 1/2, k = 1, \dots, d\}\,.$$

Define $|\gamma_i| = |\gamma \cap Q_i|$. For $k \in \mathbb{N}$, let Λ_k be the hypercube of the sidelength $2k-1$ centered at the origin in $\mathbb{R}^d$. Note that $|\Lambda_k| = l_{\Lambda_k}^d = (2k-1)^d$, $k \in \mathbb{N}$. Sometimes we will also regard Λ_k as a subset of $\mathbb{Z}^d$ by letting Λ_k represent $\Lambda_k \cap \mathbb{Z}^d$. For $i \in \mathbb{Z}^d$ let $\ln_+ \|i\| = \max\{1, \ln\|i\|\}$.

Following Ruelle[25] a measure μ is called tempered if μ is supported by the set

$$R_\infty = \bigcup_{N=1}^\infty R_N\,,$$

where $R_N = \{\gamma \in \Gamma | \sum_{i\in\Lambda_k} |\gamma_i|^2 \le N^2|\Lambda_k|, \forall\ k \ge 1\}$.

Consider two subsets of the configuration space:

$$P_\infty = \bigcup_{N=1}^\infty P_N\,,$$

where $P_N = \{\gamma \in \Gamma|\,|\gamma_{\Lambda_k}| \le N|\Lambda_k|, \forall\ k \ge 1\}$ and

$$U_\infty = \bigcup_{n=1}^\infty U_n\,,$$

where $U_n = \{\gamma \in \Gamma|\,|\gamma_i| \le n(\ln_+\|i\|)^{\frac{1}{2}}, \forall\ i \in \mathbb{Z}^d\}$.

Obviously, $R_\infty \subset P_\infty$ and for any tempered measure μ with (RPB), it is also possible to show that $\mu(U_\infty) = 1$ (see Refs. 10 and 16).

Proposition 3.1. (RPB) *implies* $\mu(P_\infty) = 1$.

Proof. Obviously,

$$\Gamma \backslash P_\infty = \bigcap_{N\geq 1} \bigcup_{k\geq 1} \{\gamma \in \Gamma | \, |\gamma_{\Lambda_k}| > N|\Lambda_k|\} \,.$$

Using σ-semi-additivity and monotonicity of the measure μ we have

$$\mu(\Gamma \backslash P_\infty) \leq \lim_{N\to\infty} \sum_{k\geq 1} \mu(\{\gamma \in \Gamma | \, |\gamma_{\Lambda_k}| > N|\Lambda_k|\}) \,. \tag{14}$$

Due to (RPB) one can show that the right-hand side of (14) equals to 0. □

Remark 3.2. Proposition 3.1 holds, if (RPB) is replaced by the following weaker probability bound:

there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ such that for any $N \geq N_0$, $N_0 \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\mu(\{\gamma | \, |\gamma_{\Lambda_k}| \geq N|\Lambda_k|\}) \leq \exp\{-(\alpha N - \delta)|\Lambda_k|\} \,. \tag{15}$$

Proposition 3.2. (RPB) *implies* $\mu(U_\infty) = 1$.

Proof. Using

$$\Gamma \backslash U_\infty = \bigcap_{n\geq 1} \bigcup_{i\in\mathbb{Z}^d} \{\gamma \in \Gamma | \, |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\} \,,$$

σ-semi-additivity and monotonicity of the measure μ we have

$$\mu(\Gamma \backslash U_\infty) \leq \lim_{n\to\infty} \sum_{i\in\mathbb{Z}^d} \mu(\{\gamma \in \Gamma | \, |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\}) \,. \tag{16}$$

Due to (RPB) we estimate (16) by

$$\lim_{n\to\infty} \sum_{i\in\mathbb{Z}^d} e^{-(\alpha n^2(\ln_+ \|i\|)-\delta)} \leq \lim_{n\to\infty} 2^{2d-1} d e^\delta \sum_{i=3}^{\infty} i^{d-1} e^{-\alpha n^2 \ln i} = 0 \,.$$

□

Remark 3.3. We will say that a measure μ satisfy $(\mathrm{RPB})^p$, $p > 0$ if the following holds:

- $(\mathrm{RPB})^p$: For any $g > 0$ there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ (may be g-dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ and $N \geq N_0$ for some $N_0 \in \mathbb{N}$

$$\mu(\{\gamma | \, |\gamma_\Lambda| \geq N\}) \leq \exp\left\{-\alpha \frac{N^p}{l_\Lambda^d} + \delta l_\Lambda^d\right\} \,. \tag{17}$$

Similar to the proof of Proposition 3.2 one can show that for any $p > 0$ the fulfillment of $(\mathrm{RPB})^p$ on the sets Q_i, $i \in \mathbb{Z}^d$ implies $\mu(U^p_\infty) = 1$. Here

$$U^p_\infty = \bigcup_{n=1}^{\infty} U^p_n\,,$$

$$U^p_n = \{\gamma \in \Gamma|\, |\gamma_i| \le n(\ln_+ \|i\|)^{\frac{1}{p}},\ \forall\, i \in \mathbb{Z}^d\}\,.$$

4. Stronger Consequences of Generalized Ruelle Bound

In this section we describe further conclusions which follow from $(\mathrm{GRB})_V$.

As before one can consider the partition of $\mathbb{R}^d$ on cubes, but now with sidelength equal to $g > 0$. Namely, for each $i \in \mathbb{Z}^d$ and any $g > 0$ let

$$Q^g_i = \{r \in \mathbb{R}^d | g(i_k - 1/2) < r_k \le g(i_k + 1/2), k = 1, \ldots, d\}$$

and $|\gamma_{i,g}| = |\gamma \cap Q^g_i|$.

By $\mathcal{J}_g(\mathbb{R}^d)$ we denote all finite unions of cubes of the form Q^g_i (such sets are used in the construction of the Jordan measure). Sometimes we will regard $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ as a subset of $\mathbb{Z}^d$ by letting Λ represent $\{i \in \mathbb{Z}^d | Q^g_i \subset \Lambda\}$.

Let $W : \Gamma_0 \to \mathbb{R}$ be a measurable increasing function, i.e. for γ, $\gamma' \in \Gamma_0$ such that $\gamma \subset \gamma'$: $W(\gamma) \le W(\gamma')$.

We will say that a measure μ satisfies the $(\mathrm{RPB})^W_V$ if the following holds:

- $(\mathrm{RPB})^W_V$: For any $g > 0$ there exist constants $B > 0$ and $\delta \in \mathbb{R}$ (may be g-dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$, any configuration $\gamma \in \Gamma_\Lambda$ and $L \in \mathbb{R}_+$

$$\mu(\{\gamma | W(\gamma_\Lambda) \ge L\}) \le \exp\{-L + \delta|\Lambda|\}\,, \tag{18}$$

and

$$\sum_{\{x,y\}\subset\gamma} V(x,y) - W(\gamma) \ge -B|\gamma|\,. \tag{19}$$

Proposition 4.1. *Suppose that there exists a measurable increasing function* W : $\Gamma_0 \to \mathbb{R}$ *which satisfies* (19). *Then* $(\mathrm{GRB})_V$ *implies* $(\mathrm{RPB})^W_V$.

Proof. Let $g > 0$ and $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ be arbitrary. Define $S := \{\gamma \in \Gamma | W(\gamma_\Lambda) \ge L\}$. Then using (19) we have

$$1\!\!1_S\,(\eta) e^{-\sum_{\{x,y\}\subset\eta} V(x,y)} \le 1\!\!1_S\,(\eta) e^{-W(\eta)+B|\eta|} \le e^{-L+B|\eta|}, \eta \in \Gamma_\Lambda\,.$$

Therefore, similarly to the proof of Theorem 3.1(i) we obtain

$$\mu(\{\gamma | W(\gamma_\Lambda) \ge L\}) \le e^{-L} \int_{\Gamma_\Lambda} (2C)^{|\eta|} e^{B|\eta|} \lambda_{z\sigma}(d\eta) = e^{-L+2zCe^B|\Lambda|}\,. \qquad \square$$

Remark 4.1. For any $0 \le \varepsilon < 1$ inequality (18) implies

$$\int_\Gamma e^{W(\gamma_\Lambda)^{1-\varepsilon}} \mu(d\gamma) < C_\Lambda, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

with some $C_\Lambda > 0$.

Indeed, let $g > 0$ be given. We increase any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ to a set $\Lambda_{\mathcal{J}} \in \mathcal{J}_g(\mathbb{R}^d)$ which is a union of all cubes $Q_{i,g}$, which have nonempty intersection with Λ. Then using Lemma 3.1 and the fact that the function W is increasing we have

$$\int_\Gamma e^{W(\gamma_\Lambda)^{1-\varepsilon}}\mu(d\gamma) \le \int_0^\infty \mu(\{\gamma | W(\gamma_{\Lambda_{\mathcal{J}}}) > (\ln y)^{\frac{1}{1-\varepsilon}}\})dy\,.$$

Inequality (18) implies the following bound for the latter integral:

$$\int_0^{\exp[2^{(1-\varepsilon)\varepsilon^{-1}}]} 1dy + \int_{\exp[2^{(1-\varepsilon)\varepsilon^{-1}}]}^\infty e^{-(\ln y)^{(1-\varepsilon)^{-1}}} + \delta|\Lambda_{\mathcal{J}}|dy\,. \tag{20}$$

A direct calculation gives us bound for (20):

$$2\exp[\delta|\Lambda_{\mathcal{J}}| + 2^{(1-\varepsilon)\varepsilon^{-1}}]\,.$$

In the literature different non-equivalent versions of the Ruelle's probability bound are known. The definition of (RPB) we used here can be found in Refs. 10 and 16. Besides this bound, Ruelle in Ref. 25 used also another one. Namely, we will say that a measure μ satisfies the strong Ruelle's probability bound if the following holds:

- (SRPB): For any $g > 0$ there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ (may be g-dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and $N \in \mathbb{N}$

$$\mu\left(\left\{\gamma \middle| \sum_{i\in\Lambda} |\gamma_{i,g}|^2 \ge N^2|\Lambda|\right\}\right) \le \exp\{-(\alpha N^2 - \delta)|\Lambda|\}\,. \tag{21}$$

As shown in Ref. 25 (SRPB) implies (RPB).

Definition 4.1. A potential V is called superstable in the sense of Ruelle[25] if for any $g > 0$ there exist $A > 0$, $B \ge 0$ (may be g-dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\gamma \in \Gamma_\Lambda$

$$\sum_{\{x,y\}\subset\gamma} V(x,y) \ge \sum_{i\in\Lambda}[A|\gamma_{i,g}|^2 - B|\gamma_{i,g}|]\,.$$

Lemma 4.1. *Ruelle's superstability implies Ginibre's superstability.*

Proof. Let $g > 0$ be given. We first increase, as before, any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ to a set $\Lambda_{\mathcal{J}} \in \mathcal{J}_g(\mathbb{R}^d)$ which is a union of all cubes $Q_{i,g}$, which have nonempty intersection with Λ. Then for any $\gamma \in \Gamma_\Lambda \subset \Gamma_{\Lambda_{\mathcal{J}}}$, Ruelle's superstability gives

$$\sum_{\{x,y\}\subset\gamma} V(x,y) \ge \sum_{i\in\Lambda_{\mathcal{J}}} A|\gamma_{i,g}|^2 - B|\gamma| \ge A\frac{g^d|\gamma|^2}{|\Lambda_{\mathcal{J}}|} - B|\gamma|\,.$$

Because $l_\Lambda \ge g$, one can show that for large $\kappa > 1$ the following inequality holds

$$|\Lambda_{\mathcal{J}}| \le \kappa l_\Lambda^d$$

and the assertion of the lemma is now obvious. □

Proposition 4.2. *Let V be superstable in the sense of Ruelle. Then*

$$(\mathrm{GRB})_V \to (\mathrm{SRPB}).$$

Proof. It follows immediately from Proposition 4.1 by choosing $W(\gamma) = A\sum_{i\in\Lambda}|\gamma_{i,g}|^2$. □

Proposition 4.3. (SRPB) *implies* $\mu(R_\infty) = 1$.

Proof. Let, as above, Λ_k denote the hypercube of sidelength $2k-1$ centered at the origin in $\mathbb{R}^d$. Using the equality

$$\Gamma\backslash R_\infty = \bigcap_{N\geq 1}\bigcup_{k\geq 1}\left\{\gamma\in\Gamma|\sum_{i\in\Lambda_k}|\gamma_i|^2 > N^2|\Lambda_k|\right\},$$

σ-semi-additivity and monotonicity of the measure μ, we have

$$\mu(\Gamma\backslash R_\infty) \leq \lim_{N\to\infty}\sum_{k\geq 1}\mu\left(\left\{\gamma\in\Gamma|\sum_{i\in\Lambda_k}|\gamma_i|^2 > N^2|\Lambda_k|\right\}\right). \tag{22}$$

Due to (SRPB) for $g=1$ we bound (22) by

$$\lim_{N\to\infty}\sum_{k\geq 1}e^{-(\alpha N^2-\delta)|\Lambda_k|} = \lim_{N\to\infty}\sum_{k\geq 1}e^{-(\alpha N^2-\delta)(2k-1)^d} = 0.$$

□

Remark 4.2. Proposition 4.3 holds if in (SRPB) we substitute (21) by the following weaker probability bound:

for any $N\geq N_0$, $N_0\in\mathbb{N}$,

$$\mu\left(\left\{\gamma\left|\sum_{i\in\Lambda}|\gamma_{i,g}|^2 \geq N^2|\Lambda|\right.\right\}\right) \leq \exp\{-(\alpha N-\delta)|\Lambda|\}. \tag{23}$$

Corollary 4.1. *Let V be superstable in the sense of Ruelle. Then*

$$(\mathrm{GRB})_V \Rightarrow \mu(R_\infty) = \mu(P_\infty) = \mu(U_\infty) = 1.$$

5. Examples

In this section we consider some class of examples known from the statistical physics to which the results of this paper can be applied. One of them is related to the so-called Gibbs states (see Ref. 8 for more details) and another with states constructed by a given family of correlation functions (see Ref. 3).

Example 5.1. (Gibbs states with pair potentials). The Hamiltonian $E^V:\Gamma_0\to\mathbb{R}$ which corresponds to the potential V (even function on $\mathbb{R}^d$) is defined by

$$E^V(\eta) = \sum_{\{x,y\}\subset\eta}V(x-y), \eta\in\Gamma_0, |\eta|\geq 2.$$

Having in mind applications in mathematical physics, we will always assume positivity of V for small distances. More precisely, we suppose that there exists g, $0 < g < \infty$, such that $V(x) \geq 0$ for $|x| \leq g$.

For fixed V we will write for short $E = E^V$ and for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ we will sometimes write $E_\Lambda(\eta)$ instead of $E(\eta)$.

For a given $\bar{\gamma} \in \Gamma$ define the interaction energy between $\eta \in \Gamma_\Lambda$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \backslash \Lambda$ as

$$W_\Lambda(\eta|\bar{\gamma}) = \sum_{x \in \eta, y \in \bar{\gamma} \cap \Lambda^c} V(x - y)\,. \tag{24}$$

Define

$$E_\Lambda(\eta|\bar{\gamma}) = E_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma})\,.$$

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\gamma} \in \Gamma$. The finite volume Gibbs state with boundary configuration $\bar{\gamma}$ for E and $z > 0$ is

$$\mu_\Lambda(d\eta|\bar{\gamma}) = \frac{\exp\{-E_\Lambda(\eta|\bar{\gamma})\}}{Z_\Lambda(\bar{\gamma})} \lambda_{z\sigma}(d\eta)\,,$$

where

$$Z_\Lambda(\bar{\gamma}) = \int_{\Gamma_\Lambda} \exp\{-E_\Lambda(\eta|\bar{\gamma})\} \lambda_{z\sigma}(d\eta)\,.$$

This finite volume Gibbs state is well defined if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma$ the interaction energy $W_\Lambda(\eta|\bar{\gamma})$ does not become $-\infty$ and partition function $Z_\Lambda(\bar{\gamma})$ is finite. The assumptions, under which these conditions hold true will be introduced later.

When $\bar{\gamma} = \emptyset$, let $\mu_\Lambda(d\eta|\emptyset) \equiv \mu_\Lambda(d\eta)$.

Let $\{\pi_\Lambda\}$ denote the specification associated with z and the Hamiltonian E (see Ref. 23), which is defined on Γ by

$$\pi_\Lambda(A|\bar{\gamma}) = \int_{A'} \mu_\Lambda(d\eta|\bar{\gamma})\,,$$

where $A' = \{\eta \in \Gamma_\Lambda : \eta \cup (\bar{\gamma}_{\Lambda^c}) \in A\}$, $A \in \mathcal{B}(\Gamma)$.

A probability measure μ on Γ is called a Gibbs state for E and z if

$$\mu(\pi_\Lambda(A|\bar{\gamma})) = \mu(A)$$

for every $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is the well-known (DLR)-equation (Dobrushin–Lanford–Ruelle equation), see Ref. 8 for more details. We denote the class of all Gibbs states by $\mathcal{G}(V, z)$.

About the potential V we will assume:

Assumption 5.1. 1. Regularity:

$$\int_{\mathbb{R}^d} |1 - e^{-V(x)}| \sigma(dx) < \infty\,.$$

2. V is superstable in the sense of Ruelle.
3. V is lower regular, e.g. there exists a positive function ψ on the non-negative integers such that $\psi(m) \leq Km^{-\lambda}$ for $m \geq 1$, and for any Λ_1 and Λ_2 which are each finite unions of unit cubes of the form Q_i, with $\gamma \subset \Lambda_1$ and $\bar{\gamma} \subset \Lambda_2$,

$$W(\gamma|\bar{\gamma}) \geq -\sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2} \psi(\|i-j\|)|\gamma_i|\,|\bar{\gamma}_j|$$

where $K > 0$, $\lambda > d$ are fixed.

Let

$$V^+(x) = \inf_{\tilde{x}:0<|\tilde{x}|\leq|x|} V(\tilde{x})\,, \quad V^-(x) = \min\left(0, \inf_{\tilde{x}:|x-\tilde{x}|\leq\frac{3}{2}g} V(\tilde{x})\right)\,,$$

$$\bar{V}(x) = \max\left(0, \sup_{\tilde{x}:|x-\tilde{x}|\leq\frac{3}{2}g} V(\tilde{x})\right)\,,$$

where the symbol $|\cdot|$ represents Euclidean norm in $\mathbb{R}^d$, and let

$$C_1 = \frac{1}{2}(v_d)^{-1}\int_{0<|x|<g} V^+(x)[1+g^{-1}|x|]^{-d-1}dx\,,$$

$$C_2 = -n^{n/2}\int_{\mathbb{R}^d} V^-(x)dx\,,$$

where v_d is the volume of a d-dimensional sphere of radius 1.

Assumption 5.2.[6] 1. The inequalities $C_2 < C_1$, $C_2 < \infty$ hold.

2. For some $D < \infty$: $\int_{x:|x|\geq D} \bar{V}(x)dx < \infty$.

It is well known from Ref. 25 that under Assumption 5.1 the set of tempered Gibbs states is nonempty. Let us denote this set by $\mathcal{G}_t(V,z)$.

Analogous existence result for Gibbs states under Assumption 5.2 can be found in Ref. 6.

The following propositions collect some known results concerning Gibbs measures.

Proposition 5.1.[2] *Suppose that Assumption* 5.1 *is fulfilled. Then for any* $\mu \in \mathcal{G}_t(V,z)$ *the correlation functions* $k_\mu^{(n)}(x_1,\ldots,x_n)$ *satisfy the following inequality*:

$$k_\mu^{(n)}(x_1,\ldots,x_n) \leq C^n \exp\left[-\sum_{i<j} V(x_i-x_j)\right] \tag{25}$$

with some $C > 0$.

Proposition 5.2.[10] *Suppose that Assumptions* 5.1.2, 5.1.3 *hold. Let* Λ *be a finite union of unit cubes of the form* Q_i. *Suppose* $\tilde{\Lambda} \supset \Lambda$, $\tilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)$. *For any* $\mu \in \mathcal{G}_t(V,z)$ *there exist constants* $\alpha > 0$ *and* δ, *depending only on* z (*independent of* $\tilde{\Lambda}$ *and* Λ), *such that for any* $N \in \mathbb{N}_0$

$$\mu_{\tilde{\Lambda}}(\{\gamma|\,|\gamma_\Lambda| \geq N|\Lambda|\}) \leq \exp\{-(\alpha N^2-\delta)|\Lambda|\}\,. \tag{26}$$

Proposition 5.3.[6] *Suppose that Assumption* 5.2 *holds and let* $\varphi(y)$, $0 < y < \infty$, *be a positive monotonically increasing convex function is such that for some* $h > 0$, $L < \infty$

$$\varphi(m) \leq L \exp\{m^2(H(m) - g^{-d}C_2 - h)\}, \quad m = 0, 1, \dots,$$

where

$$H(m) = \frac{1}{2} g^{-d} (v_d)^{-1} \int_{x : g(m^{\frac{1}{d}} - 1)^{-1} \leq |x| \leq g} V^+(x)[1 + g^{-1}|x|]^{-d-1} dx.$$

Then for any $\mu \in \mathcal{G}(V, z)$ *there exists a constant* $C_\Lambda(\varphi)$ *such that for any* $\tilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)$ *the following inequality holds*

$$\int_{\Gamma_{\tilde{\Lambda}}} \varphi(|\gamma_\Lambda|)\mu_{\tilde{\Lambda}}(d\gamma) < C_\Lambda(\varphi), \text{ for all } \Lambda \subset \tilde{\Lambda},\ \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{27}$$

The conditions on function φ *are satisfied if*

$$\varphi(m) = \exp\{dm^2\}, \quad 0 < d < (C_1 - C_2 g^{-d}).$$

Corollary 5.1. *Under Assumption* 5.1 *for any* $\mu \in \mathcal{G}_t(V, z)$ *we have*:

- (RPB) (*Ruelle's probability bound* (26)).
- *Dobrushin's bound* (27) *for all bounded* $\Lambda \subset \tilde{\Lambda}$ *such that* $l_\Lambda \leq gd^{-\frac{1}{2}}$,
- *Dobrushin's bound* (27) *for function* $\varphi(x) = e^{\lambda x^p}$, $\lambda > 0$, $p \in (0, 2)$.

Under Assumptions 5.1.2, 5.1.3 *for any* $\mu \in \mathcal{G}_t(V, z)$ *we have Dobrushin's bound for function* $\varphi(x) = e^{\lambda x^p}$, $\lambda > 0$, $p \in (0, 2)$.

The conditions of Proposition 5.3 *imply* (RPB).

Proof. We will prove only that under Assumption 5.1 for any $\mu \in \mathcal{G}_t(V, z)$ we have Dobrushin's bound for every bounded $\Lambda \subset \tilde{\Lambda}$ such that $l_\Lambda \leq gd^{-\frac{1}{2}}$ and that the conditions of Proposition 5.3 imply (RPB). The proof of the remaining statements in this corollary is a direct consequence of Theorem 3.1 for measure $\mu = \mu_{\tilde{\Lambda}}$.

Using (1) and an estimate of the function φ we have

$$\int_{\Gamma_\Lambda} |K^{-1}[\varphi(|\eta|)]|\rho_\mu(d\eta) \leq \int_{\Gamma_\Lambda} \sum_{\xi \subset \eta} \varphi(|\xi|)\rho_{\mu_\Lambda}(d\eta)$$

$$\leq L \int_{\Gamma_\Lambda} \sum_{\xi \subset \eta} \exp\{|\xi|^2 H(|\xi|) - g^{-d}C_2 - h\}\rho_{\mu_\Lambda}(d\eta). \tag{28}$$

The estimate on the correlation functions implies the bound for (28)

$$L \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1, \dots, x_n\}} e^{|\xi|^2 H(|\xi|) - \sum_{i<j} V(x_i - x_j)} C^n dx_1 \cdots dx_n. \tag{29}$$

We bound $e^{-\sum_{i<j} V(x_i - x_j)}$ using the following result from Ref. 6: there exists $m_0 \geq 2^d$ s.t. for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \leq gd^{-\frac{1}{2}}$ and $\eta \in \Gamma_\Lambda$, $|\eta| \geq m_0$ holds

$$E^V(\eta) \geq |\eta|^2 H(|\eta|).$$

Therefore, we can estimate (29) by

$$Le^{m_0^2 H(|m_0|)} \sum_{n=0}^{m_0} \frac{(2zC|\Lambda|)^n}{n!} + L \sum_{n=m_0+1}^{\infty} \frac{(2zC|\Lambda|)^n}{n!} \le Le^{2zC|\Lambda| + m_0^2 H(|m_0|)}.$$

The equalities (2) and (4) give

$$\int_{\Gamma_{\tilde{\Lambda}}} \varphi(|\eta_\Lambda|)\mu_{\tilde{\Lambda}}(d\eta) = \int_{\Gamma_\Lambda} K^{-1}[\varphi(|\eta_\Lambda|)]\rho_\mu(d\eta) \le Le^{2zC|\Lambda| + m_0^2 H(|m_0|)}.$$

To show that conditions of Proposition 5.3 imply (RPB) one should take in the proof of Theorem 3.1(d) the constant $\lambda = C_1 - C_2 g^{-d}$ and use the fact from Ref. 6 that $C_\Lambda \le e^{\delta|\Lambda|}$ for some $\delta > 0$. □

Remark 5.1. Let us note that the Poisson measure $\pi_{z\sigma}$ satisfy (15). Really, we have

$$\pi_{z\sigma}(\{\gamma |\, |\gamma_\Lambda| \ge N|\Lambda|\})$$

$$= \sum_{n \ge N|\Lambda|} e^{-z|\Lambda|} \frac{(z|\Lambda|)^n}{n!} = e^{-z|\Lambda|}(z|\Lambda|)^{n_0} \sum_{n=0}^{\infty} \frac{(z|\Lambda|)^n}{n!} \frac{n!}{(n+n_0)!}, \quad (30)$$

where n_0 is the smallest integer greater than or equal to $N|\Lambda|$. Using Stirling formula and considering $N \ge e^2 z$ we can bound (30) by $e^{-N|\Lambda|}$.

Moreover, this implies $\pi_{z\sigma}(P_\infty) = \pi_{z\sigma}(U^1_\infty) = 1$.

Remark 5.2. The Poisson measure $\pi_{z\sigma}$ does not satisfy (RPB). Indeed, suppose that (RPB) for $\pi_{z\sigma}$ holds. Then from *Theorem 3.1* we have that $\pi_{z\sigma}$ satisfy $(\mathrm{DEB})_{(1,2-\varepsilon)}$, where $0 < \varepsilon < 1$. But by the definition of the Poisson measure

$$\int_\Gamma e^{|\gamma_\Lambda|^{2-\varepsilon}} \pi_{z\sigma}(d\gamma) = e^{-z|\Lambda|} \sum_{n=0}^{\infty} e^{n^{2-\varepsilon}} \frac{(z|\Lambda|)^n}{n!},$$

where the latter series obviously diverges.

So, our assumption that the Poisson measure satisfies (RPB) is false.

Example 5.2. Let $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a non-negative pair potential and the function $k_\alpha^V : \Gamma_0 \to \mathbb{R}$ defined by

$$k_\alpha^V(\eta) = \alpha^{|\eta|} e^{-E^V(\eta)} = \alpha^{|\eta|} e^{-\sum_{\{x,y\}\subset\eta} V(x,y)}, \eta \in \Gamma_0, |\eta| \ge 2, \quad (31)$$

$$k_\alpha^V(\eta) = \alpha, |\eta| = 1,$$

$$k_\alpha^V(\emptyset) = 1,$$

with some constant $\alpha > 0$.

Assume that $c := \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (1 - e^{-V(x,y)})dy < \infty$. As shown in Ref. 3 under assumption $\alpha ce < 1$ there exists probability measure μ on $\mathcal{B}(\Gamma)$ s.t.

$$k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda_\sigma}(\eta) = k_\alpha^V(\eta), \eta \in \Gamma_0,$$

where σ denotes the Lebesgue measure on $\mathbb{R}^d$. Moreover, the bound $0 \leq k_\mu(\eta) \leq \alpha^{|\eta|}$, $\eta \in \Gamma_0$ implies the uniqueness (c.f. Ref. 11).

The measure μ is not Gibbs state associated with a pair potential. Moreover, it is difficult to show that μ corresponds to a potential in an explicit form. Even if this is true, such a potential should include interactions of all orders. In spite of this, we know that correlation functions of μ satisfy $(\mathrm{GRB})_V$. Therefore, all results of this paper connected with $(\mathrm{GRB})_V$ are applicable to this measure. In particular, we have information about support properties and probability bounds depending on the behavior of V on the diagonal.

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