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GENERALIZED FEYNMAN–KAC FORMULA WITH STOCHASTIC POTENTIAL

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In this paper we study the solution of the stochastic heat equation where the potential V and the initial condition f are generalized stochastic processes. We construct explicitly the solution and we prove that it belongs to the generalized function space $\mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{N}')$.

Keywords: Generalized functions; convolution product; stochastic heat equation; generalized stochastic processes.

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1. Introduction

In the last years new classes of spaces of generalized and test functions were introduced by many authors, see e.g., Refs. 4 and 9. Let $\mathcal{N}$ be a complex Fréchet nuclear space with topology given by an increasing family of Hilbertian norms $\{|\cdot|_n, n \in \mathbb{N}\}$. It is well known that $\mathcal{N}$ may be represented as $\mathcal{N} = \cap_{n\in\mathbb{N}}\mathcal{N}_n$, where the Hilbert space $\mathcal{N}_n$ is the completion of $\mathcal{N}$ with respect to $|\cdot|_n$, see e.g., Refs. 11 and 35. By the general duality theory $\mathcal{N}'$ is given by $\mathcal{N}' = \cup_{n\in\mathbb{N}}\mathcal{N}_{-n}$, where $\mathcal{N}_{-n} = \mathcal{N}_n'$ is the topological dual of $\mathcal{N}_n$. Let θ: $\mathbb{R}_+ \to \mathbb{R}_+$ be a continuous convex strictly increasing function such that

$$\lim_{x\to\infty}\frac{\theta(x)}{x} = \infty, \qquad \theta(0) = 0\,.$$

Such functions are called Young functions, see Ref. 17. The test function space $\mathcal{F}_\theta(\mathcal{N}')$ is defined as the space of all holomorphic functions on $\mathcal{N}'$ with an exponential growth condition of order θ. More precisely, for any $m > 0$ and $n \in \mathbb{N}$ we denote by $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$ the Banach space of holomorphic functions on the Hilbert space $\mathcal{N}_{-n}$ with the following growth condition

$$\|f\|_{\theta,m} := \sup_{z\in\mathcal{N}_{-n}} |f(z)|\exp(-\theta(m|z|_{-n})) < \infty\,. \tag{1}$$

Then $\mathcal{F}_\theta(\mathcal{N}') = \cap_{m>0,n\in\mathbb{N}}\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$ equipped with the projective limit topology is our test function space. The corresponding topological dual, equipped with the inductive limit topology, is denoted by $\mathcal{F}_\theta^*(\mathcal{N}')$ which is the generalized function space, see Ref. 9 for more details. In particular, if $\mathcal{N} = S_\mathbb{C}(\mathbb{R})$ (the complexified of the Schwartz test function space $S(\mathbb{R})$) and $\theta(x) = x^2$, then $\mathcal{F}_\theta(\mathcal{N}')$ is nothing but the analytic version of the Kubo–Takenaka test functions space and the corresponding topological dual is the Hida distribution space, see e.g., Refs. 12, 22 and 23. The test function space of Kondratiev–Streit type $(S)_\beta$, $\beta \in [0,1)$ are obtained by choosing $\theta(x) = x^{\frac{2}{1+\beta}}$, see Refs. 19, 24, 25, 29 and 30.

More recently, a multivariable version of the above spaces was introduced, see Ref. 31. In fact, we can replace the nuclear space $\mathcal{N}$ by a Cartesian product $\mathcal{N}_1 \times \cdots \times \mathcal{N}_k$, $k \in \mathbb{N}$ and θ by $(\theta_1,\ldots,\theta_k)$ where θ_i are Young functions and $\mathcal{N}_i$ is a complex nuclear Fréchet space, $1 \le i \le k$, then it is possible to extend all the results obtained in Ref. 9. In particular, the Laplace transform $\mathcal{L}$ is a topological isomorphism between the generalized function space $\mathcal{F}_\theta^*(\mathcal{N}_1' \times \cdots \times \mathcal{N}_k')$ and $\mathcal{G}_{\theta^*}(\mathcal{N}_1 \times \cdots \times \mathcal{N}_k)$, where $\mathcal{G}_{\theta^*}(\mathcal{N}_1 \times \cdots \times \mathcal{N}_k)$ is the space of entire functions on $\mathcal{N}_1 \times \cdots \times \mathcal{N}_k$ which verify an exponential growth condition similar to (1) with respect to $\theta^* = (\theta_1^*,\ldots,\theta_k^*)$, where $\theta_i^*(x) = \sup_{t>0}(tx - \theta_i(t))$ is the polar function corresponding to θ_i, see Ref. 17 for this notion. Another important result in Ref. 31 is the characterization theorem for convergent sequences of distributions in $\mathcal{F}_\theta^*(\mathcal{N}_1' \times \cdots \times \mathcal{N}_k')$, see Proposition 10 below. In fact, we can directly define for any given continuous stochastic process $X(t) \in \mathcal{F}_\theta^*(\mathcal{N}_1' \times \cdots \times \mathcal{N}_k')$ the integral

$$\int_0^t X(s)ds = \mathcal{L}^{-1}\int_0^t \mathcal{L}X(s)ds\,.$$

The convolution product on $\mathcal{F}_\theta^*(\mathcal{N}')$ is very useful in applications, see Refs. 3 and 8 for details. In fact, we define the convolution of two distributions $\Phi, \Psi \in \mathcal{F}_\theta^*(\mathcal{N}')$ by

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi \cdot \mathcal{L}\Psi)$$

which is well defined because $\mathcal{G}_{\theta^*}(\mathcal{N})$ is an algebra under pointwise multiplication. We can define for any generalized function $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ the convolution exponential of Φ denoted by $\exp^* \Phi$ as a generalized function on $\mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{N}')$. Note that for a generalized function on the Kondratiev–Streit space $\Phi \in (S)'_\beta$ the Wick exponential of Φ, $\exp^\diamond \Phi$ does not belong to $(S)'_\beta$, but it belongs to a bigger space of distributions $(S)^{-1}$, so-called Kondratiev distribution space, see Ref. 16.

Now we consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}X(t,x,\omega) = a\Delta X(t,x,\omega) + X(t,x,\omega) * V(t,x,\omega)\,, \\ X(0,x,\omega) = f(x,\omega)\,, \end{cases} \tag{2}$$

where $a \in \mathbb{R}_+$, $t \in [0,\infty)$ is the time parameter, $x = (x_1,\ldots,x_r) \in \mathbb{R}^r$ is the spatial variable, $r \in \mathbb{N}$, and $\Delta = \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^r$, $\omega = (\omega_1,\ldots,\omega_d)$ is the stochastic vector variable in the tempered Schwartz distribution space $S'_d := S'(\mathbb{R},\mathbb{R}^d)$, $d \in \mathbb{N}$, and $*$ is the convolution product between generalized functions on $\mathcal{F}^*_\theta(S'_d \times \mathbb{R}^r)$. The problem (2) was analyzed by many authors from different point of view with the Wick exponential instead of the convolution product, see e.g. Refs. 14 and 32 and references therein.

Combining the convolution calculus and the multivariable version of the above tools we give an explicit solution of the Cauchy problem (2), see Proposition 12, (14) below. In particular when V is a positive distribution, then there exists a unique Radon measure ν (see e.g., Ref. 28) on the real part of $\mathcal{N}'_\mathbb{R} =: \mathcal{M}'$ which represents V and therefore the Fourier transform of ν is given by

$$\langle V, \exp(i\xi)\rangle = \hat{\nu}(\xi) = \int_{\mathcal{M}'} \exp(i\langle y,\xi\rangle)d\nu(y)\,.$$

Moreover, if ν is a Radon measure on $\mathcal{M}'$ such that there exists $n \in \mathbb{N}$ with $\nu(\mathcal{M}_{-n}) = 1$ and ν satisfies some integrability condition, e.g.,

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n}))d\nu(y) < \infty\,,$$

for some $m > 0$, then ν is in the Albeverio–Høegh-Krohn class (Ref. 1).

We would also like to mention the work of Asai *et al.*[2,21,38] for related considerations on Feynman integrals for the Albeverio–Høegh-Krohn class of potentials.

2. Preliminaries

In this section we introduce the framework needed later on. We start with a real Hilbert space $\mathcal{H} = L^2(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. More precisely, if $(f,x) = ((f_1,\ldots,f_d),(x_1,\ldots,x_r)) \in \mathcal{H}$, then the Hilbertian norm of (f,x) is given by

$$|(f,x)|^2 := \sum_{i=1}^d \int_\mathbb{R} f_i^2(u)du + \sum_{i=1}^r x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}\,.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r = \mathcal{M}\,. \tag{3}$$

The pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle(\omega,x),(\xi,y)\rangle := (\omega,\xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x,y)_{\mathbb{R}^r}$, $(\omega,x) \in \mathcal{M}'$ and $(\xi,y) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^\infty S_n(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r = \bigcap_{n=0}^\infty \mathcal{M}_n\,,$$

where $S_n(\mathbb{R},\mathbb{R}^d) \times \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2_{\mathbb{R}^r}$, see e.g., Ref. 13 and references therein. We will consider the complexification of the triple (3) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}\,, \tag{4}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos theorem via its characteristic functional for every $(\xi, p) \in \mathcal{M}$ by

$$C_\gamma(\xi, p) = \int_{\mathcal{M}'} \exp(i\langle(\omega, x), (\xi, p)\rangle) d\gamma((\omega, x)) = \exp\left(-\frac{1}{2}(|\xi|^2 + |p|^2)\right).$$

In order to solve the Cauchy problem (2) we need to introduce an appropriate space of generalized functions for which we follow closely the construction in Ref. 31. Let $\theta = (\theta_1, \theta_2)\colon \mathbb{R}_+^2 \to \mathbb{R}$, $(t_1, t_2) \mapsto \theta_1(t_1) + \theta_2(t_2)$ such that

$$\lim_{t\to\infty} \frac{\theta_1(t)}{t} = \infty, \qquad \lim_{t\to\infty} \frac{\theta_2(t)}{t} = \infty,$$

where θ_1, θ_2 are two Young functions. For every pair $m = (m_1, m_2)$ where m_1, m_2 are strictly positive real numbers, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{N}$ by

$$\begin{aligned}\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f\colon \mathcal{N}_{-n} \to \mathbb{C}, \text{entire}, &\|f\|_{\theta,m,n} \\ &= \sup_{z\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta(m|z|_{-n})) < \infty\},\end{aligned}$$

where for each $z = (\omega, x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Now we consider as test function space the space of entire functions on $\mathcal{N}'$ of (θ_1, θ_2)-exponential growth and minimal type given by

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to construct the triplet of the complex Hilbert space $L^2(\mathcal{N}', \gamma)$ by $\mathcal{F}_\theta(\mathcal{N}')$, to this end we need an additionally condition on the pair of Young functions (θ_1, θ_2). Namely, $\lim_{t\to\infty} \frac{\theta_i(t)}{t^2} < \infty$, $i = 1, 2$. This is enough to obtain the following Gelfand triplet

$$\mathcal{F}_\theta^*(\mathcal{N}') \supset L^2(\mathcal{N}', \gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{5}$$

where $\mathcal{F}_\theta^*(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{N}', \gamma)$ endowed with the inductive limit topology which coincides with the strong topology since $\mathcal{F}_\theta(\mathcal{N}')$ is a nuclear space, see e.g., Ref. 11 or 31 for more details on this subject. We denote the duality between $\mathcal{F}_\theta^*(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ by $\langle\langle\cdot,\cdot\rangle\rangle$ which is the extension of the inner product in $L^2(\mathcal{N}', \gamma)$.

Remark 1. For every entire function $f\colon \mathcal{N}' \to \mathbb{C}$, we have the Taylor expansion

$$f(z) = \sum_{k\in\mathbb{N}_0^2} \langle z^{\otimes k}, f_k\rangle,$$

where $z^{\hat{\otimes} k} \in \mathcal{N}'^{\hat{\otimes} k}$ and this allowed us to identify each entire function f with the corresponding Taylor coefficients $\mathbf{f} = (f_k)_{k=0}^\infty$. The mapping $f \mapsto T(f) = \mathbf{f}$ is called Taylor series map.

Using the mapping T we can construct a topological isomorphism between the test function space $\mathcal{F}_\theta(\mathcal{N}')$ and the formal power series space $F_\theta(\mathcal{N})$ defined by

$$F_\theta(\mathcal{N}) = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} F_{\theta,m}(\mathcal{N}_n)\,, \tag{6}$$

where

$$F_{\theta,m}(\mathcal{N}_n) := \left\{ \mathbf{f} = (f_k)_{k\in\mathbb{N}_0^2}, f_k \in \mathcal{N}_n^{\hat{\otimes}k} \,|\, |\mathbf{f}|^2 := \sum_{k\in\mathbb{N}_0^2} \theta_k^{-2} m^{-k} |f_k|_n^2 < \infty \right\},$$

here $\theta_k^{-2} = \theta_{1,k_1}^{-2}\theta_{2,k_2}^{-2}$, with $\theta_{i,k_i} := \inf_{u>0} \frac{\exp(\theta_i(u))}{u^{k_i}}$, $i = 1, 2$. In the case where $\theta(x) = x^2$, then $F_{\theta,1}(\mathcal{N}_n)$ is nothing but the usual Bosonic Fock space associated to $\mathcal{N}_n$, see Ref. 13 for more details.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}_\theta^*(\mathcal{N}')$. This will be done in Theorem 2 with the help of the Laplace transform. Therefore, let us first define the Laplace transform of an element in $\mathcal{F}_\theta^*(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ we define the exponential function $\exp((\xi, p))$ by

$$\mathcal{N}' \ni (\omega, x) \mapsto \exp(\langle\omega, \xi\rangle + (p, x))\,. \tag{7}$$

It is not hard to verify that for every element $(\xi, p) \in \mathcal{N}$ $\exp((\xi, p)) \in \mathcal{F}_\theta(\mathcal{N}')$. With the help of this function we can define the Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\!\langle \Phi, \exp((\xi, p))\rangle\!\rangle\,. \tag{8}$$

The Laplace transform is well defined because $\exp((\xi, p))$ is a test function. In order to obtain the characterization theorem we need to introduce another space of entire functions on $\mathcal{N}$ with θ^*-exponential growth and arbitrary type, where θ^* is another Young function defined by

$$\theta^*(x) := \sup_{t>0}(tx - \theta(t))\,.$$

The next characterization theorem is essentially based on the topological dual of the formal power series space $F_\theta(\mathcal{N})$ defined in (6) and the inverse Taylor series map T^{-1}, see e.g., Ref. 9 or 31 for details. In the white noise analysis framework this theorem is known as Potthoff–Streit characterization theorem, see Refs. 15 and 33 for details and historical remarks.

Theorem 2. *The Laplace transform is a topological isomorphism between $\mathcal{F}_\theta^*(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n)\,,$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g, |g(\xi, p)| \le k \exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)), (\xi, p) \in \mathcal{N}_n\,.$$

3. The Convolution Product $*$

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator, see e.g., Ref. 6. This notion generalizes the differential equations with constant coefficients in finite dimensional case. If we consider the space of test functions $\mathcal{F} = \mathcal{F}_\theta(\mathcal{N}')$, then we can show that each convolution operator is associated with a generalized function from $\mathcal{F}_\theta^*(\mathcal{N}')$ and vice-versa, see e.g., Ref. 8.

Let us define the convolution between a generalized and a test function on $\mathcal{F}_\theta^*(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$, respectively. Let $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi \rangle\rangle\,,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega+\eta, x+y)\,.$$

It is not hard to see that $\Phi*\varphi$ is an element of $\mathcal{F}_\theta(\mathcal{N}')$, see (Proposition 2.3 of Ref. 8). Note that the dual pairing between $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is given in terms of the convolution product of Φ and φ applied at $(0,0)$, i.e. $(\Phi * \varphi)(0,0) = \langle\langle \Phi, \varphi\rangle\rangle$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}_\theta^*(\mathcal{N}')$ be given. Then $\Phi * \Psi$ is defined as

$$\langle\langle \Phi * \Psi, \varphi \rangle\rangle := \langle\langle \Phi, \Psi * \varphi \rangle\rangle, \qquad \forall\, \varphi \in \mathcal{F}_\theta(\mathcal{N}')\,. \tag{9}$$

This definition of convolution product for generalized functions will be used in Sec. 4 in order to solve the heat stochastic equation. We have the following connection between the Laplace transform and the convolution product.

Proposition 3. *Let $(\xi, p) \in \mathcal{N}$ be given and consider the exponential function $\exp((\xi,p))$ defined on (7). Then for every $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ we have*

$$\Phi * \exp((\xi,p)) = (\mathcal{L}\Phi)(\xi,p)\exp((\xi,p))\,.$$

Proof. In fact for every $(\omega, x) \in \mathcal{N}'$ we have

$$\tau_{(-w,-x)}\exp((\xi,p)) = \exp(\langle \omega, \xi\rangle + (p,x))\exp((\xi,p))\,,$$

then

$$\begin{aligned}\Phi * \exp((\xi,p))(\omega,x) &= \langle\langle \Phi, \tau_{(-w,-x)}\exp((\xi,p))\rangle\rangle \\ &= \exp(\langle \omega,\xi\rangle + (p,x))\langle\langle \Phi, \exp((\xi,p))\rangle\rangle \\ &= (\mathcal{L}\Phi)(\xi,p)\exp((\xi,p))(\omega,x)\,,\end{aligned}$$

which is just the required result. □

As a consequence of the above proposition and the definition in (9) we obtain the following corollary which says that the Laplace transform maps the convolution product in $\mathcal{F}_\theta^*(\mathcal{N}')$ into the usual pointwise in the function space $\mathcal{G}_{\theta^*}(\mathcal{N})$.

Corollary 4. *For every generalized functions* $\Phi, \Psi \in \mathcal{F}_\theta^*(\mathcal{N}')$

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi\,.$$

Remark 5. Since $\mathcal{G}_{\theta^*}(\mathcal{N})$ is an algebra under the usual pointwise product and applying Theorem 2 we may define convolution product between two generalized functions as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi\mathcal{L}\Psi)\,.$$

We stress the fact that the convolution product is commutative and associative operation on $\mathcal{F}_\theta^*(\mathcal{N}')$.

Using Corollary 4 we may define the convolution exponential of a generalized function which will be the main object in solving the stochastic partial differential equation in (2), see (14). First we need the following useful lemma which is a consequence of the characterization theorem.

Lemma 6. *Let* Φ *be an element on* $\mathcal{F}_\theta^*(\mathcal{N}^*)$, *then* $\exp(\mathcal{L}\Phi)$ *belongs to* $\mathcal{G}_{e^{\theta^*}}(\mathcal{N})$.

Using the inverse Laplace transform and the fact that any Young function θ verify the property $(\theta^*)^* = \theta$ we obtain that $\mathcal{L}^{-1}(\mathcal{G}_{e^{\theta^*}}(\mathcal{N})) = \mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{N}')$.

Definition 7. For every generalized function $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ we define $\exp^* \Phi$, the convolution exponential functional of Φ, by

$$\mathcal{L}(\exp^* \Phi) = \exp(\mathcal{L}\Phi)\,.$$

A direct consequence of Lemma 6 and the above definition is the following corollary.

Corollary 8. *For every* $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ *the convolution functional* $\exp^* \Phi$ *is an element of the space* $\mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{N}')$.

Remark 9. In the next section we will handle a special kind of stochastic process on $\mathcal{F}_\theta^*(\mathcal{N}')$, namely deterministic process. In order to apply our general framework we need the following identification: if $\Phi = \Phi_1 \otimes \Phi_2$ is a generalized function, where $\Phi_1 \in \mathcal{F}^*_{\theta_1}(S'_{d,\mathbb{C}})$ and $\Phi_2 \in \mathcal{F}^*_{\theta_2}(\mathbb{C}^r)$, then we have

$$\begin{aligned}(\mathcal{L}\Phi_1 \otimes \Phi_2)(\xi, p) &= \langle\!\langle \Phi_1 \otimes \Phi_2, \exp((\xi, p))\rangle\!\rangle \\ &= \langle\!\langle \Phi_1 \otimes \Phi_2, \exp(\xi) \otimes \exp(p)\rangle\!\rangle \\ &= (\mathcal{L}_1\Phi_1)(\xi)(\mathcal{L}_2\Phi_2)(p)\,,\end{aligned}$$

where $\mathcal{L}_1$ is the Laplace transform with respect to the first variable and $\mathcal{L}_2$ with respect to the second one. If $\mathbf{1}$ is the function such that $\mathbf{1}(\omega) = 1$, $\forall\ \omega \in S'_{d,\mathbb{C}}$, then every element $V \in \mathcal{F}^*_{\theta_2}(\mathbb{C}^r)$ can be identified with $V = \mathbf{1} \otimes V$ and moreover $(\mathcal{L}V)(\xi, p) = (\mathcal{L}_2 V)(p)$. The same reasoning can be applied to the convolution product, i.e. the convolution product $V * f$, $f \in \mathcal{F}^*_{\theta_2}(\mathbb{C}^r)$ coincides with the usual convolution product with respect to the spatial variable.

4. Applications to Stochastic Partial Differential Equations

A one-parameter generalized stochastic process with values in $\mathcal{F}^*_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi_t, t \in I\} \subset \mathcal{F}'_\theta(\mathcal{N}')$, where I is an interval from $\mathbb{R}$, without loss of generality we may assume that $0 \in I$. The process Φ_t is said to be continuous if the map $t \longmapsto \Phi_t$ is continuous. In order to introduce generalized stochastic integrals, we need the following result proved in Ref. 27 or Theorem 18 of Ref. 31.

Proposition 10. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of generalized functions on $\mathcal{F}^*_\theta(\mathcal{N}')$. Then the following two conditions are equivalent*:

(1) *The sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}^*_\theta(\mathcal{N}')$ strongly.*
(2) *The sequence $(\hat{\Phi}_n = \mathcal{L}(\Phi_n))_{n\in\mathbb{N}}$ of Laplace transform of $(\Phi_n)_{n\in\mathbb{N}}$ satisfies the following two conditions*:
   (a) *There exists $p \in \mathbb{N}$ and $m \in (\mathbb{R}^*_+)^2$ such that the sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{G}_{\theta^*,m}(\mathcal{N}_p)$ and is bounded in this Banach space.*
   (b) *For every point $z \in \mathcal{N}$, the sequence of complex numbers $(\hat{\Phi}_n(z))_{n=0}^\infty$ converges.*

Let $\{\Phi_t\}_{t\in I}$ be a continuous $\mathcal{F}^*_\theta(\mathcal{N}')$-process and put

$$\Phi_n = \frac{t}{n}\sum_{k=0}^{n-1} \Phi_{\frac{tk}{n}}, \qquad n \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}, \quad t \in I\,.$$

It is easy to prove that the sequence $(\hat{\Phi}_n)$ is bounded in $\mathcal{G}_{\theta^*}(\mathcal{N})$ and for every $\xi \in \mathcal{N}$, $p \in \mathbb{C}^r(\hat{\Phi}_n(\xi, p))_n$ converges to $\int_0^t \hat{\Phi}_s(\xi, p)ds$. Thus we conclude by Proposition 10 that (Φ_n) converges in $\mathcal{F}^*_\theta(\mathcal{N}')$. We denote its limit by

$$\int_0^t \Phi_s ds := \lim_{n\to\infty} \Phi_n \qquad \text{in } \mathcal{F}^*_\theta(\mathcal{N}')\,.$$

Proposition 11. *For a given continuous generalized stochastic process X we define the generalized function*

$$Y_t(x, \omega) = \int_0^t X_s(x, \omega)ds \in \mathcal{F}^*_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X_s(x, \omega)ds\right)(\xi, p) := \int_0^t \mathcal{L}X_s(p, \xi)ds\,.$$

Moreover, the generalized stochastic process $Y_t(x,\omega)$ is differentiable in $\mathcal{F}_\theta^*(\mathcal{N}')$ and we have $\frac{\partial}{\partial t}Y_t(x,\omega) = X_t(x,\omega)$.

Proof. Since the map $s \longmapsto \hat{X}_s \in \mathcal{G}_{\theta^*}(\mathcal{N})$ is continuous, $\{\hat{X}_s, s \in [0,t]\}$ becomes a compact set, in particular it is bounded in $\mathcal{G}_{\theta^*}(\mathcal{N})$, i.e. there exist $n \in \mathbb{N}, m = (m_1, m_2) \in (\mathbb{R}_+^*)^2$ and $C_t > 0$ such that for every $\xi \in \mathcal{N}_n$, $p \in \mathbb{C}^r$ we have

$$|\hat{X}_s(\xi,p)| \leq C_t e^{\theta_1^*(m_1|\xi|_n)+\theta_2^*(m_2|p|)}, \qquad \forall\ s \in [0,t]\,. \tag{10}$$

The inequality (10) shows that the function $\xi \longmapsto \int_0^t \hat{X}_s(\xi,p)ds$ belongs to $\mathcal{G}_{\theta^*}(\mathcal{N})$. Consequently the pointwise convergence of the sequence of functions $(\hat{X}_n)_{n=0}^\infty$ to $\int_0^t \hat{X}_s ds$ becomes a convergence sequence in $\mathcal{G}_{\theta^*}(\mathcal{N})$ and we get

$$\widehat{\int_0^t X_s ds} = \int_0^t \hat{X}_s ds\,.$$

Let $t_0 \in I$ and $\varepsilon > 0$ be such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. It follows from (10) that

$$\begin{aligned}\|\hat{Y}_t - \hat{Y}_{t_0}\|_{\theta^*,n,m} &\leq \int_{t_0}^t \|\hat{X}_s\|_{\theta^*,n,m} ds \\ &\leq |t - t_0| C_{t_0,\varepsilon}\,.\end{aligned}$$

This proves the continuity of the map $I \ni t \longmapsto \hat{Y}_t \in \mathcal{G}_{\theta^*}(\mathcal{N})$ which is equivalent to the continuity of the process Y_t on $\mathcal{F}_\theta^*(\mathcal{N}')$. By the same argument we prove the differentiability of Y_t. □

We are now ready to solve the Cauchy problem in (2). Let us recall again this problem for the reader convenience. Let f be a given generalized function in $\mathcal{F}_\theta^*(\mathcal{N}')$ and V a $\mathcal{F}_\theta^*(\mathcal{N}')$-valued continuous generalized stochastic process given. Consider the following stochastic differential equation with initial condition f

$$\begin{cases} \dfrac{\partial}{\partial t} X_t(\omega,x) = a\Delta X_t(\omega,x) + X_t(\omega,x) * V_t(\omega,x)\,, \\ X_0(\omega,x) = f(\omega,x)\,, \end{cases} \tag{11}$$

where a is a constant and Δ is the usual Laplacian with respect to the spacial variable $x \in \mathbb{R}^r$. To solve this SPDE we apply the Laplace transform to (11) and obtain

$$\begin{cases} \dfrac{\partial}{\partial t} \hat{X}_t(\xi,p) = ap^2 \hat{X}_t(\xi,p) + \hat{X}_t(\xi,p)\hat{V}_t(\xi,p)\,, \\ \hat{X}_0(\xi,p) = \hat{f}(\xi,p)\,. \end{cases} \tag{12}$$

The solution of (12) is given by

$$\hat{X}_t(\xi,p) = \hat{f}(\xi,p) \exp\left(ap^2 t + \int_0^t \hat{V}_s(\xi,p) ds \right). \tag{13}$$

Now the solution of the system (11) is given using Proposition 11, Corollary 8 and the characterization theorem, Theorem 2. We give it on the next proposition.

Proposition 12. *The Cauchy problem* (11) *has a unique solution* X_t *which is a generalized* $\mathcal{F}^*_\beta(\mathcal{N}')$ *-valued stochastic process, where the Young function* β *is given by* $\beta = (e^{\theta^*})^*$. *Moreover, the solution* X_t *is given explicitly by*

$$X_t(\omega, x) = f(\omega, x) * \exp^* \left(\int_0^t V_s(\omega, x) ds \right) * \gamma_{2at} \,, \tag{14}$$

where γ_{2at} *is Gaussian measure on* $\mathbb{R}^r$ *with variance* $2at$.

Deterministic case

In the special case when the potential V and the initial condition f do not depend on the random parameter ω the Cauchy problem (11) is solved as follows:

Corollary 13. *Let* V *and* f *be independent of* ω, *then the Cauchy problem* (11) *reduces to*

$$\begin{cases} \dfrac{\partial}{\partial t} X(t,x) = a\Delta X(t,x) + X(t,x) * V(t,x) \,, \\ X(0,x) = f(x) \,, \end{cases} \tag{15}$$

using Remark 9 *to interpret the convolution product* $*$. *The solution of* (15) *is given by*

$$X(t,x) = (g(t,\cdot) * \gamma_{2at})(x) \,, \tag{16}$$

where g *is equal to*

$$g(t,x) = f(x) \exp \left(\int_0^t V(s,x) ds \right) .$$

Remark 14. In the particular case when $a = 1/2$ and $r = 1$, it is well known that the solution of (15) when V does not depend on the time parameter t and the convolution product $*$ is replaced by the usual pointwise product given by the famous Feynman–Kac formula, see e.g., Ref. 18 or 34

$$u(t,x) = \int_{\Omega_\alpha} f(\omega(t)) \exp \left(\int_0^t V(\omega(s)) ds \right) d\mu_x(\omega) \,, \tag{17}$$

where μ_x is supported on the set Ω_α of Hölder continuous paths ω of order α with $0 < \alpha < 1/2$. The measure μ_x has the property

$$\int_{\Omega_\alpha} h(\omega(t)) d\mu_x(\omega) = (4\pi t)^{-1/2} \int_{\mathbb{R}} h(y) \exp(-|x-y|^2/(4t)) dy \,, \tag{18}$$

where h is any measurable function on $\mathbb{R}$ such that $h(\cdot)\exp(-|x-\cdot|^2/(4t))$ is integrable.

In our framework, we can write the solution (16) and as

$$\begin{aligned} X(t,x) &= \int_{\mathbb{R}} (g(t,x+y)d\gamma_t(y) \\ &= \int_{S'(\mathbb{R})} g(t,x+\langle \mathbb{1}_{[0,t]},\omega\rangle)d\gamma(\omega) \\ &= \int_{S'(\mathbb{R})} f(x+\langle \mathbb{1}_{[0,t]},\omega\rangle)\exp(tV(x+\langle \mathbb{1}_{[0,t]},\omega\rangle))d\gamma(\omega)\,, \qquad (19) \end{aligned}$$

where γ is the standard Gaussian measure on $S'(\mathbb{R})$.

It is clear that the solutions (17) and (19) coincides when V is constant. In fact, if we interpret $V(x) = V(x)\delta_0$ (here δ_0 is the Dirac measure at 0) the convolution: $V * u(t,\cdot)$ is given by

$$((V\delta_0) * u(t,\cdot))(x) = V(0)u(t,x).$$

Therefore the convolution product $(V * u(t,\cdot))(x)$ coincides with the pointwise product $V(x)u(t,x)$ when V is constant.

Moreover, if the potential V is given by a measure ν on $\mathbb{R}$ which verifies a certain integrability condition, e.g., there exists $m > 0$ and a Young function θ such that

$$\int_{\mathbb{R}} \exp(\theta(m|x|))d\nu(x) < \infty\,,$$

which implies that $\nu \in \mathcal{F}'_\theta(\mathbb{R})$, then we have

$$(V * u(t,\cdot))(x) = \int_{\mathbb{R}} u(t,x+y)d\nu(y)\,.$$

In this case we can also apply our method. A special case of such potentials was investigated by Albeverio *et al.*[1,21] (see also references therein for more details and historical remarks) and more recently by Asai *et al.*[2] For a general potential V it is still an open question whether or not exists a distribution Φ_V such that

$$(\Phi_V * u(t,\cdot))(x) = V(x)u(t,x)\,.$$

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