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MALLIAVIN CALCULUS AND ANTICIPATIVE ITÔ FORMULAE FOR LÉVY PROCESSES

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We introduce the forward integral with respect to a pure jump Lévy process and prove an Itô formula for this integral. Then we use Mallivin calculus to establish a relationship between the forward integral and the Skorohod integral and apply this to obtain an Itô formula for the Skorohod integral.

Keywords: Lévy processes; Poisson random measures; white noise; Skorohod integrals; forward integrals; stochastic derivatives; Wick product; anticipative Itô formulae.

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1. Introduction

The original infinite dimensional calculus developed by Malliavin[34] was designed to study the smoothness of the densities of the solutions of stochastic differential equations. Although this technique was developed further by many researchers, this application remained the only one known for several years. This situation changed in 1991, when Karatzas and Ocone[30] showed how the representation theorem that Ocone had formulated earlier in terms of the Malliavin derivative could be used in finance. Now this theorem is often known as the Clark–Haussmann–Ocone (CHO) formula. More precisely, the CHO theorem gives a method of finding replicating portfolios in complete markets driven by

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Brownian motion. This discovery led to an enormous increase in the interest in the Malliavin calculus both among mathematicians and finance researchers and since then the theory has been generalized and new applications have been found. In particular, Malliavin calculus for Brownian motion has been applied to compute the greeks in finance, see e.g. Refs. 2, 16 and 17. Moreover, anticipative stochastic caluclus for Brownian motion involving the forward integral (beyond the semimartingale context) has been applied to give a general approach to optimal portfolio and consumption problems for insiders in finance, see e.g. Refs. 9, 10 and 31.

An extension of the Malliavin method to processes with *discontinuous* trajectories was carried out in 1987 by Bichteler, Gravereaux and Jacod.[11] However, their work is focused on the original problem of the smoothness of the densities of the solutions of stochastic differential equations, a question that does not deal with the other more recent aspects of the Malliavin calculus. For related works on stochastic calculus for stochastic measures generated by a Poisson process on the real line see Refs. 13, 28, 29, 38, 44 and 45, for example.

Recently two types of Malliavin derivative operators $D_t^{(m)}$ and $D_{t,z}$ have been introduced for *Lévy processes* and the corresponding CHO representation theorems have been obtained. See Refs. 7, 15, 32, 33 and 42. Although markets driven by Lévy processes are not in general complete, the corresponding CHO theorem is still important for financial applications. For example, it can be used to find explicitly the minimal variance portfolio (see e.g. Ref. 7) and to compute the greeks in certain jump diffusion market models (see e.g. Ref. 12).

There has also been an increased interest in anticipative integration with respect to a Lévy process and this is partly due to its application to insider trading in finance (see e.g. Refs. 14, 41 and 43). In Sec. 4 of this paper we introduce the *forward integral* with respect to a pure jump Lévy process and we prove an Itô formula for such integrals (see Theorem 4.1). Then we use a relation between forward integrals and Skorohod integrals (see Lemma 4.1) to obtain an Itô formula for Skorohod integrals with respect to a pure jump Lévy process (see Theorem 4.2).

Since Malliavin calculus plays a crucial role in our achievements, we give a review of the main results of this theory in Sec. 3. Various versions of those results have already been obtained and are known to the public. Nevertheless we think that it is of intrest to have a unified approach based on white noise theory.

For completeness and convenience of the reader we recall the basic theory of white noise for pure jump Lévy processes in Sec. 2.

2. Framework

In this paper we deal with *pure jump* Lévy processes with no drift defined on a certain probability space $(\Omega, \mathcal{F}, P)$ and the time horizon $\mathbb{R}_+ = [0, \infty)$. General information about Lévy processes can be found in Refs. 8, 54 and 56, for example. However we recall briefly our framework. Cf. Refs. 15 and 42.

Let $\Omega = \mathcal{S}'(\mathbb{R})$ be the Schwartz space of tempered distributions equipped with its Borel σ-algebra $\mathcal{F} = \mathfrak{B}(\Omega)$. The space $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ of test functions, i.e. the rapidly decreasing smooth functions on $\mathbb{R}$. We denote the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ applied to $f \in \mathcal{S}(\mathbb{R})$ by $\langle \omega, f \rangle = \omega(f)$. See Ref. 18, for example.

Thanks to the Bochner–Milnos–Sazonov theorem, the white noise probability measure P can be defined by the relation

$$\int_\Omega e^{i\langle \omega, f\rangle} dP(\omega) = e^{\int_\mathbb{R} \psi(f(x))dx - i\alpha \int_\mathbb{R} f(x)dx}, \qquad f \in \mathcal{S}(\mathbb{R}),$$

where the real constant α and

$$\psi(u) = \int_\mathbb{R} (e^{iuz} - 1 - iuz1_{\{|z|<1\}})\nu(dz)$$

are the elements of the exponent in the characteristic functional of a pure jump Lévy process with the Lévy measure $\nu(dz)$, $z \in \mathbb{R}$, which, we recall, is such that

$$\int_\mathbb{R} 1 \wedge z^2 \nu(dz) < \infty. \tag{2.1}$$

Assuming that

$$M := \int_\mathbb{R} z^2 \nu(dz) < \infty, \tag{2.2}$$

we can set $\alpha = \int_\mathbb{R} z1_{\{|z|>1\}}\nu(dz)$ and then we obtain that

$$E[\langle \cdot, f\rangle] = 0 \quad \text{and} \quad E[\langle \cdot, f\rangle^2] = M \int_\mathbb{R} f(x)dx, \qquad f \in \mathcal{S}(\mathbb{R}).$$

Accordingly the *pure jump Lévy process with no drift*

$$\eta = \eta(\omega, t), \qquad \omega \in \Omega, t \in \mathbb{R}_+,$$

that we do consider here and in the sequel, is the cadlag modification of $\langle \omega, \chi_{(0,t]}\rangle$, $\omega \in \Omega$, $t > 0$, where

$$\chi_{(0,t]}(x) = \begin{cases} 1, & 0 < x \le t \\ 0, & \text{otherwise}, \qquad x \in \mathbb{R}, \end{cases} \tag{2.3}$$

with $\eta(\omega, 0) := 0$, $\omega \in \Omega$. We remark that, for all $t \in \mathbb{R}_+$, the values $\eta(t)$ belong to $L^2(P) := L^2(\Omega, \mathcal{F}, P)$.

The Lévy process η can be expressed by

$$\eta(t) = \int_0^\infty \int_\mathbb{R} z\tilde{N}(dt, dz), \qquad t \in \mathbb{R}_+, \tag{2.4}$$

where $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is the *compensated Poisson random measure* associated with η, cf. Ref. 24, for example. Recall that the Poisson process is the most important representative among the pure jump Lévy processes and it corresponds to the specific case in which the measure ν is a point mass at 1.

Let $\mathcal{F}_t$, $t \in \mathbb{R}_+$, be the completed filtration generated by the Lévy process in (2.4). We fix $\mathcal{F} = \mathcal{F}_\infty$.

Aiming to treat the Malliavin calculus by means of chaos expansions, we now recall the required spaces and the corresponding complete orthonormal systems.

In the space $L^2(\lambda) = L^2(\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+), \lambda)$ of the square integrable functions on $\mathbb{R}_+$ equipped with the Borel σ-algebra and the standard Lebesgue measure $\lambda(dt)$, $t \in \mathbb{R}_+$, we consider the complete orthonormal system ξ_j $(j = 1, 2, \dots)$ of the Laguerre functions of order $1/2$, i.e.

$$\xi_j(t) = \left(\frac{\Gamma(j)}{\Gamma(j+1/2)}\right)^{1/2} e^{-t} t^{1/4} L_{j-1}^{1/2}(t) 1_{(0,\infty)}(t)\,, \quad t \in \mathbb{R}_+\,, \quad j = 1, 2, \dots\,, \tag{2.5}$$

where Γ is the Gamma functions and $L_j^{1/2}$ are the Laguerre polynomials of order $1/2$ defined by

$$e^{-t} t^{1/2} L_j^{1/2}(t) = \frac{1}{j!} \frac{d^j}{dt^j} (e^{-t} t^{j+1/2})\,, \qquad j = 0, 1, \dots\,.$$

Cf. Ref. 58, for example.

In the space $L^2(\nu) := L^2(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$ of the square integrable functions on $\mathbb{R}$ equipped with the Borel σ-algebra and the Lévy measure ν, we fix a complete orthonormal system ψ_i $(i = 1, 2, \dots)$. In particular we can choose a complete system of polynomials as it was suggested in Refs. 15 and 37, provided that the moments of order greater than or equal to 2 of the measure ν are finite.

To simplify the notation we call $\mathcal{J}$ the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \dots)$ which have only finitely many nonzero values. We denote $\mathrm{Index}\,(\alpha) = \max\{n : \alpha_n \neq 0\}$ and $|\alpha| = \sum_n \alpha_n$, for $\alpha \in \mathcal{J}$.

By δ_k $(k = 1, 2, \dots)$ we identify the product

$$\delta_k(t, z) := \xi_j(t) \psi_i(z)\,, \qquad t \in \mathbb{R}_+, z \in \mathbb{R}\,, \tag{2.6}$$

where $k = \gamma(i, j)$, and $\gamma : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ is a bijective map. Note that any bijective map can be applied, e.g. we could consider the so-called "Cantor diagonalization" of the Cartesian product $\mathbb{N} \times \mathbb{N}$. We set

$$\epsilon^k(n) = \begin{cases} 1\,, & n = k\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \qquad k = 1, 2, \dots\,. \tag{2.7}$$

Now, for any $\alpha \in \mathcal{J}$ with $\mathrm{Index}\,(\alpha) = n$ and $|\alpha| = m$ we define the tensor product $\delta^{\otimes \alpha}$ as

$$\begin{aligned} \delta^{\otimes \alpha} &= \delta_1^{\otimes \alpha_1} \otimes \cdots \otimes \delta_n^{\otimes \alpha_n}((t_1, x_1), \dots, (t_m, x_m)) \\ &:= \delta_1(t_1, x_1) \cdot \ldots \cdot \delta_1(t_{\alpha_1}, x_{\alpha_1}) \cdot \ldots \\ &\quad \times \delta_n(t_{\alpha_1 + \cdots + \alpha_{n-1} + 1}, x_{\alpha_1 + \cdots + \alpha_{n-1} + 1}) \cdot \ldots \cdot \delta_n(t_m, x_m)\,, \end{aligned}$$

with $\delta_k^{\otimes 0} := 1$. Then we denote $\delta^{\hat{\otimes} \alpha}$ the symmetrization of the functions $\delta_k^{\otimes \alpha}$.

In the space $L^2(P) := L^2(\Omega, \mathcal{F}, P)$ of the square integrable random variables we consider the following complete orthogonal system K_α $(\alpha \in \mathcal{J})$:

$$K_\alpha := I_{|\alpha|}(\delta^{\hat{\otimes}\alpha}), \qquad \alpha \in \mathcal{J}, \tag{2.8}$$

where

$$I_m(f) := m! \int_0^\infty \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, x_1, \ldots, t_m, x_m) \tilde{N}(dt_1, dx_1) \cdots \tilde{N}(dt_m, dx_m), \quad m = 1, 2, \ldots$$

for the symmetric function $f \in L^2((\lambda \times \nu)^m)$ $(m = 1, 2, \ldots)$ and $I_0(f) := f$ for $f \in \mathbb{R}$. Cf. Refs. 15 and 42. In particular $K_{\varepsilon^k} = I_1(\delta_k)$. Note that $\|K_\alpha\|^2_{L^2(P)} = \alpha! = \alpha_1!\alpha_2! \cdots$ $(\alpha \in \mathcal{J})$.

The following result is given in Ref. 15, see also Ref. 42.

Theorem 2.1. (Chaos expansion I) *Every $F \in L^2(P)$ admits the unique representation in the form*

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha, \tag{2.9}$$

where $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{J}$, and $c_0 = EF$. Moreover, we have

$$\|F\|^2_{L^2(P)} = \sum_{\alpha \in \mathcal{J}} c_\alpha \alpha!.$$

If we consider the symmetric functions

$$f_m = \sum_{\alpha : |\alpha| = m} c_\alpha \delta^{\hat{\otimes}\alpha}, \qquad m = 1, 2, \ldots \tag{2.10}$$

then we obtain

$$\sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha = \sum_{m=0}^\infty \sum_{\alpha : |\alpha| = m} c_\alpha I_m(\delta^{\hat{\otimes}\alpha}) = \sum_{m=0}^\infty I_m(f_m).$$

The expansion here above is actually a result which was first proved in Ref. 25. We can state it as follows.

Theorem 2.2. (Chaos expansion II) *Every $F \in L^2(P)$ admits the (unique) representation*

$$F = \sum_{m=0}^\infty I_m(f_m) \tag{2.11}$$

via the unique sequence of symmetric functions $f_m \in L^2((\lambda \times \nu)^m)$, $m = 0, 1, \ldots$.

For any formal expansion $f = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha$ we define the norm

$$\|f\|^2_{0,k} := \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{k\alpha}, \qquad k = 0, 1, \ldots$$

where $(2\mathbb{N})^{k\alpha} = (2 \cdot 1)^{k\alpha_1}(2 \cdot 2)^{k\alpha_2}(2 \cdot 3)^{k\alpha_3} \cdots$. Now setting

$$(\mathcal{S})_{0,k} := \{f : \|f\|_{0,k} < \infty\}$$

we define

$$(\mathcal{S}) := \bigcap_{k=0}^{\infty} (\mathcal{S})_{0,k}$$

with the projective topology, and

$$(\mathcal{S})^* := \bigcup_{k=0}^{\infty} (\mathcal{S})_{0,k}$$

with the inductive topology. The space $(\mathcal{S})^*$ is the dual of $(\mathcal{S})$ and the action of $G = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha \in (\mathcal{S})^*$ applied to $f = \sum_{\alpha \in \mathcal{J}} b_\alpha K_\alpha \in (\mathcal{S})$ is

$$\langle G, f \rangle = \sum_{\alpha \in \mathcal{J}} a_\alpha b_\alpha \alpha!$$

Note that

$$(\mathcal{S}) \subset L^2(P) \subset (\mathcal{S})^* .$$

We refer to Refs. 19 and 22, for example, for the above definitions in the setting of the Gaussian and Poissonian white noise. See also Refs. 5, 21, 53 and references therein. For the Lévy case, we refer to Ref. 15.

Definition 2.1. The *white noise* $\overset{\bullet}{\tilde{N}}(t,x)$ *of the Poisson random measure* $\tilde{N}(dt,dz)$ is defined by the following formal expansion:

$$\overset{\bullet}{\tilde{N}}(t,z) = \sum_{i,j=1}^{\infty} \xi_i(t)\psi_j(z) \cdot K_{\varepsilon^{\gamma(i,j)}} . \tag{2.12}$$

It can be proved that the white noise takes values in $(\mathcal{S})^*$, $(\lambda \times \nu)$-a.e. See Ref. 42. The justification of the name "white noise" comes from the fact that, for any $B \in \mathfrak{B}(\mathbb{R})$ such that its closure does not contain 0, we have

$$I_1(\chi_{(0,t]} 1_B) = \sum_{i,j=1}^{\infty} c_{\gamma(i,j)} K_{\varepsilon^{\gamma(i,j)}}$$

with $c_{\gamma(i,j)} = \int_0^t \int_B \xi_i(s)\psi_j(z)\nu(dz)ds \cdot K_{\varepsilon^{\gamma(i,j)}}$. Cf. (2.8) and (2.10). Then

$$\tilde{N}(t,B) = \int_0^t \int_B \left(\sum_{i,j=1}^{\infty} \xi_i(s)\psi_j(z) K_{\varepsilon^{\gamma(i,j)}} \right) \nu(dz)ds .$$

So *formally* we have

$$\overset{\bullet}{\tilde{N}}(t,z) = \frac{\tilde{N}(dt,dz)}{dt\nu(dz)}$$

which is the analog of the Radon–Nikodym derivative in $(\mathcal{S})^*$.

Definition 2.2. The *white noise* $\overset{\bullet}{\eta}(t)$ *for the Lévy process* is defined by the following formal expansion

$$\overset{\bullet}{\eta}(t) = \sum_{i=1}^{\infty} \xi_i(t) K_{\varepsilon^{\gamma(i,1)}}, \tag{2.13}$$

for a specific choice of the basis ψ_i, $i = 1, 2, \ldots$, in (2.6) — cf. Ref. 15.

The Lévy white noise takes values in $(\mathcal{S})^*$ for *all* $t \in \mathbb{R}_+$. Here the boundedness of the Laguerre functions can be exploited, cf. Ref. 58.

Note that the Lévy white noise (2.13) is related to the white noise for the Poisson random measure (2.12) by the following formula which involves Bochner integrals with respect to ν:

$$\overset{\bullet}{\eta}(t) = \int_{\mathbb{R}} z \overset{\bullet}{\tilde{N}}(t,z)\nu(dz). \tag{2.14}$$

Definition 2.3. The *Wick product* $F \diamond G$ of two elements $F = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha$ and $G = \sum_{\beta \in \mathcal{J}} b_\beta K_\beta$ in $(\mathcal{S})^*$ is defined by

$$F \diamond G = \sum_{\alpha,\beta \in \mathcal{J}} a_\alpha b_\beta K_{\alpha+\beta}. \tag{2.15}$$

The spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ are topological algebras with respect to the Wick product.

3. Some Anticipative Calculus Formulae

In this section we present some known formulae for the Malliavin calculus in the case of pure jump Lévy processes. We will need these results in Sec. 4. These formulae generalize the known results for the Malliavin calculus in the case of Brownian motion, cf. Refs. 34–36 and 40, for example. First we recall the Skorohod integration and the Malliavin type stochastic derivative we are dealing with.

Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a random field taking values in $L^2(P)$. Then, for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}$, Theorem 2.2 provides the chaos expansion via symmetric functions

$$X(t,z) = \sum_{m=0}^{\infty} I_m(f_m(t_1, z_1, \ldots, t_m, z_m; t, z)).$$

Let $\hat{f}_m = \hat{f}_m(t_1, z_1, \ldots, t_{m+1}, z_{m+1})$ be the symmetrization of $f_m(t_1, z_1, \ldots, t_m, z_m; t, z)$ as a function of the $m+1$ variables $(t_1, z_1), \ldots, (t_{m+1}, z_{m+1})$ with $t_{m+1} = t$ and $z_{m+1} = z$.

The following concept was first introduced by Y. Kabanov — see Refs. 26 and 27, for example.

Definition 3.1. The random field $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, is *Skorohod integrable* if $\sum_{m=0}^{\infty}(m+1)!\|\hat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})} < \infty$. Then its *Skorohod integral with respect to*

$\tilde{N}$, i.e.

$$I(X) := \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)\,,$$

is defined by

$$I(X) := \sum_{m=0}^{\infty} I_{m+1}(\hat{f}_m)\,. \tag{3.1}$$

The Skorohod integral is an element of $L^2(P)$ and

$$\left\| \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz) \right\|^2_{L^2(P)} = \sum_{m=0}^{\infty} (m+1)! \|\hat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})}\,. \tag{3.2}$$

Moreover,

$$E\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz) = 0\,. \tag{3.3}$$

The Skorohod integral can be regarded as an extension of the Itô integral to *anticipative* integrands. In fact, the following result can be proved. Cf. Ref. 39. See also Refs. 6, 15 and 42.

Proposition 3.1. *Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a non-anticipative (adapted) integrand. Then the Skorohod integral and the Itô integral coincide in $L^2(P)$, i.e.*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\tilde{N}(dt, dz)\,.$$

Another remarkable property of the Skorohod integral is given in connection to the Wick product. See Ref. 42 for the details and the proof. Before stating the result, we remind that a random field Y taking values in $(\mathcal{S})^*$ is said to be $(\mathcal{S})^*$*-integrable* whenever $\langle Y, f\rangle \in L^1(\lambda \times \nu)$ for all $f \in (\mathcal{S})$.

Definition 3.2. For any $(\mathcal{S})^*$-integrable random field $Y(t,z)$ the $(\mathcal{S})^*$*-integral* is the unique element in $(\mathcal{S})^*$ such that

$$\left\langle \int_{\mathbb{R}_+} \int_{\mathbb{R}} Y(t,z)\nu(dz)dt, f \right\rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \langle Y(t,z), f\rangle \nu(dz)dt\,, \qquad f \in (\mathcal{S})\,. \tag{3.4}$$

Theorem 3.1. *Let $Y(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be Skorohod integrable and $\int_a^b \int_{\mathbb{R}} E[Y(t,z)^2]\nu(dz)dt < \infty$ for some $0 \le a < b$. Then $Y \diamond \overset{\bullet}{\tilde{N}}$ is $(\mathcal{S})^*$-integrable over $[a,b] \times \mathbb{R}$ and we obtain the following relationship*

$$\int_a^b \int_{\mathbb{R}} Y(t,z)\tilde{N}(\delta t, dz) = \int_a^b \int_{\mathbb{R}} Y(t,z) \diamond \overset{\bullet}{\tilde{N}}(t,z)\nu(dz)dt\,. \tag{3.5}$$

Thanks to the relation (2.4), we can easily recognize the Skorohod integral with respect to the very Lévy process $\eta(t)$, $t \in \mathbb{R}_+$, as a particular case of the Skorohod integration with respect to the compensated Poisson random measure $\tilde{N}$. See Ref. 15. In fact, for the integrands $X(t,z) = z \cdot \varphi(t)$: $X(t,z) = \sum_{m=0}^{\infty} I_m(z \cdot f(t_1, z_1, \ldots, t_m, z_m; t))$, we have

$$\int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz) = \int_{\mathbb{R}} \varphi(t)\delta\eta(t)\,. \tag{3.6}$$

Now we consider the definition of the Malliavin type derivative $D_{t,z}$ for compensated Poisson random measures which was initially given in Ref. 33. Other definitions have also been studied by several authors for the same case of pure jump Lévy processes, the particular case of Poisson random processes and for the case of the general Lévy process with no drift, see for instance Refs. 4, 7, 15, 32, 37–39, 44 and 45.

Definition 3.3. The space $\mathbb{D}_{1,2}$ is the set of all elements $F \in L^2(P)$ admitting the chaos expansion (2.11): $F = E[F] + \sum_{m=1}^{\infty} I_m(f_m)$, such that

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{m=1}^{\infty} m \cdot m! \|f_m\|^2_{L^2((\lambda\times\nu)^m)} < \infty\,.$$

The *Malliavin derivative* $D_{t,z}$ is an operator defined on $\mathbb{D}_{1,2}$ with values in the standard L^2-space $L^2(P \times \lambda \times \nu)$ given by

$$D_{t,z}F := \sum_{m=1}^{\infty} mI_{m-1}(f_m(\cdot, t, z))\,, \tag{3.7}$$

where $f_m(\cdot, t, z) = f_m(t_1, z_1, \ldots, t_{m-1}, z_{m-1}; t, z)$.

Note that if $F \in \mathbb{D}_{1,2}$, then

$$E\left[\int_0^\infty \int_{\mathbb{R}} (D_{t,z}F)^2 \nu(dz)dt\right] = \|F\|^2_{\mathbb{D}_{1,2}}\,.$$

The operator $D_{t,z}$ is proved to be closed and to coincide with a certain difference operator defined in Ref. 44.

The above operator can be extended to the whole space $(\mathcal{S})^*$ thanks to the chaos expansion (2.9).

Definition 3.4. For any $F = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \in (\mathcal{S})^*$ the *Malliavin derivative* $D_{t,z}F$ is defined as

$$D_{t,z}F := \sum_{\alpha\in\mathcal{J}} c_\alpha \sum_{i,j=1}^{\infty} \alpha_{\gamma(i,j)} K_{\alpha-\epsilon^{\gamma(i,j)}} \cdot \xi_i(t)\psi_j(z)\,. \tag{3.8}$$

It can be proved that $D_{t,z}F \in (\mathcal{S})^*$, $\lambda \times \nu$-a.e., for all $F \in (\mathcal{S})^*$. Moreover, it can also be shown that if $F = \lim_{n\to\infty} F_n$ in $(\mathcal{S})^*$, then there exists a subsequence $F_{n_k} \in (\mathcal{S})^*$ such that $D_{t,z}F = \lim_{n\to\infty} D_{t,z}F_{n_k}$ in $(\mathcal{S})^*$, $\lambda \times \nu$-a.e. See Ref. 42.

We remark that in general the stochastic derivative $D_{t,z}$, being essentially a difference operator, does not satisfy a "chain rule" as in the case of the Malliavin derivative for the Brownian motion setting. Cf. Refs. 35, 36 and 40, for example. Nevertheless a "chain rule" can still be formulated in terms of the Wick product.

Proposition 3.2. (Chain rule via Wick product) *Let $F \in (\mathcal{S})^*$ and let $g(z) = \sum_{n\geq 0} a_n z^n$ be an analytic function in the whole complex plane. Then $\sum_{n\geq 0} a_n F^{\diamond n}$ is convergent in $(\mathcal{S})^*$. Furthermore, for $g^\diamond(F) = \sum_{n\geq 0} a_n F^{\diamond n}$, the following Wick chain rule is valid*

$$D_{s,x} g^\diamond(F) = \left(\frac{d}{dz} g\right)^\diamond (F) \diamond D_{s,x} F\,. \tag{3.9}$$

Proof. The first statement can be derived following similar proofs as in Theorems 2.6.12 and 2.8.1 in Ref. 42. For what concerns the chain rule, it can be easily shown that it holds for polynomials. Then the result follows by the closedness of $D_{t,z}$ and the continuity of the Wick product. □

Now we turn our attention to the calculus and present some basic explicit formulae. First of all we recall the following result proved in Ref. 6. See also Theorem 2.6 of Ref. 12.

Theorem 3.2. (Duality formula) *Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be Skorohod integrable and $F \in \mathbb{D}_{1,2}$. Then*

$$E\left[\int_0^\infty \int_{\mathbb{R}} X(t,z) D_{t,z} F \nu(dz)dt\right] = E\left[F \int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)\right]. \tag{3.10}$$

Corollary 3.1. (Closability of Skorohod integral) *Suppose that $X_n(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, is a sequence of Skorohod integrable random fields and that the corresponding sequence of integrals*

$$I(X_n) := \int_0^\infty \int_{\mathbb{R}} X_n(t,z)\tilde{N}(\delta t, dz)\,, \qquad n = 1, 2, \dots$$

converges in $L^2(P)$. Moreover, suppose that

$$\lim_{n\to\infty} X_n = 0 \quad \text{in} \quad L^2(P \times \lambda \times \nu)\,.$$

Then we have

$$\lim_{n\to\infty} I(X_n) = 0 \quad \text{in} \quad L^2(P)\,.$$

Proof. By Theorem 3.2 we have that

$$(I(X_n), F)_{L^2(P)} = (X_n, D_{t,z}F)_{L^2(P\times\lambda\times\nu)} \longrightarrow 0\,, \qquad n \to \infty\,,$$

for all $F \in \mathbb{D}_{1,2}$. Then we conclude that $\lim_{n\to\infty} I(X_n) = 0$ weakly in $L^2(P)$. And since the sequence $I(X_n)$, $n = 1, 2, \dots$, is convergent in $L^2(P)$, the result follows. □

In view of Corollary 3.1 we can extend the definition of Skorohod integral as follows.

Definition 3.5. Let X_n, $n = 1, 2, \ldots$, be a sequence of Skorohod integrable random fields such that

$$X = \lim_{n\to\infty} X_n \quad \text{in} \quad L^2(P \times \lambda \times \nu)\,.$$

Then we define the *Skorohod integral of* X as

$$I(X) := \int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz) = \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}} X_n(t,z)\tilde{N}(\delta t, dz) =: \lim_{n\to\infty} I(X_n)\,,$$

provided that this limit exists in $L^2(P)$.

The following result is Lemma 6.1 in Ref. 38 (obtained in a more general setting).

Lemma 3.1. *Let* F, $G \in \mathbb{D}_{1,2}$ *with* G *bounded. Then* $F \cdot G \in \mathbb{D}_{1,2}$ *and we have*

$$D_{t,z}(F \cdot G) = F \cdot D_{t,z}G + G \cdot D_{t,z}F + D_{t,z}F \cdot D_{t,z}G \qquad \lambda \times \nu\text{-}a.e. \tag{3.11}$$

Proof. With the help of Lemma 9 in Ref. 33 the result can be verified for F and G of the form $g(\eta(t_1), \ldots, \eta(t_k))$, where g is a smooth function with compact support. Then, by using a limit argument, the proof follows from the closedness of $D_{t,z}$. □

Remark 3.1. For an extension of this result to normal martingales, see for example Proposition 1 in Ref. 48 or Proposition 5 in Ref. 51.

The following result is Theorem 7.1 in Ref. 38 (obtained in a more general setting).

Theorem 3.3. (Integration by parts) *Let* $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, *be a Skorohod integrable stochastic process and* $F \in \mathbb{D}_{1,2}$ *such that the product* $X(t,z)\cdot(F + D_{t,z}F)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, *is Skorohod integrable. Then*

$$F \int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)$$
$$= \int_0^\infty \int_{\mathbb{R}} X(t,z)(F + D_{t,z}F)\tilde{N}(\delta t, dz) + \int_0^\infty \int_{\mathbb{R}} X(t,z) D_{t,z}F \nu(dz)dt\,. \tag{3.12}$$

Proof. Let $G \in \mathbb{D}_{1,2}$ be bounded. Then we obtain by Theorem 3.2 and Lemma 3.1

$$E\left[G \int_0^\infty \int_{\mathbb{R}} FX(t,z)\tilde{N}(\delta t, dz)\right] = E\left[\int_0^\infty \int_{\mathbb{R}} FX(t,z) D_{t,z}G\nu(dz)dt\right]$$
$$= E\left[GF \int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)\right] - E\left[G \int_0^\infty \int_{\mathbb{R}} X(t,z) D_{t,z}F\nu(dz)dt\right]$$
$$- E\left[G \int_0^\infty \int_{\mathbb{R}} X(t,z) D_{t,z}F\tilde{N}(\delta t, dz)\right]$$

$$= E\left[G\left(F\int_0^\infty\int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t,dz) - \int_0^\infty\int_{\mathbb{R}} X(t,z)D_{t,z}F\nu(dz)dt\right.\right.$$
$$\left.\left.- \int_0^\infty\int_{\mathbb{R}} X(t,z)D_{t,z}F\tilde{N}(\delta t,dz)\right)\right].$$

The proof then follows by a density argument applied to G. □

Remark 3.2. Using the Poisson interpretation of Fock space, the formula (3.12) has been shown to be an expression of the multiplication formula for Poisson stochastic integrals. See Refs. 27 and 57, Proposition 2 and Relation (6) of Ref. 49, Definition 7 and Proposition 6 of Ref. 52, Proposition 2 of Ref. 51 and Proposition 1 of Ref. 47.

Formula (3.12) has been known for some time to quantum probabilists in identical or close formulations. See Proposition 21.6 and Proposition 21.8 in Ref. 46, Proposition 18 in Ref. 3 and Relation (5.6) in Ref. 1.

The following result is Theorem 4.2 in Ref. 38 (obtained in a more general setting).

Theorem 3.4. (Fundamental theorem of calculus) *Let $X \in L^2(P\times\lambda\times\nu)$. Assume that $X(s,y) \in \mathbb{D}_{1,2}$ for all (s,y), and $D_{t,z}X(s,y)$, $s \in \mathbb{R}_+$, $y \in \mathbb{R}$, for (t,z) $\lambda\times\nu$-a.e. is Skorohod integrable and that*

$$E\left[\int_0^\infty\int_{\mathbb{R}}\left(\int_0^\infty\int_{\mathbb{R}} D_{t,z}X(s,y)\tilde{N}(\delta s,dy)\right)^2\nu(dz)dt\right] < \infty\,.$$

Then $\int_0^\infty\int_{\mathbb{R}} X(s,y)\tilde{N}(\delta s,dy) \in \mathbb{D}_{1,2}$ and

$$D_{t,z}\left(\int_0^\infty\int_{\mathbb{R}} X(s,y)\tilde{N}(\delta s,dy)\right) = \int_0^\infty\int_{\mathbb{R}} D_{t,z}X(s,y)\tilde{N}(\delta s,dy) + X(t,z)\,. \quad (3.13)$$

Proof. First suppose that

$$X(s,y) = I_n(f_n(\cdot,s,y))\,,$$

where $f_n(t_1,z_1,\ldots,t_n,z_n,s,y)$ is symmetric with respect to $(t_1,z_1),\ldots,(t_n,z_n)$. By Definition 3.1 we have

$$\int_0^\infty\int_{\mathbb{R}} X(s,y)\tilde{N}(\delta s,dy) = I_{n+1}(\widehat{f_n})\,, \quad (3.14)$$

where

$$\widehat{f_n}(t_1, z_1, \ldots, t_n, z_n, t_{n+1}, z_{n+1})$$
$$= \frac{1}{n+1}[f_n(t_{n+1}, z_{n+1}, \ldots, t_1, z_1) + \cdots + f_n(t_{n+1}, z_{n+1}, \cdot, t_n, z_n)$$
$$+ f_n(t_1, z_1, \cdot, t_{n+1}, z_{n+1})]$$

is the symmetrization of f_n with respect to the variables $(t_1, z_1), \ldots, (t_n, z_n)$, $(t_{n+1}, z_{n+1}) = (s, y)$. Therefore we get

$$D_{t,z}\left(\int_0^\infty \int_{\mathbb{R}} X(s, y)\tilde{N}(\delta s, dy)\right)$$
$$= I_n(f_n(t, z, \cdot, t_1, z_1) + \cdots + f_n(t, z, \cdot, t_n, z_n) + f_n(\cdot, t, z)).$$

On the other hand, we see that

$$\int_0^\infty \int_{\mathbb{R}} D_{t,z}X(s, y)\tilde{N}(\delta s, dy)$$
$$= \int_0^\infty \int_{\mathbb{R}} nI_{n-1}(f_n(\cdot, t, z, s, y))\tilde{N}(\delta s, dy) = nI_n(\widehat{f_n}(\cdot, t, z, \cdot)), \qquad (3.15)$$

where

$$\widehat{f_n}(t_1, z_1, \ldots, t_{n-1}, t_{n-1}, t, z, t_n, z_n) = \frac{1}{n}[f_n(t, z, \cdot, t_1, z_1) + \cdots + f_n(t, z, \cdot, t_n, z_n)]$$

is the symmetrization of $f_n(t_1, z_n, \ldots, t_{n-1}, z_{n-1}, t, z, t_n, z_n)$ with respect to $(t_1, z_1), \ldots, (t_{n-1}, z_{n-1})$, $(t_n, z_n) = (s, y)$. A comparison of (3.14) and (3.15) yields formula (3.13).

Next consider the general case

$$X(s, y) = \sum_{n \geq 0} I_n(f_n(\cdot, s, y)).$$

Define

$$X_m(s, y) = \sum_{n=0}^{m} I_n(f_n(\cdot, s, y)), \qquad m = 1, 2, \ldots.$$

Then (3.13) holds for X_m. Since

$$\left\|\int_0^\infty \int_{\mathbb{R}} D_{t,z}X_m(s, y)\tilde{N}(\delta s, dy) - \int_0^\infty \int_{\mathbb{R}} D_{t,z}X(s, y)\tilde{N}(\delta s, dy)\right\|^2_{L_2(P\times\lambda\times\nu)}$$
$$= \sum_{n \geq m+1} n^2 n! \|\widehat{f_n}\|^2_{L_2((\lambda\times\nu)^{n+1})} \longrightarrow 0, \qquad m \longrightarrow \infty$$

the proof follows by the closedness of $D_{t,z}$. □

The following result is Theorem 4.1 in Ref. 38 (obtained in a more general setting).

Theorem 3.5. (The Itô–Lévy–Skorohod isometry) *Let* $X \in L^2(P \times \lambda \times \nu)$ *and* $DX \in L^2(P \times (\lambda \times \nu)^2)$. *Then the following isometry holds*

$$E\left[\left(\int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)\right)^2\right]$$

$$= E\left[\int_0^\infty \int_{\mathbb{R}} X^2(t,z)\nu(dz)dt\right]$$

$$+ E\left[\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} D_{t,z}X(s,y)D_{s,y}X(t,z)\nu(dy)ds\nu(dz)dt\right]. \quad (3.16)$$

Proof. Consider

$$X(t,z) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t,z)K_\alpha .$$

Define

$$S_1 = \sum_{\alpha \in \mathcal{J}} \alpha! \|c_\alpha\|^2_{L^2(\lambda \times \nu)}, \qquad S_2 = \sum_{\alpha \in \mathcal{J}, i,j \in \mathbb{N}} \alpha_{\gamma(i,j)} \alpha! (c_\alpha, \xi_j \psi_i)^2$$

and

$$S_3 = \sum_{\substack{\alpha,\beta \in \mathcal{J}, i,j,k,l \in \mathbb{N} \\ (i,j) \neq (k,l)}} (\alpha_{\gamma(i,j)} + 1)\alpha! (c_\alpha, \xi_j \psi_i)(c_\beta, \xi_k \psi_l) 1_{\{\alpha + \varepsilon^{\gamma(i,j)} = \varepsilon^{\gamma(k,l)}\}},$$

where $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\lambda \times \nu)}$. Note that by the assumption and Lemma 3.12 in Ref. 42 the sums above are convergent. First it follows that

$$E\left[\left(\int_0^\infty \int_{\mathbb{R}} X(t,z)\tilde{N}(\delta t, dz)\right)^2\right]$$

$$= E\left[\left(\int_0^\infty \int_{\mathbb{R}} X(t,z) \diamond \overset{\bullet}{\tilde{N}}(t,z)\nu(dz)dt\right)^2\right]$$

$$= E\left[\left(\int_0^\infty \int_{\mathbb{R}} \left(\sum_{\alpha \in \mathcal{J}} c_\alpha(t,z)K_\alpha\right) \diamond \left(\sum_{i,j} \xi_j(t)\psi_i(z)K_{\varepsilon^{\gamma(i,j)}}\right) \nu(dz)dt\right)^2\right]$$

$$= E\left[\left(\sum_{\alpha \in \mathcal{J}, i,j} (c_\alpha, \xi_j \psi_i) K_{\alpha + \varepsilon^{\gamma(i,j)}}\right)^2\right]$$

$$= \sum_{\substack{\alpha,\beta \in \mathcal{J}, i,j,k,l \in \mathbb{N} \\ (i,j) \neq (k,l)}} (\alpha + \varepsilon^{\gamma(i,j)})! (c_\alpha, \xi_j \psi_i)(c_\beta, \xi_k \psi_l) 1_{\{\alpha + \varepsilon^{\gamma(i,j)} = \varepsilon^{\gamma(k,l)}\}}$$

$$= S_1 + S_2 + S_3 ,$$

since $(\alpha + \varepsilon^{\gamma(i,j)})! = (\alpha_{\gamma(i,j)} + 1)\alpha!$.

Next, we have

$$E\left[\int_0^\infty \int_{\mathbb{R}} X^2(t,z)\nu(dz)dt\right] = E\left[\int_0^\infty \int_{\mathbb{R}} \left(\sum_{\alpha\in\mathcal{J}} c_\alpha(t,z)K_\alpha\right)^2 \nu(dz)dt\right]$$

$$= \sum_{\alpha\in\mathcal{I}} \int_0^\infty \int_{\mathbb{R}} c_\alpha^2(t,z)\alpha!\nu(dz)dt = S_1\,.$$

Finally, we get

$$E\left[\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} D_{t,z}X(s,y)D_{s,y}X(t,z)\nu(dy)ds\nu(dz)dt\right]$$

$$= E\Bigg[\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \left(\sum_{\alpha,k,l} c_\alpha(s,y)\xi_k(t)\psi_l(z)\alpha_{\varepsilon^{\gamma(k,l)}}K_{\alpha-\varepsilon^{\gamma(k,l)}}\right)$$

$$\cdot\left(\sum_{\beta,i,j} c_\alpha(t,z)\xi_k(s)\psi_l(y)\alpha_{\varepsilon^{\gamma(i,j)}}K_{\beta-\varepsilon^{\gamma(i,j)}}\right)\nu(dy)ds\nu(dz)dt\Bigg]$$

$$= \sum_{\alpha,\beta\in\mathcal{J},i,j,k,l\in\mathbb{N}} (c_\alpha,\xi_j\psi_i)(c_\beta,\xi_k\psi_l)\beta_{\gamma(i,j)}\alpha!1_{\{\alpha+\varepsilon^{\gamma(i,j)}=\varepsilon^{\gamma(k,l)}\}} = S_2+S_3\,.$$

Combining the three steps of the proof the desired result follows. □

Remark 3.3. Formula (3.16) can also be obtained as a consequence of the Poisson interpretation of Fock space. See Proposition 17 in Ref. 3 and Proposition 1 in Ref. 52. For an isometry of this type which is not based on Fock space, see Proposition 3.3 in Ref. 50.

4. Forward Integrals and Generalized Itô Formulae

In this section we introduce the *forward integral* with respect to the Poisson random measure $\tilde{N}$. Then we prove an Itô formula for the corresponding forward processes and we apply this to obtain an Itô formula for processes driven by Skorohod integrals. Here we can refer to Refs. 36, 55 and 20, for example, where these topics are developed for the Brownian motion.

Definition 4.1. The *forward integral*

$$J(\theta) := \int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz)$$

with respect to the Poisson random measure $\tilde{N}$, of a caglad stochastic function $\theta(t,z)$, $t\in\mathbb{R}_+$, $z\in\mathbb{R}$, with

$$\theta(t,z) := \theta(\omega,t,z)\,, \qquad \omega\in\Omega\,,$$

is defined as

$$J(\theta) = \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}} \theta(t,z) 1_{U_m} \tilde{N}(dt,dz)$$

if the limit exists in $L^2(P)$. Here U_m, $m = 1,2,\ldots$, is an increasing sequence of compact sets $U_m \subseteq \mathbb{R}\setminus\{0\}$ with $\nu(U_m) < \infty$ such that $\lim_{m\to\infty} U_m = \mathbb{R}\setminus\{0\}$.

Remark 4.1. Note that if G is a random variable, then

$$G \cdot \int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz) = \int_0^T \int_{\mathbb{R}} G\cdot\theta(t,z)\tilde{N}(d^-t,dz)\,, \tag{4.1}$$

a property that does not hold for the Skorohod integrals.

Definition 4.2. In the sequel we let $\mathcal{M}$ denote the set of the stochastic functions $\theta(t,z)$, $t\in\mathbb{R}_+$, $z\in\mathbb{R}$, such that
(i) $\theta(\omega,t,z) = \theta_1(\omega,t)\theta_2(\omega,t,z)$ where $\theta_1(\omega,t)\in\mathbb{D}_{1,2}$ is caglag and $\theta_2(\omega,t,z)$ is adapted and such that

$$E\left[\int_0^T \int_{\mathbb{R}} \theta_2^2(t,z)\nu(dz)dt\right] < \infty\,,$$

(ii) $D_{t^+,z}\xi = \lim_{s\to t^+} D_{s,z}\xi$ exists in $L^2(P\times\lambda\times\nu)$,
(iii) $\theta(t,z) + D_{t^+,z}\theta(t,z)$ is Skorohod integrable.
We let $\mathbb{M}_{1,2}$ be the closure of the linear span of $\mathcal{M}$ with respect to the norm given by

$$\|\theta\|^2_{\mathbb{M}_{1,2}} := \|\theta\|^2_{L^2(P\times\lambda\times\nu)} + \|D_{t^+,z}\theta(t,z)\|^2_{L^2(P\times\lambda\times\nu)}\,.$$

We can now show the relation between the forward integral and the Skorohod integral.

Lemma 4.1. *If $\theta\in\mathbb{M}_{1,2}$, then its forward integral exists and*

$$\int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz) = \int_0^T \int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\nu(dz)dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z) + D_{t^+,z}\theta(t,z))\tilde{N}(\delta t,dz)\,.$$

Proof. First consider the case when $\theta(\omega,t,z) = \theta_1(\omega,t)\theta_2(\omega,t,z)$. Let us take a sequence of partitions of $[0,T]$ of the form $0 = t_0^n < t_1^n < \cdots < t_{J_n}^n = T$ with $|\Delta t| := \max(t_j^n - t_{j-1}^n) \to 0$, for $n\to\infty$, into account. By Theorem 3.3 we have

$$\begin{aligned} F\cdot &\int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} \theta(t,z)\tilde{N}(\delta t,dz) \\ &= \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} F\theta(t,z)\tilde{N}(\delta t,dz) \\ &\quad + \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} \theta(t,z) D_{t,z}F\nu(dz)dt + \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} \theta(t,z)D_{t,z}F\tilde{N}(\delta t,dz)\,. \end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz) \\
&\quad = \lim_{|\Delta t|\to 0} \sum_{i=1}^{J_n} \theta_1(t_{i-1}^n) \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} \theta_2(t,z)\tilde{N}(dt,dz) \\
&\quad = \lim_{|\Delta t|\to 0} \sum_{i=1}^{J_n} \theta_1(t_{i-1}^n) \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} \theta_2(t,z)\tilde{N}(\delta t,dz) \\
&\quad = \lim_{|\Delta t|\to 0} \sum_{i=1}^{J_n} \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} [\theta_1(t_{i-1}^n) + D_{t,z}\theta_1(t_{i-1}^n)]\theta_2(t,z)\tilde{N}(\delta t,dz) \\
&\qquad + \lim_{|\Delta t|\to 0} \sum_{i=1}^{J_n} \int_{t_{i-1}^n}^{t_i^n} \int_{\mathbb{R}} D_{t,z}\theta_1(t_{i-1}^n) \cdot \theta_2(t,z)\nu(dz)dt \\
&\quad = \int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(\delta t,dz) + \int_0^T \int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\nu(dz)dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\tilde{N}(\delta t,dz)\,.
\end{aligned}$$

The proof is then completed by a limit argument in view of Definitions 3.5 and 4.2. □

Corollary 4.1. *If the forward integral exists in* $L^2(P)$*, then*

$$E\int_0^T \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz) = E\int_0^T \int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\nu(dz)dt\,. \tag{4.2}$$

Proof. This follows from (3.3) and Lemma 4.1. □

Definition 4.3. A *forward process* is a measurable stochastic function $X(t) = X(t,\omega)$, $t \in \mathbb{R}_+$, $\omega \in \Omega$, that admits the representation

$$X(t) = x + \int_0^t \int_{\mathbb{R}} \theta(s,z)\tilde{N}(d^-s,dz) + \int_0^t \alpha(s)ds\,, \tag{4.3}$$

where $x = X(0)$ is a constant. A shorthand notation for (4.3) is

$$d^-X(t) = \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz) + \alpha(t)dt\,; \qquad X(0) = x\,. \tag{4.4}$$

We call $d^-X(t)$ the *forward differential* of $X(t)$, $t \in \mathbb{R}_+$.

Theorem 4.1. (Itô formula for forward integrals) *Let $X(t)$, $t \in \mathbb{R}_+$, be a forward process of the form* (4.3) *where $\theta(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, is locally bounded in z near $z = 0$ $P \times \lambda$-a.e. and such that*

$$\int_0^T \int_{\mathbb{R}} |\theta(t,z)|^2 \nu(dz)dt < \infty \qquad P\text{-a.s.}$$

Suppose also that $|\theta(t,z)|$, $t\in\mathbb{R}_+$, $z\in\mathbb{R}$, *is forward integrable. For any function* $f\in C^2(\mathbb{R})$, *the forward differential of* $Y(t)=f(X(t))$, $t\in\mathbb{R}_+$, *is given by the following formula*:

$$\begin{aligned} d^-Y(t) &= f'(X(t))\alpha(t)dt \\ &\quad + \int_{\mathbb{R}} (f(X(t^-)+\theta(t,z)) - f(X(t^-)) - f'(X(t^-))\theta(t,z))\nu(dz)dt \\ &\quad + \int_{\mathbb{R}} (f(X(t^-)+\theta(t,z)) - f(X(t^-)))\tilde{N}(d^-t,dz)\,. \end{aligned} \tag{4.5}$$

Proof. The proof follows the same line as in the classical Itô formula (see Ref. 23 Chap. 2, Sec. 5). For simplicity we assume $x=0$ and $\alpha\equiv 0$. We can write

$$X_m(t) := \int_0^t\int_{\mathbb{R}} \theta(s,z)1_{U_m}(z)N(ds,dz) - \int_0^t\int_{\mathbb{R}} \theta(s,z)1_{U_m}(z)\nu(dz)ds\,.$$

We denote by $0=\sigma_0<\sigma_1<\cdots$ the stopping times for which the jumps of the Lévy process occur. Thus we obtain

$$\begin{aligned} f(X_m(t)) - f(X_m(0)) &= \sum_i [f(X_m(\sigma_i\wedge t)) - f(X_m(\sigma_i\wedge t^-))] \\ &\quad + \sum_i [f(X_m(\sigma_i\wedge t^-)) - f(X_m(\sigma_{i-1}\wedge t))] \\ &=: \mathcal{J}_1(t) + \mathcal{J}_2(t)\,, \end{aligned}$$

with

$$f(X_m(\sigma_i\wedge t^-)) = \begin{cases} f(X_m(\sigma_i^-))\,, & \sigma_i\le t\,,\\ f(X_m(t))\,, & \sigma_i>t\,. \end{cases}$$

By the change of variable formula for finite variation processes, it follows that

$$\mathcal{J}_2(t) = -\int_0^t\int_{\mathbb{R}} f'(X_m(s))\theta(s,z)1_{U_m}(z)\nu(dz)ds\,.$$

Moreover, it is

$$\begin{aligned} \mathcal{J}_1(t) &= \sum_i [f(X_m)(\sigma_i) - f(X_m)(\sigma_i^-)]1_{\{\sigma_i\le t,\theta(\sigma_i,\eta(\sigma_i))\neq 0\}} \\ &= \int_0^t\int_{\mathbb{R}} [f(X_m(s^-)+\theta(s,z)1_{U_m}(z)) - f(X_m)(s^-)]N(ds,dz) \end{aligned}$$

$$= \int_0^t \int_{\mathbb{R}} [f(X_m(s^-) + \theta(s,z)1_{U_m}(z)) - f(X_m)(s^-)]\tilde{N}(d^-s, dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [f(X_{m,n}(s^-) + \theta(s,z)1_{U_m}(z)) - f(X_m)(s^-)]\nu(dz)ds\,.$$

By letting $m \to \infty$, formula (4.6) follows. □

In order to state an Itô formula for Skorohod integrals we need to combine Lemma 4.1 and Theorem 4.1. To this end we go into the technical step of solving equations of the following type: given a random variable G, find the stochastic function $F(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, such that

$$F(t,z) + D_{t^+,z}F(t,z) = G\,, \tag{4.6}$$

for almost all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$. For example, if $G = g(\eta(T))$, for some measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}$ and

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(dt, dz)\,, \qquad t \in [0,T]\,,$$

then

$$F(t,z) := g(\eta(T) - z\chi_{[0,T)}(t))$$

does the job. In fact, with this choice of $F(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, we have

$$F(t,z) + D_{t^+,z}F(t,z) = g(\eta(T) - z\chi_{[0,T)}) + g(\eta(T)) - g(\eta(T) - z\chi_{[0,T)}) = G\,.$$

The above observation motivates the following definition.

Definition 4.4. The linear operator S is defined on the space of all $\mathcal{F}_T$-measurable random variables G as follows. If $G = \prod_{i=1}^k g_i(\eta(t_i))$, for some $t_i \in [0,T]$, $i = 1,\ldots,k$, we define

$$S_{t,z}\left(\prod_{i=1}^k g_i(\eta(t_i))\right) = \prod_{i=1}^k g_i(\eta(t_i) - z\chi_{[0,t_i)}(t))\,. \tag{4.7}$$

Note that via this definition the solution of Eq. (4.6) can be written as $F(t,z) = S_{t,z}G$, i.e.

$$S_{t,z}G + D_{t^+,z}(S_{t,z}G) = G\,. \tag{4.8}$$

Combining the above facts with Lemma 4.1 and Theorem 4.1, we obtain the following result.

Theorem 4.2. (Itô formula for Skorohod integrals) *Let*

$$X(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(\delta s, dz) + \int_0^t \alpha(s)ds\,, \qquad t \in [0,T]\,,$$

or, in shorthand notation,

$$\delta X(t) = \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(\delta t, dz) + \alpha(t)dt\,, \qquad t \in [0,T]\,.$$

Let $f \in C^2(\mathbb{R})$ *and let* $Y(t) = f\big(X(t)\big)$*. Set*

$$\theta(t,z) := S_{t,z}\gamma(t,z) \tag{4.9}$$

for all $t \in [0,T]$, $z \in \mathbb{R}$, *and assume* $\theta \in \mathbb{M}_{1,2}$*. Then*

$$\begin{aligned}\delta Y(t) = f'(X(t))\alpha(t)dt + \int_{\mathbb{R}} &\{f(X(t^-) + \theta(t,z)) - f(X(t^-)) \\ &+ D_{t^+,z}[f(X(t^-) + \theta(t,z)) - f(X(t^-))]\}\tilde{N}(\delta t, dz) \\ &+ \int_{\mathbb{R}} \{f(X(t^-) + \theta(t,z)) - f(X(t^-)) - f'(X(t^-))\theta(t,z) \\ &+ D_{t^+,z}[f(X(t^-) + \theta(t,z)) - f(X(t^-))] - f'(X(t^-))D_{t^+,z}\theta(t,z)\}\nu(dz)dt\,.\end{aligned} \tag{4.10}$$

Remark 4.2. Note that if γ and α are adapted, then $\theta(t,z) = \gamma(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, and

$$D_{t^+,z}\theta(t,z) = D_{t^+,z}[f(X(t^-) + \theta(t,z)) - f(X(t^-))]\,.$$

Therefore Theorem 4.2 reduces to the classical adapted Itô formula.

Proof. For simplicity we assume $\alpha \equiv 0$. By (4.8) we have

$$\theta(t,z) + D_{t^+,z}\theta(t,z) = \gamma(t,z)\,.$$

Hence by Lemma 4.1 we have

$$X(t) = \int_0^t \int_{\mathbb{R}} \theta(s,z)\tilde{N}(d^-s, dz) - \int_0^t \int_{\mathbb{R}} D_{s^+,z}\theta(s,z)\nu(dz)ds\,.$$

We can therefore apply Theorem 4.1 and get

$$\begin{aligned}Y(t) - Y(0) = &\int_0^t f'(X(s))\left(-\int_{\mathbb{R}} D_{s^+,z}\theta(s,z)\nu(dz)\right)ds \\ &+ \int_0^t \int_{\mathbb{R}} \{f(X(s^-)+\theta(s,z)) - f(X(s^-)) - f'(X(s^-))\theta(s,z)\}\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}} \{f(X(s^-) + \theta(s,z)) - f(X(s^-))\}\tilde{N}(d^-s, dz) \\ &- \int_0^t \int_{\mathbb{R}} f'(X(s^-))D_{s^+,z}\theta(s,z)\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}} \{f(X(s^-)+\theta(s,z)) - f(X(s^-)) - f'(X(s^-))\theta(s,z)\}\nu(dz)ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \int_{\mathbb{R}} D_{s^+,z}\{f(X(s^-) + \theta(s,z)) - f(X(s^-))\}\nu(dz)dt \\
&+ \int_0^t \int_{\mathbb{R}} \{f(X(s^-) + \theta(s,z)) - f(X(s^-)) + D_{s^+,z}\{f(X(s^-) \\
&+ \theta(s,z)) - f(X(s^-))\}\}\tilde{N}(\delta s, dz) \\
= &\int_0^t \int_{\mathbb{R}} \{f(X(s^-) + \theta(s,z)) - f(X(s^-)) - f'(X(s^-))\theta(s,z) \\
&+ D_{s^+,z}[f(X(s^-) + \theta(s,z)) - f(X(s^-))] \\
&- f'(X(s^-))D_{s^+,z}\theta(s,z)\}\nu(dz)ds \\
&+ \int_0^t \int_{\mathbb{R}} \{f(X(s^-) + \theta(s,z)) - f(X(s^-)) + D_{s^+,z}[f(X(s^-) \\
&+ \theta(s,z)) - f(X(s^-))]\}\tilde{N}(\delta s, dz)\,.
\end{aligned}
$$

This completes the proof. □

Remark 4.3. In Ref. 47 a different anticipative Itô formula is obtained, valid for polynomials f.

Remark 4.4. The Itô formula can be extended to cover the mixed case, involving a combination of Gaussian and compensated Poisson random measures.

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