Madeira Math Encounters XXXIV

Networks: Real World Networks, Structure, Dynamics, Models

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Graph Partitioning Algorithms

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Outline

1. The Graph Bisection Problem
   - Problem statement
   - A greedy algorithm
   - The Condon–Karp algorithm
   - Semidefinite programming
   - A spectral method
2. A more general perspective
   - The cut norm
   - Approximation by cut matrices

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1. The Graph Bisection Problem
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   - A greedy algorithm
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   - Semidefinite programming
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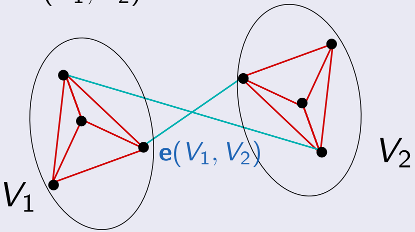
2. A more general perspective
   - The cut norm
   - Approximation by cut matrices

The Min Bisection Problem

Min Bisection

Given: A graph $G = (V, E)$ with n vertices (n even).

Goal: Partition V into two sets V_1, V_2 of equal size so as to *minimize* $e(V_1, V_2)$.



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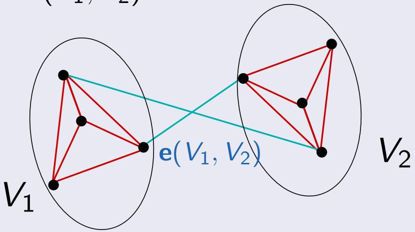

Of course...

...the Min Bisection problem is NP-hard.

Naive algorithm: try all partitions; running time (about) 2^n.

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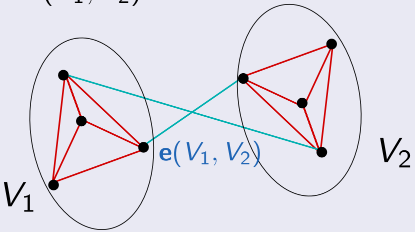


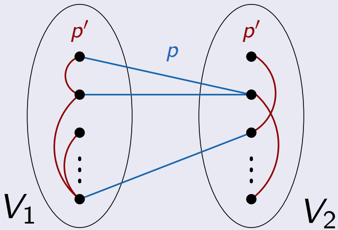
Applications

Parallel computing, VLSI, simulations...

A Simple Random Graph Model

The random graph $G_n(p, p')$

1. Split the vertex set $V = \{1, \ldots, n\}$ randomly into two sets V_1, V_2 of size $n/2$.
2. Connect any two vertices inside V_i with probability p' $(i = 1, 2)$.
3. Insert each V_1–V_2-edge with probability p.



Let $d = \frac{n}{2}(p + p') =$ *expected degree*.

A Simple Random Graph Model (ctd)

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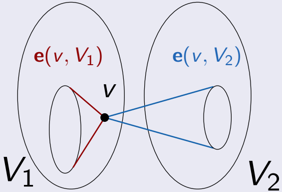

$$\mathrm{E}(\mathbf{e}(v, V_1) - \mathbf{e}(v, V_2)) = \frac{n}{2}(p' - p).$$

$$\sigma(\mathbf{e}(v, V_1) - \mathbf{e}(v, V_2)) \approx \sqrt{d} = \sqrt{n(p + p')/2}.$$

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Random partition vs. "planted" bisection

- $\#$ crossing edges in a *random* solution: $\frac{1}{4}nd = \frac{n^2}{8}(p + p')$.
- $\#$ crossing edges in the *planted* solution : $\frac{n^2}{4}p$.

A phase transition

How many optimal solutions are there?

Fix p so that $np = 100$, say.
Increasing p', we go through the...

A phase transition

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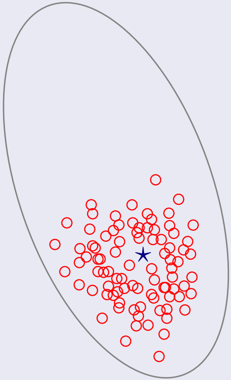
Fix p so that $np = 100$, say.
Increasing p', we go through the...

... subcritical phase: $\sqrt{d \log(d)} \leq n(p' - p) \ll \sqrt{d \log n}$.

- $\exp(n^{1-o(1)})$ optimal solutions("$\circ$").
- planted bisection is *not* optimal („$\star$“).



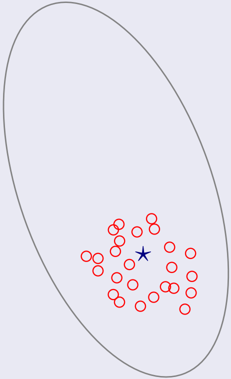
A phase transition

How many optimal solutions are there?

Fix p so that $np = 100$, say.
Increasing p', we go through the...

...critical phase: $n(p' - p) = \sqrt{d \cdot \gamma \log n}$.

- $\exp(n^{1-\gamma})$ optimal solutions ("$\circ$").
- planted bisection is *not* optimal („$\star$“).



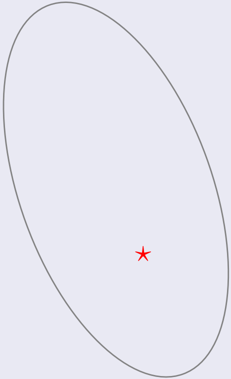
A phase transition

How many optimal solutions are there?

Fix p so that $np = 100$, say.
Increasing p', we go through the...

...supercritical phase: $n(p' - p) \gg \sqrt{d \log n}$.

- Exactly *one* optimal solution,
- namely the planted one ("$\star$").



Graph bisection algorithms: overview

Author	Method	$n(p' - p) \geq \cdots$
ACO 2005	spectral	$\sqrt{d \log d}$
Boppana 1987	*SDP*	$\sqrt{d \log n}$
McSherry 2001	spectral	$\sqrt{d \log n}$
Bollobás, Scott 2004	randomized	$\sqrt{d \log n}$
Condon, Karp 2001	*combinatorial*	$\sqrt{n}$
Kučera 1995	greedy	$\sqrt{n \log n}$
Carson, Impagliazzo 2001	hill-climbing	$\sqrt{n} \log^3 n$
Dimitriou, Impagliazzo 1998	simulated annealing	$\sqrt{dn/\log n}$
Jerrum, Sorkin 1998	metropolis	$n^{5/6}$
Dyer, Frieze 1989	*greedy*	$\Omega(n)$
Leighton et al. 1987	flows	$\Omega(d)$

A Greedy Algorithm

Algorithm `Greedy`

1. Pick an arbitrary vertex v.
2. Compute for each vertex w the number of *common neighbors* of v and w.
3. Let $V_1 =$ the $n/2$ vertices w sharing the most neighbors with v, and $V_2 = V \setminus V_1$.

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Theorem (Dyer, Frieze 1989)

If $n(p' - p) \geq \delta n$ for an arbitrarily small but fixed $\delta > 0$, then

$$\lim_{n \to \infty} P(\texttt{Greedy finds an optimal solution}) = 1.$$

That is, `Greedy` finds an optimal solution *almost surely*.

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Proof.

If $v, w \in V_1$, then $E(\#\text{common neighbors}) = \frac{n}{2}(p'^2 + p^2)$.
If $v \in V_1$, $w \in V_2$, then

$$E(\#\text{common neighbors}) = npp' < \frac{n}{2}(p'^2 + p^2) - \frac{\delta^2}{2}n.$$

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Generalization (Blum, Spencer 1995)

Count vertices at a *given distance k* from v, w.
$\rightsquigarrow$ works if $n(p' - p) \geq \sqrt{np'} \cdot n^{\varepsilon}$.

Condon–Karp 2001

Algorithm `Condon-Karp`

1. Construct a partition L_1, R_1 of $n^{1-\varepsilon/2}$ vertices *greedily*.
2. Use L_1, R_1 as a *scale* to construct a partition L_2, R_2 of $n/2$ further vertices *greedily*.
3. Use L_2, R_2 as a *scale* to partition all remaining vertices *greedily*. Let L_3, R_3 be the resulting sets.
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Theorem (Condon, Karp 2001)

If $n(p' - p) \geq n^{\frac{1}{2}+\varepsilon}$ for an arbitrarily small but fixed $\varepsilon > 0$, then `Condon-Karp` finds an optimal solution *almost surely*.

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Proof.

- $L_1,\ R_1$ has an *imbalance* of $n^{1-\varepsilon}$.
- $L_2,\ R_2$ has an *imbalance* of $\Omega(n)$.
- $L_3,\ R_3$ *coincides* with the planted partition on $L_3 \cup R_3$.
- Therefore, the final partition is correct.

Boppana 1987/Goemans–Williamson 1995

Min Bisection as an Integer Program

Given: a graph $G = (V, E)$ with n vertices.

$$\min \quad \sum_{v-w\in E} \frac{1 - x_v \cdot x_w}{2}$$

$$\text{s.t.} \quad \sum_{v,w\in V} x_v \cdot x_w = 0, \;\; x_u \in \{-1, 1\} \text{ for all } u \in V.$$

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But...

...computing an optimal solution to this is *just as hard* as computing an optimal bisection.

Idea: relax the integrality condition.

Boppana 1987/Goemans–Williamson 1995

Min Bisection as an Integer Program

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Min Bisection as a Semidefinite Program

$$\mathrm{SDP} = \begin{aligned} \min \quad & \sum_{v-w \in E} \frac{1 - x_v \cdot x_w}{2} \\ \text{s.t.} \quad & \sum_{v,w \in V} x_v \cdot x_w = 0,\ x_u \in \mathbb{R}^n, \|x_u\| = 1 \text{ for all } u \in V. \end{aligned}$$

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Min Bisection as an Integer Program

Given: a graph $G = (V, E)$ with n vertices.

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$$\text{s.t.} \quad \sum_{v,w \in V} x_v \cdot x_w = 0, \;\; x_u \in \{-1, 1\} \text{ for all } u \in V.$$

Good news

The *Semidefinite Program* can be solved efficiently (well, . . .).

Boppana 1987/Goemans–Williamson 1995 (ctd.)

Algorithm Boppana

1. Compute an optimal (vector) solution to

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2. Sample a random unit vector $z \in \mathbb{R}^n$.
3. Let $V_1 = \{v : x_v \cdot z < 0\}$, $V_2 = \{v : x_v \cdot z \geq 0\}$.

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Theorem (Boppana 1987)

If $n(p' - p) \geq \sqrt{d \log n}$, then Boppana finds an optimal solution *almost surely*.

Boppana 1987/Goemans–Williamson 1995 (ctd.)

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Proof (main idea).

Show that *any* optimal solution to SDP is integral (details are long & difficult)!

Boppana 1987/Goemans–Williamson 1995 (ctd.)

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Practical issue

Solving the SDP is a highly non-trivial numerical problem.
Feasible up to $n = 1000$ or so.

Boppana 1987/Goemans–Williamson 1995 (ctd.)

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Generalization

Worst-case approximation algorithm for *Max Cut/Max Bisection* (Goemans, Williamson 1995); inspired a bulk of further work.

A spectral approach

Basic idea

Consider $G = G_n(p, p')$ and let A =adjacency matrix.
Each vertex $v \in V_i$ *expects*

$$\begin{aligned} d_{\text{own}} &= \frac{n}{2}p' \text{ neighbors in its \textit{own} class,} \\ d_{\text{other}} &= \frac{n}{2}p \text{ neighbors in the \textit{opposite} class.} \end{aligned}$$

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Therefore, letting $\xi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$, we *expect*

$$A\xi \approx (d_{\text{own}} - d_{\text{other}})\xi = \frac{n}{2}(p' - p)\xi.$$

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Technical problem: tail of the degree distribution.

A spectral approach (ctd)

Algorithm `Spectral`

1. Obtain G' from G by *removing* all vertices of degree $> 10d$.
   Compute the *2nd eigenvector* ξ of $A(G')$.
   Let $S_1 = \{\text{vertices with } \textit{positive} \text{ entries}\}$ and $S_2 = V \setminus S_1$.

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Local improvment heuristic

1. Initially, let $T_1 = S_1,\ T_2 = S_2$.
2. While there is $v \in T_i$ with more neighbors in the *opposite* class, move v to that class.
3. Let $H =$all vertices for which the *difference* is $> 10\sqrt{d}$.

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Theorem (ACO 2005)

If $n(p' - p) > \sqrt{d \log d}$, then `Spectral` finds the optimum almost surely.

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Theorem (ACO 2005)

If $n(p' - p) > \sqrt{d \log d}$, then Spectral finds the optimum almost surely.

Hence, Spectral works in the *critical* and the *subcritical phase*.

Outline

The cut norm

Goal

Approximate a given 0/1 matrix A by a *low-rank matrix* combinatorially!

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The *cut norm* of a $m \times n$ matrix M is

$$\|M\|_\square = \max_{R\subset[m],C\subset[n]} \left| \sum_{(i,j)\in R\times C} M_{ij} \right|.$$

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Can we find a *low rank matrix* D such that $\|A - D\|_{\square} < \varepsilon \|A\|_{\square}$?

The cut norm (ctd)

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Theorem (Grothendieck 1953; Alon, Naor 2004)

We can compute sets S, T such that

$$|M(S,T)| = \left| \sum_{(i,j) \in S \times T} M_{ij} \right| \geq 0.56 \cdot \|M\|_\square$$

via *Semidefinite Programming*.

Approximation by cut matrices

Algorithm `FriezeKannan`

1. Set $A_0 = A$.
2. For $j = 0, 1, 2, \ldots$ do
3. Compute $S_{j+1},\ T_{j+1}$ s.t. $|A_j(S_{j+1}, T_{j+1})| \geq 0.56\|A_j\|_\square$.

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4. Let $d_{j+1} = \frac{A_j(S_{j+1}, T_{j+1})}{|S_{j+1}| \cdot |T_{j+1}|}$ and set

$$D_{j+1} = d_{j+1} \cdot \mathbf{1}_{S_{j+1} \times T_{j+1}},\ A_{j+1} = A_j - D_{j+1}.$$

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4. Let $d_{j+1} = \frac{A_j(S_{j+1}, T_{j+1})}{|S_{j+1}|\cdot|T_{j+1}|}$ and set
$$D_{j+1} = d_{j+1} \cdot \mathbf{1}_{S_{j+1}\times T_{j+1}},\ A_{j+1} = A_j - D_{j+1}.$$
5. If $\|A_{j+1}\|_\square < \varepsilon\|A\|_\square$, output $D_1 + \cdots + D_{j+1}$ and halt.

Approximation by cut matrices

Algorithm `FriezeKannan`

1. Set $A_0 = A$.
2. For $j = 0, 1, 2, \ldots$ do
3. Compute S_{j+1}, T_{j+1} s.t. $|A_j(S_{j+1}, T_{j+1})| \geq 0.56\|A_j\|_\Box$.
4. Let $d_{j+1} = \frac{A_j(S_{j+1}, T_{j+1})}{|S_{j+1}| \cdot |T_{j+1}|}$ and set
$$D_{j+1} = d_{j+1} \cdot \mathbf{1}_{S_{j+1} \times T_{j+1}}, \; A_{j+1} = A_j - D_{j+1}.$$
5. If $\|A_{j+1}\|_\Box < \varepsilon \|A\|_\Box$, output $D_1 + \cdots + D_{j+1}$ and halt.

Theorem (Frieze, Kannan 1995)

If `FriezeKannan` halts, then the matrix $D = D_1 + \cdots + D_j$ satisfies $\|A - D\|_\Box < 2\varepsilon \|A\|_\Box$.

The boundedness condition

Question

For which inputs does `FriezeKannan` halt, and after *how many iterations*?

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For which inputs does `FriezeKannan` halt, and after *how many iterations*?

Boundedness condition

Call a graph $G = (V, E)$ *(C, γ)-bounded* if

$$e(S) \leq C \cdot \frac{|S|^2}{|V|^2} \cdot |E| \text{ for all } S \subset V, \ |S| \geq \gamma |V|.$$

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Examples

- The graph $G_n(p, p')$.
- More generally, subgraphs of random graphs.
- Expander graphs.
- ...

The boundedness condition

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Theorem (ACO, Cooper, Frieze 2008)

For any $\varepsilon > 0$ there is $\gamma > 0$ such that `FriezeKannan` halts after $(C/\varepsilon)^2$ rounds on (C, γ)-bounded graphs.

Comprises best prior results on $G_n(p, p')$.

Conclusion

- *Planted bisection model* as a benchmark for graph partitioning.
- Various techniques for graph partitioning: *greedy*, *combinatorial*, *SDP*, *spectral*.
- *Local* methods like *greedy* in contrast to *global* methods like *SDP*, *spectral*.
- *Local* methods are more efficient, but need more "evidence".
- *Global* methods are restricted to graphs with a simple "solution space".
- *Cut norm approximations* generalize various graph partitioning problems.

Some ideas on the fractional Dirac Equation

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Abstract

I present some general ideas on the fractional Dirac equation and its properties, specifically, what is related to conservation laws. I start with an introduction to the original Dirac equation.

Contents

1 Introduction

1.1 The original Dirac equation

The Dirac equation is a relativistic covariant version of the Schrödinger equation of motion for a particle [1]. In natural units, the equation for a particle with mass m moving freely in one spatial dimension is given by

$$i\gamma^0\partial_t\psi + i\gamma^1\partial_x\psi - m\psi = 0\,, \tag{1}$$

where i is the imaginary unit,

$$\psi(t,x) = \begin{pmatrix}\psi_1(t,x)\\ \psi_2(t,x)\end{pmatrix}$$

is a complex 2-dimensional vector, called "spinor", t is the time, x is the space, and the γ's are 2×2 real matrices that satisfy the Pauli's algebra:

$$\gamma^0\gamma^0 = I\,, \quad \gamma^0\gamma^1 + \gamma^1\gamma^0 = \mathcal{O}\,, \quad \gamma^1\gamma^1 = -I\,. \tag{2}$$

There are two independent possible choices for the γ's, each one leading to a somewhat different formulation, but for the massless case a linear (complex) transformation passes from one to the other. One of the possibilities is called the *Dirac base*:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

the other is the *chiral base*:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

Note that γ^1 is the same in both cases. The simpler case of a massless[a] particle corresponds to

$$\gamma^0 \partial_t \psi + \gamma^1 \partial_x \psi = 0\,, \tag{5}$$

We will suppose that the equation is well posed and that a solution exists for all times. The spatial region is, $x \in \Omega = (-\infty, +\infty)$, but we may also consider a finite or a semi-finite interval.

[a]Massless fermions are widely considered. See, for instance, [2, 3].

The Dirac equation, with a mass term, has been considered as a "square root" of the Klein-Gordon equation. In this case, we can see that the solutions of the massless Dirac equation (5) are also solutions of the Wave equation (that is, the massless Klein-Gordon equation). We apply ∂_t to (5) and manipulate formally:

$$\begin{aligned} \gamma^0\psi_t + \gamma^1\psi_x = 0 \;&\Longrightarrow\; \partial_t(\gamma^0\psi_t + \gamma^1\psi_x) = 0 \\ &\Longleftrightarrow\; \gamma^0\psi_{tt} + \gamma^1\psi_{xt} = 0 \\ &\Longleftrightarrow\; \psi_{tt} + \gamma^0\gamma^1\partial_x(\psi_t) = 0 \\ &\Longleftrightarrow\; \psi_{tt} + \gamma^0\gamma^1\partial_x(-\gamma^0\gamma^1\psi_x) = 0 \;\Longleftrightarrow\; \psi_{tt} = \psi_{xx}\,. \end{aligned} \tag{6}$$

1.2 Conservation laws

Equation (5) has several conserved quantities.

A conservation law is obtained whenever we have a “conserved current”, that is, an equation of the form:

$$\partial_t(A) = \partial_x(B)\,, \tag{7}$$

since integration in x over Ω provides

$$\partial_t \int_\Omega A\,dx = [B]_{\partial\Omega}\;, \tag{8}$$

if B is zero on $\partial\Omega$, or if we have periodic boundary conditions there.

In fact, (5) is already a conservent current:

$$\gamma^0\partial_t\psi + \gamma^1\partial_x\psi = 0 \iff \partial_t\psi = -\gamma^0\gamma^1\partial_x\psi$$
$$\Longrightarrow \partial_t \int_\Omega \psi\,dx = -\gamma^0\gamma^1\left[\psi\right]_{\partial\Omega}. \tag{9}$$

We have the following conservation laws (all but the first one are scalars):

$\Psi \equiv \int_\Omega \psi \, dx$	$\partial_t \Psi = -\gamma^0\gamma^1 \left[\psi\right]_{\partial\Omega}$
$Q \equiv \int_\Omega \psi^+ \psi \, dx$	$\partial_t Q = -\left[\psi^+ \gamma^0\gamma^1 \psi\right]_{\partial\Omega}$
$C \equiv \int_\Omega \psi^+ \gamma^0\gamma^1 \psi \, dx$	$\partial_t C = -\left[\psi^+ \psi\right]_{\partial\Omega}$
$E \equiv \int_\Omega \left(\psi^+ \gamma^0\gamma^1 \psi_x - \psi_x^+ \gamma^0\gamma^1 \psi\right) dx$	$\partial_t E = \left[\psi^+ \gamma^0\gamma^1 \psi_t - \psi_t^+ \gamma^0\gamma^1 \psi\right]_{\partial\Omega}$
$P \equiv \int_\Omega \left(\psi^+ \psi_x - \psi_x^+ \psi\right) dx$	$\partial_t P = \left[\psi^+ \psi_t - \psi_t^+ \psi\right]_{\partial\Omega}$

ψ^+ is the hermitian conjugate of ψ (the transposed, complex-conjugate of ψ).

2 The fractional Dirac equation

I will consider the case where the fractional derivative is only in the time variable. The equation is now

$$\gamma^0 \partial_t^\alpha \psi + \gamma^1 \partial_x \psi = 0\,, \quad \alpha \in (0,1]\,, \tag{10}$$

where ∂_t^α is a fractional derivative of order α, either the Riemann-Liouville one or Caputo's.

If we try to reproduce the conservation laws, we find problems since we no longer have the Leibniz rule for the derivative of a product in the fractional calculus.

We may try to reproduce what we had for the Dirac equation.

We end with the following integral relations, for the equivalents of Ψ, Q and C:

$$\boxed{\partial_t^\alpha \int_\Omega \psi \, dx = -\gamma^0\gamma^1 \left[\psi\right]_{\partial\Omega}}$$

$$\boxed{\int_\Omega \left(\psi^+ \partial_t^\alpha \psi + \partial_t^\alpha \psi^+ \psi\right) dx = -\left[\psi^+ \gamma^0\gamma^1 \psi\right]_{\partial\Omega}}$$

$$\boxed{\int_\Omega \left(\psi^+ \gamma^0\gamma^1 \partial_t^\alpha \psi + \partial_t^\alpha \psi^+ \gamma^0\gamma^1 \psi\right) dx = -\left[\psi^+ \psi\right]_{\partial\Omega}}$$

while for the equivalents of E and P we get the following “differential” relations:

$$\boxed{\partial_t^\alpha \psi^+ \gamma^0\gamma^1 \psi_x - \psi_x^+ \gamma^0\gamma^1 \partial_t^\alpha \psi = 0}$$

$$\boxed{\partial_t^\alpha \psi^+ \psi_x - \psi_x^+ \partial_t^\alpha \psi = 0}$$

Again, the first relation is spinorial, while the others are scalar.

If we consider Caputo's fractional derivative we have $\partial_t^\alpha \Psi = 0 \Rightarrow \Psi$ constant, but not for Riemann-Liouville.

For Riemann-Liouville a different approach can be used in the case where α is a rational number. Let us suppose that

$$\alpha = \frac{m}{n}, \quad m, n \in \mathbb{N}, \tag{11}$$

and let us apply n times the operator ∂_t^α to (10). For the Riemann-Liouville derivative we have that

$$\overbrace{\partial_t^{m/n} \circ \cdots \circ \partial_t^{m/n}}^{n \text{ times}} = \partial_t^m, \tag{12}$$

but this is not true for Caputo's. On the other hand we have from (10)

$$\overbrace{\left(\partial_t^{m/n} \circ \cdots \circ \partial_t^{m/n}\right)}^{p \text{ times}} \psi = \left(-\gamma^0\gamma^1\right)^p \partial_x^p \psi. \tag{13}$$

Combining this with (12), we obtain the conserved current:

$$\partial_t \left(\partial_t^{m-1}\psi\right) = \partial_x \Big[\left(-\gamma^0\gamma^1\right)^n \partial_x^{n-1}\psi \Big], \tag{14}$$

with the corresponding integral conservation law

$$\partial_t \int_\Omega \left(\partial_t^{m-1}\psi\right) dx = \left(-\gamma^0\gamma^1\right)^n \left[\partial_x^{n-1}\psi\right]_{\partial\Omega} . \tag{15}$$

This conservation law does not grant that the solution is not going to zero pointwise, as we see in the following example:

Case $\alpha = 1/2$

For instance, $\alpha = 1/2$ gives the diffusion equation for ψ:

$$\psi_t = \psi_{xx} \, , \tag{16}$$

which is a conserved current, with the conservation law for ψ

$$\partial_t \int_\Omega \psi \, dx = [\psi_x]_{\partial\Omega} \, . \tag{17}$$

In this case, working with (16), we get the relation for the L_2 norm:

$$\partial_t \int_\Omega \psi^+ \psi \, dx = [\psi^+ \psi_x + \psi_x^+ \psi]_{\partial\Omega} - 2 \int_\Omega \psi_x^+ \psi_x \, dx \, . \tag{18}$$

This corresponds to a decay, for instance, if ψ is zero on $\partial\Omega$, since the integral is positive. This is not suprising, because this case corresponds to a diffusion equation and the solution decays to zero, or get "thermalized", pointwise.

Reversible case $\alpha = 1/3$

Since we are looking for conservation laws, that is, time-preserved expressions, it seems reasonable to look for them in the time-reversible cases. According to Pierantozzi and Vázquez [4], one of the time-reversible cases corresponds to $\alpha = 1/3$. Here $m = 1$, $n = 3$ and we get the equation:

$$\psi_t = -\gamma^0\gamma^1\psi_{xxx}\,, \tag{19}$$

and the conservation law for the L_2 norm:

$$\partial_t \int_\Omega \psi^+\psi\,dx = -\Big[\psi^+\gamma^0\gamma^1\psi_{xx} + \psi^+_{xx}\gamma^0\gamma^1\psi - \psi^+_x\gamma^0\gamma^1\psi_x\Big]_{\partial\Omega}.$$

The simplest way to ensure the conservation law is to ask for periodic boundary conditions, especially if Ω is a bounded interval. Besides that, (15) gives

$$\partial_t \int_\Omega \psi\,dx = \left(-\gamma^0\gamma^1\right)\left[\psi_{xx}\right]_{\partial\Omega}\,, \tag{20}$$

but if the L_2 norm is preserved, the solution cannot tend to zero, pointwise.

L_2 norm: let us consider $m = 1$ and n arbitrary. We have

$$\partial_t \int_\Omega \psi^+ \psi \, dx = \begin{cases} [R]_{\partial\Omega} \,, & \text{if } n \text{ odd,} \\ [R]_{\partial\Omega} + 2(-1)^{n/2} \displaystyle\int_\Omega \left(\partial_x^{n/2} \psi^+ \, \partial_x^{n/2} \psi \right) dx, & \text{if } n \text{ even,} \end{cases} \tag{21}$$

where

$$R = (-1)^{(n+1)/2} \partial_x^{(n-1)/2} \psi^+ \gamma^0 \gamma^1 \partial_x^{(n-1)/2} \psi + \sum_{k=(n+1)/2}^{n-1} (-1)^{k+1} (\partial_x^{n-1-k} \psi^+ \gamma^0 \gamma^1 \partial_x^k \psi + \partial_x^k \psi^+ \gamma^0 \gamma^1 \partial_x^{n-1-k} \psi), \quad \text{if } n \text{ odd,}$$

$$R = \sum_{k=(n/2)-1}^{n-1} (-1)^{k+1} (\partial_x^{n-1-k} \psi^+ \partial_x^k \psi + \partial_x^k \psi^+ \partial_x^{n-1-k} \psi), \quad \text{if } n \text{ even.}$$

We have thus conservation of the L_2 norm if n is odd, decay if $n/2$ is odd and growth if $n/2$ is even. This agrees with the fact that if $m = 1$, only the values of n odd correspond to time-reversible cases.

If, in the general case, we consider $m \geq 2$, it is not clear how to get a conservation/variation law for the L_2 norm, although the cases might be time-reversible (for instance $\alpha = 2/3$, or $\alpha = 3/5$).

3 Some numerical simulations

3.1 Caputo derivative

I have considered the Caputo fractional derivative and the implicit numerical scheme given by [5]:

$$\psi_l^n = \psi_l^0 - \frac{\Delta t \gamma^0 \gamma^1}{2\Gamma(\alpha+1)} \sum_{k=0}^{n-1} \left(\frac{\psi_{l+1}^{k+1} - \psi_{l-1}^{k+1}}{2\Delta x} + \frac{\psi_{l+1}^{k} - \psi_{l-1}^{k}}{2\Delta x} \right) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t}, \quad (22)$$

with the usual notation in discrete variables: $\psi_l^n = \psi(n\Delta t, l\Delta x)$.

The numerical results show that, as expected, Ψ is preserved in all cases considered, the L^2 norm is not preserved and the solution spreads out.

There is not a significant difference of behaviour between the $\alpha = 1/2$ and the $\alpha = 1/3$ cases, or even with the more general case $\alpha = (\sqrt{5}-1)/2$. In fact it would seem that there is always some decay, the more important as α is smaller.

3.2 Riemann-Liouville derivative

We have combined Diethelm's method [6] for the fractional derivative with a centered finite difference for the spatial derivative to get the implicit numerical scheme:

$$\psi_l^n = \frac{-1}{\omega_{nn}} \left(\frac{\Delta t^\alpha}{\Delta x} \gamma^0 \gamma^1 \frac{\psi_{l+1}^n - \psi_{l-1}^n}{2} + \sum_{j=0}^{n-1} \omega_{nj} \psi_l^j \right), \tag{23}$$

where ω_{nj} are the weights of Diethelm's method:

$$\omega_{nj} = \frac{1}{\Gamma(2-\alpha)} \times \begin{cases} 1 & \text{if } j = n, \\ (n-1-j)^{1-\alpha} - 2(n-j)^{1-\alpha} + (n+1-j)^{1-\alpha} & \text{if } 1 \leq j \leq n-1, \\ (n-1)^{1-\alpha} + (1-\alpha)n^{-\alpha} - n^{1-\alpha} & \text{if } j = 0. \end{cases}$$

The study is not finished, but preliminary results show the predicted behaviour of the L^2 Norm.

Conclusions

We have build conservation/variation laws for the fractional Dirac equation, either with Riemann-Liouville or Caputo fractional derivative and tested them numerically.

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[6] K. Diethelm, “An algorithm for the numerical solution of differential equations of fractional order”, *Elect. Transact. Numer. Anal.* **5**, pp 1-6 (1997).

OBRIGADO!

Nemo

Overview of the Model

by Ramon Scholz and Andreas Pyka

Madeira, March 7th – 8th 2008

I. Kenes
II. Distances
III. Modification of knowledge
IV. Proposals
V. Projects
VI. Networks
VII. Graphical User Interface
VIII. Collected Data
IX. Possible Extensions

- Renewed Structure
  - KO - Knowledge Orientation
  - C - Capabilities
  - A - Abilities
  - E - Expertise
- To avoid confusion we renamed the "old" RD (Research Direction)

- We changed the internal structure of the Kenes to be more consistent with the other WP's
- New Ranges:
  - KO: integer from 0 to 9
  - C: integer from 1 to 900
  - A: float from 1 to 10
  - E: integer from 0 to 40

- To calculate distances between two Kene-elements we use the following algorithm:

$$d = \kappa \cdot | KO_1 - KO_2 | + \gamma \cdot | C_1 - C_2 | + \alpha \cdot | A_1 - A_2 |$$

  - Where κ, γ, α are used to weight the single distances in the components of the Kene-elements

- "Shortest" – Distance:
  - Shortest distance of two Kene elements in the Kenes of two agents
  - measures the relatedness of the knowledge of agents
  - measures the absorptive capacities of two agents
- "Longest" – Distance:
  - Longest distance of two Kene elements in the Kenes of two agents
  - Measures the unrelatedness of the knowledge of agents

- Agents can modify their knowledge through:
  - Cooperation in projects (learning from partners)
  - “Normal” research – deepening of knowledge (increasing the expertise)
  - “Radical” research – creation of new knowledge (variation of the KO,C or A of a Kene – item)

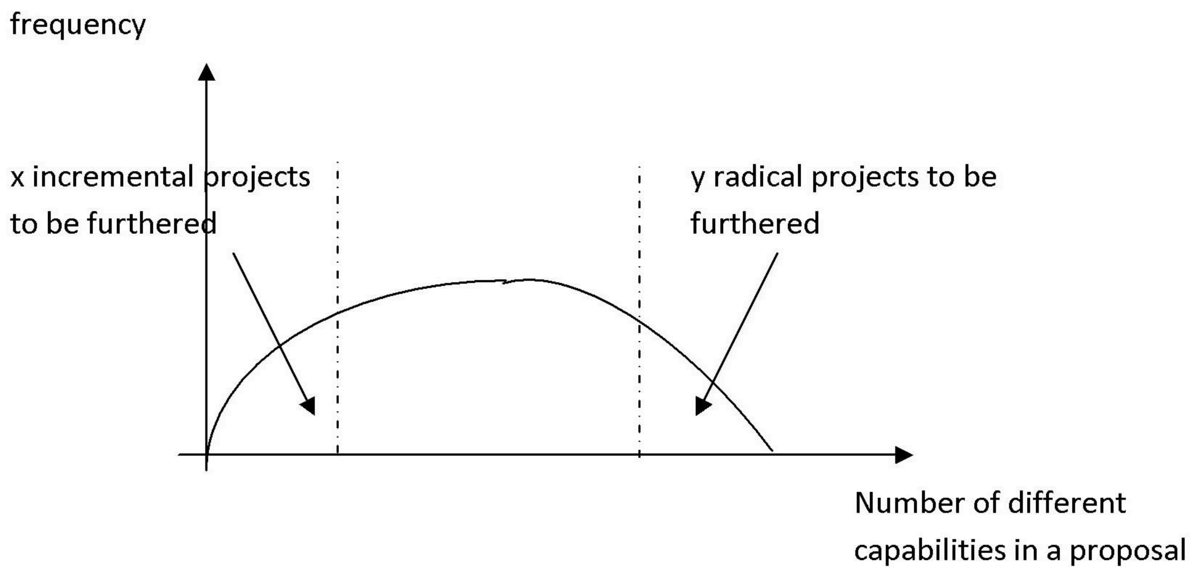
- Changing the Knowledge Orientation
  - Agents try to orientate towards a stronger applied or a stronger basic research orientation
- Changing the Ability
  - Agents try to use existing knowledge in a new way
- Changing the Capability
  - The most radical form of research: the agent tries to explore new frontiers

- Every actor can initialize every period one proposal only
- All actors have the opportunity to join proposals which were set up by other actors
- Calls for proposals and evaluations take place every period (4 periods equals a year)

- To initialize a proposal, actors search for possible partners; the research projects of partners have to fit the strategy of the initiator
- Two possible strategies:
  - Conservative
  - Progressive

- The search routine of actors
  - In a first step, actors search in their “network” for partners with a “positive history” (maximum 80% of proposal-partners).
  - Then proposal partners are asked for additional members which might join the consortium.
  - Finally, random actors are asked to join the proposal.

- From all proposals sent to the European Commission only a limited number is positively evaluated.
- The selection process:



- If a proposal is rejected, the attractiveness-values of the contributing partners decrease.
- If a proposal is successful, the attractiveness-values increase and the consortium begins with the project.

- Every project has a time horizon between 12 and 16 periods. The consortium dissolves at the end.
- The Project-Kene contains the knowledge the actors are contributing.
- Every period the consortium presents its work to the European-Commission in the form of a new project hypothesis (e.g. milestones, deliverables etc.).

- The project hypothesis is a written idea (e.g. working papers, journal articles or patents)
  - The quality is measured depending on the strategy:

conservative $$\upsilon = \frac{\min d}{\tau_1} + \frac{\max d}{\tau_2} + \sigma_1$$

progressive $$\upsilon = \frac{\min d}{\tilde{\tau}_1} + \frac{\max d}{\tilde{\tau}_2} + \sigma_2$$

with d as the distance in the knowledge-space between the Kene elements in the project hypothesis, $\tau_1, \tau_2, \tilde{\tau}_1, \tilde{\tau}_2$ to standardize the output and σ_1, σ_2 are random variables.

- The success of a project hypothesis depends on:
  - the type of research
  - the experience of actors concerning the used knowledge
  - the cooperation history of the actors
  - the absorptive capacities of the participating actors
  - the past experience with the project hypothesis

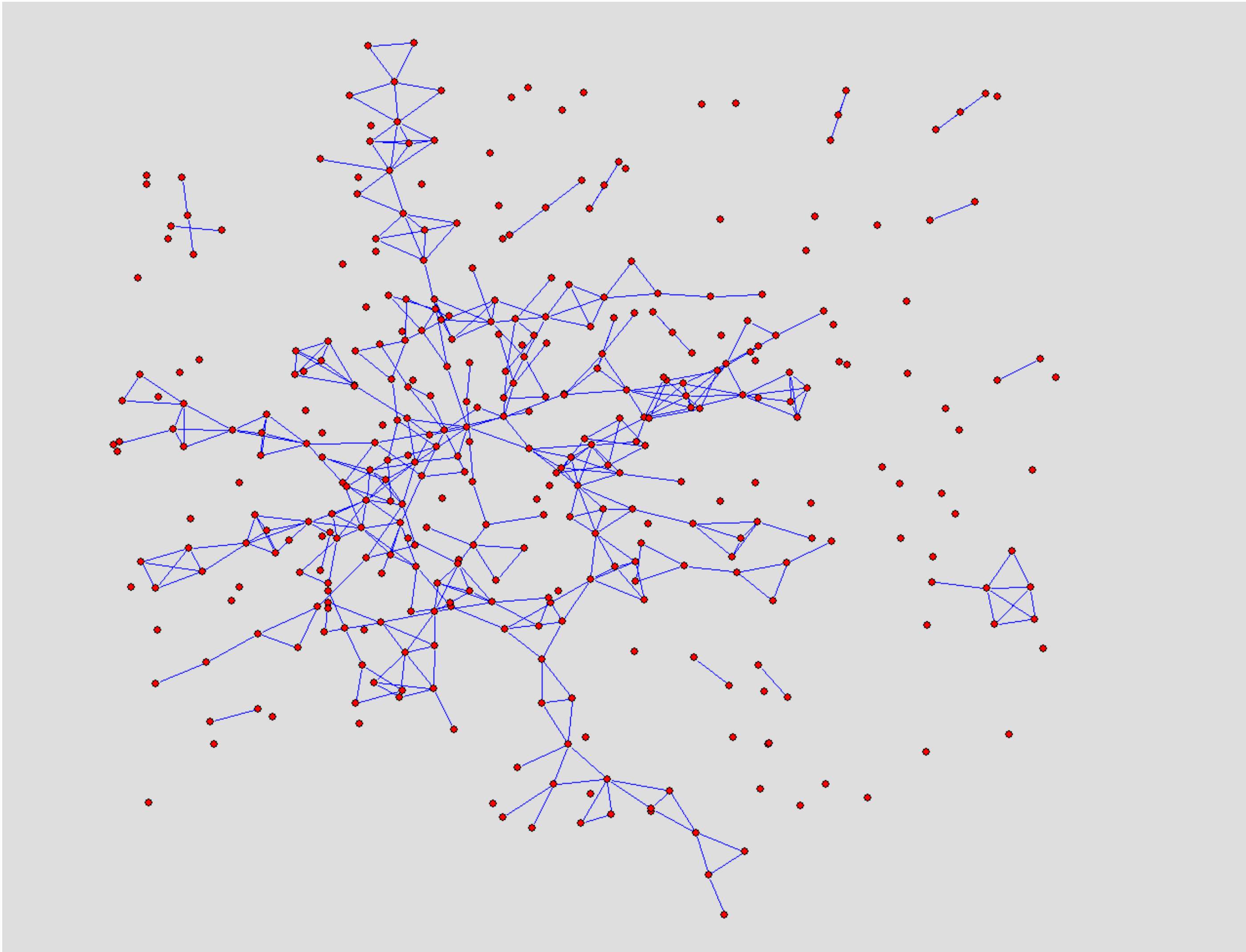
- Projects continue the work and keep their “old” project hypothesis:
  - It is possible to modify an “old” project hypothesis with successfully acquired new knowledge,
  - or to increase the value of an already successful “old” project hypothesis.

- If a project or project hypothesis fails, the attractiveness-value of the contributing partners decreases.
- If a project or project hypothesis is successful, the attractiveness-value increases and the partners are more likely to cooperate as well as have better chances to cooperate successfully in the future.

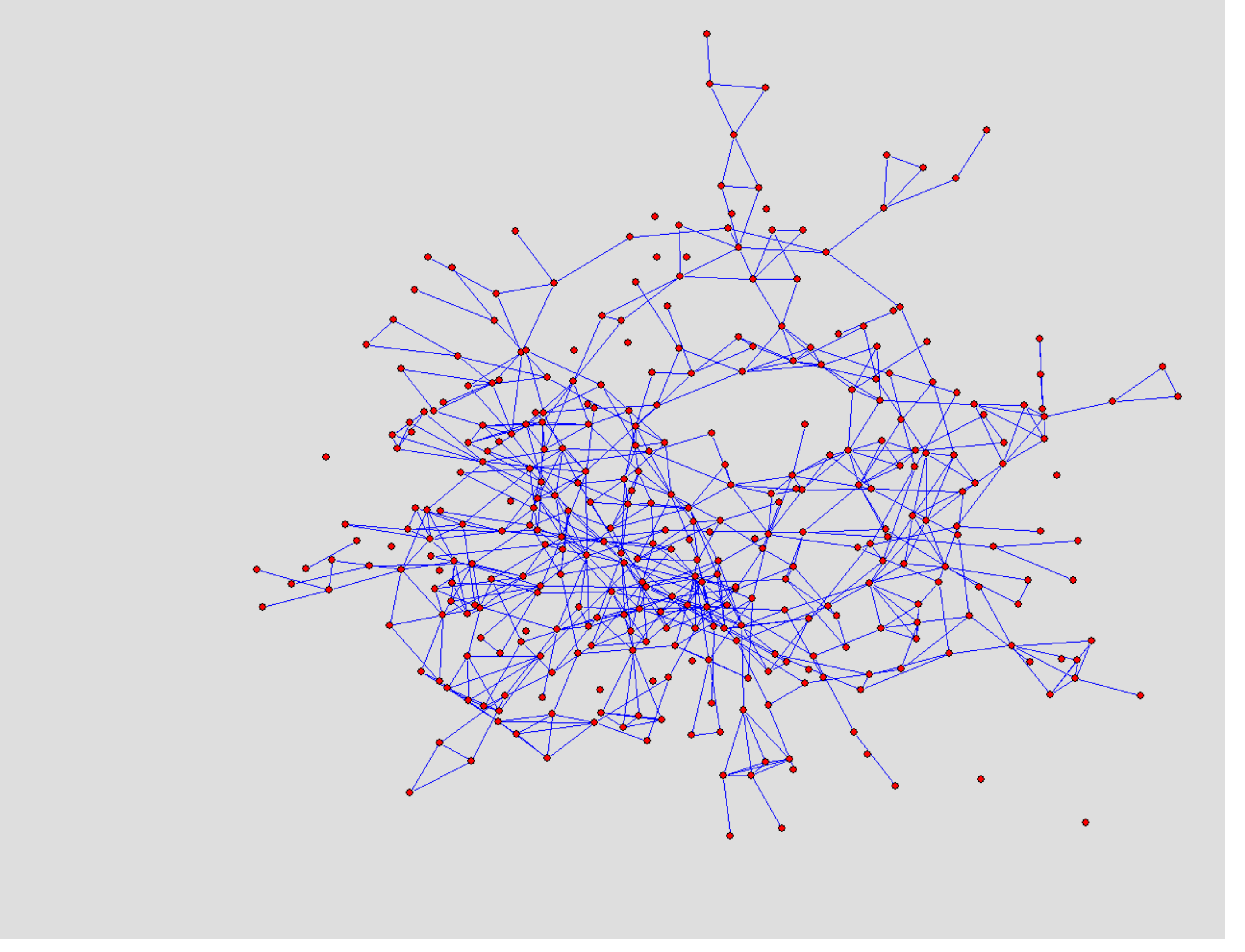
- bilateral network linkages,
- weighted edges,
- neutral value of unity,
- possibility of a positive history between two nodes (value > 1),
- or possibility of negative history between two nodes (value < 1).

- The strength of the edges depends on
  - the success of the active cooperation, and
  - the time two agents do not cooperate.
- Positive network connections lower over time
- Over time negative network connections return to unity.

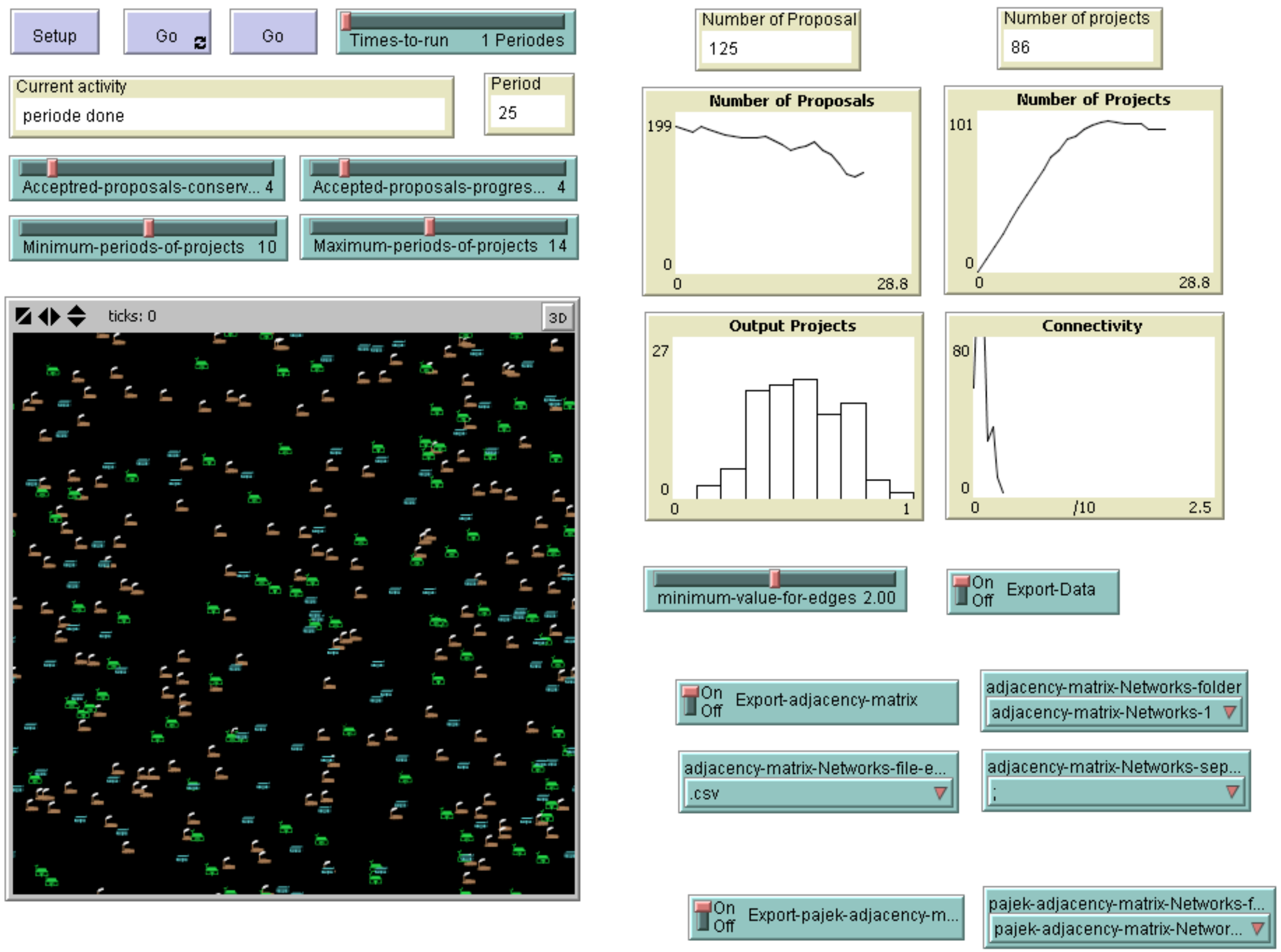
VI Network – Example of the Structure I



VI Network – Example of the Structure II



VII Graphical User Interface - Example



- Users can vary some policy rules by setting new values with the sliders or enable them through switches.
- Data can be exported.

- The influence of changes by users on the most important variables can directly be observed in the monitors.
- The plots show time series of important variables and the degree distribution of agents.

- Export of Pajec-conform adjacency matrices
  - Single file format to visualize the network structure by one click
- Export of Ucinet-conform adjacency matrices
  - CVS – Sheets to import and analyze the matrix with Ucinet

- Possibility to export all values chronological for
  - a single agent,
  - the mean values of the agents,
  - a project and
  - global variables.

- Expansion of the proposal building rules
- Expansion of the rules for the proposal evaluation
- Expansion of the rules for the project and project hypothesis evaluation
- Expansion of the model structure

- Modification of the Graphical User Interface
  - possibilities to influence the behavior of the model
  - possibilities to include new plots for the observation of real time results
- Modification of the agent set
  - E.g. introduction of new agent types with new responsibilities.

Thank you for your attention

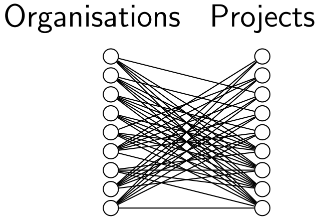
Partitioning Bipartite Graphs Using a Projection Graph?

André Lanka

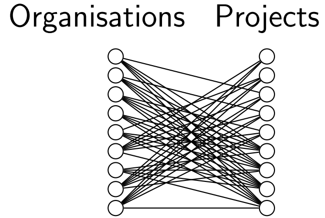
(joint work with Marc Dietzschkau)

March 7, 2008

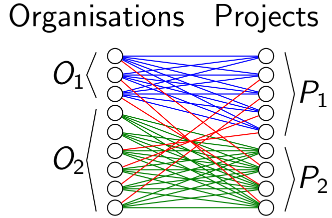
Task: Given a bipartite graph $G = (V_1 \dot{\cup} V_2, E)$ having some "hidden" structure



Task: Given a bipartite graph $G = (V_1 \dot\cup V_2, E)$ having some "hidden" structure



find that structure



How can we efficiently split O into O_1 and O_2?

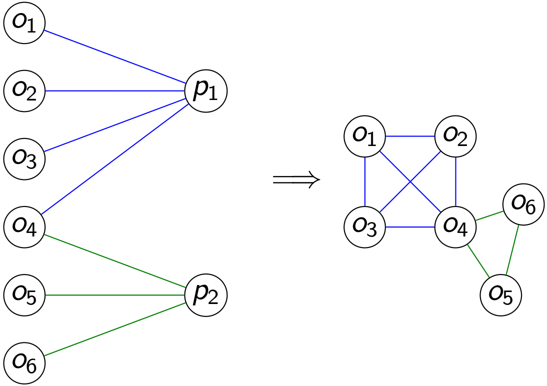
Idea 1: The projection

1. Construct the graph $G_O(O, E_O)$ with

$$\begin{aligned}\{o_1, o_2\} \in E_O \quad &\iff \quad o_1, o_2 \text{ have a common project } p \\ &\iff \quad \{o_1, p\} \in E \text{ and } \{o_2, p\} \in E \\ &\qquad\ \text{for some } p \in P\end{aligned}$$

2. Partition G_O using spectral methods.

What happens in the first step?



- We construct many cliques.
  $\Rightarrow$ Maybe, large cliques "confuse" the spectral method.

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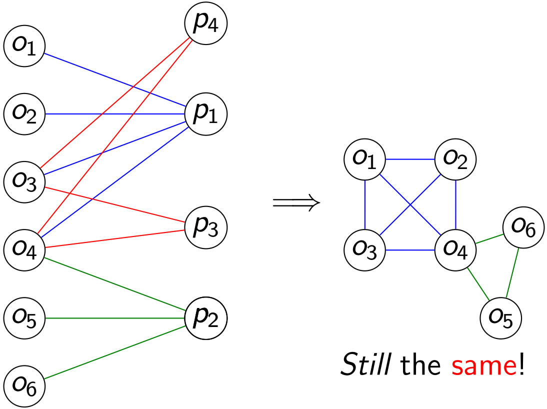
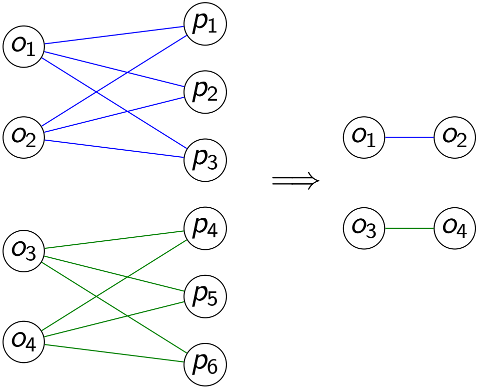


Still the same!

- We construct many cliques.
  $\Rightarrow$ Maybe, large cliques "confuse" the spectral method.
- We do not use all the information.
  $\Rightarrow$ Maybe, we can do better.

What happens in the first step?

- We construct many cliques.
- We do not use all the information.



What happens in the first step?

- We construct many cliques.
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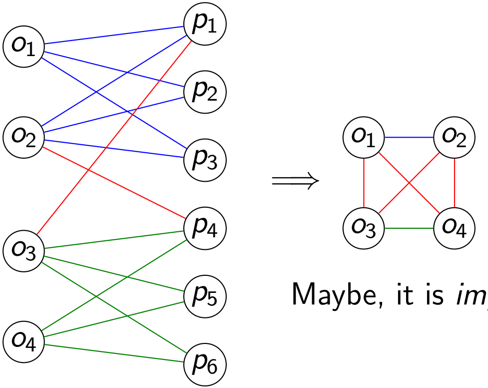


Maybe, it is *impossible* to partition!

Idea 2: Weighted projection graph

1. The weight of $\{u, v\}$ is the number of projections both u and v are involved in.
2. Partition the graph using spectral methods.

Advantage: More robust than the simple projection.

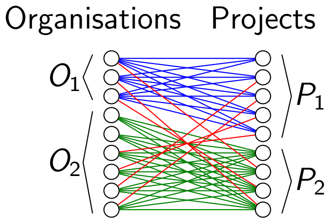
Idea 3: No projection

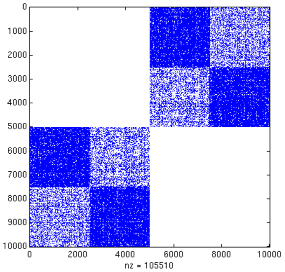
1. Partition G itself by spectral methods.

Advantage: No additional combinatorical structure introduced.

Random bipartite graphs with planted clusters:

- Choose O_1, O_2, P_1 and P_2 arbitrary.
- Omit edges inside $O_1 \cup O_2$ and inside $P_1 \cup P_2$.
- Insert the remaining edges independently.
- Prefer edges between O_1 and P_1 (resp. O_2 and P_2).
- Let the degrees nearly follow a power law.





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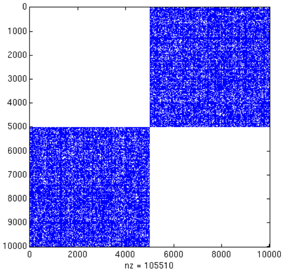
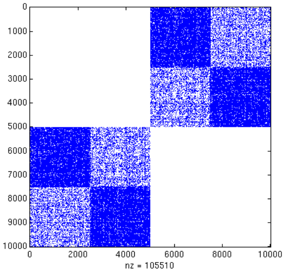




An eigenvector e of a matrix A fulfills

$$A \cdot e = \lambda \cdot e$$

for some scalar λ (=eigenvalue).

An eigenvector e of a matrix A fulfills

$$A \cdot e = \lambda \cdot e$$

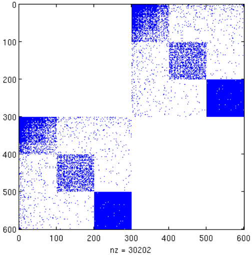
for some scalar λ (=eigenvalue).

If A is "clever", then its eigenvectors reflect G's structure.

The normalized Laplacian $\mathcal{L} = (l_{uv})$:

$$l_{uv} = \begin{cases} 1 & \text{if } u = v \\ -1/\sqrt{d_u \cdot d_v} & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

1. Delete vertices of very small degree.
2. Take $\mathcal{L}$'s eigenvector e to the second-smallest eigenvalue.
3. Divide each entry e_u by $\sqrt{d_u}$.



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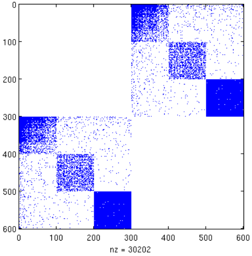

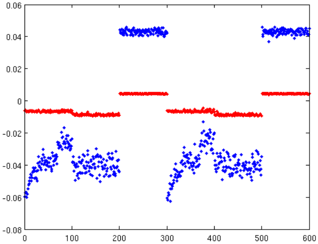

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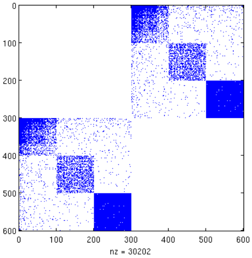

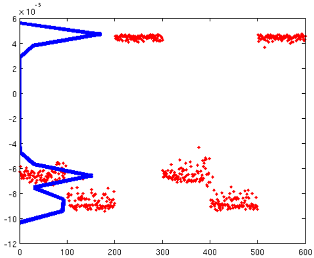

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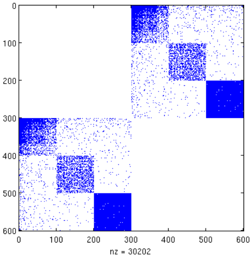
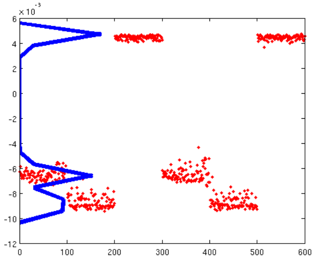
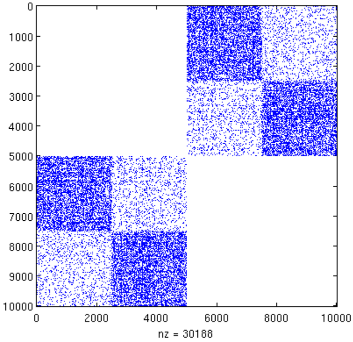




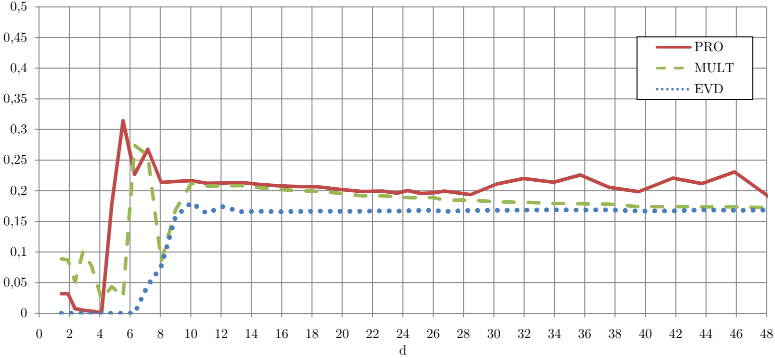
For refinement:
Split the clusters found using additional eigenvectors.

Planted two communities (each of size 2 500) so that 1/6 of all edges are between them.



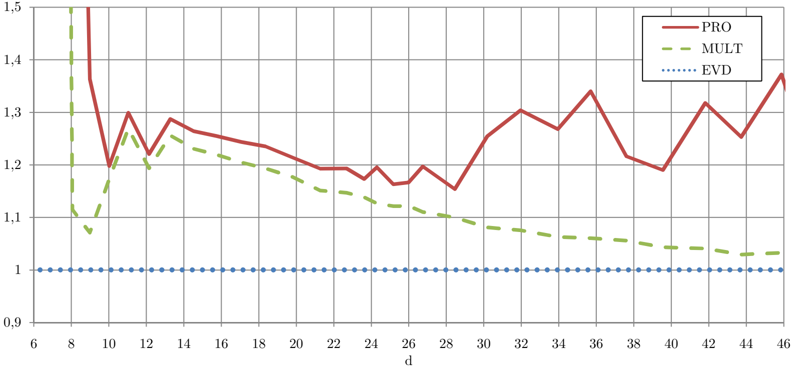
Planted two communities (each of size 2 500) so that 1/6 of all edges are between them.

Fraction of edges between the communities found:



Planted two communities (each of size 2 500) so that 1/6 of all edges are between them.

Relative performance:



Model extends Chung-Lu model (2002)

1. Split $V = O \mathbin{\dot\cup} P$ into O_1, O_2, P_1 and P_2.
2. Choose non-negative constants d_{ij} for $i, j \in \{1, 2\}$.
3. Choose weights $(w_1, \ldots, w_{|O|})$ and $(w'_1, \ldots, w'_{|P|})$ such that $\sum_m w_m = \sum_m w'_m$.
4. Insert edge between $u \in O_i$ and $v \in P_j$ independently with probability

$$d_{ij} \cdot \frac{w_u \cdot w'_v}{\sum_m w_m}$$

To be concrete:

1. O_i and P_i are chosen randomly, each has size 2 500.
2. $d_{ii} = 5$, $d_{12} = d_{21} = 1$
3. The weights follow a power law with exponent 2.5.

   To get denser graphs, we multiplied each weight with some constant. (This gives no power law anymore, but is similar to one and still highly skewed).

FROM THE NONLOCAL PROBLEMS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Funchal, 6-11 March 2008

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NONLOCALITY:

IN SPACE : Long Range Interactions (Many Space Scales)

IN TIME : Effects with Memory / Delay (Many Time Scales)

INTEGRODIFFERENTIAL // INTEGRAL EQUATIONS

Scenarios of Integral Equations

- **Potential Theory**: Newton's inverse square law of gravitational attraction and Coulomb's law in electromagnetism.
- **Problems in Geophysics**: Three dimensional map of Earth's interior. Gravimetric methods.
- **Problems in Electricity and Magnetism.**
- **Hereditary Phenomena in Physics (***materials with memory; hysteresis***) and Biology (***ecological processes: accumulation of metals).*
- **Problems in Population Growth and Industrial Replacement.**
- **Radiation Problems.**
- **Optimization, Automatic Control Systems.**
- **Communication Theory.**
- **Mathematical Economics.**

PHYSICAL CONTEXTS WITH THE SAME EQUATION

(*) Henry Darcy, "Les Fontaines Publiques de la Ville de Dijon" (1856)

	DARCY (*) LAW ***q=-K Grad h***	***FOURIER*** LAW ***Q=-κ Grad T***	***FICK*** LAW ***f=-D Grad C***	***OHM*** LAW ***j=-σ Grad V***
Flux of	***Groundwater q***	***Heat: Q***	***Solute: f***	***Charge: j***
Potential	***Head: h***	***Temperature T***	***Concentration C***	***Voltage: V***
Medium Property	***K: Hydraulic Conductivity***	***κ: Thermal Conductivity***	***D: Diffusion Coefficient***	σ: ***Electrical Conductivity***

WAVES + FRACTALS → FRACTIONAL CALCULUS (1)

- XIX Century: James Clerk Maxwell and Lor Rayleigh studied the interaction of electromagnetic waves with Euclidean regular structures (cilinders, spheres,…).
- There are either nonregular artificial structures or from Nature that show many lenght scales and they are no suitable to be studied in the Euclidean context:
  - Nonregular surfaces, disordered media, structures with ***specific properties of scattering***,..etc.
  - Relation between the geometrical parameters (structure descriptors) and the physical quantities that characterize electromagnetically the system.
  - Tecnology: New space and time scales.

WAVES + FRACTALS → FRACTIONAL CALCULUS (2)

- **Geometrical Optics:**
  - **Wave length $\lambda <<<<$ Dimension of any change in the media. The eikonal is not longer valid.**

- **The Geometrical Optics cannot be applied in fractal media.**

- **Stationary eigenvalue problem:**
  - **Wave equation in a fractal potential.**
  - **Wave equation with fractal boundary conditions:**

Ex. $Lu = \lambda u$

-L is a linear differential operator on R^n with boundary conditions $u_0(x)$ on a non-differentiable surface but which admits the fractional derivative D^β with $\beta < 1$.

- If we define $\Phi = D^{\beta-1} u$, we have the problem $L \Phi = \lambda \Phi$ with the boundary condition $\Phi_0(x)$, being Φ differentiable

The new boundary problem is smooth!

Application: Distribution of Suspended Particles in the Atmosphere + Radiation Effects

- The family of fractional differential equations

$$D_t^{\alpha} u(t,x) - a\, D_x^{\beta} u(t,x) = 0 \quad , \quad t > 0,\ x \in R,$$

associated to diffusion pocesses allows to define a set of probability distributions which are an analytic instrument to approximate the study of problems as particles suspended in the atmosphere, radiation,..etc

- ***To characterize the influence on the radiation arriving to the Earth surface (dispersion + absorption → Optical Depth $\tau(\lambda)$ is a measure of the radiation damping)***

- Example: Junge Distribution N(z,a)= C(z,a) $a^{-(1+\nu)}$ where
  - ***z is the high in the atmosphere; a is the size particles (tipically for aerosols 0.01-10 μm); and $2 < \nu < 4$***
  - ***C(z,a) is a scale factor depending on the particle concentration.***
- $\boldsymbol{\tau(\lambda)=k\, \lambda^{(2-\nu)}}$

Mars Exploration

- ***REMS-MSL Project (Approved)***

  (Rover Environmental Monitoring Station – Mars Science Laboratory)

  NASA Mission to Mars (2009, 2011?)

  → Models of the Boundary Layer and Martian Atmosphere

  Pressure, Humidity, Temperature (Air and Ground), UV Radiation and Wind.

  →M.P. Zorzano, A.M. Mancho and L. Vazquez: Appl. Math. and Comp. 164, 263-274 (2005).

  M. P. Zorzano and L. Vázquez: Optics Letters 31, 1420-1423 (2006).

  L. Vázquez, M.P. Zorzano and S. Jiménez: Optics Letters 32, 2596-2598 (2007)

- ***MiniHUM Project (Approved)***

  ESA Mission to Mars (2011,2013?)

  →Models of diffusion processes in the Martian Ground

- ***METNET (Meteorological Network) Project***
  - ***Precursor: 2 Stations (2009, 2011?) (Approved)***
  - ***Global: 15 Stations (2015?) (Evaluation Process)***

Basic Considerations (1)

- Fundamental Theorem of Calculus:
  - $dX/dt = F(t)$, $X(0) = Xo$

$$\downarrow$$

$$X(t) = Xo + \int_o^t \; 1 \; F(\tau)\, d\tau$$

$$\downarrow$$

$$X(t) = Xo + \int_o^t K(t-\tau) \; F(\tau)\, d\tau$$

Question:

Integral Transform $\leftrightarrow$ ***Fractional Derivative ?***

- Roots in the Complex Plane:

$$x^3 = 1 \rightarrow R_1, R_2, R_3$$

Basic Considerations (2)

- Numerical Schemes for Systems of first and second order:
  - $dX/dt = F(X)$, $X(0) = Xo$

    $\downarrow$
  - $d^2X/dt^2 = F(X)\, dX/dt = F(X)\ F'(X) = 1/2\ dF(X)^2/dX$

    $\downarrow$

Newton Equation: $d^2X/dt^2 = G(X) = -\, dU(X)/dX$

» ***$U(X)$ = Potential Energy $\rightarrow U(X) = -1/2\ F(X)^2$***

» ***Conservative Schemes, Symplectic Schemes.***

CONTINUOUS MEDIA THEORY:
TIMOSHENKO EQUATION-(1)

$$\partial^4\varphi/\partial x^4-(a^2+b^2)\,\partial^4\varphi/\partial x^2\partial t^2+a^2 b^2\,\partial^4\varphi/\partial t^4+a^2 c^2\,\partial^2\varphi/\partial t^2=0$$

- **Flexural vibrations of an infinite uniform beam free from lateral loading and including the shear deflection of the beam:**
  - **1/a has the dimension of a velocity.**
  - **1/b has the dimension of a velocity and it is related to the shear modulus of elesticity.**
  - **c=1/R, R is the radius of gyration of the cross section.**

- **The Timoshenko equation was introduced to avoid the unphysical behaviour of the Rayleigh equation $a^2 c^2\,\partial^2\varphi/\partial t^2+\partial^4\varphi/\partial x^4=0$, which is not accurate to describe the effect of impact loads on a beam: the phase and group velocities tend to infinity as the wave length tend to zero.**
      - » $\omega = k^2/ac$

CONTINUOUS MEDIA THEORY:
TIMOSHENKO EQUATION-(2)

$$\partial^4\varphi/ \partial x^4-(a^2 +b^2)\, \partial^4\varphi / \partial x^2 \partial t^2 +a^2 b^2\, \partial^4\varphi/ \partial t^4+ a^2 c^2\, \partial^2\varphi/ \partial t^2=0$$

- **If a=b the square root of Timoshenko equation has a simple algebraic structure:**
  - $i\, a\, c\, \partial\varphi/ \partial t = a^2\, \partial^2\varphi/ \partial t^2 - \partial^2\varphi/ \partial x^2$
- **We can name this equation:**

  Schrödinger—Klein-Gordon equation

  - **The dispersion relation is:** $\omega = (k^2- a^2 \omega^2) /ac$

- **Relativistic and nonrelativistic properties.**

- **If a≠b the algebraic structure is more complicated.**

Fractional Diffusion Equation

$$A\frac{\partial^{\alpha}\Psi}{\partial t^{\alpha}} + B\frac{\partial\Psi}{\partial x} = 0$$

$$\begin{tikzcd}
A\frac{\partial\Psi}{\partial t} + B\frac{\partial\Psi}{\partial x} = 0 \arrow[rr] & & \boxed{A\frac{\partial^{1/2}\Psi}{\partial t^{1/2}} + B\frac{\partial\Psi}{\partial x} = 0} \\
\frac{\partial^{2}u}{\partial t^{2}} - \frac{\partial^{2}u}{\partial x^{2}} = 0 \arrow[u] \arrow[rr] & & \frac{\partial u}{\partial t} - \frac{\partial^{2}u}{\partial x^{2}} = 0 \arrow[u]
\end{tikzcd}$$

$$A^{2} = I$$

$$B^{2} = -I$$

$$\{A, B\} = 0$$

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\gamma = 2\alpha$$

$$\frac{\partial^{\gamma}u}{\partial t^{\gamma}} - \frac{\partial^{2}u}{\partial x^{2}} = 0$$

$$A\frac{\partial^{\frac{1}{2}}\Psi}{\partial t^{\frac{1}{2}}}+B\frac{\partial\Psi}{\partial x}=0 \qquad \Psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

- We can interpret it as a system with two coupled diffusion processes or a diffusion process with internal degrees of freedom.
- The components φ and χ satisfy the classical diffusion equation and they are named *difunors* in analogy with the *spinors* of Quantum Mechanics.
- It is other panoramic view of the possible interpolations between the hyperbolic operator of the wave equation and the parabolic one of the classical diffusion equation.
- According to the representation of the Pauli algebra of A and B, we have either an uncoupled system or a coupled system of equations.

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad \begin{cases}\partial_t^{\alpha}\varphi=\varphi\\ \partial_t^{\alpha}\chi=-\chi\end{cases}$$

$$A\frac{\partial^\alpha \Psi}{\partial t^\alpha} + B\frac{\partial \Psi}{\partial x} = 0 \qquad \overset{\gamma = 2\alpha}{\Leftrightarrow} \qquad \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0$$

Time Inversion (t—>-t)

- If α=1 we have the Dirac and wave equations which are invariant under time inversion.
- If α=½ the classical diffusion equation and its square root *are not* invariants under time inversion.
- Interpolation for : 0< α < 1. The invariance under time inversion is satisfied for
  - Dirac Fractional Equation:

$$\alpha = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \frac{3}{5}, \frac{3}{7}, \frac{3}{9}, \ldots, \frac{5}{7}, \frac{5}{9}, \frac{5}{11}, \ldots$$

  - Diffusion Fractional Equation:

$$\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}, \frac{1}{9}, \ldots,$$

$$A\frac{\partial^\alpha \Psi}{\partial t^\alpha} + B\frac{\partial \Psi}{\partial x} = 0 \qquad \overset{\gamma = 2\alpha}{\Leftrightarrow} \qquad \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0$$

Space-Time Inversion (x—>-x, t—>-t)

- *Both equations are invariants under space inversion.*
- Interpolation : $0 < \alpha < 1$. The invariance under space-time inversion is satisfied for the same values of α in both equations:

$$\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}, \frac{1}{9}, \ldots,$$

The fractional Dirac equation is not invariant under time traslations due to the nonlocal behaviour of the fractional time derivative.

OTHER FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTERNAL DEGREES OF FREEDOM:

The 1/3-root of the Wave and Diffusion Equations

- **Wave Equation:**
  $$P\,\partial_t^{2/3}\varphi + Q\,\partial_x^{2/3}\varphi = 0$$
- **Diffusion Equation:**
  $$P\,\partial_t^{1/3}\varphi + Q\,\partial_x^{2/3}\varphi = 0$$
- **$P^3 = I$, $Q^3 = -I$**
- **$PPQ + PQP + QPP = 0$; $QQP + QPQ + PQQ = 0$,**
- **A possible realization is in terms of the matrices 3x3 associated to the *Silvester Algebra:***

Where:
$$P = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \qquad Q = \Omega \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}$$

being ω a cubic root of unity and Ω a cubic root of the negative unity.

- **φ has three components**

Mars Exploration Rover

Communication Index

a study of a new efficiency measure for networks – *work in progress !!!*

involved so far:

Andreas Krueger

Madeleine Sirugue-Collin

Philippe Blanchard

Sascha Delitzscher

Tyll Krueger

6.3.2008

Math Encounters 34

Funchal, Madeira, PT

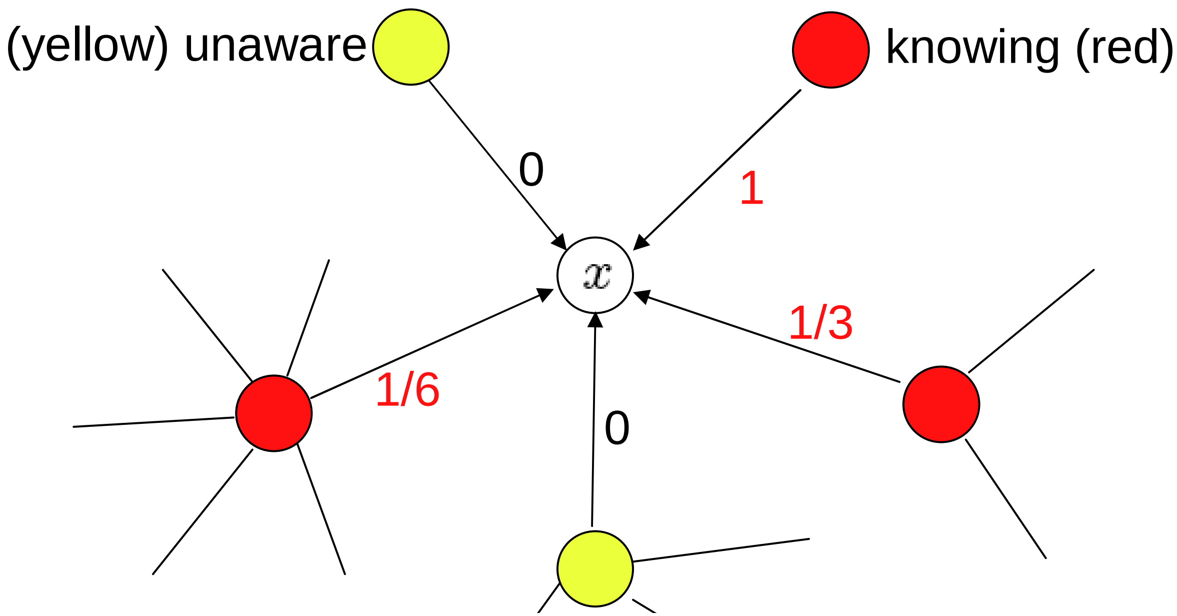
Where does it come from ...?

GEP modell for knowledge diffusion on networks ...
... the local interaction depends on how busy s.o. is

local observables

$$\Omega_t(x) = \sum_{x\sim y} \omega(y) \qquad \text{number of knowing neighbours of } x \quad = 3$$

$$\Phi_t(x) = \sum_{x\sim y} \frac{1}{d(y)}\omega(y) \qquad \text{local knowledge inflow} \quad =1+1/3+1/6 =1.5$$



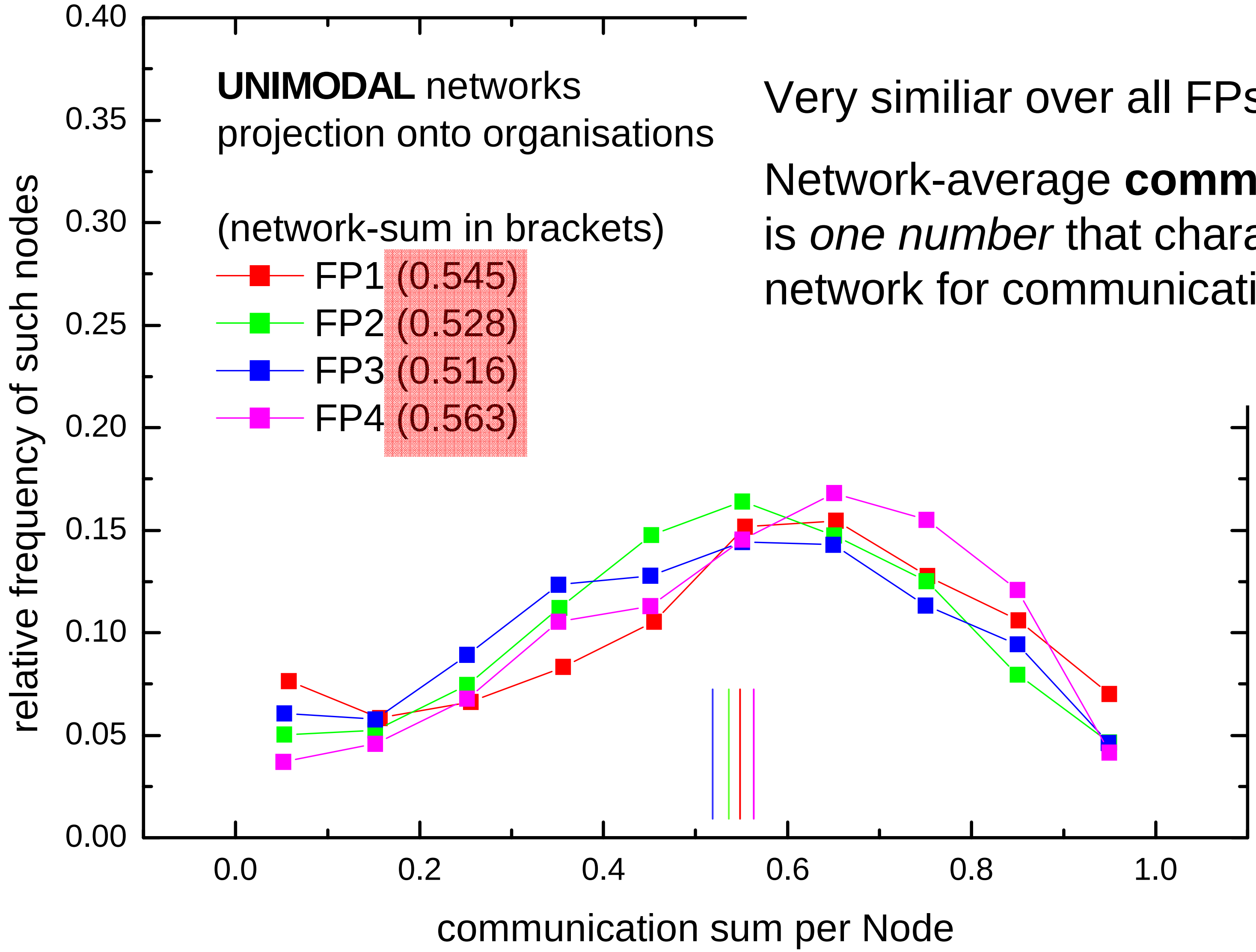
1/degree weighing of the knowing neighbours

From that *process* to a *static* measure

- Let an existing edge x~y symbolize *communication* between node x and y
- Time someone *can* spend with neighbours be equally divided among them → 1/degree
- BUT: Relevant for the time that is *actually spent* …
- … is the *more busy* of both nodes:
  → edgeweight(x,y) = min [1/degree(x), 1/degree(y)]
- Sum of all such edgeweights around each x: communication „capacity utilisation" („workload") of x
- then Sum this over all nodes / take the average =: „<u>communication index</u>" of whole network

UNIMODAL ORGs Projection (unweighted)

Node statistics of „capacity utilisation“ („workload“)

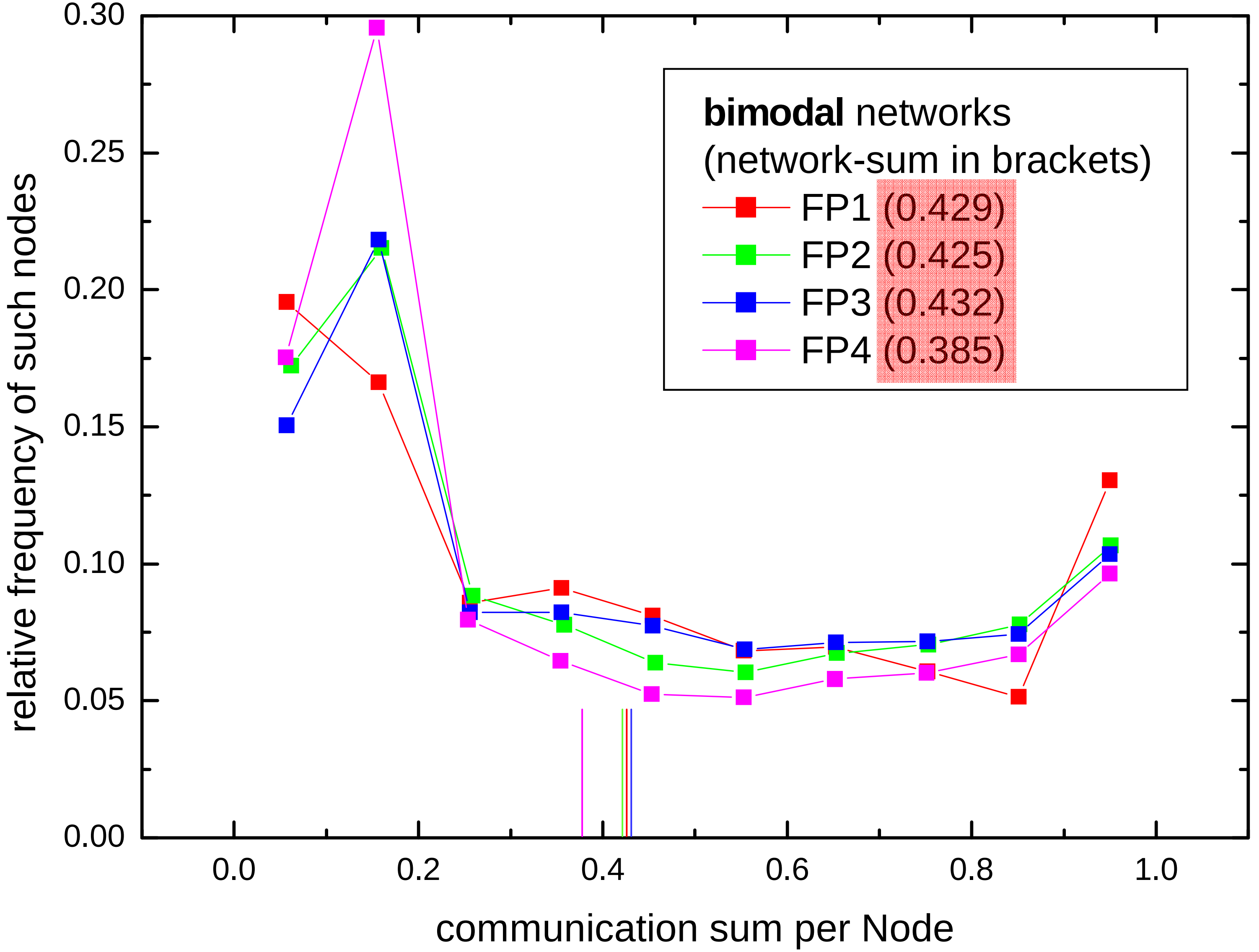


Very similiar over all FPs !!!

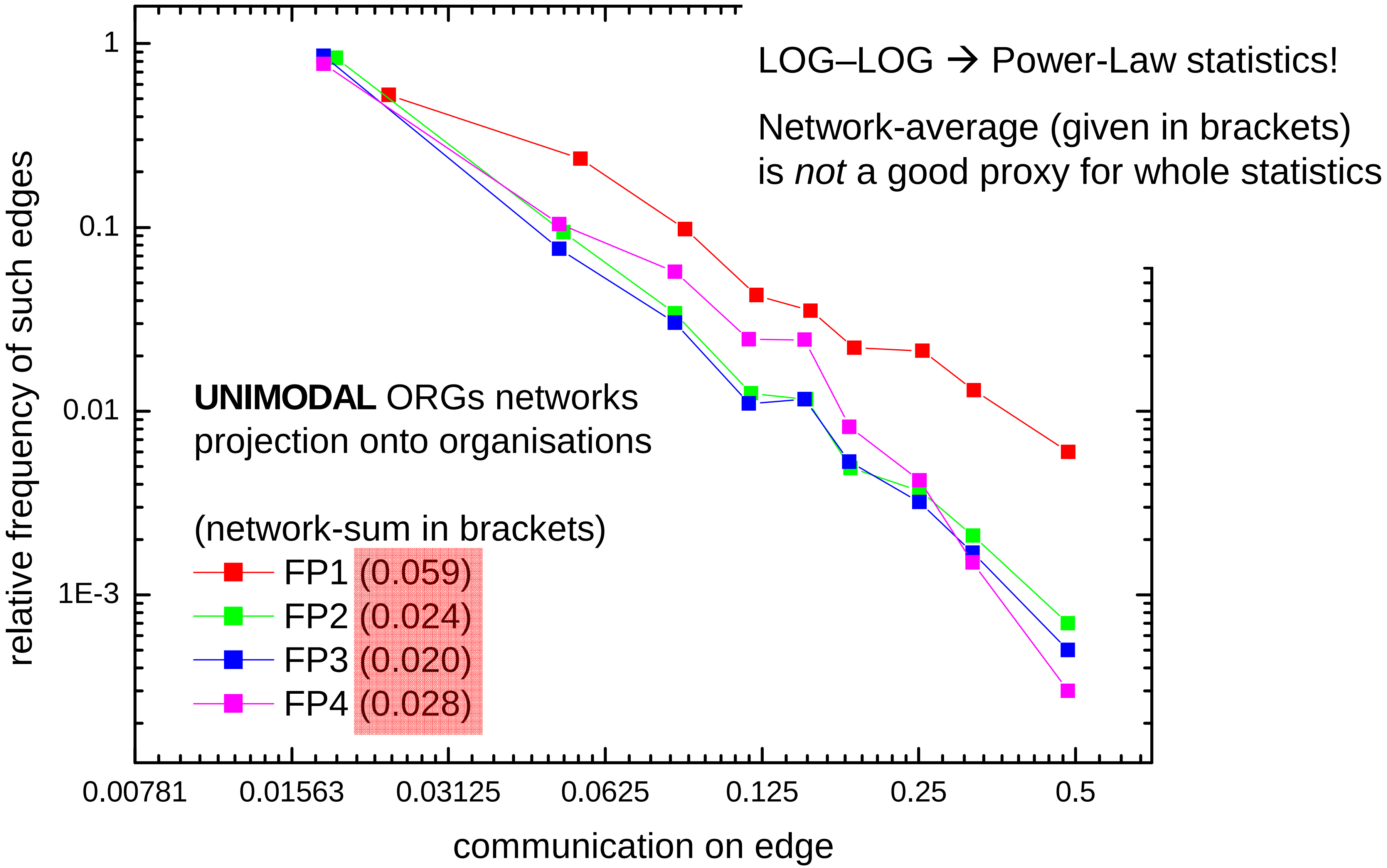
Network-average **communication-index** is *one number* that characterizes the network for communication purposes

<u>BI</u>MODAL (projects also treated as actors!)

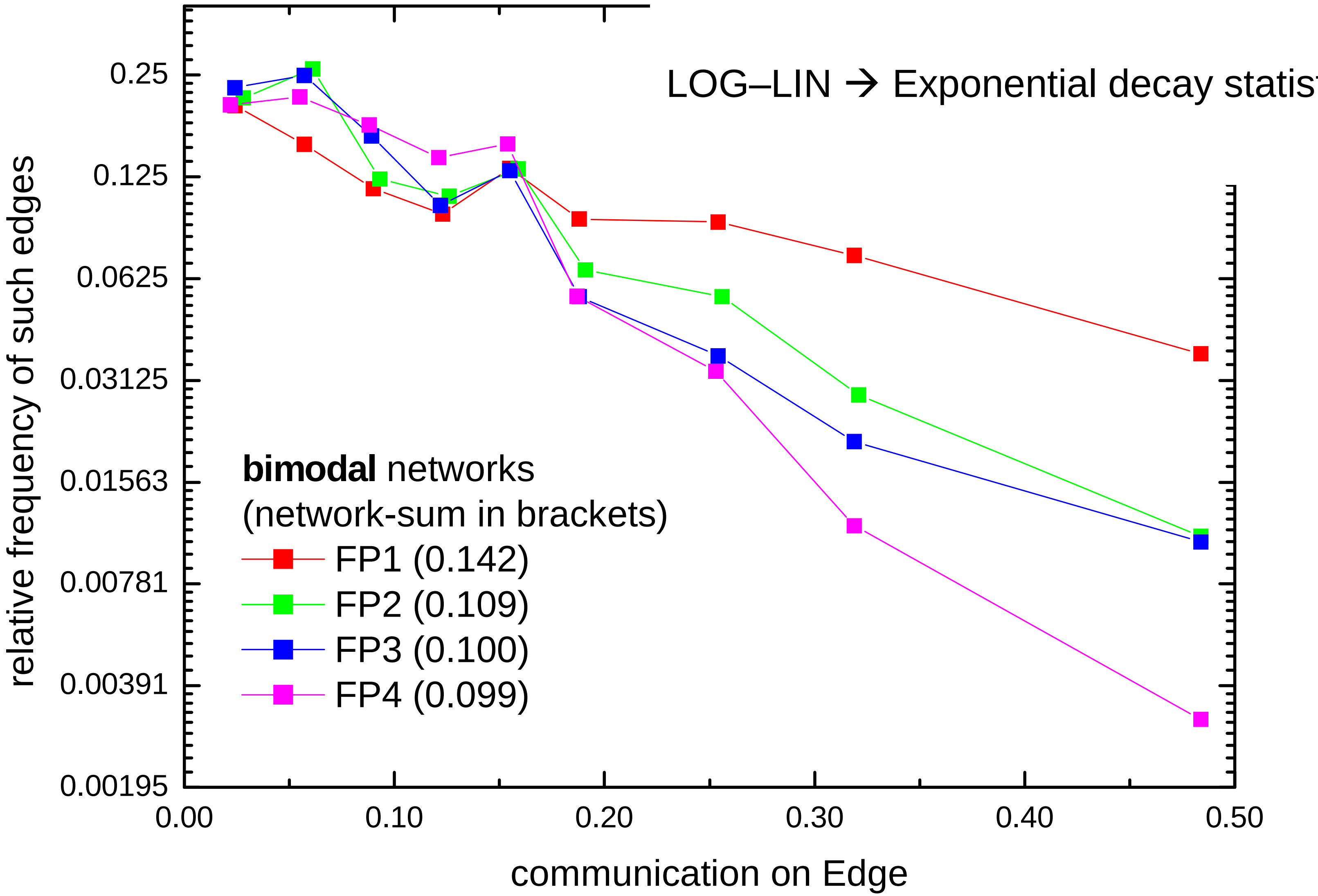
Node statistics of „capacity utilisation“ („workload“)



UNIMODAL ORGs Projection (unweighted) communication-edgeweights statistics



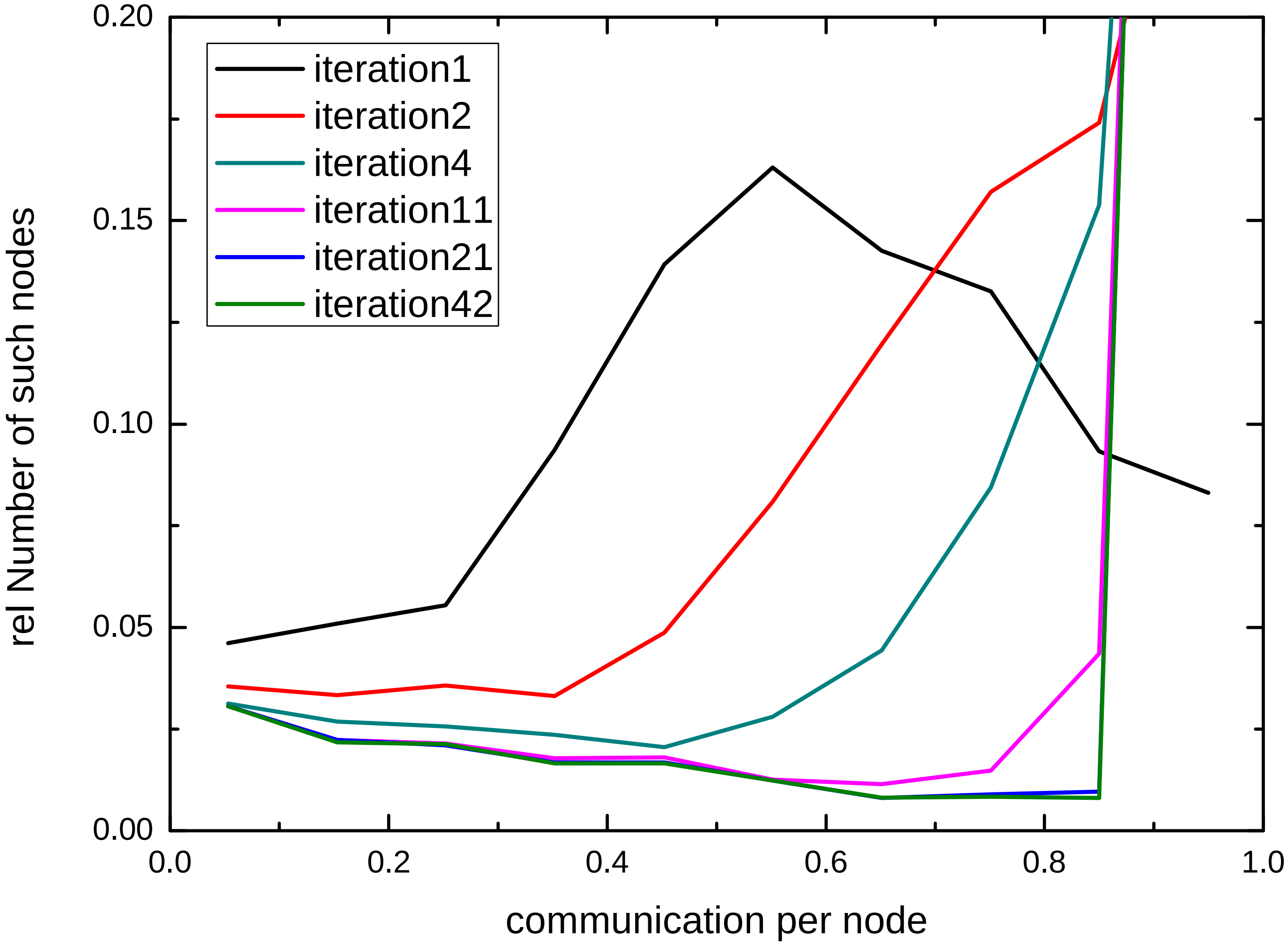
BIMODAL (projects also treated as actors!) communication-edges statistics



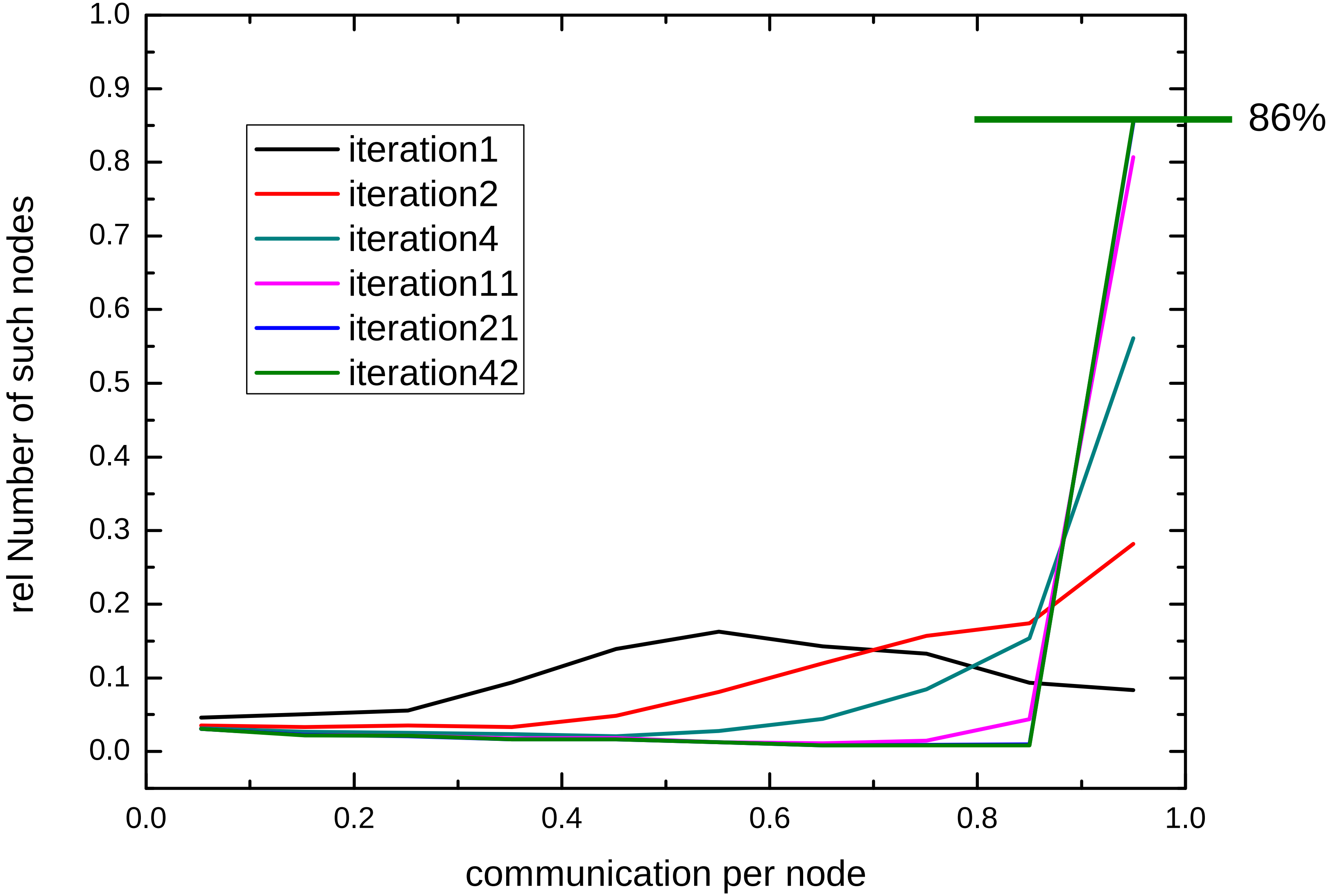
Further iterations

- The unbusy nodes still have
  free communication capacity
  among each other
- The busy nodes (nodeSum=1.0) are taken out of the game … then it is iterated
- At some iteration, it stagnates.
- Interesting question: How many of the nodes have ~100% communication after stagnation

Iterations until stagnation (FP2_ORGS)



Iterations until stagnation (FP2_ORGS)



Analytically tractable model !

→ Bollobas-Riordan Kernel Method

→ Sascha, Tyll, Madeleine, Philippe

→ Andreas: Mathematica numerics, EVs and plots

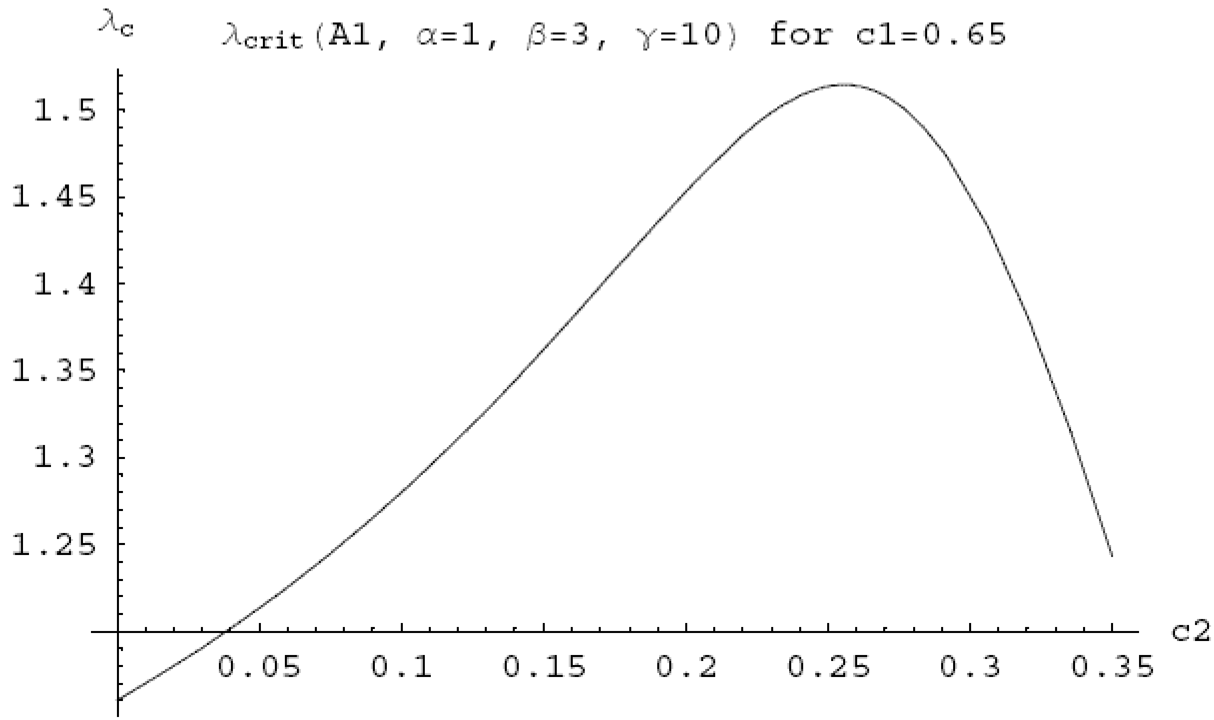
e.g. 3 node types society with mixture of hubs, middle-degree, low-degree :

1) Setup the kernel for 1/degree communication with a knowledge transmission probability λ
2) If Operator-norm of that kernel reaches 1 → birth of giant component
3) For which λ_{crit} does it happen?

Resulting plot, *very* preliminary:

critical transmission probability λ_{crit}

ratios of the *degrees* of the 3 node types = **α:β:γ**

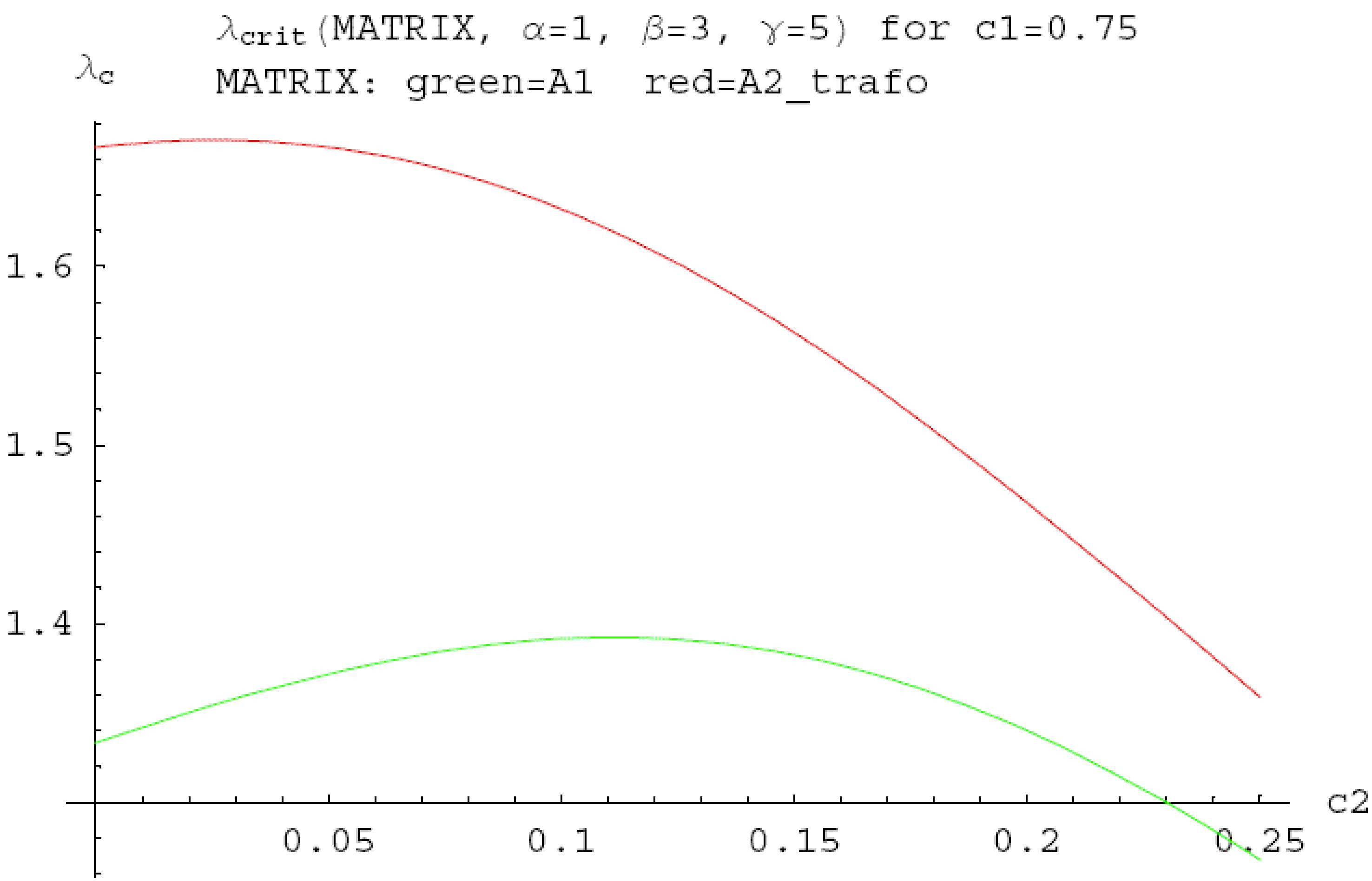


c1 of α-degree-type c2 of β-degree-type c3 of γ-degree-type

c1 fixed to 65% of nodes plot over c2 → c3 = 1 - c1 - c2

So to the right: more hubs, to the left: more middle-degree nodes

Resulting plot, *very* preliminary
Multiplicative vs additive coupling
(green) (red)



Resulting plot, *very* preliminary
Multiplicative vs additive coupling
(green) (red)