External and Internal Geometry on Configuration Spaces

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Dedicated to Professor Ludwig Streit on the occasion of his 60th birthday

Abstract

In this talk we will show a transparent relation between the intrinsic pre-Dirichlet form $\mathcal{E}_\mu^\Gamma$ and the extrinsic one $\mathcal{E}^P_{\mu,H_\sigma^X}$ corresponding to the Gibbs measure μ on the configuration space Γ_X. This extends the result obtained in [1] (see also [2]) for Poisson measure π_σ. As a consequence we prove the closability of $\mathcal{E}_\mu^\Gamma$ on $L^2(\Gamma_X,\mu)$ under very general assumptions on the interaction potential of the Gibbs measures μ, see also. [3]

1 Introduction

In the recent papers, [2] [4] [1] and [5] analysis and geometry on configuration spaces Γ_X over a Riemannian manifold X, i.e,

$$\Gamma := \Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}, \qquad (1)$$

was developed. They realized that the Dirichlet form of the Poisson measure π_σ with intensity measure σ on $\mathcal{B}(X)$ describes the well-known equilibrium process on configuration spaces, moreover this form is canonically associated with the introduced geometry on configuration spaces and is called *intrinsic Dirichlet form* of the measure π_σ.

On the other hand there is a well-known realization of the Hilbert space $L^2(\Gamma_X,\pi_\sigma)$ and the corresponding Fock space

$$\mathrm{Exp}L^2(X,\sigma) := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n L^2(X,\sigma), \qquad (2)$$

where $\mathrm{Exp}_n L^2(X,\sigma)$ denotes the n-fold symmetric tensor product of $L^2(X,\sigma)$ and $\mathrm{Exp}_0 L^2(X,\sigma) := \mathbf{R}$. This isomorphism gives us the possibility to produce natural operations in $L^2(\Gamma,\pi_\sigma)$ as images of the standard Fock space operators, see e.g., [6] and references therein. In particular, the image of the annihilation operator on the Fock space produces a natural version of a "gradient" operator in $L^2(\Gamma, \pi_\sigma)$. The differentiable structure in $L^2(\Gamma, \pi_\sigma)$ which appears in this way we consider as *external*.

In [1] the authors shown that the intrinsic Dirichlet form $\mathcal{E}^{\Gamma}_{\pi_\sigma}$ of the measure π_σ can be represented also in terms of the external Dirichlet form $\mathcal{E}^{P}_{\pi_\sigma, H^X_\sigma}$ with coefficient H^X_σ (the Dirichlet operator associated with σ on X) which uses this external differentiable structure, i.e.,

$$\int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma) = \int_\Gamma (\nabla^P F(\gamma), H^X_\sigma \nabla^P G(\gamma))_{L^2(X,\sigma)} d\pi_\sigma(\gamma). \tag{3}$$

If we change the Poisson measure π_σ to a Gibbs measure μ on the configuration space Γ which describes the equilibrium of interacting particle systems, the corresponding intrinsic Dirichlet form can still be used for constructing the corresponding stochastic dynamics (cf. [5]) or for constructing a quantum infinite particle Hamiltonian in models of quantum fields theory, see. [7]

The aim of this talk is to show that even for the interacting case there is a transparent relation between the intrinsic Dirichlet form and the extrinsic one, see Theorem 5.1. The proof is based on the Nguyen-Zessin characterization of Gibbs measure (cf. [8] or Proposition 5.2 below) which on a heuristic level can be considered as a consequence of the Mecke identity (cf. [9]).

As a consequence of the mentioned relation we prove the closability of the pre-Dirichlet form $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ on $L^2(\Gamma_X, \mu)$, where μ is a tempered grand canonical Gibbs measure, see Section 2 for this notion. We would like to emphasize that we achieve this result under a general condition (see (35) below) on the potential Φ which is not covered by condition (A.6) in. [10] Finally we mention the closability of the Dirichlet form $\mathcal{E}^\Gamma_\mu$ which is crucial (for physical reasons, see, [7] and) for applying the general theory of Dirichlet forms including the construction of a corresponding diffusion process (cf. [11]) which models an infinite particle system with (possibly) very singular interactions (cf. [5]).

2 Framework

In this section we describe some facts about probability measures on configuration spaces which are necessary later on.

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by T_xX; and the tangent bundle will be denoted by $TX = \cup_{x\in X} T_xX$. The Riemannian metric on X associates to each point $x \in X$ an inner product on T_xX which we denote by $\langle \cdot, \cdot \rangle_{T_xX}$ and the associated norm will be denoted by $|\cdot|_{T_xX}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

Let $\Gamma := \Gamma_X$ be the set of all locally finite subsets in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}. \tag{4}$$

We will identify γ with the positive integer-valued measure $\sum_{x\in\gamma} \varepsilon_x$. Then for any $\varphi \in C_0(X)$ we have a functional $\Gamma \ni \gamma \mapsto \langle \varphi, \gamma\rangle = \sum_{x\in\gamma} \varphi(x) \in \mathbf{R}$. Here $C_0(X)$ is the set of all real-valued continuous functions on X with compact support. The space Γ is endowed with the vague topology. Let $\mathcal{B}(\Gamma)$ denote the corresponding Borel σ-algebra. For $\Lambda \subset X$ we sometimes use the shorthand γ_Λ for $\gamma \cap \Lambda$.

For any $B \in \mathcal{B}(X)$ we define, as usual, $\Gamma \ni \gamma \mapsto N_B(\gamma) := \gamma(B) \in \mathbf{Z}_+ \cup \{+\infty\}$. Then $\mathcal{B}(\Gamma) = \sigma(\{N_\Lambda | \Lambda \in \mathcal{O}_c(X)\})$. For any $A \in \mathcal{B}(X)$ we also define $\mathcal{B}_A(\Gamma) := \sigma(\{N_B | B \in \mathcal{B}_c(X), B \subset A\})$.

Let $d\sigma(x) = \rho(x)dm(x)$, where $\rho > 0$ m-a.e. be such that $\rho^{\frac{1}{2}} \in H^{1,2}_{loc}(X)$ (the Sobolev space of order 1 in $L^2(X,m)$) and $\rho \notin L^1(X,m)$. We recall that the Poisson measure π_σ (with intensity measure σ) on $(\Gamma, \mathcal{B}(\Gamma))$ is defined via its Laplace transform by

$$\int_\Gamma e^{\langle\gamma,\varphi\rangle} d\pi_\sigma(\gamma) = \exp\left(\int_X (e^{\varphi(x)} - 1) d\sigma(x)\right),\ \varphi \in C_0(X), \tag{5}$$

see e.g.,[1] [12] [13]. Let us mention that if $\rho \in L^1(X,m)$, then we have a finite intensity measure σ on X, and in this case the corresponding measure π_σ will be concentrated on finite configurations. The latter can be considered as a degenerated case which can be reduced to finite dimensional analysis on every subset of n-particle configurations.

Let us briefly recall the definition of grand canonical Gibbs measures on $(\Gamma, \mathcal{B}(\Gamma))$. We adopt the notation in, [5] and refer the interested reader to the beautiful work by C. Preston, [14], but also. [15] [16]

A function $\Phi : \Gamma \to \mathbf{R} \cup \{+\infty\}$ will be called a *potential* iff for all $\Lambda \in \mathcal{B}_c(X)$ we have $\Phi(\emptyset) = 0$, $\Phi = 1\!\!1_{\{N_X<\infty\}}\Phi$, and $\gamma \mapsto \Phi(\gamma_\Lambda)$ is $\mathcal{B}_\Lambda(\Gamma)$-measurable.

For $\Lambda \in \mathcal{B}_c(X)$ the *conditional energy* $E^\Phi_\Lambda : \Gamma \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$E^\Phi_\Lambda(\gamma) := \begin{cases} \sum_{\gamma'\subset\gamma, \gamma'(\Lambda)>0} \Phi(\gamma') & \text{if } \sum_{\gamma'\subset\gamma,\gamma'(\Lambda)>0} |\Phi(\gamma')| < \infty, \\ +\infty & \text{otherwise,} \end{cases} \tag{6}$$

where the sum of the empty set is defined to be zero.

Later on we will use conditional energies which satisfy an additional assumption, namely, the *stability condition*, i.e., there exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(X)$ and for all $\gamma \in \Gamma_\Lambda$

$$E^\Phi_\Lambda(\gamma) \geq -B|\gamma|.$$

Definition 2.1 *For any $\Lambda \in \mathcal{O}_c(X)$ define for $\gamma \in \Gamma$ the measure $\Pi^\Phi_\Lambda(\gamma, \cdot)$ by*

$$\Pi^{\sigma,\Phi}_\Lambda(\gamma,\Delta) := 1\!\!1_{\{Z^{\sigma,\Phi}_\Lambda<\infty\}}(\gamma)[Z^{\sigma,\Phi}_\Lambda(\gamma)]^{-1} \int_\Gamma 1\!\!1_\Delta(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda) \cdot \exp[-E^\Phi_\Lambda(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] d\pi_\sigma(\gamma'),\ \Delta \in \mathcal{B}(\Gamma), \tag{7}$$

where

$$Z_\Lambda^{\sigma,\Phi}(\gamma) := \int_\Gamma \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] d\pi_\sigma(\gamma'). \tag{8}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called grand canonical Gibbs measure with interaction potential Φ if for all $\Lambda \in \mathcal{O}_c(X)$

$$\mu\Pi_\Lambda^\Phi = \mu. \tag{9}$$

Let $\mathcal{G}_{gc}(\sigma, \Phi)$ denote the set of all such probability measures μ. The Equations (9) are called Dobrushin-Landford-Ruelle (DLR) equations.

3 Intrinsic geometry on Poisson space

We recall some results to be used below from [1] [4] to which we refer for the corresponding proofs and more details.

A homeomorphism $\psi : X \to X$ defines a transformation of Γ by $\psi(\gamma) = \{\psi(x) | x \in \gamma\}$. Any vector field $v \in V_0(X)$ (i.e., the set of all smooth vector fields on X with compact support) defines a one-parameter group ψ_t^v, $t \in \mathbf{R}$, of diffeomorphisms on X.

Definition 3.1 *For $F : \Gamma \to \mathbf{R}$ we define the directional derivative along the vector field v as (provided the right hand side exists)*

$$(\nabla_v^\Gamma F)(\gamma) := \frac{d}{dt} F(\psi_t^v(\gamma))|_{t=0}.$$

This definition applies to F in the following class $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ of so-called smooth cylinder functions. Let $\mathcal{D} := C_0^\infty(X)$ (the set of all smooth functions on X with compact support). We define $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as the set of all functions on Γ of the form

$$F(\gamma) = g_F(\langle\gamma, \varphi_1\rangle, \ldots, \langle\gamma, \varphi_N\rangle), \ \gamma \in \Gamma, \tag{10}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and g_F is from $C_b^\infty(\mathbf{R}^N)$. Clearly, $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is dense in $L^2(\pi_\sigma) := L^2(\Gamma, \pi_\sigma)$. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have

$$(\nabla_v^\Gamma F)(\gamma) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle\gamma, \varphi_1\rangle, \ldots, \langle\gamma, \varphi_N\rangle)\langle\gamma, \nabla_v^X \varphi_i\rangle, \tag{11}$$

where $x \mapsto (\nabla_v^X \varphi)(x) = \langle\nabla^X \varphi(x), v(x)\rangle_{TX}$ is the usual directional derivative on X along the vector field v and ∇^X denotes the gradient on X.

The logarithmic derivative of the measure σ is given by the vector field $\beta^\sigma := \nabla^X \log\rho = \nabla^X \rho/\rho$ (where $\beta^\sigma = 0$ on $\{\rho = 0\}$). Then the logarithmic derivative of σ along v is the function $x \mapsto \beta_v^\sigma(x) = \langle\beta^\sigma(x), v(x)\rangle_{T_xX} + \mathrm{div}^X v(x)$, where div^X denotes the divergence on X w.r.t. the volume element m. Analogously, we define div_σ^X as the divergence on X w.r.t. σ, i.e., div_σ^X is the dual operator on $L^2(\sigma) := L^2(X, \sigma)$ of ∇^X.

Definition 3.2 *For any $v \in V_0(X)$ we define the logarithmic derivative of π_σ along v as the following function on Γ :*

$$\Gamma \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) := \langle \beta_v^\sigma, \gamma \rangle = \int_X [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div}^X v(x)] d\gamma(x). \tag{12}$$

Theorem 3.3 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and any $v \in V_0(X)$ the following integration by parts formula for π_σ holds:*

$$\int_\Gamma \nabla_v^\Gamma F G d\pi_\sigma = -\int_\Gamma F \nabla_v^\Gamma G d\pi_\sigma - \int_\Gamma F G B_v^{\pi_\sigma} d\pi_\sigma, \tag{13}$$

or $(\nabla_v^\Gamma)^* = -\nabla_v^\Gamma - B_v^{\pi_\sigma}$, as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ in $L^2(\pi_\sigma)$.

Definition 3.4 *We introduce the tangent space $T_\gamma \Gamma$ to the configuration space Γ at the point $\gamma \in \Gamma$ as the Hilbert space of γ-square-integrable sections (measurable vector fields) $V : X \to TX$ with scalar product $\langle V^1, V^2 \rangle_{T_\gamma \Gamma} = \int_X \langle V^1(x), V^2(x) \rangle_{T_x X} d\gamma(x)$, $V^1, V^2 \in T_\gamma \Gamma = L^2(X \to TX; \gamma)$. The corresponding tangent bundle is denoted by $T\Gamma$.*

The intrinsic gradient of a function $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is a mapping $\Gamma \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma \Gamma$ such that $(\nabla_v^\Gamma F)(\gamma) = \langle \nabla^\Gamma F(\gamma), v \rangle_{T_\gamma \Gamma}$ for any $v \in V_0(X)$. Furthermore, by (11), if F is given by (10), we have for $\gamma \in \Gamma$, $x \in X$

$$(\nabla^\Gamma F)(\gamma; x) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \nabla^X \varphi_i(x). \tag{14}$$

Definition 3.5 *For a measurable vector field $V : \Gamma \to T\Gamma$ the divergence $\mathrm{div}_{\pi_\sigma}^\Gamma V$ is defined via the duality relation for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by*

$$\int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma \Gamma} d\pi_\sigma(\gamma) = -\int_\Gamma F(\gamma)(\mathrm{div}_{\pi_\sigma}^\Gamma V)(\gamma) d\pi_\sigma(\gamma), \tag{15}$$

provided it exists (i.e., provided $F \mapsto \int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma \Gamma} d\pi_\sigma(\gamma)$ is continuous on $L^2(\pi_\sigma)$).

Proposition 3.6 *For any vector field $V = G\,v$, where $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, $v \in V_0(X)$ we have*

$$(\mathrm{div}_{\pi_\sigma}^\Gamma V)(\gamma) = \langle (\nabla^\Gamma G)(\gamma), v \rangle_{T_\gamma \Gamma} + G(\gamma) B_v^{\pi_\sigma}(\gamma). \tag{16}$$

For any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we introduce the *pre-Dirichlet form* which is generated by the intrinsic gradient ∇^Γ as

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F, G) = \int_\Gamma \langle (\nabla^\Gamma F)(\gamma), (\nabla^\Gamma G)(\gamma) \rangle_{T_\gamma \Gamma} d\pi_\sigma(\gamma). \tag{17}$$

We will also need the *classical pre-Dirichlet* form for the intensity measure σ which is given as $\mathcal{E}_\sigma^X(\varphi, \psi) = \int_X \langle \nabla^X \varphi, \nabla^X \psi \rangle_{TX} d\sigma$ for any $\varphi, \psi \in \mathcal{D}$. This form is associated with the *Dirichlet operator* H_σ^X which is given on $\mathcal{D}$ by $H_\sigma^X\varphi(x) := -\triangle^X\varphi(x) - \langle\beta^\sigma(x), \nabla^X\varphi(x)\rangle_{T_xX}$ and which satisfies $\mathcal{E}_\sigma^X(\varphi,\psi) = (H_\sigma^X\varphi,\psi)_{L^2(\sigma)}$, $\varphi,\psi\in\mathcal{D}$, see e.g. [17] and [11].

For any $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, $(\nabla^\Gamma\nabla^\Gamma F)(\gamma,x,y)\in T_\gamma\Gamma\otimes T_\gamma\Gamma$ and we can define the Γ-*Laplacian* $(\triangle^\Gamma F)(\gamma) := \mathrm{Tr}(\nabla^\Gamma\nabla^\Gamma F)(\gamma)\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. We introduce a differential operator in $L^2(\pi_\sigma)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ by the formula

$$(H_{\pi_\sigma}^\Gamma F)(\gamma) = -\triangle^\Gamma F(\gamma) - \langle\mathrm{div}_\sigma^X(\nabla^\Gamma F)(\gamma,\cdot),\gamma\rangle. \tag{18}$$

Theorem 3.7 *The operator $H_{\pi_\sigma}^\Gamma$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_\sigma}^\Gamma$, i.e.,*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F,G) = (H_{\pi_\sigma}^\Gamma F,G)_{L^2(\pi_\sigma)}, \tag{19}$$

or $H_{\pi_\sigma}^\Gamma = -\mathrm{div}_{\pi_\sigma}^\Gamma\nabla^\Gamma$ on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. We call $H_{\pi_\sigma}^\Gamma$ the intrinsic Dirichlet operator of the measure π_σ.

4 Extrinsic geometry on Poisson space

We recall the extrinsic geometry on $L^2(\pi_\sigma)$ based on the isomorphism with the Fock space. Our approach is based on [18] but we should also mention [19] [20] [6] [21] [22] for related considerations and references therein. For proofs of the results stated below in this section, we refer to. [1]

Let us define another "gradient" on functions $F:\Gamma\to\mathbf{R}$. This gradient ∇^P has specific useful properties on Poissonian spaces. We will call ∇^P the *Poissonian gradient*. To this end the tangent space to Γ at any point $\gamma\in\Gamma$ we consider the same space $L^2(\sigma)$ and define a mapping $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)\ni F\mapsto\nabla^P F\in L^2(\pi_\sigma)\otimes L^2(\sigma)$ by

$$(\nabla^P F)(\gamma,x) := F(\gamma+\varepsilon_x) - F(\gamma),\ \gamma\in\Gamma, x\in X. \tag{20}$$

We stress that the transformation $\Gamma\ni\gamma\mapsto\gamma+\varepsilon_x\in\Gamma$ is π_σ-a.e. well-defined because $\pi_\sigma(\{\gamma\in\Gamma|x\in\gamma\}) = 0$ for any $x\in X$. The directional derivative is then defined as

$$(\nabla_\varphi^P F)(\gamma) = \int_X [F(\gamma+\varepsilon_x) - F(\gamma)]\varphi(x)d\sigma(x),\ \varphi\in\mathcal{D}. \tag{21}$$

The Poissonian gradient ∇^P yields an orthogonal system of Charlier polynomials on the Poisson space $(\Gamma,\mathcal{B}(\Gamma),\pi_\sigma)$.

For any $n\in\mathbf{N}$ and all $\varphi\in\mathcal{D}$ we introduce a function in $L^2(\pi_\sigma)$ by

$$Q_n^{\pi_\sigma}(\gamma;\varphi^{\otimes n}) := ((\nabla_\varphi^P)^{*n}1)(\gamma), \tag{22}$$

and define $Q_0^{\pi_\sigma} := 1$. Due to the kernel theorem [17] these functions have the representation $Q_n^{\pi_\sigma}(\gamma;\varphi^{\otimes n}) = \langle Q_n^{\pi_\sigma}(\gamma),\varphi^{\otimes n}\rangle$, with generalized symmetric kernels $\Gamma\ni\gamma\mapsto Q_n^{\pi_\sigma}(\gamma)\in\mathrm{Exp}_n\mathcal{D}'$, $n\in\mathbf{N}$. Here and below by Exp_nE we denote the n-th symmetric tensor power of a linear space E. Then for any smooth kernel $\varphi^{(n)}\in\mathrm{Exp}_n\mathcal{D}^{\otimes n}$ we introduce the function $Q_n^{\pi_\sigma}(\gamma;\varphi^{(n)}) := \langle Q_n^{\pi_\sigma}(\gamma),\varphi^{(n)}\rangle$ such that for all $\varphi^{(n)}\in\mathrm{Exp}_n\mathcal{D}^{\otimes n}$, $\psi^{(m)}\in\mathrm{Exp}_m\mathcal{D}^{\otimes m}$

$$\int_\Gamma Q_n^{\pi_\sigma}(\gamma;\varphi^{(n)})Q_m^{\pi_\sigma}(\gamma;\psi^{(m)})d\pi_\sigma(\gamma) = \delta_{nm}n!(\varphi^{(n)},\psi^{(m)})_{L^2(\sigma^{\otimes n})}. \tag{23}$$

Hence (22) extends to the case of kernels from the so-called n-particle Fock space $\mathrm{Exp}_n L^2(\sigma)$, $n \in \mathbf{N}$, and we set $\mathrm{Exp}_0 L^2(\sigma) := \mathbf{R}$.

As usual the symmetric Fock space over the Hilbert space $L^2(\sigma)$ is defined as $\mathrm{Exp}L^2(\sigma) := \oplus_{n=0}^{\infty} \mathrm{Exp}_n L^2(\sigma)$, see e.g. [17] and [23]. It is well-known that there exists an isomorphism between $\mathrm{Exp}L^2(\sigma)$ and $L^2(\pi_\sigma)$ given by

$$\mathrm{Exp}L^2(\sigma) \ni (f^{(n)})_{n=0}^{\infty} \leftrightarrow F(\gamma) = \sum_{n=0}^{\infty} Q_n^{\pi_\sigma}(\gamma; f^{(n)}).$$

The following proposition shows that the operators ∇_φ^P and ∇_φ^{P*} play the role of the annihilation resp. creation operators in the Fock space $\mathrm{Exp}L^2(\sigma)$.

Proposition 4.1 *For all $\varphi, \psi \in \mathcal{D}$, $n \in \mathbf{N}$ the following formulas hold*

$$\nabla_\psi^P Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) = n(\varphi, \psi)_{L^2(\sigma)} Q_{n-1}^{\pi_\sigma}(\gamma; \varphi^{\otimes(n-1)}) \tag{24}$$

$$\nabla_\psi^{P*} Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) = Q_{n+1}^{\pi_\sigma}(\gamma; \varphi^{\otimes n} \hat{\otimes} \psi), \ \gamma \in \Gamma, \tag{25}$$

where $\varphi^{\otimes n} \hat{\otimes} \psi$ means the symmetric tensor product of $\varphi^{\otimes n}$ and ψ.

Next we give an explicit expression for the adjoint of the Poissonian gradient ∇^{P*}.

Proposition 4.2 *For any function $F \in L^1(\pi_\sigma) \otimes L^1(\sigma)$ we have $F \in D(\nabla^{P*})$ and the following equality holds*

$$(\nabla^{P*} F)(\gamma) = \int_X F(\gamma - \varepsilon_x, x) d\gamma(x) - \int_X F(\gamma, x) d\sigma(x), \ \gamma \in \Gamma, \tag{26}$$

provided the right hand side of (26) is in $L^2(\pi_\sigma)$.

Proof. For $X = \mathbf{R}^d$ this proposition was proved in [6]. The general case follows from (20) and the Mecke identity, see e.g., [9]

$$\int_\Gamma \left(\int_X h(\gamma, x) d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_X \int_\Gamma h(\gamma + \varepsilon_x, x) d\pi_\sigma(\gamma) d\sigma(x), \tag{27}$$

where h is any non-negative, $\mathcal{B}(\Gamma) \times \mathcal{B}(X)$-measurable function.

For any contraction B in $L^2(\sigma)$ it is possible to define an operator $\mathrm{Exp}B$ as a contraction in $\mathrm{Exp}L^2(\sigma)$ which in any n-particle subspace $\mathrm{Exp}_n L^2(\sigma)$ is given by $B \otimes \cdots \otimes B$ (n times). For any positive self-adjoint operator A in $L^2(\sigma)$ (with $\mathcal{D} \subset D(A)$) we have a contraction semigroup e^{-tA}, $t \geq 0$, hence it is possible to introduce the second quantization operator $d\mathrm{Exp}A$ as the generator of the semigroup $\mathrm{Exp}(e^{-tA})$, $t \geq 0$, i.e., $\mathrm{Exp}(e^{-tA}) = \exp(-td\mathrm{Exp}A)$, see e.g. [24] We denote by H_A^P the image of the operator $d\mathrm{Exp}A$ in the Poisson space $L^2(\pi_\sigma)$ under the described isomorphism.

Proposition 4.3 *Let $\mathcal{D} \subset D(A)$. Then the symmetric bilinear form corresponding to the operator H_A^P has the following form, ($F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$)*

$$(H_A^P F, G)_{L^2(\pi_\sigma)} = \int_\Gamma (\nabla^P F(\gamma), A \nabla^P G(\gamma))_{L^2(\sigma)} d\pi_\sigma(\gamma). \tag{28}$$

The right hand side of (28) is called the "Poissonian pre-Dirichlet form" with coefficient operator A and is denoted by $\mathcal{E}_{\pi_\sigma, A}^P$.

Let us consider the special case of the second quantization operator $d\mathrm{Exp}A$, where the one-particle operator A coincides with the Dirichlet operator H^X_σ generated by the measure σ on X. Then we have the following theorem which relates the intrinsic Dirichlet operator $H^\Gamma_{\pi_\sigma}$ and the operator $H^P_{H^X_\sigma}$.

Theorem 4.4 *$H^\Gamma_{\pi_\sigma} = H^P_{H^X_\sigma}$ on $\mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$. In particular, for all $F,G \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$*

$$\int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma) = \int_\Gamma (\nabla^P F(\gamma), H^X_\sigma \nabla^P G(\gamma))_{L^2(\sigma)} d\pi_\sigma(\gamma). \tag{29}$$

5 Relation between intrinsic and extrinsic Dirichlet forms

Here we consider the class of measures $\mathcal{G}^1_{gc}(\sigma,\Phi)$ consisting of all $\mu \in \mathcal{G}_{gc}(\sigma,\Phi)$ such that

$$\int_\Gamma \gamma(K) d\mu(\gamma) < \infty \text{ for all compact } K \subset X.$$

We define for any $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$ the pre-Dirichlet form $\mathcal{E}^\Gamma_\mu$ by

$$\mathcal{E}^\Gamma_\mu(F,G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\mu(\gamma),\ F,G \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma). \tag{30}$$

After all our preparations we are now going to prove an analogue of (29) for $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$. We would like to emphasize that the corresponding formula (31) is not obtained from (29) by just replacing π_σ by $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$. The essential difference is, in addition, an extra factor involving the conditional energy E^Φ_Λ.

Theorem 5.1 *For any $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$, we have for all $F,G \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$*

$$\mathcal{E}^\Gamma_\mu(F,G) = \int_\Gamma \int_X \langle \nabla^X \nabla^P F(\gamma,x), \nabla^X \nabla^P G(\gamma,x)\rangle_{T_xX}\, e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x) d\mu(\gamma). \tag{31}$$

Proof. Let us take any $F \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$ of the form (10). Then given $\gamma \in \Gamma$ and $x \in X$ (20) implies that

$$\nabla^X \nabla^P F(\gamma,x) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \varphi_1,\gamma\rangle + \varphi_1(x), \dots, \langle \varphi_N,\gamma\rangle + \varphi_N(x)) \nabla^X \varphi_i(x).$$

Let us define $\hat{F}_i(\gamma) := \frac{\partial g_F}{\partial s_i}(\langle \varphi_1,\gamma\rangle, \dots, \langle \varphi_N,\gamma\rangle)$, $i = 1,\dots,N$. Obviously, it is enough to prove the equality (31) for $F = G$. Thus, inserting the result of $\nabla^X\nabla^P F(\gamma,x)$ into the right hand side of (31) we obtain

$$\int_\Gamma \int_X \sum_{i,j=1}^N \langle \nabla^X \varphi_i(x), \nabla^X \varphi_j(x)\rangle_{T_xX} \hat{F}_i(\gamma+\varepsilon_x) \hat{F}_j(\gamma+\varepsilon_x) e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x) d\mu(\gamma). \tag{32}$$

Then we need the following useful proposition which generalizes the Mecke identity to measures in $\mathcal{G}_{gc}(\sigma,\Phi)$, see. [8] [25]

Proposition 5.2 *Let $h : \Gamma \times X \to \mathbf{R}_+$ be $\mathcal{B}(\Gamma)\times\mathcal{B}(X)$-measurable, and let $\mu \in \mathcal{G}_{gc}(\sigma,\Phi)$. Then we have*

$$\int_\Gamma \left(\int_X h(\gamma,x) d\gamma(x) \right) d\mu(\gamma) = \int_X \int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x)} d\mu(\gamma) d\sigma(x). \tag{33}$$

Using this proposition we transform (32) into

$$\int_\Gamma \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle \nabla^X \varphi_i(\cdot), \nabla^X \varphi_j(\cdot)\rangle_{TX}, \gamma\rangle d\mu(\gamma).$$

On the other hand using (14) we obtain

$$\langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T\Gamma} = \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle \nabla^X \varphi_i(\cdot), \nabla^X \varphi_j(\cdot)\rangle_{TX}, \gamma\rangle.$$

Therefore the equality on the dense $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is valid which proves the theorem.

6 Closability of intrinsic Dirichlet forms

In this section we will prove the closability of the intrinsic Dirichlet form $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu) := L^2(\Gamma,\mu)$ for all $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$, using the integral representation (31) in Theorem 5.1. The closability of $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ over Γ is implied by the closability of an appropriate family of pre-Dirichlet forms over X. Let us describe this more precisely. We define new intensity measures on X by $d\sigma_\gamma(x) := \rho_\gamma(x) dm(x)$, where

$$\rho_\gamma(x) := e^{-E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x)} \rho(x),\ x \in X, \gamma \in \Gamma \tag{34}$$

It was shown in [26, Theorem 5.3] (in the case $X = \mathbf{R}^d$) that the components of the Dirichlet form $(\mathcal{E}^X_{\sigma_\gamma}, \mathcal{D}^{\sigma_\gamma})$ corresponding to the measure σ_γ are closable on $L^2(\mathbf{R}^d, \sigma_\gamma)$ if and only if σ_γ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^d$ and the Radon-Nikodym derivative satisfies some condition, see (35) below for details. This result allows us to prove the closability of $(\mathcal{E}^\Gamma_\mu, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu)$. Let us first recall the above mentioned result.

Theorem 6.1 *(cf. Theorem 5.3 in [26]) Let ν by a probability measure on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$, $d \in \mathbf{N}$ and let $\mathcal{D}^\nu$ denote the ν-classes determined by $\mathcal{D}$. Then the forms $(\mathcal{E}_{\nu,i}, \mathcal{D}^\nu)$ defined by*

$$\mathcal{E}_{\nu,i}(u,v) := \int_{\mathbf{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} d\nu,\ u, v \in \mathcal{D},$$

are well-defined and closable on $L^2(\mathbf{R}^d, \nu)$ for $1 \le i \le d$ if and only if ν is absolutely continuous with respect to Lebesgue measure λ^d on $\mathbf{R}^d$, and the Radon-Nikodym derivative $\rho = d\nu/d\lambda^d$ satisfies the condition:

$$\text{for any } 1 \le i \le d \text{ and } \lambda^{d-1}-\text{a.e. } x \in \left\{ y \in \mathbf{R}^{d-1} | \int_{\mathbf{R}} \rho_y^{(i)}(s) d\lambda^1(s) > 0 \right\},$$
$$\rho_x^{(i)} = 0 \ \lambda^1-\text{a.e. on } \mathbf{R}\backslash \mathbf{R}(\rho_x^{(i)}), \text{ where } \rho_x^{(i)}(s) := \rho(x_1, \ldots, x_{i-1}, s, x_i, \ldots, x_d),$$
$$s \in \mathbf{R}, \text{ if } x = (x_1, \ldots, x_{d-1}) \in \mathbf{R}^{d-1}, \text{ and where} \quad (35)$$

$$R(\rho_x^{(i)}) := \left\{ t \in \mathbf{R} | \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho_x^{(i)}(s)} ds < \infty \text{ for some } \varepsilon > 0 \right\}. \quad (36)$$

There is an obvious generalization of Theorem 6.1 to the case where a Riemannian manifold X is replacing $\mathbf{R}^d$, to be formulated in terms of local charts. Since here we are only interested in the “if part” of Theorem 6.1, we now recall a slightly weaker sufficient condition for closability in the general case where X is a manifold as before.

Theorem 6.2 *Suppose $\sigma_1 = \rho_1 \cdot m$, where $\rho_1 : X \to \mathbf{R}_+$ is $\mathcal{B}(X)$-measurable such that*

$$\rho_1 = 0 \ m-\text{a.e. on } X \backslash \left\{ x \in X | \int_{\Lambda_x} \frac{1}{\rho_1} dm < \infty \text{ for some open neighbourhood } \Lambda_x \text{ of } x \right\}. \quad (37)$$

Then $(\mathcal{E}_{\sigma_1}^X, \mathcal{D}^{\sigma_1})$ defined by

$$\mathcal{E}_{\sigma_1}^X(u, v) := \int_X \langle \nabla^X u(x), \nabla^X v(x) \rangle_{T_x X} \, d\sigma_1(x); \ u, v \in \mathcal{D},$$

is closable on $L^2(\sigma_1)$.

The proof is a straightforward adaptation of the line of arguments in [11] (Chap. II, Subsection 2a), see also Theorem 4.2 in [27] for details. We emphasize that (37) e.g. always holds, if ρ_1 is lower semicontinuous, and that neither ν in Theorem 6.1 nor σ_1 in Theorem 6.2 is required to have full support, so e.g. ρ_1 is not necessarily strictly positive m-a.e. on X.

We are now ready to prove the closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ on $L^2(\mu)$ under the above assumption.

Theorem 6.3 *Let $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$. Suppose that for μ-a.e. $\gamma \in \Gamma$ the function ρ_γ defined in (34) satisfies (37) (resp. (35) in case $X = \mathbf{R}^d$). Then the form $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ is closable on $L^2(\mu)$.*

We address the interested reader to [3] for the details of the proof.

Remark 6.4 *The above method to prove closability of pre-Dirichlet forms on configuration spaces Γ_X extends immediately to the case where X is replaced by an infinite dimensional “manifold” such as the loop space (cf. [28]).*

Example 6.5 *Let $X = \mathbf{R}^d$ with the Euclidean metric and $\sigma := z \cdot m$, $z \in (0, \infty)$. A pair potential is a $\mathcal{B}(\mathbf{R}^d)$-measurable function $\phi : \mathbf{R}^d \to \mathbf{R} \cup \{\infty\}$ such that $\phi(-x) = \phi(x)$. Any pair potential ϕ defines a potential*

$\Phi = \Phi_\phi$ in the sense of Section 2 as follows: we set $\Phi(\gamma) := 0$, $|\gamma| \neq 2$ and $\Phi(\gamma) := \phi(x-y)$ for $\gamma = \{x, y\} \subset \mathbf{R}^d$. For such pair potentials ϕ the condition in Theorem 6.3 ensuring closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ on $L^2(\mu)$ for $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$ can be now easily formulated as follows: for μ-a.e. $\gamma \in \Gamma$ and m-a.e. $x \in \{y \in \mathbf{R}^d | \sum_{y' \in \gamma \setminus \{y\}} |\phi(y-y')| < \infty\}$ it holds that $\int_{V_x} e^{\sum_{y' \in \gamma \setminus \{y\}} \phi(y-y')} m(dy) < \infty$ for some open neighborhood V_x of x. This condition trivially holds e.g. if suppϕ is compact, $\{\phi < \infty\}$ is open, and $\phi^+ \in L_{loc}^\infty)(\{\phi < \infty\}; m)$. If even $\mu \in \mathcal{G}_{gc}^t(z, \phi)$ and ϕ satisfies the assumptions in Proposition 7.1, then it suffices to merely assume that $\{\phi < \infty\}$ is open and $\phi^+ \in L_{loc}^\infty(\{\phi < \infty\}; m)$. This follows by an elementary consideration.

Remark 6.6 *1. We emphasize that Example 6.5 generalizes the closability result in [10], though an a-priori bigger domain for $\mathcal{E}_\mu^\Gamma$ is considered there. However, Theorems 6.1-6.3 are also valid for this bigger domain. The proofs are exactly the same.*

2. We also like to emphasize that similarly to Example 6.5 one proves the closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ (or with a larger domain in [10]) on $L^2(\mu)$ for $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$ in the case of multi-body potentials ϕ.

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