On a relation between intrinsic and extrinsic Dirichlet forms for interacting particle systems

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Abstract

In this paper we extend the result obtained in [AKR98a] (see also [AKR96a]) on the representation of the intrinsic pre-Dirichlet form $\mathcal{E}^{\Gamma}_{\pi_\sigma}$ of the Poisson measure π_σ in terms of the extrinsic one $\mathcal{E}^{\Gamma}_{\pi_\sigma,H^X_\sigma}$. More precisely, replacing π_σ by a Gibbs measure μ on the configuration space Γ_X we derive a relation between the intrinsic pre-Dirichlet form $\mathcal{E}^{\Gamma}_{\mu}$ of the measure μ and the extrinsic one $\mathcal{E}^{P}_{\mu,H^X_\sigma}$. As a consequence we prove the closability of $\mathcal{E}^{\Gamma}_{\mu}$ on $L^2(\Gamma_X,\mu)$ under very general assumptions on the interaction potential of the Gibbs measures μ.

1 Introduction

In the recent papers [AKR96a], [AKR96b], [AKR98a], and [AKR98b] analysis and geometry on configuration spaces Γ_X over a Riemannian manifold X, i.e,

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\},$$

was developed. One of the consequences of the discussed approach was a description of the well-known equilibrium process on configuration spaces as the Brownian motion associated with a Dirichlet form of the Poisson measure π_σ with intensity measure σ on $\mathcal{B}(X)$. This form is canonically associated with the introduced geometry on configuration spaces and is called *intrinsic Dirichlet form* of the measure π_σ.

On the other hand there is a well-known realization (canonical isomorphism) of the Hilbert space $L^2(\Gamma_X, \pi_\sigma)$ and the corresponding Fock space

$$\mathrm{Exp} L^2(X,\sigma) := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n L^2(X,\sigma),$$

where $\mathrm{Exp}_n L^2(X,\sigma)$ denotes the n-fold symmetric tensor product of $L^2(X,\sigma)$ and $\mathrm{Exp}_0 L^2(X,\sigma) := \mathbb{C}$. This isomorphism produces natural operations in $L^2(\Gamma_X, \pi_\sigma)$ as images of the standard Fock space operators, see e.g., [KSSU98] and references therein. In particular, we can consider the image of the annihilation operator from the Fock space as a natural version of a "gradient" operator in $L^2(\Gamma_X, \pi_\sigma)$. The differentiable structure in $L^2(\Gamma_X, \pi_\sigma)$ which appears in this way we consider as *external* because it is produced via transportation from the Fock space.

As was shown in [AKR98a, Section 5] the intrinsic Dirichlet form of the measure π_σ can be represented also in terms of the external Dirichlet form $\mathcal{E}^P_{\pi_\sigma, H^X_\sigma}$ with coefficient H^X_σ (the Dirichlet operator associated with σ on X) which uses this external differentiable structure, i.e.,

$$\int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma \Gamma} d\pi_\sigma(\gamma) = \int_\Gamma (\nabla^P F(\gamma), H^X_\sigma \nabla^P G(\gamma))_{L^2(X,\sigma)} d\pi_\sigma(\gamma).$$

As a result we have a full spectral description of the corresponding Dirichlet operator $H^\Gamma_{\pi_\sigma}$ which is the generator of the equilibrium process on Γ_X.

If we change the Poisson measure π_σ to a Gibbs measure μ on the configuration space Γ_X which describes the equilibrium of interacting particle systems, the corresponding intrinsic Dirichlet form can still be used for constructing the corresponding stochastic dynamics (cf. [AKR98b, Section 5]) or for constructing a quantum infinite particle Hamiltonian in models of quantum fields theory, see [AKR97].

The aim of this paper is to show that even for the interacting case there is a transparent relation between the intrinsic Dirichlet form and the extrinsic one, see Theorem 5.1. The proof is based on the Nguyen-Zessin characterization of Gibbs measure (cf. [NZ79, Theorem 2] or Proposition 5.2 below) which on a heuristic level can be considered as a consequence of the Mecke identity (cf. [Mec67, Satz 3.1]), see Remark 5.3 below for more details.

As a consequence of the mentioned relation we prove the closability of the pre-Dirichlet form $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C^{\infty}_{b}(\mathcal{D}, \Gamma))$ on $L^2(\Gamma_X, \mu)$, where μ is a tempered grand canonical Gibbs measure, see Section 2 for this notion. It turns out that this result is obtained as a "lifting" of the closable Dirichlet forms on X. We would like to emphasize that we achieve this result under a general condition (see (6.2) below) on the potential Φ which is not covered by condition (A.6) in [Osa96]. The closability is crucial (for physical reasons, see [AKR97], and) for applying the general theory of Dirichlet forms including the construction of a corresponding diffusion process (cf. [MR92]) which models an infinite particle system with (possibly) very singular interactions (cf. [AKR98b]).

Another motivation for deriving Theorem 5.1 is to use this result for studying spectral properties of Hamiltonians of intrinsic Dirichlet forms associated with Gibbs measures. This will be implemented in a forthcoming paper.

2 Preliminaries and Framework

In this section we describe some facts about probability measures on configuration spaces which are necessary later on.

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by T_xX; and the tangent bundle will be denoted by $TX = \cup_{x\in X} T_xX$. The Riemannian metric on X associates to each point $x \in X$ an inner product on T_xX which we denote by $\langle \cdot, \cdot \rangle_{T_xX}$ and the associated norm will be denoted by $| \cdot |_{T_xX}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

Let $\Gamma := \Gamma_X$ be the set of all locally finite subsets in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}.$$

We will identify γ with the positive integer-valued measure $\sum_{x\in\gamma} \varepsilon_x$. Then for any $\varphi \in C_0(X)$ we have a functional $\Gamma \ni \gamma \mapsto \langle \varphi, \gamma \rangle = \sum_{x\in\gamma} \varphi(x) \in \mathbb{R}$. Here $C_0(X)$ is the set of all real-valued continuous functions on X with compact support. The space Γ is endowed with the vague topology. Let $\mathcal{B}(\Gamma)$ denote the corresponding Borel σ-algebra. For $\Lambda \subset X$ we sometimes use the shorthand γ_Λ for $\gamma \cap \Lambda$.

For any $B \in \mathcal{B}(X)$ we define, as usual, $\Gamma \ni \gamma \mapsto N_B(\gamma) := \gamma(B) \in \mathbb{Z}_+ \cup \{+\infty\}$. Then $\mathcal{B}(\Gamma) = \sigma(\{N_\Lambda | \Lambda \in \mathcal{O}_c(X)\})$. For any $A \in \mathcal{B}(X)$ we also define $\mathcal{B}_A(\Gamma) := \sigma(\{N_B | B \in \mathcal{B}_c(X), B \subset A\})$.

Let $d\sigma(x) = \rho(x)dm(x)$, where $\rho > 0$ m-a.e. such that $\rho^{\frac{1}{2}} \in H^{1,2}_{loc}(X)$ (the Sobolev space of order 1 in $L^2(X,m)$) and $\rho \notin L^1(X,m)$. We recall that the Poisson measure π_σ (with intensity measure σ) on $(\Gamma, \mathcal{B}(\Gamma))$ is defined via its Laplace transform by

$$\int_\Gamma e^{\langle \gamma, \varphi \rangle} d\pi_\sigma(\gamma) = \exp\left(\int_X (e^{\varphi(x)} - 1) d\sigma(x) \right), \; \varphi \in C_0(X), \tag{2.1}$$

see e.g. [AKR98a], [GV68], and [Shi94]. Let us mention that if $\rho \in L^1(X,m)$, then we have a finite intensity measure σ on X, and in this case the corresponding measure π_σ will be concentrated on finite configurations. The latter can be considered as a degenerated case which can be reduced to finite dimensional analysis on every subset of n-particle configurations.

Let us briefly recall the definition of grand canonical Gibbs measures on $(\Gamma, \mathcal{B}(\Gamma))$. We adopt the notation in [AKR98b], and refer the interested reader to the beautiful work by C. Preston, [Pre79], but also [Pre76], and [Geo79].

A function $\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ will be called a *potential* iff for all $\Lambda \in \mathcal{B}_c(X)$ we have $\Phi(\emptyset) = 0$, $\Phi = 1\!\!1_{\{N_X < \infty\}} \Phi$, and $\gamma \mapsto \Phi(\gamma_\Lambda)$ is $\mathcal{B}_\Lambda(\Gamma)$-measurable.

For $\Lambda \in \mathcal{B}_c(X)$ the *conditional energy* $E^\Phi_\Lambda : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E^\Phi_\Lambda(\gamma) := \begin{cases} \displaystyle\sum_{\gamma' \subset \gamma, \gamma'(\Lambda) > 0} \Phi(\gamma') & \text{if } \displaystyle\sum_{\gamma' \subset \gamma, \gamma'(\Lambda) > 0} |\Phi(\gamma')| < \infty, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.2}$$

where the sum of the empty set is defined to be zero.

Later on we will use conditional energies which satisfy an additional assumption, namely, the *stability condition*, i.e., there exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(X)$ and for all $\gamma \in \Gamma_\Lambda$

$$E^{\Phi}_{\Lambda}(\gamma) \geq -B|\gamma|.$$

Definition 2.1 *For any $\Lambda \in \mathcal{O}_c(X)$ define for $\gamma \in \Gamma$ the measure $\Pi^{\Phi}_{\Lambda}(\gamma, \cdot)$ by*

$$\begin{aligned}\Pi^{\sigma,\Phi}_{\Lambda}(\gamma, \Delta) \;&:=\; 1\!\!1_{\{Z^{\sigma,\Phi}_{\Lambda}<\infty\}}(\gamma)[Z^{\sigma,\Phi}_{\Lambda}(\gamma)]^{-1}\int_{\Gamma} 1\!\!1_{\Delta}(\gamma_{X\setminus\Lambda} + \gamma'_{\Lambda}) \\ &\quad \cdot \exp[-E^{\Phi}_{\Lambda}(\gamma_{X\setminus\Lambda} + \gamma'_{\Lambda})]d\pi_{\sigma}(\gamma'),\ \Delta \in \mathcal{B}(\Gamma),\end{aligned} \tag{2.3}$$

where

$$Z^{\sigma,\Phi}_{\Lambda}(\gamma) := \int_{\Gamma} \exp[-E^{\Phi}_{\Lambda}(\gamma_{X\setminus\Lambda} + \gamma'_{\Lambda})]d\pi_{\sigma}(\gamma'). \tag{2.4}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called grand canonical Gibbs measure with interaction potential Φ if for all $\Lambda \in \mathcal{O}_c(X)$

$$\mu\Pi^{\Phi}_{\Lambda} = \mu. \tag{2.5}$$

Let $\mathcal{G}_{gc}(\sigma, \Phi)$ denote the set of all such probability measures μ.

Remark 2.2 *1. It is well-known that $(\Pi^{\sigma,\Phi}_{\Lambda})_{\Lambda\in\mathcal{O}_c(X)}$ is a $(\mathcal{B}_{X\setminus\Lambda}(\Gamma))_{\Lambda\in\mathcal{O}_c(X)}$-specification in the sense of [Pre76, Section 6] or [Pre79].*

2. For any $\gamma \in \Gamma$ the measure $\mu\Pi^{\Phi}_{\Lambda}$ in (2.5) is defined by

$$(\mu\Pi^{\sigma,\Phi}_{\Lambda})(\Delta) := \int_{\Gamma} d\mu(\gamma)\Pi^{\sigma,\Phi}_{\Lambda}(\gamma, \Delta),\ \Delta \in \mathcal{B}(\Gamma) \tag{2.6}$$

and (2.5) are called Dobrushin-Landford-Ruelle (DLR) equations.

3 Intrinsic geometry on Poisson space

We recall some results to be used below from [AKR98a], [AKR96b] to which we refer for the corresponding proofs and more details.

A homeomorphism $\psi : X \to X$ defines a transformation of Γ by

$$\Gamma \ni \gamma \mapsto \psi(\gamma) = \{\psi(x) | x \in \gamma\} = \sum_{x \in \gamma} \varepsilon_{\psi(x)}.$$

Any vector field $v \in V_0(X)$ (i.e., the set of all smooth vector fields on X with compact support) defines (via the exponential mapping) a one-parameter group ψ_t^v, $t \in \mathbb{R}$, of diffeomorphisms of X.

Definition 3.1 *For $F : \Gamma \to \mathbb{R}$ we define the directional derivative along the vector field v as (provided the right hand side exists)*

$$(\nabla_v^\Gamma F)(\gamma) := \frac{d}{dt} F(\psi_t^v(\gamma))|_{t=0}.$$

This definition applies to F in the following class $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ of so-called smooth cylinder functions. Let $\mathcal{D} := C_0^\infty(X)$ (the set of all smooth functions on X with compact support). We define $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as the set of all functions on Γ of the form

$$F(\gamma) = g_F(\langle \gamma, \varphi_1 \rangle, \ldots, \langle \gamma, \varphi_N \rangle), \ \gamma \in \Gamma, \tag{3.1}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and g_F is from $C_b^\infty(\mathbb{R}^N)$. Clearly, $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is dense in $L^2(\pi_\sigma) := L^2(\Gamma, \pi_\sigma)$. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have

$$(\nabla_v^\Gamma F)(\gamma) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \gamma, \varphi_1 \rangle, \ldots, \langle \gamma, \varphi_N \rangle) \langle \gamma, \nabla_v^X \varphi_i \rangle, \tag{3.2}$$

where $x \mapsto (\nabla_v^X \varphi)(x) = \langle \nabla^X \varphi(x), v(x) \rangle_{TX}$ is the usual directional derivative on X along the vector field v and ∇^X denotes the gradient on X.

The logarithmic derivative of the measure σ is given by the vector field $\beta^\sigma := \nabla^X \log \rho = \nabla^X \rho / \rho$ (where $\beta^\sigma = 0$ on $\{\rho = 0\}$). Then the logarithmic derivative of σ along v is the function $x \mapsto \beta_v^\sigma(x) = \langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \operatorname{div}^X v(x)$, where div^X denotes the divergence on X w.r.t. the volume element m. Analogously, we define $\operatorname{div}_\sigma^X$ as the divergence on X w.r.t. σ, i.e., $\operatorname{div}_\sigma^X$ is the dual operator on $L^2(\sigma) := L^2(X, \sigma)$ of ∇^X.

Definition 3.2 *For any $v \in V_0(X)$ we define the logarithmic derivative of π_σ along v as the following function on Γ :*

$$\Gamma \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) := \langle \beta_v^\sigma, \gamma \rangle = \int_X [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \operatorname{div}^X v(x)] d\gamma(x). \tag{3.3}$$

Theorem 3.3 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and any $v \in V_0(X)$ the following integration by parts formula for π_σ holds:*

$$\int_\Gamma \nabla_v^\Gamma F G d\pi_\sigma = -\int_\Gamma F \nabla_v^\Gamma G d\pi_\sigma - \int_\Gamma F G B_v^{\pi_\sigma} d\pi_\sigma, \tag{3.4}$$

or $(\nabla_v^\Gamma)^* = -\nabla_v^\Gamma - B_v^{\pi_\sigma}$, as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ in $L^2(\pi_\sigma)$.

Definition 3.4 *We introduce the tangent space $T_\gamma\Gamma$ to the configuration space Γ at the point $\gamma \in \Gamma$ as the Hilbert space of γ-square-integrable sections (measurable vector fields) $V : X \to TX$ with scalar product $\langle V^1, V^2 \rangle_{T_\gamma\Gamma} = \int_X \langle V^1(x), V^2(x) \rangle_{T_xX} d\gamma(x)$, $V^1, V^2 \in T_\gamma\Gamma = L^2(X \to TX; \gamma)$. The corresponding tangent bundle is denoted by $T\Gamma$.*

The intrinsic gradient of a function $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is a mapping $\Gamma \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma\Gamma$ such that $(\nabla_v^\Gamma F)(\gamma) = \langle \nabla^\Gamma F(\gamma), v \rangle_{T_\gamma\Gamma}$ for any $v \in V_0(X)$. Furthermore, by (3.2), if F is given by (3.1), we have for $\gamma \in \Gamma$, $x \in X$

$$(\nabla^\Gamma F)(\gamma; x) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \nabla^X \varphi_i(x). \tag{3.5}$$

Definition 3.5 *For a measurable vector field $V : \Gamma \to T\Gamma$ the divergence $\operatorname{div}_{\pi_\sigma}^\Gamma V$ is defined via the duality relation for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by*

$$\int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma) = -\int_\Gamma F(\gamma)(\operatorname{div}_{\pi_\sigma}^\Gamma V)(\gamma) d\pi_\sigma(\gamma), \tag{3.6}$$

provided it exists (i.e., provided

$$F \mapsto \int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma\Gamma} d\pi_\sigma(\gamma)$$

is continuous on $L^2(\pi_\sigma)$).

Proposition 3.6 *For any vector field $V = G\,v$, where $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, $v \in V_0(X)$ we have*

$$(\operatorname{div}_{\pi_\sigma}^\Gamma V)(\gamma) = \langle (\nabla^\Gamma G)(\gamma), v \rangle_{T_\gamma\Gamma} + G(\gamma) B_v^{\pi_\sigma}(\gamma). \tag{3.7}$$

For any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we introduce the *pre-Dirichlet form* which is generated by the intrinsic gradient ∇^Γ as

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F,G) = \int_\Gamma \langle (\nabla^\Gamma F)(\gamma), (\nabla^\Gamma G)(\gamma)\rangle_{T_\gamma \Gamma} d\pi_\sigma(\gamma). \tag{3.8}$$

We will also need the *classical pre-Dirichlet* form for the intensity measure σ which is given as $\mathcal{E}_\sigma^X(\varphi,\psi) = \int_X \langle \nabla^X \varphi, \nabla^X \psi\rangle_{TX} d\sigma$ for any $\varphi, \psi \in \mathcal{D}$. This form is associated with the *Dirichlet operator* H_σ^X which is given on $\mathcal{D}$ by $H_\sigma^X \varphi(x) := -\triangle^X \varphi(x) - \langle \beta^\sigma(x), \nabla^X \varphi(x)\rangle_{T_x X}$ and which satisfies $\mathcal{E}_\sigma^X(\varphi,\psi) = (H_\sigma^X \varphi, \psi)_{L^2(\sigma)}$, $\varphi, \psi \in \mathcal{D}$, see e.g. [BK95] and [MR92].

For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, $(\nabla^\Gamma \nabla^\Gamma F)(\gamma, x, y) \in T_\gamma\Gamma \otimes T_\gamma \Gamma$ and we can define the Γ-*Laplacian* $(\triangle^\Gamma F)(\gamma) := \mathrm{Tr}(\nabla^\Gamma\nabla^\Gamma F)(\gamma) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. We introduce a differential operator in $L^2(\pi_\sigma)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by the formula

$$(H_{\pi_\sigma}^\Gamma F)(\gamma) = -\triangle^\Gamma F(\gamma) - \langle \mathrm{div}_\sigma^X (\nabla^\Gamma F)(\gamma, \cdot), \gamma\rangle. \tag{3.9}$$

Theorem 3.7 *The operator $H_{\pi_\sigma}^\Gamma$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_\sigma}^\Gamma$, i.e.,*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F,G) = (H_{\pi_\sigma}^\Gamma F, G)_{L^2(\pi_\sigma)}, \tag{3.10}$$

or $H_{\pi_\sigma}^\Gamma = -\mathrm{div}_{\pi_\sigma}^\Gamma \nabla^\Gamma$ on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. We call $H_{\pi_\sigma}^\Gamma$ the intrinsic Dirichlet operator of the measure π_σ.

4 Extrinsic geometry on Poisson space

We recall the extrinsic geometry on $L^2(\pi_\sigma)$ based on the isomorphism with the Fock space. Our approach is based on [KSS97] but we should also mention [BLL95], [IK88], [KSSU98], [NV95], [Pri95] for related considerations and references therein. For proofs of the results stated below in this section, we refer to [AKR98a, Sect. 5].

Let us define another "gradient" on functions $F : \Gamma \to \mathbb{R}$ This gradient ∇^P has specific useful properties on Poissonian spaces. We will call ∇^P the *Poissonian gradient.* To this end we consider as the tangent space to Γ at any point $\gamma \in \Gamma$ the same space $L^2(\sigma)$ and define a mapping $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \ni F \mapsto \nabla^P F \in L^2(\pi_\sigma) \otimes L^2(\sigma)$ by

$$(\nabla^P F)(\gamma, x) := F(\gamma + \varepsilon_x) - F(\gamma),\ \gamma \in \Gamma, x \in X. \tag{4.1}$$

We stress that the transformation $\Gamma \ni \gamma \mapsto \gamma + \varepsilon_x \in \Gamma$ is π_σ-a.e. well-defined because $\pi_\sigma(\{\gamma \in \Gamma | x \in \gamma\}) = 0$ for any $x \in X$. The directional derivative is then defined as

$$(\nabla^P_\varphi F)(\gamma) = (\nabla^P F(\gamma, \cdot), \varphi)_{L^2(\sigma)} = \int_X [F(\gamma + \varepsilon_x) - F(\gamma)]\varphi(x) d\sigma(x),\ \varphi \in \mathcal{D}. \tag{4.2}$$

The Poissonian gradient ∇^P yields (via a corresponding "integration by parts" formula) an orthogonal system of Charlier polynomials on the Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$.

For any $n \in \mathbb{N}$ and all $\varphi \in \mathcal{D}$ we introduce a function in $L^2(\pi_\sigma)$ by

$$Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) := ((\nabla^P_\varphi)^{*n} 1)(\gamma), \tag{4.3}$$

and define $Q_0^{\pi_\sigma} := 1$. Due to the kernel theorem [BK95, Chap. 1] these functions have the representation $Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) = \langle Q_n^{\pi_\sigma}(\gamma), \varphi^{\otimes n}\rangle$, with generalized symmetric kernels $\Gamma \ni \gamma \mapsto Q_n^{\pi_\sigma}(\gamma) \in \mathrm{Exp}_n \mathcal{D}'$, $n \in \mathbb{N}$. Here and below by $\mathrm{Exp}_n E$ we denote the n-th symmetric tensor power of a linear space E. Then for any smooth kernel $\varphi^{(n)} \in \mathrm{Exp}_n \mathcal{D}^{\otimes n}$ we introduce the function $Q_n^{\pi_\sigma}(\gamma; \varphi^{(n)}) := \langle Q_n^{\pi_\sigma}(\gamma), \varphi^{(n)}\rangle$ such that for all $\varphi^{(n)} \in \mathrm{Exp}_n \mathcal{D}^{\otimes n}$, $\psi^{(m)} \in \mathrm{Exp}_m \mathcal{D}^{\otimes m}$

$$\int_\Gamma Q_n^{\pi_\sigma}(\gamma; \varphi^{(n)}) Q_m^{\pi_\sigma}(\gamma; \psi^{(m)}) d\pi_\sigma(\gamma) = \delta_{nm} n! (\varphi^{(n)}, \psi^{(m)})_{L^2(\sigma^{\otimes n})}. \tag{4.4}$$

Hence (4.3) extends to the case of kernels from the so-called n-particle Fock space $\mathrm{Exp}_n L^2(\sigma)$, $n \in \mathbb{N}$, and we set $\mathrm{Exp}_0 L^2(\sigma) := \mathbb{R}$.

As usual the symmetric Fock space over the Hilbert space $L^2(\sigma)$ is defined as $\mathrm{Exp} L^2(\sigma) := \oplus_{n=0}^\infty \mathrm{Exp}_n L^2(\sigma)$, see e.g. [BK95] and [HKPS93]. The square of the norm of a vector $(f^{(n)})_{n=0}^\infty \in \mathrm{Exp} L^2(\sigma)$ is given by $\sum_{n=0}^\infty n! \left\| f^{(n)} \right\|^2_{L^2(\sigma^{\otimes n})}$. For any $F \in L^2(\pi_\sigma)$ there exists such a Fock vector, so that we have the following chaos decomposition

$$F(\gamma) = \sum_{n=0}^\infty Q_n^{\pi_\sigma}(\gamma; f^{(n)}), \tag{4.5}$$

and moreover $\|F\|^2_{L^2(\pi_\sigma)} = \sum_{n=0}^\infty n! \left\| f^{(n)} \right\|^{2.}_{L^2(\sigma^{\otimes n})}$. And vice versa, by (4.5) any Fock vector generates a function from $L^2(\pi_\sigma)$. This produces an isomorphism between $L^2(\pi_\sigma)$ and $\mathrm{Exp} L^2(\sigma)$.

Remark 4.1 *In probability theory the functions $Q_n^{\pi_\sigma}(\gamma; f^{(n)})$ are called the n-multiple stochastic integrals of $f^{(n)}$ with respect to the compensated Poisson process generated by the Poisson measure π_σ, see e.g. [NV95].*

There is an alternative approach to the chaos decomposition on the Poisson space which uses the concept of generalized Appell systems, see e.g. [KSS97].

The following proposition shows that the operators ∇_φ^P and ∇_φ^{P*} play the role of the annihilation resp. creation operators in the Fock space $\mathrm{Exp}L^2(\sigma)$.

Proposition 4.2 *For all $\varphi, \psi \in \mathcal{D}$, $n \in \mathbb{N}$ the following formulas hold*

$$\nabla_\psi^P Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) = n(\varphi, \psi)_{L^2(\sigma)} Q_{n-1}^{\pi_\sigma}(\gamma; \varphi^{\otimes(n-1)}) \tag{4.6}$$

$$\nabla_\psi^{P*} Q_n^{\pi_\sigma}(\gamma; \varphi^{\otimes n}) = Q_{n+1}^{\pi_\sigma}(\gamma; \varphi^{\otimes n} \hat{\otimes} \psi), \ \gamma \in \Gamma, \tag{4.7}$$

where $\varphi^{\otimes n} \hat{\otimes} \psi$ means the symmetric tensor product of $\varphi^{\otimes n}$ and ψ.

Next we give an explicit expression for the adjoint of the Poissonian gradient ∇^{P*}.

Proposition 4.3 *For any function $F \in L^1(\pi_\sigma) \otimes L^1(\sigma)$ we have $F \in D(\nabla^{P*})$ and the following equality holds*

$$(\nabla^{P*}F)(\gamma) = \int_X F(\gamma - \varepsilon_x, x) d\gamma(x) - \int_X F(\gamma, x) d\sigma(x), \ \gamma \in \Gamma, \tag{4.8}$$

provided the right hand side of (4.8) is in $L^2(\pi_\sigma)$.

Proof. For $X = \mathbb{R}^d$ this proposition was proved in [KSSU98]. Let $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ be given. Then an application of (4.1) gives

$$\begin{aligned}(\nabla^P G, F)_{L^2(\pi_\sigma)\otimes L^2(\sigma)} &= \int_X \int_\Gamma G(\gamma + \varepsilon_x) F(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) \\ &\quad - \int_X \int_\Gamma G(\gamma) F(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x).\end{aligned} \tag{4.9}$$

Now we use the Mecke identity, see e.g., [Mec67, Satz 3.1]

$$\int_\Gamma \left(\int_X h(\gamma, x) d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_X \int_\Gamma h(\gamma + \varepsilon_x, x) d\pi_\sigma(\gamma) d\sigma(x), \tag{4.10}$$

where h is any non-negative, $\mathcal{B}(\Gamma)\times\mathcal{B}(X)$-measurable function. By (4.10) the right hand side of (4.9) transforms into

$$\int_\Gamma G(\gamma)\left[\int_X F(\gamma-\varepsilon_x,x)d\gamma(x)-\int_X F(\gamma,x)d\sigma(x)\right]d\pi_\sigma(\gamma),$$

which proves the proposition. ■

For any contraction B in $L^2(\sigma)$ it is possible to define an operator Exp$B$ as a contraction in Exp$L^2(\sigma)$ which in any n-particle subspace $\mathrm{Exp}_n L^2(\sigma)$ is given by $B\otimes\cdots\otimes B$ (n times). For any positive self-adjoint operator A in $L^2(\sigma)$ (with $\mathcal{D}\subset D(A)$) we have a contraction semigroup e^{-tA}, $t\ge 0$, hence it is possible to introduce the second quantization operator dExp$A$ as the generator of the semigroup $\mathrm{Exp}(e^{-tA})$, $t\ge 0$, i.e., $\mathrm{Exp}(e^{-tA})=\exp(-td\mathrm{Exp}A)$, see e.g., [RS75]. We denote by H^P_A the image of the operator dExp$A$ in the Poisson space $L^2(\pi_\sigma)$ under the described isomorphism.

Proposition 4.4 *Let $\mathcal{D}\subset D(A)$. Then the symmetric bilinear form corresponding to the operator H^P_A has the following form*

$$(H^P_AF,G)_{L^2(\pi_\sigma)}=\int_\Gamma(\nabla^PF(\gamma),A\nabla^PG(\gamma))_{L^2(\sigma)}d\pi_\sigma(\gamma),\ F,G\in\mathcal{F}C^\infty_b(\mathcal{D},\Gamma). \tag{4.11}$$

The right hand side of (4.11) is called the "Poissonian pre-Dirichlet form" with coefficient operator A and is denoted by $\mathcal{E}^P_{\pi_\sigma,A}$.

Let us consider the special case of the second quantization operator dExp$A$, where the one-particle operator A coincides with the Dirichlet operator H^X_σ generated by the measure σ on X. Then we have the following theorem which relates the intrinsic Dirichlet operator $H^\Gamma_{\pi_\sigma}$ and the operator $H^P_{H^X_\sigma}$.

Theorem 4.5 $H^\Gamma_{\pi_\sigma}=H^P_{H^X_\sigma}$ *on $\mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$. In particular, for all $F,G\in\mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$*

$$\begin{aligned}&\int_\Gamma\langle\nabla^\Gamma F(\gamma),\nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma}d\pi_\sigma(\gamma)\\ =&\int_\Gamma(\nabla^PF(\gamma),H^X_\sigma\nabla^PG(\gamma))_{L^2(\sigma)}d\pi_\sigma(\gamma)\\ =&\int_\Gamma\int_X\langle\nabla^X\nabla^PF(\gamma,x),\nabla^X\nabla^PG(\gamma,x)\rangle_{T_xX}d\sigma(x)d\pi_\sigma(\gamma).\end{aligned}\tag{4.12}$$

Remark 4.6 *Theorem 4.5 gives full information about the spectrum of the intrinsic Dirichlet operator* $H^{\Gamma}_{\pi_\sigma}$ *on the Poisson space in terms of the underlying Dirichlet operator* H^X_σ *coming from the intensity measure* σ*. In the following section we obtain an analogue of Theorem 4.5 for the class of measures* $\mathcal{G}^1_{gc}(\sigma,\Phi)$ *(see (6.5) below) which is the aim of this paper.*

5 Relation between intrinsic and extrinsic Dirichlet forms

Here we consider the class of measures $\mathcal{G}^1_{gc}(\sigma,\Phi)$ consisting of all $\mu \in \mathcal{G}_{gc}(\sigma,\Phi)$ such that

$$\int_\Gamma \gamma(K) d\mu(\gamma) < \infty \text{ for all compact } K \subset X.$$

We define for any $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$ the pre-Dirichlet form $\mathcal{E}^\Gamma_\mu$ by

$$\mathcal{E}^\Gamma_\mu(F,G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\mu(\gamma),\ F,G \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma). \tag{5.1}$$

After all our preparations we are now going to prove an analogue of (4.12) for $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$. We would like to emphasize that the corresponding formula (5.2) is not obtained from (4.12) by just replacing π_σ by $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$. The essential difference is, in addition, an extra factor involving the conditional energy E^Φ_Λ.

Theorem 5.1 *For any* $\mu \in \mathcal{G}^1_{gc}(\sigma,\Phi)$*, we have for all* $F,G \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$

$$\begin{aligned}\mathcal{E}^\Gamma_\mu(F,G) &= \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma} d\mu(\gamma) \\ &= \int_\Gamma\int_X \langle \nabla^X\nabla^P F(\gamma,x), \nabla^X\nabla^P G(\gamma,x)\rangle_{T_xX}\, e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)} d\sigma(x) d\mu(\gamma)\end{aligned} \tag{5.2}$$

Proof. Let us take any $F \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$ of the form (3.1). Then given $\gamma \in \Gamma$ and $x \in X$ (4.1) implies that

$$\begin{aligned}\nabla^X\nabla^P F(\gamma,x) &= \nabla^X F(\gamma+\varepsilon_x) \\ &= \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle\varphi_1,\gamma\rangle + \varphi_1(x), \ldots, \langle\varphi_N,\gamma\rangle+\varphi_N(x))\nabla^X\varphi_i(x).\end{aligned}$$

Let us define $\hat{F}_i(\gamma) := \frac{\partial g_F}{\partial s_i}(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle)$, $i=1,\ldots,N$. Obviously, it is enough to prove the equality (5.2) for $F=G$. Thus, inserting the result of $\nabla^X\nabla^P F(\gamma,x)$ into the right hand side of (5.2) we obtain

$$\int_\Gamma\int_X \sum_{i,j=1}^N \langle\nabla^X\varphi_i(x),\nabla^X\varphi_j(x)\rangle_{T_xX}\hat{F}_i(\gamma+\varepsilon_x)\hat{F}_j(\gamma+\varepsilon_x)e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)}d\sigma(x)d\mu(\gamma). \tag{5.3}$$

Then we need the following useful proposition which generalizes the Mecke identity to measures in $\mathcal{G}_{gc}(\sigma,\Phi)$, see [NZ79], [MMW79].

Proposition 5.2 *Let $h:\Gamma\times X\to\mathbb{R}_+$ be $\mathcal{B}(\Gamma)\times\mathcal{B}(X)$-measurable, and let $\mu\in\mathcal{G}_{gc}(\sigma,\Phi)$. Then we have*

$$\int_\Gamma\left(\int_X h(\gamma,x)d\gamma(x)\right)d\mu(\gamma) = \int_X\int_\Gamma h(\gamma+\varepsilon_x,x)e^{-E^\Phi_{\{x\}}(\gamma+\varepsilon_x)}d\mu(\gamma)d\sigma(x). \tag{5.4}$$

Using this proposition we transform (5.3) into

$$\int_\Gamma \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle\nabla^X\varphi_i(\cdot),\nabla^X\varphi_j(\cdot)\rangle_{TX},\gamma\rangle d\mu(\gamma).$$

On the other hand using (3.5) we obtain

$$\langle\nabla^\Gamma F(\gamma),\nabla^\Gamma G(\gamma)\rangle_{T\Gamma} = \sum_{i,j=1}^N \hat{F}_i(\gamma)\hat{F}_j(\gamma)\langle\langle\nabla^X\varphi_i(\cdot),\nabla^X\varphi_j(\cdot)\rangle_{TX},\gamma\rangle.$$

Therefore the equality on the dense $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ is valid which proves the theorem. ■

Remark 5.3 *For the reader's convenience let us give a heuristic proof of the Nguyen-Zessin characterization of Gibbs measures in (5.4) which really is a consequence of the Mecke identity (cf. (4.10)). Indeed, let us write (heuristically)*

$$d\mu(\gamma) = \frac{1}{Z^{\sigma,\Phi}}e^{-E^\Phi(\gamma)}d\pi_\sigma(\gamma).$$

Then the function $E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x) = E^{\Phi}(\gamma+\varepsilon_x) - E^{\Phi}(\gamma)$ informally is the variation of the potential energy $E^{\Phi}(\gamma)$ when we add to the configuration γ an additional point $x \in X$. Using this representation we have

$$\begin{aligned}
&\int_X \int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x)} d\mu(\gamma) d\sigma(x) \\
= \quad & (Z^{\sigma,\Phi})^{-1} \int_X \int_\Gamma h(\gamma+\varepsilon_x, x) e^{-E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x)} d\pi_\sigma(\gamma) d\sigma(x).
\end{aligned}$$

Then we use the Mecke identity to transform the right hand side of the above equality into

$$(Z^{\sigma,\Phi})^{-1} \int_\Gamma \left(\int_X h(\gamma, x) e^{-E^{\Phi}(\gamma)} d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_\Gamma \left(\int_X h(\gamma, x) d\gamma(x) \right) d\mu(\gamma).$$

The rigorous proof in [NZ79] is obtained as a formalization of the heuristic computations above.

6 Closability of intrinsic Dirichlet forms

In this section we will prove the closability of the intrinsic Dirichlet form $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C^{\infty}_b(\mathcal{D}, \Gamma))$ on $L^2(\mu) := L^2(\Gamma, \mu)$ for all $\mu \in \mathcal{G}^1_{gc}(\sigma, \Phi)$, using the integral representation (5.2) in Theorem 5.1. The closability of $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C^{\infty}_b(\mathcal{D}, \Gamma))$ over Γ is implied by the closability of an appropriate family of pre-Dirichlet forms over X. Let us describe this more precisely.

We define new intensity measures on X by $d\sigma_\gamma(x) := \rho_\gamma(x) dm(x)$, where

$$\rho_\gamma(x) := e^{-E^{\Phi}_{\{x\}}(\gamma+\varepsilon_x)} \rho(x), \; x \in X, \gamma \in \Gamma \tag{6.1}$$

It was shown in [AR90, Theorem 5.3] (in the case $X = \mathbb{R}^d$) that the components of the Dirichlet form $(\mathcal{E}^X_{\sigma_\gamma}, \mathcal{D}^{\sigma_\gamma})$ corresponding to the measure σ_γ are closable on $L^2(\mathbb{R}^d, \sigma_\gamma)$ if and only if σ_γ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and the Radon-Nikodym derivative satisfies some condition, see (6.2) below for details. This result allows us to prove the closability of $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C^{\infty}_b(\mathcal{D}, \Gamma))$ on $L^2(\mu)$. Let us first recall the above mentioned result.

Theorem 6.1 *(cf. [AR90, Theorem 5.3]) Let ν by a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \in \mathbb{N}$ and let $\mathcal{D}^\nu$ denote the ν-classes determined by $\mathcal{D}$. Then the forms $(\mathcal{E}_{\nu,i}, \mathcal{D}^\nu)$ defined by*

$$\mathcal{E}_{\nu,i}(u,v) := \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} d\nu,\ u,v \in \mathcal{D},$$

are well-defined and closable on $L^2(\mathbb{R}^d, \nu)$ for $1 \le i \le d$ if and only if ν is absolutely continuous with respect to Lebesgue measure λ^d on $\mathbb{R}^d$, and the Radon-Nikodym derivative $\rho = d\nu/d\lambda^d$ satisfies the condition:

$$\begin{aligned}&\text{for any } 1 \le i \le d \text{ and } \lambda^{d-1}\text{–a.e. } x \in \left\{ y \in \mathbb{R}^{d-1} | \int_{\mathbb{R}} \rho_y^{(i)}(s) d\lambda^1(s) > 0 \right\},\\ &\rho_x^{(i)} = 0\ \lambda^1\text{–a.e. on } \mathbb{R}\backslash R(\rho_x^{(i)}), \text{ where } \rho_x^{(i)}(s) := \rho(x_1, \ldots, x_{i-1}, s, x_i, \ldots, x_d),\\ &s \in \mathbb{R}, \text{ if } x = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}, \text{ and where}\end{aligned} \tag{6.2}$$

$$R(\rho_x^{(i)}) := \left\{ t \in \mathbb{R} | \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho_x^{(i)}(s)} ds < \infty \text{ for some } \varepsilon > 0 \right\}. \tag{6.3}$$

There is an obvious generalization of Theorem 6.1 to the case where a Riemannian manifold X is replacing $\mathbb{R}^d$, to be formulated in terms of local charts. Since here we are only interested in the "if part" of Theorem 6.1, we now recall a slightly weaker sufficient condition for closability in the general case where X is a manifold as before.

Theorem 6.2 *Suppose $\sigma_1 = \rho_1 \cdot m$, where $\rho_1 : X \to \mathbb{R}_+$ is $\mathcal{B}(X)$-measurable such that*

$$\rho_1 = 0\ m\text{–a.e. on } X \backslash \left\{ x \in X | \int_{\Lambda_x} \frac{1}{\rho_1} dm < \infty \text{ for some open neighbourhood } \Lambda_x \text{ of } x \right\}. \tag{6.4}$$

Then $(\mathcal{E}_{\sigma_1}^X, \mathcal{D}^{\sigma_1})$ defined by

$$\mathcal{E}_{\sigma_1}^X(u,v) := \int_X \langle \nabla^X u(x), \nabla^X v(x) \rangle_{T_x X}\, d\sigma_1(x);\ u,v \in \mathcal{D},$$

is closable on $L^2(\sigma_1)$.

The proof is a straightforward adaptation of the line of arguments in [MR92, Chap. II, Subsection 2a] (see also [ABR89, Theorem 4.2] for details). We emphasize that (6.4) e.g. always holds, if ρ_1 is lower semicontinuous, and that neither ν in Theorem 6.1 nore σ_1 in Theorem 6.2 is required to have full support, so e.g. ρ_1 is not necessarily strictly positive m-a.e. on X.

We are now ready to prove the closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu)$ under the above assumption.

Theorem 6.3 *Let $\mu \in \mathcal{G}_{gc}^1(\sigma,\Phi)$. Suppose that for μ-a.e. $\gamma \in \Gamma$ the function ρ_γ defined in (6.1) satisfies (6.4) (resp. (6.2) in case $X = \mathbb{R}^d$). Then the form $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ is closable on $L^2(\mu)$.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ such that $F_n \to 0$, $n \to \infty$ in $L^2(\mu)$ and

$$\mathcal{E}_\mu^\Gamma(F_n - F_m, F_n - F_m) \underset{n,m\to\infty}{\longrightarrow} 0. \tag{6.5}$$

We have to show that

$$\mathcal{E}_\mu^\Gamma(F_{n_k}, F_{n_k}) \underset{k\to\infty}{\longrightarrow} 0 \tag{6.6}$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$. Let $(n_k)_{k\in\mathbb{N}}$ be a subsequence such that

$$\left(\int_\Gamma F_{n_k}^2 d\mu\right)^{1/2} + \mathcal{E}_\mu^\Gamma(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

Then

$$\begin{aligned}
\infty &> \sum_{k=1}^\infty \mathcal{E}_\mu^\Gamma(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} \\
&\geq \sum_{k=1}^\infty \int_\Gamma \left(\int_X |\nabla^X\nabla^P(F_{n_{k+1}} - F_{n_k})(x,\gamma)|_{T_xX}^2\, e^{-E_{\{x\}}^\Phi(\gamma+\varepsilon_x)} d\sigma(x)\right)^{1/2} d\mu(\gamma) \\
&= \int_\Gamma \sum_{k=1}^\infty \left(\int_X |\nabla^X\nabla^P(F_{n_{k+1}} - F_{n_k})(x,\gamma)|_{T_xX}^2\, \rho_\gamma(x) dm(x)\right)^{1/2} d\mu(\gamma),
\end{aligned}$$

where we used Theorem 5.1 and (6.1). From the last expression we obtain that

$$\begin{aligned}
&\sum_{k=1}^\infty \mathcal{E}_{\sigma_\gamma}^X(u_{n_{k+1}}^{(\gamma)} - u_{n_k}^{(\gamma)}, u_{n_{k+1}}^{(\gamma)} - u_{n_k}^{(\gamma)})^{1/2} \qquad (6.7)\\
&= \sum_{k=1}^\infty \left(\int_X |\nabla^X\nabla^P(F_{n_{k+1}} - F_{n_k})(x,\gamma)|_{T_xX}^2 \rho_\gamma(x) dm(x)\right)^{1/2} < \infty \text{ for } \mu\text{–a.e. } \gamma \in \Gamma,
\end{aligned}$$

where for $k \in \mathbb{N}$, $\gamma \in \Gamma$,

$$u_{n_k}^{(\gamma)}(x) := F_{n_k}(\gamma + \varepsilon_x) - F_{n_k}(\gamma),\ x \in X.$$

Note that $u_{n_k}^{(\gamma)} \in \mathcal{D}$. (6.7) implies that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}_{\sigma_\gamma}^X(u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}, u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}) \underset{k,l\to\infty}{\longrightarrow} 0. \tag{6.8}$$

Let $\Lambda \subset \mathcal{O}_c(X)$.

Claim 1: For μ-a.e. $\gamma \in \Gamma$

$$\int_X (u_{n_k}^{(\gamma)}(x))^2 1\!\!1_\Lambda(x) d\sigma_\gamma(x) \underset{k\to\infty}{\longrightarrow} 0.$$

To prove Claim 1 we first note that for μ-a.e. $\gamma \in \Gamma$

$$\sigma_\gamma(\Lambda) < \infty,$$

as follows immediately from (5.4) (taking $h(\gamma, x) := 1\!\!1_\Lambda(x)$ for $x \in X$, $\gamma \in \Gamma$), since $\mu \in \mathcal{G}_{gc}^1(\sigma, \Phi)$. Therefore, for μ-a.e. $\gamma \in \Gamma$

$$\int_X F_{n_k}^2(\gamma) 1\!\!1_\Lambda(x) d\sigma_\gamma(x) = F_{n_k}^2(\gamma)\sigma_\gamma(\Lambda) \underset{k\to\infty}{\longrightarrow} 0. \tag{6.9}$$

Furthermore, by (5.4)

$$\begin{aligned} & \int_\Gamma \int_X F_{n_k}^2(\gamma + \varepsilon_x) 1\!\!1_\Lambda(x) d\sigma_\gamma(x)(1 + \gamma(\Lambda))^{-1} d\mu(\gamma) \\ = & \int_\Gamma F_{n_k}^2(\gamma) \int_X \frac{1\!\!1_\Lambda(x)}{1 + \gamma(\Lambda) - 1\!\!1_\Lambda(x)} \gamma(dx) d\mu(\gamma) \\ \leq & \int_\Gamma F_{n_k}^2(\gamma) d\mu(\gamma) < \frac{1}{2^k}, \end{aligned}$$

because the integral w.r.t. γ is dominated by 1 for all $\gamma \in \Gamma$. Hence

$$\begin{aligned} \infty > & \sum_{k=1}^\infty \left(\int_\Gamma \int_X F_{n_k}^2(\gamma + \varepsilon_x) 1\!\!1_\Lambda(x) d\sigma_\gamma(x)(1 + \gamma(\Lambda))^{-1} d\mu(\gamma) \right)^{1/2} \\ \geq & \int_\Gamma \sum_{k=1}^\infty \left(\int_X F_{n_k}^2(\gamma + \varepsilon_x) 1\!\!1_\Lambda(x) d\sigma_\gamma(x) \right)^{1/2} (1 + \gamma(\Lambda))^{-1} d\mu(\gamma). \end{aligned}$$

Therefore, for μ-a.e. $\gamma \in \Gamma$

$$\int_X F_{n_k}^2(\gamma + \varepsilon_x) 1\!\!1_\Lambda(x) d\sigma_\gamma(x) \underset{k\to\infty}{\longrightarrow} 0. \tag{6.10}$$

Then Claim 1 follows by (6.9) and (6.10).

Claim 2: For μ-a.e. $\gamma \in \Gamma$

$$|\nabla^X u_{n_k}^{(\gamma)}|_{TX} \underset{k\to\infty}{\longrightarrow} 0 \quad \sigma_\gamma\text{−a.e.}$$

To prove Claim 2 we first note that clearly (6.7) implies that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}^X_{1\!\!1_\Lambda \sigma_\gamma}(u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}, u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}) \underset{k,l\to\infty}{\longrightarrow} 0. \tag{6.11}$$

Hence we can apply Theorem 6.2 (resp. 6.1) to $\rho_1 := 1\!\!1_\Lambda \rho_\gamma$ and conclude by Claim 1 and (6.11) that for μ-a.e. $\gamma \in \Gamma$

$$\mathcal{E}^X_{1\!\!1_\Lambda \sigma_\gamma}(u_{n_k}^{(\gamma)}, u_{n_k}^{(\gamma)}) \underset{k\to\infty}{\longrightarrow} 0,$$

hence by (6.7)

$$1\!\!1_\Lambda |\nabla^X u_{n_k}^{(\gamma)}|_{TX} \underset{k\to\infty}{\longrightarrow} 0 \quad \sigma_\gamma\text{−a.e.}$$

Since Λ was arbitrary, Claim 2 is proven.

From Claim 2 we now easily deduce (6.6) by (5.2) and Fatou's Lemma as follows:

$$\begin{aligned} \mathcal{E}^\Gamma_\mu(F_{n_k}, F_{n_k}) &\leq \int_\Gamma \liminf_{l\to\infty} \int_X |\nabla^X (u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)})|^2_{TX}\, d\sigma_\gamma(x) d\mu(\gamma) \\ &\leq \liminf_{l\to\infty} \mathcal{E}^\Gamma_\mu(F_{n_k} - F_{n_l}, F_{n_k} - F_{n_l}), \end{aligned}$$

which by (6.5) can be made arbitrarily small for k large enough. ■

Remark 6.4 *The above method to prove closability of pre-Dirichlet forms on configuration spaces Γ_X extends immediately to the case where X is replaced by an infinite dimensional "manifold" such as the loop space (cf. [MR97]).*

Example 6.5 *Let $X = \mathbb{R}^d$ with the Euclidean metric and $\sigma := z \cdot m$, $z \in (0, \infty)$. A pair potential is a $\mathcal{B}(\mathbb{R}^d)$-measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\phi(-x) = \phi(x)$. Any pair potential ϕ defines a potential*

$\Phi = \Phi_\phi$ in the sense of Section 2 as follows: we set $\Phi(\gamma) := 0$, $|\gamma| \neq 2$ and $\Phi(\gamma) := \phi(x-y)$ for $\gamma = \{x,y\} \subset \mathbb{R}^d$. For such pair potentials ϕ the condition in Theorem 6.3 ensuring closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu)$ for $\mu \in \mathcal{G}_{gc}^1(\sigma,\Phi)$ can be now easily formulated as follows: for μ-a.e. $\gamma \in \Gamma$ and m-a.e. $x \in \{y \in \mathbb{R}^d | \sum_{y' \in \gamma\setminus\{y\}} |\phi(y-y')| < \infty\}$ it holds that $\int_{V_x} e^{\sum_{y'\in\gamma\setminus\{y\}} \phi(y-y')} m(dy) < \infty$ for some open neighbourhood V_x of x. This condition trivially holds e.g. if $\mathrm{supp}\phi$ is compact, $\{\phi < \infty\}$ is open, and $\phi^+ \in L^\infty_{loc})(\{\phi < \infty\}; m)$. If even $\mu \in \mathcal{G}_{gc}^t(z,\phi)$ and ϕ satisfies the assumptions in Proposition 7.1, then it suffices to merely assume that $\{\phi < \infty\}$ is open and $\phi^+ \in L^\infty_{loc}(\{\phi < \infty\}; m)$. This follows by an elementary consideration.

Remark 6.6

1. *We emphasize that Example 6.5 generalizes the closability result in [Osa96], though an a-priori bigger domain for $\mathcal{E}_\mu^\Gamma$ is considered there. However, Theorems 6.1-6.3 are also valid for this bigger domain. The proofs are exactly the same.*

2. *We also like to emphasize that similarly to Example 6.5 one proves the closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ (or with a larger domain in [Osa96]) on $L^2(\mu)$ for $\mu \in \mathcal{G}_{gc}^1(\sigma,\Phi)$ in the case of multi-body potentials ϕ.*

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