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The Feynman integral for time-dependent anharmonic oscillators

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Abstract

We review some basic notions and results of white noise analysis that are used in the construction of the Feynman integrand as a generalized white noise functional. We show that the Feynman integrand for the time-dependent harmonic oscillator in an external potential is a Hida distribution.

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I. Introduction

Path integrals are a useful tool in many branches of theoretical physics including quantum mechanics, quantum field theory and polymer physics. We are interested in a rigorous treatment of such path integrals. As our basic example we think of a quantum mechanical particle.

On one hand it is possible to represent solutions of the heat equation by a path integral representation, based on the Wiener measure in a mathematically rigorous way. This is stated by the famous Feynman-Kac formula. On the other hand there have been a lot of attempts to write solutions of the Schrödinger equation as a Feynman (path) integral in a useful mathematical sense. The methods are always more involved and less direct than in the Euclidean (i.e. Feynman-Kac) case. Among them are analytic continuation, limits of finite dimensional approximations and Fourier transform. Instead of enumerating a comprehensive list of publications on theories concerned with Feynman integrals we refer to the method proposed by.[1] An illustrative theory has been developed there by using Fresnel integrals, and, additionally, they provide a recommendable list of supplementary references. Here we have chosen a white noise approach.

White noise analysis is a mathematical framework which offers various generalizations of concepts known from finite dimensional analysis, among them are differential operators and Fourier transform. Although we will give a brief introduction to white noise calculus in Section II the reader unfamiliar with this topic is recommended to the monographs[2,3,4,5] and the introductory articles[6,7,8,].[9]

The idea of realizing Feynman integrals within the white noise framework goes back to.[10] The "average over all paths" is performed with a Hida distribution as the weight (instead of a measure). The existence of such Hida distributions corresponding to Feynman integrands has been established in.[11] In[12] Khandekar and Streit moved beyond the existence theorem by giving an explicit construction for a large class of potentials including singular ones. Basically they constructed a strong Dyson series for the Feynman integrand in the space of Hida distributions. This approach only works for one space dimension. Their construction was generalized to (one dimensional) time-dependent potentials of non-compact support in.[13]

In[14] those ideas have been carried over to perturbations of the harmonic oscillator. Hence, instead of constructing a Dyson series around the free particle Feynman integrand they expand around the Feynman integrand of the harmonic oscillator as obtained in.[11] The external potentials to which the oscillator is submitted correspond to the wide class of time-dependent singular potentials treated in.[13]

Here we expand the considerations of[14] to oscillators with time-dependent mass and frequency. Starting from the associated Lagrangians we use the formulas derived in[15] in order to calculate the corresponding Feynman propagators. There the polygonal approach has been applied in order to integrate "over all paths". Then we integrate this propagator into the frame of white noise analysis and in Theorem 8 we show that the associated Feynman integrand is a Hida distribution. As a final result, we proved in Theorem 11 that the Feynman integrand for the time-dependent harmonic oscillator which is submitted to the same class of potentials as in[13] exists again a Hida distribution.

For example, the associated propagators enable us to study quantum mechanics of damped oscillators in a particular form of the time-dependent mass and frequency. This is treated detailed in Example 7.

In[1] the path integral of the harmonic oscillator is defined within the theory of Fresnel integrals. Compared to our ansatz this procedure has the advantage of being manifestly independent of the space dimension. Despite the lack of a generalization to higher dimensional quantum systems our construction has some interesting features:

- The admissible potentials may be very singular.
- We are not restricted to smooth initial wave functions and may thus study the propagator directly.
- Instead of giving a meaning to the Feynman *integral* we define the Feynman *integrand* as a Hida distribution. By taking expectation we get the propagator. On the other hand one may now use the toolbox of white noise analysis and apply differential operators to derive variational relations or Ehrenfest's theorem, see[3,16]

II. White Noise Analysis

In this section we do not give an account of white noise analysis. Our aim is to introduce the tools we need in this work and we refer throughout to[2,3,17] for details and proofs.

A. The triple $(S) \subset L^2(\mu) \subset (S)'$

We start with the fundamental real Gel'fand triple:

$$S(\mathbf{R}) \subset L^2(\mathbf{R}) \subset S'(\mathbf{R}),$$

where $S(\mathbf{R})$ and $S'(\mathbf{R})$ denotes the real Schwartz space of test functions and tempered distributions respectively. Via Minlos' theorem we construct a measure space $(S'(\mathbf{R}),\mathcal{B},\mu)$ called the *white noise space* by fixing the characteristic functional in the following way:

$$C(f) = \int_{S'(\mathbf{R})} \exp(i\langle w, f\rangle)\, d\mu(w) = \exp\left(-(1/2)\,|f|_0^2\right), \quad f \in S(\mathbf{R}),$$

here the dual pairing $\langle\cdot,\cdot\rangle$ of $S'(\mathbf{R})$ and $S(\mathbf{R})$ is realized as an extension of the inner product in $L^2(\mathbf{R})$, $\langle h, f\rangle := (h,f)_0$, $h \in L^2(\mathbf{R})$, $f \in S(\mathbf{R})$ and $|\cdot|_0$ denotes the norm in $L^2(\mathbf{R})$.

Within this formalism a version of Wiener's Brownian motion is given by:

$$x(t) := \langle w, \mathbf{1}_{[0,t)}\rangle = \int_0^t w(s)\, ds.$$

We now consider the space $L^2(\mu)$, which is defined to be the complex Hilbert space $L^2(S'(\mathbf{R}),\mathcal{B},\mu)$. For applications the space $L^2(\mu)$ is often too small. A convenient way to solve this problem is to introduce a space of test functionals in $L^2(\mu)$ and to use its larger dual space.

Take a system of Hilbert norms $\{|\cdot|_p\}$ topologizing $S(\mathbf{R})$ which grows sufficiently fast. Then $S(\mathbf{R})$ is realized as a projective limit of Hilbert spaces $S_p(\mathbf{R})$:

$$S(\mathbf{R}) = \bigcap_{p\geq 0} S_p(\mathbf{R}),$$

where $S_p(\mathbf{R})$ denotes the completion of $S(\mathbf{R})$ with respect to $|\cdot|_p$. Then the space of tempered distributions is

$$S'(\mathbf{R}) = \bigcup_{p\geq 0} S_{-p}(\mathbf{R}),$$

where the dual norm $|\cdot|_{-p}$ topologizes the Hilbert space $S_{-p}(\mathbf{R})$.

One convenient choice is

$$|f|_p := |A^p f|_0 , \quad f \in S(\mathbf{R}),$$

where

$$Af(t) = -f''(t) + \left(t^2+1\right) f(t)$$

is the Hamiltonian of a harmonic oscillator with ground state eigenvalue 2. Since $L^2(\mu)$ is Segal isomorphic to the symmetric Fock space $\Gamma(L^2(\mathbf{R}))$ of $L^2_{\mathbf{C}}(\mathbf{R}) := L^2(\mathbf{R}) \oplus iL^2(\mathbf{R})$, we can identify the Fock space $\Gamma(S_p)$ with a subspace $(S)_p$ of $L^2(\mu)$ and define the nuclear space

$$(S) = \bigcap_{p\geq 0} (S)_p .$$

Thus we arrived at the Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)' .$$

Elements of the space $(S)'$ are called *Hida distributions* (or generalized Brownian functionals). It is possible to characterize the spaces (S) and $(S)'$ by their S- or T-transforms. For $\Phi \in (S)'$ and $f \in S(\mathbf{R})$ these transforms are defined as:

$$T\Phi(f) := \langle\!\langle \Phi, \exp(i\langle \cdot, f\rangle)\rangle\!\rangle = \int_{S'(\mathbf{R})} \exp(i\langle w, f\rangle)\,\Phi(w)\,d\mu(w)$$

$$S\Phi(f) := \langle\!\langle \Phi, :\exp\langle \cdot, f\rangle :\rangle\!\rangle ,$$

here $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ denotes the bilinear dual pairing between (S) and $(S)'$ and we have used the traditional notation

$$:\exp(\langle \cdot, f\rangle): \; := \exp\left(\langle \cdot, f\rangle - (1/2)\,|f|_0^2\right), \quad f \in S(\mathbf{R}) .$$

We denote by $\mathbf{E}(\Phi) := \langle\!\langle \Phi, 1\rangle\!\rangle$ the expectation of a Hida distribution Φ. S- and T-transform have extensions to the complex Schwartz space $S_{\mathbf{C}}(\mathbf{R})$ and are related by the following formula:

$$T\Phi(f) = C(f)\, S\Phi(if), \quad f \in S_{\mathbf{C}}(\mathbf{R}).$$

B. U-Functionals and the characterization theorem

Here we give the characterization theorem, which is due to Potthoff and Streit[18] and has been generalized in various ways, see e.g.[19,20,21] For a full proof of a generalized version see.[17]

Theorem 1 *The following statements are equivalent:*

1. $F : S(\mathbf{R}) \to \mathbf{C}$ *has*
   (a) *"ray-analyticity": for all* $f, g \in S(\mathbf{R})$ *the mapping* $\mathbf{C} \ni z \mapsto F(zf + g)$ *is entire, and*
   (b) *"bound": F is uniformly of order two, i.e. there exist constants* $K_1, K_2 > 0$ *and* $p \in \mathbf{N}_0$ *such that for all* $z \in \mathbf{C}$, $f \in S(\mathbf{R})$,

$$|F(zf)| \leq K_1 \exp\left(K_2 |z|^2 |f|_p^2\right).$$

2. *F is the S-transform of a unique Hida distribution* $\Phi \in (S)'$.
3. *F is the T-transform of a unique Hida distribution* $\hat{\Phi} \in (S)'$.

A functional satisfying 1 is usually called a *U-functional*.

As an application of this theorem we give the following example.

Example 2 (Donsker's delta function) *Consider the composition* $\delta_a \circ x(t)$ *of the Dirac distribution* δ_a *at* $a \in \mathbf{R}$ *with Brownian motion* $x(t)$, $t > 0$:

$$\Phi = \delta(x(t) - a) = \delta\left(\langle \cdot, \mathbf{1}_{[0,t)} \rangle - a\right), \quad a \in \mathbf{R}.$$

The S-transform of Φ is calculated to be[3]*:*

$$S\Phi(f) = (2\pi t)^{-1/2} \exp\left(-\frac{1}{2t}\left(\int_0^t f(s)\, ds - a\right)^2\right)$$

and Theorem 1 gives immediately that Φ is well defined element in $(S)'$.

Theorem 1 enables us to discuss convergence of a sequence of generalized functionals. For the proof of the following theorem we refer to[3,17,18]

Theorem 3 *Let $(F_n)_{n\in\mathbf{N}}$ denote a sequence of U-functionals with the following properties:*

1. *for all $f \in S(\mathbf{R})$, $(F_n(f))_{n\in\mathbf{N}}$ is a Cauchy sequence, and*
2. *there exist $K_1, K_2 > 0$ and $p \in \mathbf{N}_0$ such that the bound*

$$|F_n(zf)| \leq K_1 \exp\left(K_2 |z|^2 |f|_p^2\right), \quad f \in S(\mathbf{R}),\ z \in \mathbf{C},$$

   holds for almost all $n \in \mathbf{N}$.

Then there is a unique $\Phi \in (S)'$ such that $T^{-1}F_n$ converges strongly to Φ.

This theorem is also valid for S-transform.

As a second application we consider a theorem which concerns the integration of a family of generalized functionals.

Theorem 4 *Let (Ω, B, ν) denote a measure space and $\lambda \mapsto \Phi_\lambda$ a mapping from Ω to $(S)'$. We assume that the T-transform $F_\lambda = T\Phi_\lambda$ satisfies the following conditions for all $\lambda \in \Omega$:*

1. *for every $f \in S(\mathbf{R})$ the mapping $\lambda \mapsto F_\lambda(f)$ is measurable, and*
2. *there exists $p \in \mathbf{N}_0$ such that*

$$|F_\lambda(zf)| \leq K_1(\lambda) \exp\left(K_2(\lambda) |z|^2 |f|_p^2\right), \quad f \in S(\mathbf{R}),\ z \in \mathbf{C},$$

   with $K_1 \in L^1(\nu)$ and $K_2 \in L^\infty(\nu)$.

Then Φ is Bochner integrable in some $(S)_{-q}$ and thus

$$\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (S)'$$

and the T-transform and integration commute:

$$T\left(\int_\Omega \Phi_\lambda d\nu(\lambda)\right)(f) = \int_\Omega T\Phi_\lambda(f)\, d\nu(\lambda), \quad f \in S(\mathbf{R}).$$

Again the same theorem holds for the S-transform.

Example 5 *The Donsker's delta function from Example 2 given by*

$$\delta\left(x\left(t\right)-a\right)=\frac{1}{2\pi}\int_{\mathbf{R}}\exp\left(i\lambda\left(x\left(t\right)-a\right)\right)d\lambda$$

in the sense of Bochner integration, see e.g..[3]

Remark 1 *For later use we have to define pointwise products of a Hida distribution Φ with a Donsker delta function $\delta(\langle w,g\rangle-a)$, i.e.*

$$\Phi\cdot\delta\left(\langle w,g\rangle-a\right). \tag{1}$$

If the mapping $\lambda\mapsto T\Phi(f+\lambda g)$ is integrable on $\mathbf{R}$ the following formula may be used to define the product in (1) as

$$T\left(\Phi\cdot\delta\left(\langle w,g\rangle-a\right)\right)\left(f\right)=\frac{1}{2\pi}\int_{\mathbf{R}}\exp\left(-i\lambda a\right)T\Phi\left(f+\lambda g\right)d\lambda \tag{2}$$

in the case that the right hand integral is indeed a U-functional.

Before we close this section we would like to give one more example of a Hida distribution which is a first approximation of the Feynman integrand that we will introduce in next section.

Example 6 *Let us consider the following formal expression for complex $c\neq 1/2$:*

$$\exp\left(c\int_a^b w^2\left(s\right)\,ds\right).$$

Calculation of its S-transform produces a U-functional "up to an infinite constant" (for details see[3]*). So, as a renormalization, we omit this factor and get a well-defined U-functional:*

$$F\left(f\right)=\exp\left(\frac{c}{1-2c}\int_a^b f^2\left(s\right)\,ds\right),\quad f\in S(\mathbf{R}).$$

Hence, we may define

$$\text{Nexp}\left(c\int_a^b w^2\left(s\right)ds\right):=S^{-1}F,$$

or, formally,

$$\text{Nexp}\left(c\int_a^b w^2\left(s\right)ds\right)=\frac{\exp\left(c\int_a^b w^2\left(s\right)ds\right)}{\mathbf{E}\left(\exp\left(c\int_a^b w^2\left(s\right)ds\right)\right)}.$$

III. Realization of Feynman path integrals in White Noise Analysis

A. The Feynman integrand for the free particle propagator

We follow[10,11] in regarding the Feynman integral as a weighted average over Brownian paths. These paths are modeled within the white noise framework according to

$$x\left(t\right)=x_{0}+\left(\frac{\hbar}{m}\right)^{1/2}\int_{t_{0}}^{t}w\left(\tau\right)d\tau,\quad w\in S'\left(\mathbf{R}\right),$$

in the sequel we set $\hbar=1$ and in this section we also set $m=1$. In[11] the (distribution-valued) weight for free quantum mechanical propagator from $x\left(t_{0}\right)=x_{0}$ to $x\left(t\right)=x$, $t\geq t_{0}$, $x_{0},x,t_{0},t\in\mathbf{R}$, is constructed from a kinetic energy factor $\exp(\frac{i}{2}\int_{t_{0}}^{t}w^{2}(\tau)d\tau)$ and a Donsker delta function $\delta\left(x\left(t\right)-x\right)$ to fix the endpoint. Furthermore a factor $\exp((1/2)\int_{t_{0}}^{t}w^{2}(\tau)\,d\tau)$ is introduced to compensate the Gaussian fall-off of the white noise measure in order to mimic Feynman's non-existing "flat" measure $D^{\infty}x$. Thus in[11] the Feynman integrand for the free motion reads

$$I_{0}=\mathrm{N}\exp\left(\frac{i+1}{2}\int_{t_{0}}^{t}w^{2}(\tau)d\tau\right)\cdot\delta\left(x\left(t\right)-x\right),$$

where N is the normalizing pre-factor introduced in Example 6. There I_{0} is a Hida distribution, with T-transform given by

$$\begin{aligned}TI_{0}\left(f\right) &= \left(\frac{1}{2\pi i\left|\Delta\right|}\right)^{1/2}\\ &\quad\times\exp\left(-\frac{i}{2}\left|f_{\Delta}\right|^{2}-\frac{1}{2}\left|f_{\Delta^{\complement}}\right|^{2}+\frac{i}{2\left|\Delta\right|}\left(\int_{t_{0}}^{t}f\left(\tau\right)d\tau+x-x_{0}\right)^{2}\right),\end{aligned}$$

where $\Delta:=[t_{0},t]$ and $f_{\Delta}:=f\upharpoonright\Delta$, $f_{\Delta^{\complement}}:=f\upharpoonright\Delta^{\complement}$ denote the restrictions of $f\in S(\mathbf{R})$ to Δ and its complement $\Delta^{\complement}$ respectively. Furthermore the Feynman integral $\mathbf{E}\left(I_{0}\right)=TI_{0}\left(0\right)$ is indeed the free particle propagator

$$\left(\frac{1}{2\pi i\left|\Delta\right|}\right)^{1/2}\exp\left(\frac{i}{2\left|\Delta\right|}\left(x-x_{0}\right)^{2}\right).$$

Not only the expectation but also the T-transform has a physical meaning. Integrating formally by parts we find, see[3,11]:

$$\begin{aligned} TI_0(f) &= \exp\left(-\frac{1}{2}\left|f_{\Delta^c}\right|^2\right)\exp\left(ixf(t)-ix_0f(t_0)\right) \\ &\quad\times\mathbf{E}\left(I_0\exp\left(-i\int_{t_0}^t x(\tau)\,\dot f(\tau)\,d\tau\right)\right). \end{aligned} \tag{3}$$

The term

$$\exp\left(-i\int_{t_0}^t x(\tau)\,\dot f(\tau)\,d\tau\right)$$

would thus correspond to a time-dependent potential $W(x,t)=\dot f(\tau)\,x$. And indeed it is straightforward to verify that

$$TI_0(f)=\exp\left(-\frac{1}{2}\left|f_{\Delta^c}\right|^2\right)\exp\left(ixf(t)-ix_0f(t_0)\right)K_0^{(\dot f)}(x,t|x_0,t_0), \tag{4}$$

where

$$\begin{aligned} K_0^{(\dot f)}(x,t|x_0,t_0) &= \left(\frac{1}{2\pi i\left|\Delta\right|}\right)^{1/2}\exp\left(ix_0f(t_0)-ixf(t)\right) \\ &\quad\times\exp\left(-\frac{i}{2}\left|f_\Delta\right|^2+\frac{i}{2\left|\Delta\right|}\left(\int_{t_0}^t f(\tau)\,d\tau+x-x_0\right)^2\right) \end{aligned}$$

is the Green's function corresponding to the potential W, i.e. $K_0^{(\dot f)}$ obeys the Schrödinger equation

$$\left(i\partial_t+\frac{1}{2}\partial_x^2-\dot f(t)\,x\right)K_0^{(\dot f)}(x,t|x_0,t_0)=i\delta(t-t_0)\,\delta(x-x_0),$$

see[3,].[13] More generally one calculates

$$\begin{aligned} T\left(I_0\prod_{j=1}^n\delta\left(x(t_j)-x_j\right)\right)(f) &= \exp\left(-\frac{1}{2}\left|f_{\Delta^c}\right|^2\right)\exp\left(ixf(t)-ix_0f(t_0)\right) \\ &\quad\times\prod_{j=1}^{n+1}K_0^{(\dot f)}(x_j,t_j|x_{j-1},t_{j-1}), \end{aligned} \tag{5}$$

see[3,12] Here $t_0 < t_1 < \cdots < t_n < t_{n+1} := t$ and $x_{n+1} := x$. In Proposition 9 we will give a generalization of this result.

Remark 2 *Formula (5) is a version of the composition property of Feynman propagators which is well-known in quantum physics. In Subsection C we prove this property for the time-dependent harmonic oscillator.*

B. The perturbed Feynman integrand

In order to pass from a given Feynman integrand I to more general situations one has to give a rigorous definition of the formal expression

$$I_V = I \exp\left(-i\int_{t_0}^{t} V\left(x\left(\tau\right)\right) d\tau\right).$$

In[12] Khandekar and Streit accomplished this by perturbative methods in case V is a finite signed Borel measure with compact support and $I = I_0$. This construction was generalized in[13] to a wider class of potentials by allowing time-dependent potentials and a Gaussian fall-off instead of a bounded support.

The starting point is a power series expansion of

$$\exp\left(-i\int_{t_0}^{t} V\left(x\left(\tau\right),\tau\right) d\tau\right)$$

using

$$V\left(x\left(\tau\right),\tau\right) = \int_{-\infty}^{+\infty} V\left(x,\tau\right)\delta\left(x\left(\tau\right)-x\right)dx.$$

Thus we have

$$\exp\left(-i\int_{t_0}^{t} V\left(x\left(\tau\right),\tau\right) d\tau\right)$$

$$= \sum_{n=0}^{\infty}(-i)^n\int_{\mathbf{R}^n}\int_{\Lambda_n}\prod_{j=1}^{n} V\left(x_j,t_j\right)\delta\left(x\left(t_j\right)-x_j\right)dt_j dx_j\,,$$

where $\Lambda_n = \left\{(t_1,\cdots t_n)\,|\,t_0 < t_1 < \cdots < t_n < t\right\}$.

In order to consider singular potentials V is no longer taken to be a function V but a measure ν. Under suitable conditions on ν and $I = I_0$ it is proven in[12,13] that

$$I_V = I + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbf{R}^n} \int_{\Lambda_n} \left(I \cdot \prod_{j=1}^{n} \delta\left(x\left(t_j\right) - x_j\right) \right) \prod_{j=1}^{n} \nu\left(dt_j, dx_j\right)$$

exists as a well-defined element of $(S)'$ using Theorem 1 and 3. Furthermore in[14] they obtain the same result for $I = I_h$, where I_h is the Feynman integrand for the harmonic oscillator with constant mass and frequency.

In this work we generalize this result for time-dependent mass and frequency.

IV. The Feynman propagator for the time-dependent harmonic oscillator

A. Calculation of the Feynman propagator

In this subsection we calculate the Feynman propagator corresponding the Lagrangian

$$L\left(x(t),\dot{x}(t),t\right)=\frac{1}{2}\left(m(t)\dot{x}^2(t)-k^2(t)x^2(t)\right)\right)-\dot{f}(t)x(t) \tag{6}$$

where $k\in C(\mathbf{R}),\ f\in C^\infty(\mathbf{R}),\ m\in C^2(\mathbf{R})$ and $m>0$. Physically, the Lagrangian corresponds to the motion of an oscillator with a variable mass m, frequency k and a force $-\dot{m}\dot{x}$ linear in the velocity. Furthermore, the oscillator is forced by $-\dot{f}$. For us the importance of this extra force term lies in the fact that the corresponding propagator immediately gives us the T-transform of the Feynman integrand as explained in formula (4).

In order to calculate the associated Feynman propagator we use the formulas derived in[15,].[22] There the polygonal approach as been applied in order to integrate "over all paths" and the following expression has been found for the Feynman propagator:

$$K_{\text{TD}}^{(\dot{f})}(x_0,t_0|x,t)=D_{\text{TD}}\exp(iS_{cl}\left(x_0,t_0|x,t\right)),$$

where D_{TD} is a pre-factor which will be discussed later. $S_{cl}\left(x_0,t_0|x,t\right)$ is the action evaluated along the classical path:

$$S_{cl}\left(x_0,t_0|x,t\right)=\frac{1}{2}\left(m(t)x\dot{u}(t)-m(t_0)x_0\dot{u}(t_0)-\int_{t_0}^{t}u(s)\dot{f}(s)ds\right),$$

where u is the solution of the associated classical equation of motion:

$$\frac{d}{ds}\left(m(s)\dot{u}(s)\right)+k^2(s)u(s)=-\dot{f}(s), \tag{7}$$

and satisfies the boundary conditions

$$u(t_0)=x_0,\quad u(t)=x. \tag{8}$$

By means of a substitution $v=m^{1/2}u$, (7) can be cast into the simpler form

$$\ddot{v}(s)+\Omega^2(s)v(s)=-\frac{\dot{f}(s)}{m(s)^{1/2}}, \tag{9}$$

with the boundary conditions (8) as

$$v(t_0) = m(t_0)^{1/2} x_0, \quad v(t) = m(t)^{1/2} x, \tag{10}$$

and

$$\Omega^2 = \frac{\dot{m}^2}{4m^2} - \frac{\ddot{m}}{2m} + \frac{k^2}{m}.$$

Since Ω^2 is a continuous function on $\mathbf{R}$ we can conclude that there exist two linear independent solutions $\tilde{\omega}_1$ and $\tilde{\omega}_2$ on $\mathbf{R}$ of the homogeneous differential equation

$$\ddot{\omega}(s) + \Omega^2(s)\omega(s) = 0 \tag{11}$$

corresponding to (9) by using the general theory of linear differential equations. If the determinant

$$R_{\text{TD}}(t_0, t) = \begin{vmatrix} \tilde{\omega}_1(t_0) & \tilde{\omega}_2(t_0) \\ \tilde{\omega}_1(t) & \tilde{\omega}_2(t) \end{vmatrix} \neq 0$$

we can conclude from the theory of Green's functions that there exist two linear independent solutions ω_1 and ω_2 of the homogeneous differential equation (11) satisfying the initial conditions

$$\omega_1(t_0) = 0,\ \omega_1(t) = R_{\text{TD}},\ \omega_2(t_0) = R_{\text{TD}},\ \omega_2(t) = 0. \tag{12}$$

Then the Green's function of the boundary problem (9), (10) is given by

$$\begin{aligned} G(s,r) &= \begin{cases} \dfrac{\omega_1(s)\,\omega_2(r)}{Q} & \text{for} \quad s \le r \\ \dfrac{\omega_1(r)\,\omega_2(s)}{Q} & \text{for} \quad s \ge r \end{cases}, \\ Q &= \omega_1(t)\dot{\omega}_2(t), \end{aligned}$$

see e.g..[23] Hence the solution of (9) satisfying the conditions (10) is

$$v(s) = \frac{m(t_0)^{1/2} x_0 \omega_2(s)}{\omega_2(t_0)} + \frac{m(t)^{1/2} x \omega_1(s)}{\omega_1(t)} - \int_{t_0}^{t} \frac{G(s,r)\dot{f}(r)}{m(r)^{1/2}}\, dr. \tag{13}$$

Now we analyse the determinant R_{TD}. Consider

$$R_{\text{TD}}(t_0, t) = \tilde{\omega}_1(t_0)\tilde{\omega}_2(t) - \tilde{\omega}_1(t)\tilde{\omega}_2(t_0)$$

for fixed $t_0 \in \mathbf{R}$. Then $R_{\text{TD},t_0}(t) := R_{\text{TD}}(t_0, t)$ as a function of $t \in \mathbf{R}$ is a non-trivial solution of the homogeneous differential equation (11) with $R_{\text{TD},t_0}(t_0) = 0$. The solutions of (11) have isolated zeros, see e.g..[24] Hence there exists $d_{t_0} > 0$ such that $R_{\text{TD},t_0}(t) \neq 0$ for all $t > t_0$ with $0 < t - t_0 < d_{t_0}$. Thus the two independent solutions ω_1 and ω_2 of the homogeneous differential equation (11) exist and in[24] it has been proved that ω_1 and ω_2 can be written in the form

$$\begin{aligned} \omega_1(s) &= \rho(s) \sin \phi(s, t_0), \\ \omega_2(s) &= \rho(s) \sin \phi(t, s), \quad s \in [t_0, t], \end{aligned} \tag{14}$$

where ρ is strictly positive on $\mathbf{R}$ and satisfies an auxiliary equation known as Pinney's equation, see[25]:

$$\ddot{\rho} + \Omega^2 \rho - \rho^{-3} = 0.$$

ϕ is defined as

$$\phi(s, r) = \nu(s) - \nu(r) = \int_r^s \rho^{-2}(y)\, dy.$$

Note that ρ and ν may be interpreted as the amplitude and phase of the time-dependent oscillator of equation (11).

The pre-factor D_{TD} is calculated to be

$$D_{\text{TD}} = \frac{\dot{\omega}_1(t_0)^{1/2} (m(t_0) m(t))^{1/4}}{(2\pi i \omega_1(t))^{1/2}},$$

see[15],[22] Using formula (14) D_{TD} turns out to be

$$D_{\text{TD}} = \frac{(m(t_0) m(t))^{1/4}}{(2\pi i \rho\,(t_0)\, \rho\,(t) \sin \phi\,(t, t_0))^{1/2}}.$$

From (12) and (14) we obtain

$$R_{\text{TD}}\,(t_0, t) = \rho\,(t) \sin \phi\,(t, t_0)\,.$$

Therefore, $R_{\text{TD}}\,(t_0, t)$ does not vanish for $0 < \phi\,(t, t_0) < \pi$. Hence for such t_0, t the propagator exists and the denominator of D_{TD} is different from zero (note that $\phi\,(t_0, t_0) = 0$ and $\phi\,(t, t_0)$ is monotone increasing in t).

The entire propagator now can be written as

$$
\begin{aligned}
& K_{\mathrm{TD}}^{(\dot f)}(x_0,t_0|x,t) \\
= & \frac{(m(t_0)m(t))^{1/4}}{(2\pi i\rho(t_0)\rho(t)\sin\phi(t,t_0))^{1/2}} \\
& \times\exp\left(-\frac{i}{4}\left(\dot m(t)x^2-\dot m(t_0)x_0^2\right)+\frac{i}{2}\left(\frac{\dot\rho(t)m(t)x^2}{\rho(t)}-\frac{\dot\rho(t_0)m(t_0)x_0^2}{\rho(t_0)}\right)\right. \\
& +\frac{i}{2\sin\phi(t,t_0)}\left(\left(\frac{m(t)x^2}{\rho^2(t)}+\frac{m(t_0)x_0^2}{\rho^2(t_0)}\right)\cos\phi(t,t_0)\right. \\
& -\frac{(m(t_0)m(t))^{1/2}2x_0x}{\rho(t_0)\rho(t)}-\frac{m(t)^{1/2}2x}{\rho(t)}\int_{t_0}^{t}\dot f(s)\frac{\rho(s)\sin\phi(s,t_0)}{m(s)^{1/2}}ds \\
& -\frac{m(t_0)^{1/2}2x_0}{\rho(t_0)}\int_{t_0}^{t}\dot f(s)\frac{\rho(s)\sin\phi(t,s)}{m(s)^{1/2}}ds \\
& \left.\left.-2\int_{t_0}^{t}\int_{t_0}^{r}\dot f(r)\dot f(s)\frac{\rho(r)\rho(s)\sin\phi(t,r)\sin\phi(s,t_0)}{(m(r)m(s))^{1/2}}dsdr\right)\right). \qquad (15)
\end{aligned}
$$

In the following subsections we also need an expression of the propagator (15) as a function of f instead of $\dot f$, hence we use the formula of integration by parts and obtain

$$
\begin{aligned}
& K_{\mathrm{TD}}^{(\dot f)}(x_0,t_0|x,t) \\
= & \frac{(m(t_0)m(t))^{1/4}}{(2\pi i\rho(t_0)\rho(t)sin(\phi(t,t_0)))^{1/2}}\exp\left(-\frac{i}{2}\int_{t_0}^{t}\frac{f^2(s)}{m(s)^{1/2}}ds-ixf(t)\right. \\
& +x_0f(t_0)-\frac{i}{4}\left(\dot m(t)x^2-\dot m(t_0)x_0^2\right)+\frac{i}{2}\left(\frac{\dot\rho(t)m(t)x^2}{\rho(t)}-\frac{\dot\rho(t_0)m(t_0)x_0^2}{\rho(t_0)}\right) \\
& +\frac{i}{2\sin\phi(t,t_0)}\left(\left(\frac{m(t)x^2}{\rho^2(t)}+\frac{m(t_0)x_0^2}{\rho^2(t_0)}\right)\cos\phi(t,t_0)-\frac{(m(t_0)m(t))^{1/2}2x_0x}{\rho(t_0)\rho(t)}\right.
\end{aligned}
$$

$$
+\frac{m(t)^{1/2}2x}{\rho(t)}\int_{t_0}^{t} f(s)\frac{\dot\rho(s)\sin\phi(s,t_0)+\rho^{-1}(s)\cos\phi(s,t_0)}{m(s)^{1/2}}ds
$$

$$
+\frac{m(t_0)^{1/2}2x_0}{\rho(t_0)}\int_{t_0}^{t} f(s)\frac{\dot\rho(s)\sin\phi(t,s)-\rho^{-1}(s)\cos\phi(t,s)}{m(s)^{1/2}}ds
$$

$$
-\frac{m(t)^{1/2}x}{\rho(t)}\int_{t_0}^{t} f(s)\frac{\rho(s)\sin\phi(s,t_0)\dot m(s)}{m(s)^{1/2}}ds
$$

$$
-\frac{m(t_0)^{1/2}x_0}{\rho(t_0)}\int_{t_0}^{t} f(s)\frac{\rho(s)\sin\phi(t,s)\dot m(s)}{m(s)^{1/2}}ds
$$

$$
-2\int_{t_0}^{t}\int_{t_0}^{r} f(r)f(s)\frac{(\dot\rho(r)\sin\phi(t,r)-\rho^{-1}(r)\cos\phi(t,r))}{(m(r)m(s))^{1/2}}
$$

$$
\times\left(\dot\rho(s)\sin\phi(s,t_0)+\rho^{-1}(s)\cos\phi(s,t_0)\right)dsdr
$$

$$
+\int_{t_0}^{t}\int_{t_0}^{r} f(r)f(s)\frac{\rho(s)\sin\phi(s,t_0)\dot m(s)}{(m(r)m^3(s))^{1/2}}
$$

$$
\times\left(\dot\rho(r)\sin\phi(t,r)-\rho^{-1}(r)\cos\phi(t,r)\right)dsdr
$$

$$
+\int_{t_0}^{t}\int_{t_0}^{r} f(r)f(s)\frac{\rho(r)\sin\phi(t,r)\dot m(r)}{(m^3(r)m(s))^{1/2}}
$$

$$
\times\left(\dot\rho(s)\sin\phi(s,t_0)+\rho^{-1}(s)\cos\phi(s,t_0)\right)dsdr
$$

$$
-\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{r} f(r)f(s)\frac{\rho(r)\sin\phi(t,r)\rho(s)\sin\phi(s,t_0)\dot m(r)\dot m(s)}{(m(r)m(s))^{3/2}}dsdr\Bigg)\Bigg) \quad (16)
$$

In the sequel we choose fixed $t_0, t \in \mathbf{R}$ such that $0 < \phi(t, t_0) < \pi$, hence we know that the corresponding propagator $K^{(\dot f)}_{\mathrm{TD}}(x_0, t_0|x, t)$ exists and is well-defined. In addition, we are interested in the propagators $K^{(\dot f)}_{\mathrm{TD}}(x_1, t_1|x_2, t_2)$, where $t_0 < t_1 < t_2 < t$. Now we show that we again can use the formula (15) for $K^{(f)}_{\text{TD}}(x_1, t_1|x_2, t_2)$, we just have to substitute t_0, t, x_0 and x by t_1, t_2, x_1 and x_2 respectively (this is not obvious because in general ω_1 and ω_2 depend on the endpoints). We define

$$\begin{aligned} \hat{\omega}_1(s) &:= \rho(s) \sin \phi(s, t_1) \\ \hat{\omega}_2(s) &:= \rho(s) \sin \phi(t_2, s), \quad s \in [t_1, t_2], \end{aligned} \tag{17}$$

and it is easy to see that $\hat{\omega}_1$ and $\hat{\omega}_2$ are linear independent solutions of (11) with

$$\hat{\omega}_1(t_1) = 0, \quad \hat{\omega}_2(t_2) = 0.$$

From (17), $\rho > 0$ and the fact that $t_2 - t_1 < t - t_0$ we can derive that

$$\hat{\omega}_1(t_2) > 0, \quad \hat{\omega}_2(t_1) > 0.$$

Thus, we can use the linear independent solutions $\hat{\omega}_1$ and $\hat{\omega}_2$ in order to calculate the propagator $K^{(f)}_{\text{TD}}(x_1, t_1|x_2, t_2)$ the same way as we used ω_1 and ω_2 to calculate the propagator $K^{(f)}_{\text{TD}}(x_0, t_0|x, t)$, see[23] (note that the Green's function and the propagator do not vary if we multiply $\hat{\omega}_1$ or $\hat{\omega}_2$ by a constant different from zero). Therefore, we found the expression for $K^{(f)}_{\text{TD}}(x_1, t_1|x_2, t_2)$ as described above.

Example 7 *We consider the Lagrangian (6) in the particular form* $m(t) = \exp(\gamma t)$, $k^2(t) = \nu^2 \exp(\gamma t)$, $f(t) = 0$, $t, \nu \in \mathbf{R}$, $\gamma > 0$*:*

$$L(x(t), \dot{x}(t), t) = \frac{1}{2} \left(\exp(\gamma t) \dot{x}^2(t) - \nu^2 \exp(\gamma t) x^2(t) \right). \tag{18}$$

Then an application of the Euler-Lagrange differential equation to the Lagrangian (18) gives us the corresponding classical equation of motion

$$\ddot{x}(t) = -\nu^2 x(t) - \gamma \dot{x}(t) \tag{19}$$

via the Hamilton principle. Equation (19) is the equation of motion for damped harmonic oscillators. Caldirola[26] and subsequently Kanai[27] first proposed the Lagrangian (18) in order to study the quantum mechanics of damped harmonic oscillators. Therefore it is known as Caldirola-Kanai Lagrangian. The question whether the Lagrangian (18) correctly describes the quantum mechanics of damped harmonic oscillators has been discussed contradictorily, see[28,29,30,31,32]. However, it is generally accepted now that the damping in quantum mechanics can be adequately described by a time-varying mass, see[33,34].

B. The Feynman propagator in the frame of White Noise Analysis

In this subsection we put the Feynman propagator of the time-dependent harmonic oscillator into the frame of white noise analysis. For t_0, t such that $0 < \phi(t, t_0) < \pi$ we consider the propagator $K_{\text{TD}}^{(\dot{f})}(x_0, t_0|x, t)$ in the representation (16) as a function of f. For $f \in S(\mathbf{R})$ it is easy to verify that the propagator is a U-functional and therefore we know by Theorem 1 that there exists a Hida distribution which is the inverse T-transform of the propagator. But we are searching for the Feynman integrand I_{TD} of the time-dependent harmonic oscillator as discussed in Section III. A reasonable choice is to define its T-transform as:

$$TI_{\text{TD}}(f) := \exp\left(-\frac{1}{2}\left|f_{\Delta^{\complement}}\right|^2 + ixf(t) - ix_0 f(t_0)\right) K_{\text{TD}}^{(\dot{f})}(x_0, t_0|x, t), \qquad (20)$$

for $f \in S(\mathbf{R})$. But why is this choice reasonable? Is I_{TD} given by the formal expression

$$\begin{aligned} I_{\text{TD}} &= \text{Nexp}\left(\frac{i}{2}\int_{t_0}^{t} m(\tau)\omega^2(\tau)\,d\tau + \frac{1}{2}\int_{t_0}^{t}\omega^2(\tau)\,d\tau\right)\delta(x(t)-x) \quad (21)\\ &\quad \times \exp\left(-i\int_{t_0}^{t} U(\tau, x(\tau))\,d\tau\right), \qquad U(x,t) = \frac{1}{2}k^2(t)x^2, \end{aligned}$$

in the sense of Hida distributions? How is the pointwise multiplication with the interaction term defined? In[11] this product has been justified and the T-transform of (21) has been calculated to be the same as in (20) for m and k constant. Furthermore, we know that $K_{\text{TD}}^{(\dot{f})}(x_0, t_0|x, t)$ is the propagator corresponding to Lagrangian (6) and the formal integration by parts in (3) shows that the additional factor in (20) is independent of the potential to which the particle is submitted. So, (20) is a reasonable definition of I_{TD}.

As a result of this subsection we have the following theorem.

Theorem 8 *For t_0, t such that $0 < \phi(t, t_0) < \pi$ the Feynman integrand I_{TD} of the time-dependent harmonic oscillator exists as a Hida distribution, i.e. $I_{TD} \in (S)'$.*

Proof. From (20) and (16) we obtain that $TI_{\text{TD}}(f)$, $f \in S(\mathbf{R})$, contains f only in the exponent up to second order. Hence TI_{TD} is a U-functional and thus, by characterization, $I_{\text{TD}} \in (S)'$. ■

C. The composition property of the Feynman propagator

Proceeding exactly as in the case of free motion, see[12,13] we first have to define the pointwise product

$$I_{\text{TD}} \cdot \prod_{j=1}^{n} \delta(x(t_j) - x_j) \tag{22}$$

in $(S)'$. The expectation of this object can be interpreted as the propagator of a particle in a harmonic time-dependent potential, where the paths are "pinned" such that $x(t_j) = x_j$, $1 \leq j \leq n$. Following the ideas of the Remark 1 we have to apply (2) repeatedly. But due to the form of $TI_{\text{TD}}(f)$ in (16), which contains f only in the exponent up to second order, all these integrals are expected to be Gaussian. Using this we arrive at the following proposition.

Proposition 9 *For $x_0 < x_j < x$, $1 \leq j \leq n$, $x_{n+1} := x$, $t_0 < t_j < t_{j+1} < t$, $1 \leq j \leq n-1$, $t_{n+1} := t$, $I_{TD} \cdot \prod_{j=1}^{n} \delta(x(t_j) - x_j)$ is a Hida distribution and its T-transform is given by*

$$\begin{aligned} & T\left(I_{TD} \cdot \prod_{j=1}^{n} \delta(x(t_j) - x_j)\right)(f) \\ = \;& \exp\left(-\frac{1}{2}\left|f_{\Delta^c}\right|^2 + ixf(t) - ix_0 f(t_0)\right) \prod_{j=1}^{n+1} K_{TD}^{(\dot{f})}(x_{j-1}, t_{j-1}|x_j, t_j). \end{aligned}$$

Proof. For $n = 1$ we check this assertion by direct computation using formula (2). In[14] this computation has already been done for the case of constant mass and frequency. There they have used expression (16) to obtain the T-transform of I_{TD}. But in the time-dependent case this expression is much to complex to handle. So, we use expression (15) in the computation of the integral in formula (2). This expression is simpler but, unfavourably, it is a function of $\dot{f}$ and therefore we have to define the derivative of an indicator function. In order to do this we extend the formula of integration by parts and find the following rules:

$$\int_{t_0}^{t} \frac{d}{ds}\left(f+\lambda \mathbf{1}_{[t_0,t_1)}\right)(s)\, g(s)\, ds = \int_{t_0}^{t} \dot{f}(s)\, g(s)\, ds - \lambda g(t_1)$$

and

$$\int_{t_0}^{t_1}\int_{t_0}^{r} \frac{d}{dr}\left(f+\lambda \mathbf{1}_{[t_0,t_1)}\right)(r) \frac{d}{ds}\left(f+\lambda \mathbf{1}_{[t_0,t_1)}\right)(s)\, g_1(s)\, g_2(r)\, dsdr$$

$$= \int_{t_0}^{t_1}\int_{t_0}^{r} \dot{f}(r)\,\dot{f}(s)\, g_1(s)\, g_2(r)\, dsdr - \lambda g_2(t_1)\int_{t_0}^{t_1} \dot{f}(s)\, g_1(s)\, ds$$
$$-\lambda g_1(t_1)\int_{t_1}^{t} \dot{f}(r)\, g_2(r)\, dr + \frac{\lambda^2}{2} g_1(t_1)\, g_2(t_1),$$

where g, g_1 and g_2 are continuous differentiable functions. Using this rules the integral in (2) turns out to be a Gaussian integral, which can be computed with the following formula:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{i}{2}a\lambda^2 - ib\lambda\right) d\lambda = \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{ib^2}{2a}\right).$$

Then we can find the assertion for $n=1$ by reordering the obtained terms.

To perform the induction we need the following

Lemma 10 *Let $t_0 < t_1 < t_2 < t$ then*

$$K_{TD}^{(f+\lambda\mathbf{1}_{[t_2,t)})^\bullet}(x_0,t_0|x_1,t_1) = K_{TD}^{(\dot{f})}(x_0,t_0|x_1,t_1), \quad \forall \lambda \in \mathbf{R},$$

and for $t_0 < t_1 < t$ we have

$$K_{TD}^{(f+\lambda\mathbf{1}_{[t_1,t)})^\bullet}(x_0,t_0|x_1,t_1) = K_{TD}^{(\dot{f})}(x_0,t_0|x_1,t_1)\exp(-ix_1\lambda), \quad \forall \lambda \in \mathbf{R}.$$

Proof of Lemma 10. To proof this lemma we use expression (15) for the propagator. Then we substitute f by $(f+\lambda\mathbf{1}_{[t_2,t)})$ and $(f+\lambda\mathbf{1}_{[t_1,t)})$ respectively and obtain the desired result. ∎

∎

V. The Feynman integrand for the time-dependent harmonic oscillator in an external potential

In this section we construct the Feynman integrand for the time-dependent harmonic oscillator in an external potential $V(x,t)$. Thus we have to define

$$I_{\text{TD,V}} = I_{\text{TD}} \cdot \exp\left(-i\int_{t_0}^{t} V(x(\tau),\tau)d\tau\right).$$

As described in Subsection B we introduce the perturbation V via the series expansion of the exponential. Hence we have to find conditions for the measure ν corresponding to the potential V such that

$$I_{\text{TD,V}} = I_{\text{TD}} + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbf{R}^n}\int_{\Lambda_n} \left(I_{\text{TD}} \cdot \delta\left(x\left(t_j\right) - x_j\right)\right) \prod_{j=1}^{n} \nu\left(dt_j, dx_j\right)$$

exists in $(S)'$.

Since we want to study singular time-dependent potentials, we consider ν a finite signed Borel measure on $\mathbf{R}\times\Delta$. Let ν_x denote the marginal measure

$$\nu_x\left(A \in \mathcal{B}\left(\mathbf{R}\right)\right) := \nu\left(A \times \Delta\right)$$

and similarly

$$\nu_t\left(B \in \mathcal{B}\left(\Delta\right)\right) := \nu\left(\mathbf{R} \times B\right).$$

The following theorem contains conditions under which the Feynman integrand $I_{\text{TD,V}}$ exists as a Hida distribution.

Theorem 11 *Let $\nu = \nu_+ - \nu_-$ be a finite signed Borel measure on $\mathbf{R}\times\Delta$, $0 < \phi\left(t,t_0\right) < \pi$. Further, we assume that the marginal measures $|\nu|_x := (\nu_+ + \nu_-)_x$ and $|\nu|_t$ satisfy:*
***1.** there exist $R, \beta > 0$ such that*

$$\left|\nu\right|_x\left(\{x : |x| > r\}\right) < \exp(-\beta r^2)$$

for all $r > R$, and
***2.** $\left|\nu\right|_t$ has a L^∞ density.*
Then

$$I_{TD,V} = I_{TD} + \sum_{n=1}^{\infty} (-1)^n \int_{\mathbf{R}^n}\int_{\Delta_n} \left(I_{TD} \cdot \prod_{j=1}^{n} \delta\left(x\left(t_j\right) - x_j\right)\right) \prod_{j=1}^{n} \nu\left(dt_j, dx_j\right) \tag{23}$$

is a Hida distribution.

Remark 3 *Conditions 1 is satisfied for very singular potentials, e.g.*

$$V(x) = \sum_{n=1}^{\infty} e^{-n^2} \delta_n(x), \ x \in \mathbf{R}.$$

For cut-off interaction, i.e. compactly supported ν_x, condition 1 is of course valid. Furthermore all potentials V for which there exists $R' > 0$ such that

$$|V(x)| \leq A\,|x - C|\, e^{-B(x-C)^2}, \ A, B > 0, \ C \in \mathbf{R},$$

for all $|x| \geq R'$ are in the class of admissible potentials.

Note also that ν is not supposed to be a product measure, hence the time-dependence can be more intricate than simple multiplication by a function of time.

Remark 4 *In[14] this theorem has already been proved for the special case of constant mass and frequency.*

Proof. 1. part. In the first part of the proof we establish the central estimate (25). We have to use a very careful procedure to achieve that (25) survives n-fold integration and summation in the second part of the proof.

Here we need an expression for the T-transform as a function of f, so we have to use formula (16) for the Feynman propagator. Then from Proposition 9 we find

$$T\left(I_{\text{TD}} \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right) \right)(zf)$$

$$= \exp\left(-\frac{z^2}{2}\left|f_{\Delta^C}\right|^2\right) \prod_{j=1}^{n+1} \frac{(m(t_{j-i})m(t_j))^{1/4}}{(2i\pi\rho(t_{j-1})\rho(t_j)\sin\phi(t_j,t_{j-1}))^{1/2}}$$

$$\times \exp\left(-\frac{iz^2}{2}\int_{\Delta_j} \frac{f^2(s)}{m(s)^{1/2}} ds - \frac{i}{4}\left(\dot{m}(t_j)x_j^2 - \dot{m}(t_{j-1})x_{j-1}^2\right)\right.$$

$$+\frac{i}{2}\left(m(t_j)x_j^2 \left(\frac{\dot{\rho}(t_j)}{\rho(t_j)} + \frac{\cos\phi(t_j,t_{j-1})}{\rho^2(t_j)\sin\phi(t_j,t_{j-1})} \right)\right)$$

$$-m(t_{j-1})x_{j-1}^2\left(\frac{\dot{\rho}(t_{j-1})}{\rho(t_{j-1})}-\frac{\cos\phi(t_j,t_{j-1})}{\rho^2(t_{j-1})\sin\phi(t_j,t_{j-1})}\right)-\frac{2(m(t_j)m(t_{j-1}))^{1/2}x_jx_{j-1}}{\rho(t_{j-1})\rho(t_j)\sin\phi(t_j,t_{j-1})}\right)$$

$$+\frac{i}{2}\left(\frac{2zm(t_j)^{1/2}x_j}{\rho(t_j)\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\left(\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})+\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})\right)}{m\left(t\right)^{1/2}}dt\right.$$

$$+2\frac{zm(t_{j-1})^{1/2}x_{j-1}}{\rho(t_{j-1})\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\left(\dot{\rho}\left(t\right)\sin\phi(t_j,t)-\rho^{-1}\left(t\right)\cos\phi(t_j,t)\right)}{m\left(t\right)^{1/2}}dt$$

$$-\frac{zm(t_j)^{1/2}x_j}{\rho(t_j)\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\rho\left(t\right)\sin\phi(t,t_{j-1})\dot{m}\left(t\right)}{m\left(t\right)^{3/2}}dt$$

$$-\frac{zm(t_{j-1})^{1/2}x_{j-1}}{\rho(t_{j-1})\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\rho\left(t\right)\sin\phi(t_j,t)\dot{m}\left(t\right)}{m\left(t\right)^{3/2}}dt$$

$$-2z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f\left(s\right)f\left(t\right)\left(\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})+\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})\right)}{\sin\phi(t_j,t_{j-1})(m\left(s\right)m\left(t\right))^{1/2}}$$

$$\times\left(\dot{\rho}\left(s\right)\sin\phi(t_j,s)-\rho^{-1}\left(s\right)\cos\phi(t_j,s)\right)dtds$$

$$+z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f\left(s\right)f\left(t\right)\dot{m}\left(t\right)\rho\left(t\right)\sin\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m\left(s\right)^{1/2}m\left(t\right)^{3/2}}$$

$$\times\left(\dot{\rho}\left(s\right)\sin\phi(t_j,s)-\rho^{-1}\left(s\right)\cos\phi(t_j,s)\right)dtds$$

$$+z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f\left(s\right)f\left(t\right)\dot{m}\left(s\right)\rho\left(s\right)\sin\phi(t_j,s)}{\sin\phi(t_j,t_{j-1})m\left(s\right)^{3/2}m\left(t\right)^{1/2}}$$

$$\times\left(\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})+\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})\right)dtds$$

$$\left.\left.-\frac{z^2}{2}\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f\left(s\right)f\left(t\right)\dot{m}\left(s\right)\dot{m}\left(t\right)\rho\left(t\right)\sin\phi(t,t_{j-1})\rho\left(s\right)\sin\phi(t_j,s)}{\sin\phi(t_j,t_{j-1})\left(m\left(s\right)m\left(t\right)\right)^{3/2}}dtds\right)\right) \quad (24)$$

with $\Delta_j := [t_{j-1}, t_j]$.

After a straightforward but very lengthy calculation we obtain the following estimate of (24) for some $p \in \mathbf{N}_0$ and for all $\gamma > 0$:

$$\left| T\left(I_{\mathrm{TD}} \cdot \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right)\right)(zf)\right|$$

$$\leq \left(\prod_{j=1}^{n+1} \left(\frac{C_1}{|\Delta_j|} \right)^{1/2} \right) \exp\left(X^2\gamma\right) \exp\left(\left(C_2 + \frac{C_3^2}{4\gamma} \right) |z|^2 |f|_p^2 \right), \quad (25)$$

where C_1, C_2 and C_3 are constants. For the clarity of the presentation we moved the crucial steps of the estimation to the Appendix.

2. part. In this final step we use the method developed in[13] to control the convergence of (23). Although the slight modification to our case is easy we give the basic steps for the convenience of the reader.

In order to apply Theorem 4, to perform the integration, we need to show that

$$\left(\prod_{j=1}^{n+1} \left(\frac{C_1}{|\Delta_j|} \right)^{1/2} \right) \exp\left(X^2\gamma\right)$$

is integrable with respect to ν. To this end we choose $q > 2$ and $0 < \gamma < \frac{\beta}{q}$. With this choice of γ the property 1 of ν yields that $\exp\left(\gamma X^2\right) \in L^q\left(\mathbf{R}^n \times \Lambda_n, |\nu|\right)$ and with

$$Q := \left(\int_{\mathbf{R}} \int_{\Delta} \exp\left(\gamma q x^2\right) |\nu|(dt, dx) \right)^{1/q}$$

we have

$$\left(\int_{\mathbf{R^n}} \int_{\Lambda_n} \exp\left(\gamma q X^2\right) \prod_{j=1}^{n} |\nu|(dt_j, dx_j) \right)^{1/q} \leq \exp\left(\gamma\left(x_0^2 + x^2\right)\right) Q^n < \infty.$$

Now we choose p such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the property 2 of ν and the formula

$$\int_{\Lambda_n} \prod_{j=1}^{n+1} \left(\frac{C_1}{\Delta_j} \right)^{\alpha} d^n t = \left(C_1^{\alpha} \Gamma\left(1-\alpha\right)\right)^{n+1} \frac{|\Delta|^{n(1-\alpha)-\alpha}}{\Gamma\left((n+1)\left(1-\alpha\right)\right)}, \; \alpha < 1,$$

we obtain the following bound

$$\left(\int_{\mathbf{R^n}}\int_{\Lambda_n}\prod_{j=1}^{n+1}\left(\frac{C_1}{\Delta_j}\right)^{p/2}\prod_{j=1}^{n}|\nu|\left(dt_j,dx_j\right)\right)^{1/p}$$

$$\leq\quad |\nu|_{t\infty}^{n/p}\frac{C_1^{(n+1)/2}\Gamma\left(\frac{2-p}{2}\right)^{(n+1)/p}|\Delta|^{n/p-(n+1)/2}}{\Gamma\left((n+1)\frac{2-p}{2}\right)^{1/p}}<\infty,$$

$|\nu|_{t\infty}$ is shorthand notation for the essential supremum of the L^{∞}-density of $|\nu|_t$ which exists due to property 2 of ν.

Finally an application of Hölder's inequality gives

$$\left|\left(\prod_{j=1}^{n+1}\left(\frac{C_1}{\Delta_j}\right)^{1/2}\right)\exp\left(\gamma X^2\right)\right|_1$$

$$\leq\quad \exp\left(\gamma x_0^2+\gamma x^2\right)Q^n\,|\nu|_{t\infty}^{n/p}\frac{C_1^{(n+1)/2}\Gamma\left(\frac{2-p}{2}\right)^{(n+1)/p}|\Delta|^{n/p-(n+1)/2}}{\Gamma\left((n+1)\frac{2-p}{2}\right)^{1/p}}=:B_n<\infty$$

Hence Theorem 4 yields

$$I_n:=(-i)^n\int_{\mathbf{R^n}}\int_{\Lambda_n}\left(I_{\text{TD}}\cdot\prod_{j=1}^{n}\delta\left(x\left(t_j\right)-x_j\right)\right)\prod_{j=1}^{n}\nu\left(dt_j,dx_j\right)\in(S)'.$$

As the B_n are rapidly decreasing in n the hypothesis of Theorem 3 are fulfilled and hence

$$I_{\text{TD,V}}=\sum_{n=0}^{\infty}I_n\in(S)'. \tag{26}$$

In the Appendix we restricted Δ to $0<\phi\left(t,t_0\right)<\pi-\epsilon$. Since the proof works for all $\epsilon>0$ we proved (26) for all Δ such that $0<\phi\left(t,t_0\right)<\pi$. ■

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Apppendix: The estimation of the T-transform

We used the following properties of the functions inside of the expression (24) in order to estimate it. We assumed that $m \in C^2(\mathbf{R})$ and $m > 0$. Furthermore, in Subsection A we have shown that $\rho \in C^2(\mathbf{R})$ and $\rho > 0$. So we can conclude that there exist $\epsilon_1, \epsilon_2 > 0$ and $K_1, K_2, K_3, K_4 \in \mathbf{R}$ such that

$$0 < \epsilon_1 \leq m\,(s) \leq K_1, \quad \dot{m}\,(s) \leq K_2,$$

$$0 < \epsilon_3 \leq \rho\,(s) \leq K_3, \quad \dot{\rho}\,(s) \leq K_4, \quad \forall s \in [t_o, t]\,.$$

Further, it is easy to see that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\sin s \leq s, \quad \sin s \geq C_e s, \quad s \in [0, \pi - \epsilon]\,,$$

and

$$\phi(t_{j-1}, t_j) = \int_{\Delta_j} \rho^{-2}\,(u)\,du \leq |\Delta_j|\,\epsilon_3^{-2},$$

$$\phi(t_{j-1}, t_j) = \int_{\Delta_j} \rho^{-2}\,(u)\,du \geq |\Delta_j|\,K_3^{-2}.$$

Therefore we have for Δ_j such that $0 < \phi(t_j, t_{j-1}) \leq \pi - \epsilon$:

$$\sin \phi(t_j, t_{j-1}) \leq |\Delta_j|\,\epsilon_3^{-2}, \tag{27}$$

$$\sin \phi(t_j, t_{j-1}) \geq C\,|\Delta_j|\,K_3^{-2}. \tag{28}$$

Since we use inequality (27) and (28) very often in the following estimates we restrict Δ such that $0 < \phi\,(t, t_0) < \pi - \epsilon$ in order to be sure that these inequalities are valid.

In addition we need the following definitions:

$$X := \max_{0 \leq j \leq n+1} |x_j|$$

and

$$\|f\| := \sup_{\Delta} |f| + \sup_{\Delta} \left|\dot{f}\right| + |f|_0\,, \quad f \in S(\mathbf{R}).$$

Clearly there exists $p \in \mathbf{N}_0$ such that $|\cdot|_p$ estimates $\|\cdot\|$ on $S(\mathbf{R})$.

Using the properties derived above the following estimates can easily be found:

$$\Bigg|\exp\left(-\frac{z^2}{2}\left|f_{\Delta^C}\right|^2\right)\prod_{j=1}^{n+1}\frac{(m(t_{j-i})m(t_j))^{1/4}}{(2i\pi\rho(t_{j-1})\rho(t_j)\sin\phi(t_j,t_{j-1}))^{1/2}}$$

$$\times\exp\Bigg(-\frac{iz^2}{2}\int_{\Delta_j}\frac{f^2(s)}{m(s)^{1/2}}ds-\frac{i}{4}\left(\dot{m}(t_j)x_j^2-\dot{m}(t_{j-1})x_{j-1}^2\right)$$

$$+\frac{i}{2}\Bigg(m(t_j)x_j^2\left(\frac{\dot{\rho}(t_j)}{\rho(t_j)}+\frac{\cos\phi(t_j,t_{j-1})}{\rho^2(t_j)\sin\phi(t_j,t_{j-1})}\right)$$

$$-m(t_{j-1})x_{j-1}^2\left(\frac{\dot{\rho}(t_{j-1})}{\rho(t_{j-1})}-\frac{\cos\phi(t_j,t_{j-1})}{\rho^2(t_{j-1})\sin\phi(t_j,t_{j-1})}\right)-\frac{2(m(t_j)m(t_{j-1}))^{1/2}x_jx_{j-1}}{\rho(t_{j-1})\rho(t_j)\sin\phi(t_j,t_{j-1})}\Bigg)$$

$$+\frac{i}{2}\Bigg(-\frac{zm(t_j)^{1/2}x_j}{\rho(t_j)\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f(t)\,\rho(t)\sin\phi(t,t_{j-1})\dot{m}(t)}{m(t)^{1/2}}dt$$

$$-\frac{zm(t_{j-1})^{1/2}x_{j-1}}{\rho(t_{j-1})\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f(t)\,\rho(t)\sin\phi(t_j,t)\dot{m}(t)}{m(t)^{1/2}}dt$$

$$-2z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f(s)\,f(t)\left(\dot{\rho}(t)\sin\phi(t,t_{j-1})+\rho^{-1}(t)\cos\phi(t,t_{j-1})\right)}{\sin\phi(t_j,t_{j-1})(m(s)\,m(t))^{1/2}}$$

$$\times\left(\dot{\rho}(s)\sin\phi(t_j,s)-\rho^{-1}(s)\cos\phi(t_j,s)\right)dtds$$

$$+z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f(s)\,f(t)\,\dot{m}(t)\,\rho(t)\sin\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m(s)^{1/2}\,m(t)^{3/2}}$$

$$\times\left(\dot{\rho}(s)\sin\phi(t_j,s)-\rho^{-1}(s)\cos\phi(t_j,s)\right)dtds$$

$$+z^2\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f(s)\,f(t)\,\dot{m}(s)\,\rho(s)\sin\phi(t_j,s)}{\sin\phi(t_j,t_{j-1})m(s)^{3/2}\,m(t)^{1/2}}$$

$$\times \left(\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})+\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})\right)dtds$$

$$-\ \frac{z^2}{2}\int_{\Delta_j}\int_{t_{j-1}}^{s}\frac{f\left(s\right)f\left(t\right)\dot{m}\left(s\right)\dot{m}\left(t\right)\rho\left(t\right)\sin\phi(t,t_{j-1})\rho\left(s\right)\sin\phi(t_j,s)}{\sin\phi(t_j,t_{j-1})\left(m\left(s\right)m\left(t\right)\right)^{3/2}}dtds\Bigg)\Bigg)\Bigg|$$

$$\leq\ \left(\prod_{j=1}^{n+1}\left(\frac{C_1}{|\Delta_j|}\right)^{1/2}\right)\exp\left(L_1X\left|z\right|\left\|f\right\|+L_2\left|z\right|^2\left\|f\right\|^2\right),$$

where L_1 and L_2 are constants.

The estimation of the remaining factor is much more complicated. In the following we explain the basic ideas. Using the same considerations as above we find:

$$\Bigg|\prod_{j=1}^{n+1}\exp\Bigg(\frac{izm(t_j)^{1/2}x_j}{\rho(t_j)\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\left(\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})+\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})\right)}{m\left(t\right)^{1/2}}dt$$

$$+\ \frac{izm(t_{j-1})^{1/2}x_{j-1}}{\rho(t_{j-1})\sin\phi(t_j,t_{j-1})}\int_{\Delta_j}\frac{f\left(t\right)\left(\dot{\rho}\left(t\right)\sin\phi(t_j,t)-\rho^{-1}\left(t\right)\cos\phi(t_j,t)\right)}{m\left(t\right)^{1/2}}dt\Bigg)\Bigg|$$

$$\leq\ \exp\left(L_3X\left|z\right|\left\|f\right\|\right)\exp\Bigg(\left|z\right|\Bigg|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\Bigg(\int_{\Delta_j}\frac{f\left(t\right)\dot{\rho}\left(t\right)\sin\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m\left(t\right)^{1/2}}dt$$

$$+\ \int_{\Delta_{j+1}}\frac{f\left(t\right)\dot{\rho}\left(t\right)\sin\phi(t_{j+1},t)}{\sin\phi(t_{j+1},t_j)m\left(t\right)^{1/2}}dt\Bigg)\Bigg|\Bigg) \tag{29}$$

$$\times\exp\Bigg(\left|z\right|\Bigg|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\Bigg(\int_{\Delta_j}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m\left(t\right)^{1/2}}dt$$

$$-\ \int_{\Delta_{j+1}}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t_{j+1},t)}{\sin\phi(t_{j+1},t_j)m\left(t\right)^{1/2}}dt\Bigg)\Bigg|\Bigg), \tag{30}$$

where L_3 is a constant. Now we at first estimate factor (29):

$$\exp\left(|z|\left|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\left(\int_{\Delta_j}\frac{f(t)\,\dot{\rho}(t)\sin\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m(t)^{1/2}}dt\right.\right.\right.$$

$$\left.\left.\left.+\int_{\Delta_{j+1}}\frac{f(t)\,\dot{\rho}(t)\sin\phi(t_{j+1},t)}{\sin\phi(t_{j+1},t_j)m(t)^{1/2}}dt\right)\right|\right)$$

$$\leq \exp\left(\frac{|z|\,XK_4K_1^{1/2}}{\epsilon_3\epsilon_1^{1/2}}\sum_{j=1}^{n}\left(\int_{\Delta_j}|f(t)|\,dt+\int_{\Delta_{j+1}}|f(t)|\,dt\right)\right)$$

$$\leq \exp\left(L_4X\,|z|\,\|f\|\right),$$

where L_4 is a constant. To estimate the factor (30) we add and subtract a term in the exponential and divide the result into the factors (31) and (32). Factor (31) is easy to estimate but the estimation of factor (32) will be the most difficult one. For factor (31) we have

$$\exp\left(|z|\left|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\left(\int_{\Delta_{j+1}}\frac{f(t)\,\rho^{-1}(t)\cos\phi(t,t_j)}{\sin\phi(t_{j+1},t_j)m(t)^{1/2}}dt\right.\right.\right.$$

$$\left.\left.\left.-\int_{\Delta_{j+1}}\frac{f(t)\,\rho^{-1}(t)\cos\phi(t,t_{j+1})}{\sin\phi(t_{j+1},t_j)m(t)^{1/2}}dt\right)\right|\right) \tag{31}$$

$$\leq \exp\left(|z|\left|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\left(\int_{\Delta_{j+1}}\frac{f(t)\,\rho^{-1}(t)}{\sin\phi(t_{j+1},t_j)m(t)^{1/2}}\right.\right.\right.$$

$$\left.\left.\left.\times\int_{\Delta_{j+1}}\rho^{-2}(\tau)\sin\phi(t,\tau)\,d\tau dt\right)\right|\right)$$

$$\leq \exp\left(|z|\,X\sum_{j=1}^{n}\frac{K_1^{1/2}}{\epsilon_3^3\epsilon_1^{1/2}}\int_{\Delta_{j+1}}\frac{|f(t)|}{\sin\phi(t_{j+1},t_j)}\int_{\Delta_{j+1}}\sin\phi(t,\tau)\,d\tau dt\right)$$

$$\leq \quad \exp\left(L_5 X\,|z|\,\|f\|\right),$$

where L_5 is a constant.
And now we estimate the factor (32):

$$\exp\left(|z|\left|\sum_{j=1}^{n}\frac{m(t_j)^{1/2}x_j}{\rho(t_j)}\left(\int_{\Delta_j}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})}{\sin\phi(t_j,t_{j-1})m\left(t\right)^{1/2}}dt\right.\right.\right.$$

$$\left.\left.\left.-\int_{\Delta_{j+1}}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t,t_j)}{\sin\phi(t_{j+1},t_j)m\left(t\right)^{1/2}}dt\right)\right|\right). \tag{32}$$

To do this we expand

$$F(t_{j-1})=\int_{\Delta_j}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t,t_{j-1})}{m\left(t\right)^{1/2}}dt$$

and

$$G(t_{j+1})=\int_{\Delta_{j+1}}\frac{f\left(t\right)\rho^{-1}\left(t\right)\cos\phi(t_j,t)}{m\left(t\right)^{1/2}}dt$$

around t_j. This yields with $\eta_j\in\Delta_j$ and $\eta_{j+1}\in\Delta_{j+1}$ the following bound for factor (32)

$$\leq \quad \exp\left(|z|\,X\left|\sum_{j=1}^{n}\frac{f(t_j)\rho^{-1}(t_j)}{m(t_j)^{1/2}}\left(\frac{|\Delta_j|}{\sin\phi(t_j,t_{j-1})}-\frac{|\Delta_{j+1}|}{\sin\phi(t_{j+1},t_j)}\right)\right|\right) \tag{33}$$

$$\times\exp\left(|z|\,X\left|\sum_{j=1}^{n}\frac{|\Delta_{j-1}|^2}{2\sin\phi(t_j,t_{j-1})}\ddot{F}(\eta_j)-\frac{|\Delta_{j+1}|^2}{2\sin\phi(t_{j+1},t_j)}\ddot{G}(\eta_{j+1})\right|\right) \tag{34}$$

To bound the first factor we look at the function

$$r\left(y\right)=\frac{d}{dy}\left(\frac{y}{\sin\phi\left(y,0\right)}\right)=\frac{\sin\phi\left(y,0\right)-y\cos\phi\left(y,0\right)\rho^{-2}\left(y\right)}{\sin^2\phi\left(y,0\right)},\quad 0<y\leq|\Delta|$$

because

$$\frac{|\Delta_j|}{\sin\phi(t_j,t_{j-1})}-\frac{|\Delta_{j+1}|}{\sin\phi(t_{j+1},t_j)}=\int_{|\Delta_{j+1}|}^{|\Delta_j|}r\left(y\right)dy.$$

If we can show that

$$|r(y)| \leq M, \quad \forall y \in (0, |\Delta|] \tag{35}$$

then we can bound the factor (33) by

$$\begin{aligned} &\leq \exp\left(2M\epsilon_3^{-1}\epsilon_1^{-1/2}X\,|\Delta|\,|z|\,\|f\|\right) \\ &= \exp\left(L_6 X\,|z|\,\|f\|\right), \end{aligned}$$

where L_6 is a constant. If there exists $y_0 \in (0, |\Delta|]$ such that $|r(y)| \leq |r(y_0)|$ for all $y \in (0, |\Delta|]$ we are done. Hence, we only have to take care in the limit $y \to 0$ in order to show (35). Applying the rule of de L'Hôspital twice we find that

$$\lim_{y\to 0} |r(y)| = \frac{K_4}{2\epsilon_3^2}$$

and so we know that the constant required in (35) exists.

For the second factor (34) a lengthy but straightforward calculation gives us the following bound:

$$\begin{aligned} &\exp\left(\frac{|z|\,X\epsilon_3^2}{2C_\epsilon}\left(\sum_{j=1}^{n}\Delta_j\left|\ddot{F}(\eta_j)\right| + \sum_{j=1}^{n}\Delta_{j+1}\left|\ddot{G}(\eta_{j+1})\right|\right)\right) \\ \leq\ &\exp\left(\frac{X\epsilon_3^2}{2C_\epsilon}\left(\left(\left(\frac{2K_4}{\epsilon_3^4\epsilon_1^{1/2}} + \frac{1}{\epsilon_3^5\epsilon_1^{1/2}}\right)\sum_{j=1}^{n}\Delta_j^2\right.\right.\right. \\ &\left.\left.\left. + \frac{K_1^{1/2}\left(\epsilon_3^{-1} + \epsilon_3^{-2}K_4 + \epsilon_3^{-3}\right) + K_2(2\epsilon_1^{1/2}\epsilon_3)^{-1}}{\epsilon_1}\sum_{j=1}^{n}|\Delta_{j+1}|\right)|z|\,\|f\|\right) \\ \leq\ &\exp\left(L_7 X\,|z|\,\|f\|\right), \end{aligned}$$

where L_7 is a constant.
Hence we arrive at the following estimate for (30):

$$\exp\left((L_4 + L_5 + L_6 + L_7)\,X\,|z|\,\|f\|\right).$$

Putting all of this together we finally obtain

$$\left| T\left(I_{\mathrm{TD}} \cdot \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right)\right)(zf)\right|$$

$$\leq \left(\prod_{j=1}^{n+1}\left(\frac{C_1}{|\Delta_j|}\right)^{1/2}\right) \exp\left(L_2 |z|^2 \|f\|^2 + (L_3 + L_4 + L_5 + L_6 + L_7) X |z| \|f\|\right)$$

$$\leq \left(\prod_{j=1}^{n+1}\left(\frac{C_1}{|\Delta_j|}\right)^{1/2}\right) \exp\left(C_2 X |z| |f|_p + C_3 |z|^2 |f|_p\right),$$

where C_1, C_2 and C_3 are constants. For a more detailed consideration we refer to.[35]

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