DIFFERENTIAL GEOMETRY ON COMPOUND POISSON SPACE

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ABSTRACT. In this paper we carry out analysis and geometry for a class of infinite dimensional manifolds, namely, compound configuration spaces as a natural generalization of the work [AKR97]. More precisely a differential geometry is constructed on the compound configuration space Ω_X over a Riemannian manifold X. This geometry is obtained as a natural lifting of the Riemannian structure on X. In particular, the intrinsic gradient ∇^{Ω_X} divergence $\mathrm{div}^{\Omega_X}_{\pi^\tau_\sigma}$, and Laplace-Beltrami operator $H^{\Omega_X}_{\pi^\tau_\sigma} = -\mathrm{div}^{\Omega_X}_{\pi^\tau_\sigma} \nabla^{\Omega_X}$ are constructed. Therefore the corresponding Dirichlet forms $\mathcal{E}^{\Omega_X}_{\pi^\tau_\sigma}$ on $L^2(\Omega_X, \pi^\tau_\sigma)$ can be defined. Each is shown to be associated with a diffusion process on Ω_X (so called equilibrium process) which is nothing but the diffusion process on the simple configuration space Γ_X together with corresponding marks, i.e, $(X^{\gamma_\omega}_t, m_\omega)$. As another consequence of our results we obtain a representation of the Lie-algebra of compactly supported vector fields on X on compound Poisson space. Finally generalizations to the case when π^τ_σ is replaced by a marked Poisson measure $\mu_{\sigma \otimes \tau}$ easily follow from this construction.

1. INTRODUCTION

Starting with the work of Gelfand et al. [GGV75], many researchers consider representations on compound Poisson space, see also [I96]. Hence it is natural to ask about geometry and analysis on this space. On the other hand in statistical physics of continuous systems compound Poisson measures and their Gibbsian perturbation are used for the description of many concrete models, see e.g. [AGL78].

In constructing analysis and geometry in the space of simple configurations Γ_X over a manifold X, i.e.,

$$\Gamma_X = \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\},$$

an important tool is the action of the group of diffeomorphism $\mathrm{Diff}_0(X)$ on X which are equal to the identity outside a compact on the configuration space Γ_X (similar to shifts on linear spaces), cf. [AKR97].

In this paper we present a natural extension of the results obtained in [AKR97] to the case of compound configuration space Ω_X over a Riemannian manifold X, i.e., the space of $\mathbb{R}_+$-valued measures on X of the form

$$\Omega_X = \{\omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_x \in \mathcal{D}' | s_x \in \mathrm{supp}\ \tau, \gamma_\omega \in \Gamma_X\},$$

where τ is a finite measure on $\mathbb{R}_+$. This geometry is constructed via a "lifting procedure" and is completely determined by the Riemannian structure on X (cf. Subsection 3.3). In particular, we obtain the corresponding intrinsic gradient ∇^{Ω_X},

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divergence $\operatorname{div}^{\Omega_X}_{\pi^\tau_\sigma}$, and Laplace-Beltrami operator $\triangle^{\Omega_X} = \operatorname{div}^{\Omega_X}_{\pi^\tau_\sigma} \nabla^{\Omega_X}$. For details we refer to Section 3 and Section 4. Here we only mention that the "tangent bundle" $T\Omega_X$ of Ω_X is given as follows

$$T_\omega \Omega_X := L^2(X \to TX; \omega), \ \omega \in \Omega_X,$$

i.e., the space of sections in the tangent bundle TX of X which are square-integrable with respect to the Radon measure ω. Since each $T_\omega \Omega_X$ is thus a Hilbert space (endowed with the corresponding L^2-inner product $\langle \cdot, \cdot \rangle_{T_\omega \Omega_X}$ coming from the measure ω) Ω_X obtains a Riemannian-type structure which is non-trivial (i.e., varies with ω) even when $X = \mathbb{R}^d$.

The problem of analysis and geometry on infinite dimensional spaces is highly connected with the lack of a good notion of "volume element" which is due to the fact that there is no Lebesgue measure on infinite dimensional linear spaces. The volume element on X is (up to constant multiples) the unique positive Radon measure μ on X such that the gradient ∇^X and the divergence div^X become dual operators on $L^2(X, \mu)$ (w.r.t. $\langle \cdot, \cdot \rangle_{TX}$), see e.g. [C84]. In Subsection 3.2 we prove that the probability measure π^τ_σ on Ω_X for which ∇^{Ω_X} and $\operatorname{div}^{\Omega_X}_{\pi^\tau_\sigma}$ become dual operators on $L^2(\Omega_X, \pi^\tau_\sigma)$ (w.r.t. $\langle \cdot, \cdot \rangle_{T\Omega_X}$) is the right "volume element" corresponding to our differential geometry on Ω_X. Of course for completeness concerning the "volume element" one should investigate for which class of measures the above result is valid but we do not explore this question in the paper.

Let us stress that the "test" functions $\mathcal{F}C^\infty_b(\mathcal{D}, \Omega_X)$ (resp. "test" vector fields V) we consider as domains for our gradient ∇^{Ω_X} (resp. $\operatorname{div}^{\Omega_X}_{\pi^\tau_\sigma}$) above are of cylinder type, i.e., $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega_X)$ if and only if

$$\omega \mapsto F(\omega) = g_F(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle)$$

for some $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{D} := C^\infty_0(X)$, $g_F \in C^\infty_b(\mathbb{R}^N)$ (and V correspondingly, cf. (3.22)). Hence so far the analysis on Ω_X is basically finite dimensional. However, we can do generic infinite dimensional analysis on Ω_X by introducing the first order Sobolev space $H^{1,2}_0(\Omega_X, \pi^\tau_\sigma)$ by closing the corresponding Dirichlet form

$$\mathcal{E}^{\Omega_X}_{\pi^\tau_\sigma}(F, G) = \int_{\Omega_X} \langle (\nabla^{\Omega_X} F)(\omega), (\nabla^{\Omega_X} G)(\omega) \rangle_{T_\omega \Omega_X} d\pi^\tau_\sigma(\omega),$$

on $L^2(\Omega_X, \pi^\tau_\sigma)$, i.e., a function $F \in H^{1,2}_0(\Omega_X, \pi^\tau_\sigma)$ is together with its gradient $\nabla^{\Omega_X} F$ obtained as a limit in $L^2(\Omega, \pi^\tau_\sigma)$ of a sequence $F_n \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega_X)$, resp. $\nabla^{\Omega_X} F_n$, $n \in \mathbb{N}$. Thus such F really depends on infinitely many points in X (cf. Subsections 4.1, 4.2).

Since on X there is a natural diffusion process intrinsically determined by the geometry, namely distorted Brownian motion on X, it is natural to ask whether the same is true for our geometry on Ω_X. In the case of the space of simple configurations Γ_X this process is constructed using the standard theory of Dirichlet forms (cf. [MR92]), see recent results in [Y96] and [AKR97]. In the case of compound configuration space this process has a simple relation with the one mentioned above. Let us explain this more precisely. We regard every compound configuration $\omega \in \Omega_X$ as depending on two variables, namely $\omega = (\gamma_\omega, m_\omega)$ (or more general marked configuration) and this allowed us to obtain the following embedding

$$L^2(\Gamma_X, \pi_\sigma) \hookrightarrow L^2(\Omega_X, \pi^\tau_\sigma).$$

As a result we may apply operators acting on $L^2(\Gamma_X, \pi_\sigma)$, e.g. ∇^Γ, $\nabla^{\Gamma*}$ to the space $L^2(\Omega_X, \pi_\sigma^\tau)$ acting on part of the variables, see e.g. [BK88]. It turns out that the following equality holds

$$(\nabla^\Omega F)(\omega) = (\nabla^\Gamma F)((\gamma_\omega, m_\omega)), \ \omega = (\gamma_\omega, m_\omega) \in \Omega_X,$$

and from this relation it is not hard to obtain relations between the Dirichlet operators as well as between the correspondings semigroups. Therefore the process associated to our Dirichlet form is nothing but the process $X_t^{\gamma_\omega}$, $t \geq 0$, together with marks, i.e.,

$$\Xi_t = (X_t^{\gamma_\omega}, m_\omega), \ t \geq 0,$$

where $X_t^{\gamma_\omega}$ is just the equilibrium process on Γ_X, see [AKR97, Sect. 6] for details. We describe this procedure in detail in Section 5.

We would like to emphasize the contents of Subsection 3.4. It is well-known, see [GGV75, Sect. 6] that there is a canonical unitary representation on compound Poisson space, i.e., $L^2(\Omega_X, \pi_\sigma^\tau)$, of the group of diffeomorphisms $\mathrm{Diff}_0(X)$. On the basis of our results described above, we provide a corresponding representation of the associated Lie algebra of compactly supported vector fields. We also exhibit explicit formulas for the corresponding generators.

Thus the contents of Sections 3, 4, and 5 have been described. It remains to add that Section 2 consists of necessary preliminaries to the furthers sections. Finally in Section 6 we prove in detail the existence of a marked Poisson measure over the marked Poisson space Ω_X^M, where M is a complete separable metric space with a probability measure. Hence all the results obtained in this paper extend with trivial changes to marked Poisson space.

Last but not least we would like to mention that most of the results obtained in this paper extend in a natural way (along the lines of the work [AKR97a] and [AGL78]) to the case where compound Poisson measures are replaced by Gibbs measures of Ruelle type.

Part of the results of this paper were presented in the international conference "Analysis on infinite-dimensional Lie algebras and groups" in Marseille September'97.

2. MEASURES ON CONFIGURATION SPACES

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by $T_x X$; and the tangent bundle endowed with its natural differentiable structure will be denoted by $TX = \cup_{x \in X} T_x X$. The Riemannian metric on X associates to each point $x \in X$ an inner product on $T_x X$ which we denote by $\langle \cdot, \cdot \rangle_{T_x X}$. The associated norm will be denoted by $| \cdot |_{T_x X}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

2.1. The configuration space over a manifold.

The configuration space Γ_X over the manifold X is defined as the set of all locally finite subsets (simple configurations) in X :

$$\Gamma_X = \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}.$$

Here $|A|$ denotes the cardinality of a set A.

We can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure

$$\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}_p(X) \subset \mathcal{M}(X),$$

where $\sum_{x \in \emptyset} \varepsilon_x :=$ zero measure and $\mathcal{M}(X)$ (resp. $\mathcal{M}_p(X)$) denotes the set of all positive (resp. positive integer-valued) Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X such that all maps

$$\Gamma_X \ni \gamma \mapsto \langle \gamma, f \rangle := \int_X f(x) d\gamma(x) = \sum_{x \in \gamma} f(x)$$

are continuous. Here $f \in C_0(X)$ (the set of all real-valued continuous functions on X with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ-algebra. $\mathcal{B}(\Gamma_X)$ is generated by the sets

$$(2.1) \qquad\qquad C_{\Lambda,n} := \{ \gamma \in \Gamma_X \,|\, |\gamma \cap \Lambda| = n \},$$

where $\Lambda \in \mathcal{O}_c(X)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, see e.g. [GGV75] and [S94]. Note that for any $\Lambda \in \mathcal{B}(X)$ and all $n \in \mathbb{N}_0$ the set $C_{\Lambda,n}$ is, indeed, a Borel subset of Γ_X. Sets of the form (2.1) are called cylinder sets.

For any $B \subset X$ we introduce a function $N_B : \Gamma_X \to \mathbb{N}_0$ such that

$$N_B(\gamma) = |\gamma \cap B|, \ \gamma \in \Gamma_X.$$

Then $\mathcal{B}(\Gamma_X)$ is the smallest σ-algebra on Γ_X such that all the functions N_B, $B \in \mathcal{B}_c(X)$, are measurable.

2.2. Poisson measures.

For the construction of a Poisson measure on Γ_X first we need to fix an intensity measure σ on the underlying manifold X. We take a density $\rho > 0$ m-a.s. such that $\rho^{1/2} \in H^{1,2}_{loc}(X)$ and put $d\sigma(x) = \rho(x) dm(x)$. Here $H^{1,2}_{loc}(X)$ denotes the local Sobolev space of order 1 in $L^2_{loc}(X,m)$. Then σ is a non-atomic Radon measure on X, in particular, $\sigma(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_c(X)$.

There are different ways to define the Poisson measure π_σ with intensity σ on Γ_X, see e.g. [AKR97] and [GV68]. Here we characterize π_σ by its Laplace transform.

Definition 2.1. The Laplace transform of π_σ is given for $f \in C_0(X)$ by

$$(2.2) \qquad \begin{aligned} l_{\pi_\sigma}(f) &= \int_{\Gamma_X} \exp(\langle \gamma, f \rangle) d\pi_\sigma(\gamma) \\ &= \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x) \right). \end{aligned}$$

Let us mention that (2.2) defines, via Minlos' theorem, a measure π_σ on a linear space $F(X)$ of generalized functions on X, see e.g. [GV68]. An additional analysis shows that the support of the measure π_σ consists of generalized functions of the form $\sum_{x \in \gamma} \varepsilon_x$, $\gamma \in \Gamma_X$, see e.g. [O87] and [S94], and then π_σ can be considered as a measure on Γ_X.

Remark 2.2. By the same argument (2.2) holds for any $\mathcal{B}(X)$-measurable function f with compact support such that e^f is σ-integrable on $\mathrm{supp}\, f$. A simple limit-argument then implies that (2.2) holds for all f such that $e^f - 1 \in L^1(\sigma)$.

2.3. Compound Poisson measures.

Let τ be a non-negative finite measure on $\mathbb{R}_+ :=]0, \infty[$ having all moments finite and satisfying the analyticity property

$$(2.3) \qquad \exists C > 0 : \forall n \in \mathbb{N}_0 \int_0^\infty |s|^n d\tau(s) < C^n n!.$$

We denote $\mathcal{D} = C_0^\infty(X)$ (the set of C^∞-functions on X with compact support) equipped with the usual topology, see e.g. [Au82, Chap. 2].

Definition 2.3. A measure π_σ^τ on $\mathcal{D}'$ is called a compound Poisson measure if its Laplace transform is given for $\varphi \in \mathcal{D}$ by

$$(2.4) \qquad \begin{aligned} l_{\pi_\sigma^\tau}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle \omega, \varphi \rangle) d\pi_\sigma^\tau(\omega) \\ &= \exp\left(\int_X \int_0^\infty (e^{s\varphi(x)} - 1) d\tau(s) d\sigma(x) \right), \end{aligned}$$

see e.g. [GGV75].

The measure π_σ^τ has the following properties.

Proposition 2.4. *1. π_σ^τ has an analytic Laplace transform.*
2. π_σ^τ is supported on $\Omega := \Omega_X$, the space of compound configurations, i.e.,

$$\Omega = \{ \omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_x \in \mathcal{D}' | s_x \in \operatorname{supp} \tau, \ \gamma_\omega \in \Gamma_X \},$$

in other words $\pi_\sigma^\tau(\Omega) = 1$.
3. If $\operatorname{supp} \tau = \{1\}$, i.e., $d\tau(s) = \varepsilon_1(ds)$, then $\pi_\sigma^\tau = \pi_\sigma$.

For the proof of the above proposition we refer to [KSSU97], [Ob84]. A more general case so called marked Poisson measure is worked out in Section 6.

2.4. The isomorphism between Poisson and compound Poisson spaces.

Let us define a measure $\hat{\sigma}$ on $(X \times \mathbb{R}_+, \mathcal{B}(X \times \mathbb{R}_+))$ as the product measure of the measures τ and σ, i.e.,

$$d\hat{\sigma}(\hat{x}) := d\tau(s) d\sigma(x), \ \hat{x} = (x, s) \in X \times \mathbb{R}_+.$$

Denote by $\hat{\Gamma}$ the set of the locally finite configurations $\hat{\gamma} \subset X \times \mathbb{R}_+$ such that

$$\hat{\gamma} = \sum_{\hat{x}_i \in \hat{\gamma}} \varepsilon_{\hat{x}_i}, \ \hat{x}_i = (x_i, s_i) \in X \times \mathbb{R}_+, \ x_i \neq x_j, \ i \neq j$$

and define the Poisson measure $\pi_{\hat{\sigma}}$ with intensity measure $\hat{\sigma}$ on $(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ via its Laplace transform

$$(2.5) \qquad \begin{aligned} l_{\pi_{\hat{\sigma}}}(\hat{\varphi}) &= \int_{\hat{\Gamma}} \exp(\langle \hat{\gamma}, \hat{\varphi} \rangle) d\pi_{\hat{\sigma}}(\hat{\gamma}) \\ &= \exp\left(\int_{X \times \mathbb{R}_+} (e^{\hat{\varphi}(\hat{x})} - 1) d\hat{\sigma}(\hat{x}) \right), \ \hat{\varphi} \in \mathcal{D}(X \times \mathbb{R}_+). \end{aligned}$$

It follows from (2.3) that the Laplace transform $l_{\pi_{\hat{\sigma}}}$ is well defined for $\hat{\varphi}(s, x) = p(s)\varphi(x)$ where $p(s) = \sum_{k=0}^{m} p_k s^k$ ($p_0 \neq 0$) is a polynomial and $\varphi \in \mathcal{D}$ (cf. [LRS95]). Let us put $\hat{\varphi}(s, x) = s\varphi(x)$, $\varphi \in \mathcal{D}$ in (2.5). Then by (2.2) we obtain

$$l_{\pi_{\sigma}^{\tau}}(\varphi) = l_{\pi_{\hat{\sigma}}}(s\varphi), \ \varphi \in \mathcal{D}.$$

Then it follows that the compound Poisson measure π_{σ}^{τ} is the image of $\pi_{\hat{\sigma}}$ under the transformation $\Sigma : \hat{\Gamma} \to \Sigma\hat{\Gamma} = \Omega \subset \mathcal{D}'$ given by

$$(2.6) \qquad \hat{\Gamma} \ni \hat{\gamma} \mapsto (\Sigma\hat{\gamma})(\cdot) = \Sigma\left(\sum_{\hat{x_i} \in \hat{\gamma}} \varepsilon_{\hat{x_i}}\right)(\cdot) := \sum_{(s_i, x_i) \in \hat{\gamma}} s_i \varepsilon_{x_i}(\cdot) \in \Omega \subset \mathcal{D}',$$

i.e., $\forall B \in \mathcal{B}(\mathcal{D}')$

$$\pi_{\sigma}^{\tau}(B) = \pi_{\sigma}^{\tau}(B \cap \Omega) = \pi_{\hat{\sigma}}(\Sigma^{-1}(B \cap \Omega)),$$

where $\Sigma^{-1}\Delta$ is the pre-image of the set Δ.

The latter equality may be rewritten in the following form

$$\int_{\mathcal{D}'} \mathbb{1}_B(\omega) d\pi_{\sigma}^{\tau}(\omega) = \int_{\Omega} \mathbb{1}_B(\omega) d\pi_{\sigma}^{\tau}(\omega) = \int_{\hat{\Gamma}} \mathbb{1}_B(\Sigma\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}),$$

which is analogous to the well known change of variable formula for the Lebesgue integral. Namely, for any $h \in L^1(\mathcal{D}', \pi_{\sigma}^{\tau}) = L^1(\Omega, \pi_{\sigma}^{\tau})$ the function $h \circ \Sigma \in L^1(\hat{\Gamma}, \pi_{\hat{\sigma}})$ and

$$(2.7) \qquad \int_{\Omega} h(\omega) d\pi_{\sigma}^{\tau}(\omega) = \int_{\hat{\Gamma}} h(\Sigma\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}).$$

Remark 2.5. It is worth noting that there exists on Ω an inverse map $\Sigma^{-1} : \Omega \to \hat{\Gamma}$. And we obtain that $\pi_{\hat{\sigma}}$ on $\hat{\Gamma}$ is the image of π_{σ}^{τ} on Ω under the map Σ^{-1}, i.e., $\forall \hat{C} \in \mathcal{B}(\hat{\Gamma})$, $\pi_{\hat{\sigma}}(\hat{C}) = \pi_{\sigma}^{\tau}(\Sigma\hat{C})$ or after rewriting

$$\int_{\hat{\Gamma}} \mathbb{1}_{\hat{C}}(\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}) = \int_{\Omega} \mathbb{1}_{\Sigma\hat{C}}(\omega) d\pi_{\sigma}^{\tau}(\omega) = \int_{\Omega} \mathbb{1}_{\hat{C}}(\Sigma^{-1}\omega) d\pi_{\sigma}^{\tau}(\omega).$$

As before we easily can write the corresponding change of variables formula, namely for any $\hat{f} \in L^1(\hat{\Gamma}, \pi_{\hat{\sigma}})$ the function $\hat{f} \circ \Sigma^{-1} \in L^1(\Omega, \pi_{\sigma}^{\tau})$ and

$$(2.8) \qquad \int_{\hat{\Gamma}} \hat{f}(\hat{\gamma}) d\pi_{\hat{\sigma}}(\hat{\gamma}) = \int_{\Omega} \hat{f}(\Sigma^{-1}\omega) d\pi_{\sigma}^{\tau}(\omega).$$

Now we construct a unitary isomorphism U_{Σ} between the Poisson space $L^2(\pi_{\hat{\sigma}}) := L^2(\hat{\Gamma}, \pi_{\hat{\sigma}})$ and the compound Poisson space $L^2(\pi_{\sigma}^{\tau}) := L^2(\Omega, \pi_{\sigma}^{\tau})$. Namely,

$$L^2(\Omega, \pi_{\sigma}^{\tau}) \ni h \mapsto U_{\Sigma}h := h \circ \Sigma \in L^2(\hat{\Gamma}, \pi_{\hat{\sigma}})$$

and

$$L^2(\hat{\Gamma}, \pi_{\hat{\sigma}}) \ni \hat{f} \mapsto U_{\Sigma}^{-1}\hat{f} = \hat{f} \circ \Sigma^{-1} \in L^2(\Omega, \pi_{\sigma}^{\tau}).$$

The isometry of U_{Σ} and U_{Σ}^{-1} follows from (2.7) and (2.8), respectively

As a result we have established the following proposition.

Proposition 2.6. *The map U_Σ is a unitary isomorphism between the Poisson space $L^2(\pi_{\hat{\sigma}})$ and the compound Poisson space $L^2(\pi_\sigma^\tau)$.*

2.5. The group of diffeomorphisms and compound Poisson measures.

Let us denote the group of all diffeomorphisms on X by $\mathrm{Diff}(X)$ and by $\mathrm{Diff}_0(X)$ the subgroup of all diffeomorphisms $\phi : X \to X$ with compact support, i.e., which are equal to the identity outside of a compact set (depending on ϕ).

For any $f \in C_0(X)$ we have a continuous functional

$$\Omega \ni \omega \mapsto \langle \omega, f \rangle = \int_X f(x) d\omega(x) = \sum_{x \in \gamma_\omega} s_x f(x)$$

and given $\phi \in \mathrm{Diff}_0(X)$ we have

$$\langle \phi^* \omega, f \rangle = \int_X f(x) d\omega(\phi^{-1}(x))$$
$$= \sum_{x \in \gamma_\omega} s_x f \circ \phi(x)$$
$$= \langle \omega, f \circ \phi \rangle .$$

Any $\phi \in \mathrm{Diff}_0(X)$ defines (pointwise) a transformation of any subset of X and, consequently, the diffeomorphism ϕ has the following "lifting" from X to Ω :

$$\Omega \ni \omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_x \mapsto \phi^* \omega = \sum_{x \in \gamma_\omega} s_x \varepsilon_{\phi(x)} \in \Omega,$$

because for any $f \in C_0(X)$

$$\int_X f(x) d(\phi^* \omega)(x) = \int_X f(\phi(x)) d\omega(x)$$
$$= \sum_{x \in \gamma_\omega} s_x f(\phi(x))$$
$$= \int_X f(y) \sum_{x \in \gamma_\omega} s_x \varepsilon_{\phi(x)} (dy).$$

This mapping is obviously measurable and we can define the image $\phi^* \pi_\sigma^\tau$ of the measure π_σ^τ under ϕ as usually by $\phi^* \pi_\sigma^\tau = \pi_\sigma^\tau \circ \phi^{-1}$, i.e.,

$$(\phi^* \pi_\sigma^\tau)(A) = \pi_\sigma^\tau(\phi^{-1}(A)), \ A \in \mathcal{B}(\Omega).$$

The following proposition shows that this transformation is nothing but a change of the intensity measure σ, and τ is preserved.

Proposition 2.7. *For any $\phi \in \mathrm{Diff}_0(X)$ we have*

$$\phi^* \pi_\sigma^\tau = \pi_{\phi^* \sigma}^\tau.$$

Proof. Due to the characterization of the measures it is enough to compute the Laplace transform of the measure $\phi^* \pi_\sigma^\tau$, to show the property.

Let $f \in C_0(X)$ be given. Then the Laplace transform of $\phi^* \pi_\sigma^\tau$ is given by

$$
\begin{aligned}
\int_\Omega \exp(\langle \omega, f \rangle) d(\phi^* \pi_\sigma^\tau)(\omega) &= \int_\Omega \exp(\langle \omega, f \rangle) d\pi_\sigma^\tau(\phi^{-1}(\omega)) \\
&= \int_\Omega \exp(\langle \phi^* \omega, f \rangle) d\pi_\sigma^\tau(\omega) \\
&= \int_\Omega \exp(\langle \omega, f \circ \phi \rangle) d\pi_\sigma^\tau(\omega) \\
&= \exp\left(\int_X \int_0^\infty (e^{sf \circ \phi(x)} - 1) d\tau(s) d\sigma(x) \right) \\
&= \exp\left(\int_X \int_0^\infty (e^{sf(x)} - 1) d\tau(s) d(\phi^* \sigma)(x) \right) \\
&= \int_\Omega \exp(\langle \omega, f \rangle) d\pi_{\phi^* \sigma}^\tau(\omega)
\end{aligned}
$$

which is just the Laplace transform of the measure $\pi_{\phi^* \sigma}^\tau$. $\square$

For any $\phi \in \mathrm{Diff}_0(X)$ we introduce the Radon-Nikodym density of σ as

$$
(2.9) \quad
\begin{cases}
p_\phi^\sigma(x) := \dfrac{d(\phi^* \sigma)}{d\sigma}(x) = \dfrac{\rho(\phi^{-1}(x))}{\rho(x)} \dfrac{dm(\phi^{-1}(x))}{dm(x)} = \dfrac{\rho(\phi^{-1}(x))}{\rho(x)} J_m^\phi(x), \\
\\
\qquad \text{if } x \in \{0 < \rho < \infty\} \cap \{0 < \rho \circ \phi^{-1} < \infty\}; \\
\\
p_\phi^\sigma(x) := 1, \text{ otherwise,}
\end{cases}
$$

where J_m^ϕ is the Jacobian determinant of ϕ (with respect to the Riemannian volume m), see e.g. [Boo75]. Note that $p_\phi^\sigma(x) \equiv 1$ outside a compact.

The next proposition is a consequence of the Proposition 2.6, Skorokhod's theorem on absolute continuity of Poisson measures, see e.g. [Sk57], [T90] and also [S94]. It shows that π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$.

Proposition 2.8. *The compound Poisson measure π_σ^τ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ and for any $\phi \in \mathrm{Diff}_0(X)$ we have $p_\phi^{\pi_\sigma^\tau} = p_\phi^{\pi_{\lambda_\tau \sigma}}$, where $\lambda_\tau = \tau(\mathbb{R}_+)$, i.e.,*

$$
p_\phi^{\pi_\sigma^\tau}(\omega) = \frac{d(\phi^* \pi_\sigma^\tau)}{d\pi_\sigma^\tau}(\omega) = \prod_{x \in \gamma_\omega} p_\phi^\sigma(x) \exp\left(\lambda_\tau \int_X (1 - p_\phi^\sigma(x)) d\sigma(x) \right).
$$

Proof. Given $\phi \in \mathrm{Diff}_0(X)$ then $\hat{\phi} := \phi \otimes \mathrm{id} \in \mathrm{Diff}(X \times \mathbb{R}_+)$. Hence having in mind the isomorphism described in Subsection 2.4 the Radon-Nikodym density of

π_σ^τ with respect to the group $\mathrm{Diff}_0(X)$ is given by

$$
\begin{aligned}
p_\phi^{\pi_\sigma^\tau}(\omega) &= U_\Sigma^{-1} p_{\phi\otimes\mathrm{id}}^{\pi_{\hat\sigma}}(\omega) \\
&= \prod_{\hat x\in\hat\gamma_\omega} \frac{d\hat\sigma\circ(\phi\otimes\mathrm{id})^{-1}}{d\hat\sigma}(\hat x)\exp\left(\int_{X\times\mathbb{R}_+}\left(1-\frac{d\hat\sigma\circ(\phi\otimes\mathrm{id})^{-1}}{d\hat\sigma}(\hat x)\right)d\hat\sigma(\hat x)\right) \\
&= \prod_{\hat x\in\hat\gamma_\omega} p_\phi^\sigma(x)\exp\left(\lambda_\tau\int_X(1-p_\phi^\sigma(x))d\sigma(x)\right) \\
&= \prod_{x\in\gamma_\omega} p_\phi^\sigma(x)\exp\left(\lambda_\tau\int_X(1-p_\phi^\sigma(x))d\sigma(x)\right) \\
&= p_\phi^{\pi_{\lambda_\tau\sigma}}(\gamma_\omega),
\end{aligned}
$$

where we have used [AKR97, Proposition 2.2]. $\square$

3. Differential geometry on compound Poisson space

The underlying differentiable structure on X has a natural lifting to the configuration space Ω. As a result there appear in Ω objects such as the gradient, the tangent space etc. Below we describe the corresponding constructions in details.

3.1. The tangent bundle of Ω.

Let $V(X)$ be the set of all C^∞-vector fields on X (i.e., smooth sections of TX). We will use a subset $V_0(X)\subset V(X)$ consisting of all vector fields with compact support. $V_0(X)$ can be considered as an infinite dimensional Lie algebra which corresponds to the group $\mathrm{Diff}_0(X)$ in the following sense: for any $v\in V_0(X)$ we can construct the flow of this vector field as a collection of mappings $\phi_t^v:X\to X$, $t\in\mathbb{R}$ obtained by integrating the vector field.

More precisely, for any $x\in X$ the curve

$$\mathbb{R}\ni t\longmapsto \phi_t^v(x)\in X$$

is defined as the solution to the following Cauchy problem

$$
\begin{cases}
\dfrac{d}{dt}\phi_t^v(x) = v(\phi_t^v(x)) \\[4mm]
\phi_0^v(x) = x
\end{cases}
.
$$

That no explosion is possible and ϕ_t^v is well-defined for each $t\in\mathbb{R}$, is a consequence of $v\in V_0(X)$ (the latter implies that v is a complete vector field). The mappings $\{\phi_t^v, t\in\mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in the group $\mathrm{Diff}_0(X)$ (see e.g. [Boo75]), that is,

$$1)\ \forall t\in\mathbb{R}\ \phi_t^v\in\mathrm{Diff}_0(X)$$

$$2)\ \forall t,s\in\mathbb{R}\ \phi_t^v\circ\phi_s^v=\phi_{t+s}^v.$$

Let us fix $v\in V_0(X)$. Having the group ϕ_t^v, $t\in\mathbb{R}$, we can consider for any $\omega\in\Omega$ the curve

$$\mathbb{R}\ni t\longmapsto \phi_t^v(w)\in\Omega.$$

Definition 3.1. For a function $F : \Omega \to \mathbb{R}$ we define the directional derivative along the vector field $v \in V_0(X)$ as

$$(\nabla_v^\Omega F)(\omega) := \frac{d}{dt} F(\phi_t^{v*} \omega)|_{t=0},$$

provided the right hand side exists.

We note that $\nabla_v^\Omega F$ is closely related to the concept of the Lie derivative corresponding to a special class of vector fields on Ω, see below.

Let us introduce a special class of smooth functions on Ω which play an important role in our considerations below. We introduce $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ as the set of all functions $F : \Omega \to \mathbb{R}$ of the form

$$(3.10) \qquad F(\omega) = g_F(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle), \ \omega \in \Omega,$$

where (generating directions) $\varphi_1, \dots, \varphi_N \in \mathcal{D}$ and $g_F(s_1, \dots, s_N)$ (generating function for F) is from $C_b^\infty(\mathbb{R}^N)$.

For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ of the form (3.10) and given $v \in V_0(X)$ we have

$$F(\phi_t^{v*} \omega) = g_F(\langle \phi_t^{v*} \omega, \varphi_1 \rangle, \dots, \langle \phi_t^{v*} \omega, \varphi_N \rangle)$$
$$= g_F(\langle \omega, \varphi_1 \circ \phi_t^v \rangle, \dots, \langle \omega, \varphi_N \circ \phi_t^v \rangle)$$

and, therefore, an application of Definition 3.1 gives

$$(3.11) \qquad (\nabla_v^\Omega F)(\omega) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \langle \omega, \nabla_v^X \varphi_i \rangle,$$

where $\nabla_v^X \varphi$ is the directional (or Lie) derivative of $\varphi : X \to \mathbb{R}$ along the vector field $v \in V_0(X)$, i.e.,

$$(\nabla_v^X \varphi)(x) = \langle \nabla^X \varphi(x), v(x) \rangle_{T_x X},$$

where ∇^X denotes the gradient on X.

The expression of ∇_v^Ω on smooth cylinder functions given by (3.11) motivates the following definition.

Definition 3.2. We introduce the tangent space $T_\omega \Omega$ to the configuration space Ω at the point $\omega \in \Omega$ as the Hilbert space of measurable ω - square - integrable sections (measurable vector fields) $V_\omega : X \to TX$ with the scalar product

$$(3.12) \qquad \langle V_\omega^1, V_\omega^2 \rangle_{T_\omega \Omega} = \int_X \langle V_\omega^1(x), V_\omega^2(x) \rangle_{T_x X} \, d\omega(x)$$

$V_\omega^1, V_\omega^2 \in T_\omega \Omega$. The corresponding tangent bundle is

$$T\Omega = \bigcup_{\omega \in \Omega} T_\omega \Omega.$$

Let us stress that any $v \in V_0(X)$ can be considered as a "constant" vector field on Ω such that

$$\Omega \ni \omega \longmapsto V_\omega(\cdot) = v(\cdot) \in T_\omega \Omega,$$
$$\langle v, v \rangle_{T_\omega \Omega} = \int_X |v(x)|_{T_x X}^2 \, d\omega(x) < \infty.$$

Usually in Riemannian geometry, having the directional derivative and a Hilbert space as the tangent space we can introduce the gradient.

Definition 3.3. We define the intrinsic gradient of a function $F : \Omega \to \mathbb{R}$ as the mapping

$$\Omega \ni \omega \longmapsto (\nabla^{\Omega} F)(\omega) \in T_{\omega}\Omega$$

such that for any $v \in V_0(X)$

$$(3.13) \qquad (\nabla_v^{\Omega} F)(\omega) = \left\langle (\nabla^{\Omega} F)(\omega), v \right\rangle_{T_{\omega}\Omega}.$$

Note that (3.13), in particular, implies that $\nabla_v^{\Omega} F$ is the directional derivative along the "constant" vector field v on Ω. Furthermore, by (3.11) for any $F \in \mathcal{F}C_b^{\infty}(\mathcal{D}, \Omega)$ of the form (3.10) gives

$$(3.14) \qquad (\nabla^{\Omega} F)(\omega; x) = \sum_{i=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \nabla^X \varphi_i(x), \ \omega \in \Omega, x \in X.$$

3.2. Integration by parts and divergence on compound Poisson space.

Let the configuration space Ω be equipped with the compound Poisson measure π_{σ}^{τ} (cf. Subsection 2.3). The set $\mathcal{F}C_b^{\infty}(\mathcal{D}, \Omega)$ is a dense subset in the space $L^2(\Omega, \mathcal{B}(\Omega), \pi_{\sigma}^{\tau}) =: L^2(\pi_{\sigma}^{\tau})$. For any vector field $v \in V_0(X)$ we have a differential operator in $L^2(\pi_{\sigma}^{\tau})$ on the domain $\mathcal{F}C_b^{\infty}(\mathcal{D}, \Omega)$ given by

$$\mathcal{F}C_b^{\infty}(\mathcal{D}, \Omega) \ni F \longmapsto \nabla_v^{\Omega} F \in L^2(\pi_{\sigma}^{\tau}).$$

Our aim now is to compute the adjoint operator $\nabla_v^{\Omega*}$ in $L^2(\pi_{\sigma}^{\tau})$. It corresponds, of course, to an integration by parts formula with respect to the measure π_{σ}^{τ}.

To this end we recall first of all the integration by parts formula for the measure σ. The logarithmic derivative of σ is given by the vector field

$$X \ni x \longmapsto \beta^{\sigma}(x) := \frac{\nabla^X \rho(x)}{\rho(x)} \in T_x X.$$

(where as usual $\beta^{\sigma} := 0$ on $\{\rho = 0\}$). For all $\varphi_1, \varphi_2 \in \mathcal{D}$ we have

$$(3.15) \qquad \begin{aligned} &\int_X (\nabla_v^X \varphi_1)(x) \varphi_2(x) d\sigma(x) \\ &= -\int_X \varphi_1(x)(\nabla_v^X \varphi_2)(x) d\sigma(x) - \int_X \varphi_1(x) \varphi_2(x) \beta_v^{\sigma}(x) d\sigma(x), \end{aligned}$$

where

$$(3.16) \qquad \beta_v^{\sigma}(x) := \langle \beta^{\sigma}(x), v(x) \rangle_{T_x X} + \operatorname{div}^X v(x)$$

is the so-called logarithmic derivative of the measure σ along the vector field v and $\operatorname{div}^X := \operatorname{div}_m^X$ is the divergence on X with respect to m. Analogously, we define $\operatorname{div}_{\sigma}^X$ as the divergence on X with respect to σ, i.e., $\operatorname{div}_{\sigma}^X$ is the dual operator on $L^2(X, \sigma) =: L^2(\sigma)$ of ∇^X. Then on the one hand we can rewrite (3.15) as an operator equality on the domain $\mathcal{D} \subset L^2(\sigma)$:

$$\nabla_v^{X*} = -\nabla_v^X - \beta_v^{\sigma},$$

where the adjoint operator is considered with respect to $L^2(\sigma)$. Note that, obviously, $\beta_v^{\sigma} \in L^2(\sigma)$ for all $v \in V_0(X)$. On the other hand we have

$$(3.17) \qquad \operatorname{div}_{\sigma}^X = \beta^{\sigma}.$$

Having the logarithmic derivative β_v^{σ} we introduce an analogous object for the compound Poisson measure.

Definition 3.4. For any $v \in V_0(X)$ we define the logarithmic derivative of the compound Poisson measure π_σ^τ along v as the following function on Ω :

$$(3.18) \quad \Omega \ni \omega \mapsto B_v^{\pi_\sigma^\tau}(\omega) := \langle \gamma_\omega, \beta_v^\sigma \rangle = \int_X [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div}^X v(x)] d\gamma_\omega(x).$$

A motivation for this definition is given by the following integration by parts formula.

Theorem 3.5. *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ and any $v \in V_0(X)$ we have*

$$(3.19) \quad \begin{aligned} &\int_\Omega (\nabla_v^\Omega F)(\omega) G(\omega) d\pi_\sigma^\tau(\omega) \\ &= -\int_\Omega F(\omega)(\nabla_v^\Omega G)(\omega) d\pi_\sigma^\tau(\omega) - \int_\Omega F(\omega) G(\omega) B_v^{\pi_\sigma^\tau}(\omega) d\pi_\sigma^\tau(\omega), \end{aligned}$$

or

$$(3.20) \qquad\qquad \nabla_v^{\Omega *} = -\nabla_v^\Omega - B_v^{\pi_\sigma^\tau}$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ in $L^2(\pi_\sigma^\tau)$.

Proof. Due to Proposition 2.7 we have that

$$\int_\Omega F(\phi_t^v(\omega)) G(\omega) d\pi_\sigma^\tau(\omega) = \int_\Omega F(\omega) G(\phi_{-t}^v \omega) d\pi_{\phi_t^{v*}\sigma}^\tau(\omega).$$

Differentiating this equation with respect to t and interchanging $\frac{d}{dt}$ with the integrals, by Definition 3.1 the left hand side becomes (3.19). To see that the right hand side also coincides with (3.19) we note that

$$\frac{d}{dt} G(\phi_{-t}^v(\omega))|_{t=0} = -(\nabla_v^\Omega G)(\omega)$$

and (by Proposition 2.8)

$$\begin{aligned} &\frac{d}{dt} \left[\frac{d\pi_{\phi_t^{v*}\sigma}^\tau}{d\pi_\sigma^\tau}(\omega) \right]\Bigg|_{t=0} \\ &= \frac{d}{dt} \left[\prod_{x \in \gamma_\omega} \frac{\rho(\phi_t^{v*}(x))}{\rho(x)} J_m^{\phi_t^v}(x) \right]\Bigg|_{t=0} \\ &+ \frac{d}{dt} \left[\exp\left\{ \lambda_\tau \int_X \left(1 - \frac{\rho(\phi_t^{v*}(x))}{\rho(x)} J_m^{\phi_t^v}(x) \right) d\sigma(x) \right\} \right]\Bigg|_{t=0}. \end{aligned}$$

Using (3.16) and the formula $\frac{d}{dt}[J_m^{\phi_t^v}(x)]|_{t=0} = -\mathrm{div}^X v(x)$, the latter expressions becomes equal to

$$\begin{aligned} &- \sum_{x \in \gamma_\omega} [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div}^X v(x)] \\ &+ \lambda_\tau \int_X [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div}^X v(x)] d\sigma(x) \\ &= - \sum_{x \in \gamma_\omega} \beta_v^\sigma(x) + \lambda_\tau \int_X \beta_v^\sigma(x) d\sigma(x) = -B_v^{\pi_\sigma^\tau}(\omega), \end{aligned}$$

where we have used the equality

$$\int_X \beta_v^\sigma(x) d\sigma(x) = -\int_X (\nabla_v^{X*} 1)(x) d\sigma(x) = 0.$$

This completes the proof. $\square$

Definition 3.6. For a vector field

$$V : \Omega \ni \omega \longmapsto V_\omega \in T_\omega \Omega$$

the intrinsic divergence $\mathrm{div}^\Omega_{\pi^\tau_\sigma} V$ is defined via the duality relation

$$(3.21) \qquad \int_\Omega \left\langle V_\omega, (\nabla^\Omega F)(\omega) \right\rangle_{T_\omega \Omega} d\pi^\tau_\sigma(\omega) = - \int_\Omega F(\omega)(\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) d\pi^\tau_\sigma(\omega)$$

for all $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$, provided it exists (i.e., provided

$$F \longmapsto \int_\Omega \left\langle V_\omega, (\nabla^\Omega F)(\omega) \right\rangle_{T_\omega \Omega} d\pi^\tau_\sigma(\omega)$$

is continuous on $L^2(\pi^\tau_\sigma)$).

The existence of the divergence, of course, requires some smoothness of the vector field. A class of smooth vector fields on Ω for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.7. *For any vector field*

$$(3.22) \qquad V_\omega(x) = \sum_{j=1}^N G_j(\omega) v_j(x), \ \omega \in \Omega, \ x \in X,$$

with $G_j \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$, $v_j \in V_0(X)$, $j = 1, \dots, N$, we have

$$(3.23) \qquad \begin{aligned} (\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) &= \sum_{j=1}^N (\nabla^\Omega_{v_j} G_j)(\omega) + \sum_{j=1}^N B^{\pi^\tau_\sigma}_{v_j}(\omega) G_j(\omega) \\ &= \sum_{j=1}^N \left\langle (\nabla^\Omega G_j)(\omega), v_j \right\rangle_{T_\omega \Omega} + \sum_{j=1}^N \left\langle \gamma, \beta^\sigma_{v_j} \right\rangle G_j(\omega). \end{aligned}$$

Proof. Due to the linearity of ∇^Ω it is sufficient to consider the case $N = 1$, i.e., $V_\omega(x) = G(\omega) v(x)$. Then for all $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Omega)$ and the Definition 3.6 we have

$$\begin{aligned} \int_\Omega (\mathrm{div}^\Omega_{\pi^\tau_\sigma} V)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) &= - \int_\Omega \left\langle V_\omega, (\nabla^\Omega F)(\omega) \right\rangle_{T_\omega \Omega} d\pi^\tau_\sigma(\omega) \\ &= - \int_\Omega G(\omega) \left\langle v, (\nabla^\Omega F)(\omega) \right\rangle_{T_\omega \Omega} d\pi^\tau_\sigma(\omega) \\ &= - \int_\Omega G(\omega)(\nabla^\Omega_v F)(\omega) d\pi^\tau_\sigma(\omega) \\ &= - \int_\Omega (\nabla^{\Omega*}_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\ &= \int_\Omega (\nabla^\Omega_v G)(\omega) F(\omega) d\pi^\tau_\sigma(\omega) \\ &\quad + \int_\Omega B^{\pi^\tau_\sigma}_v(\omega) G(\omega) F(\omega) d\pi^\tau_\sigma(\omega), \end{aligned}$$

where we have used (3.20). Hence

$$(\mathrm{div}^{\Omega}_{\pi_\tau} V)(\omega) = (\nabla^{\Omega}_v G)(\omega) + B^{\pi_\sigma^\tau}_v(\omega)G(\omega)$$
$$= \left\langle (\nabla^{\Omega} G)(\omega), v \right\rangle_{T_\omega \Omega} + \left\langle \gamma, \beta^\sigma_v \right\rangle G(\omega).$$

$\square$

In the next subsection we give an equivalent description via a "lifting rule" of the above differential objects on Ω. Before we would like to consider the special case when $\mathrm{supp}\,\tau = \{1\}$, i.e., $d\tau(s) = \varepsilon_1(ds)$, cf. Proposition 2.4-3. A detailed description can be found in [AKR97]. In this case our intrinsic gradient ∇^{Ω} is nothing but the intrinsic gradient ∇^{Γ} on $L^2(\Gamma_X, \pi_\sigma)$, i.e., for any $F \in \mathcal{F}C^{\infty}_b(\mathcal{D}, \Gamma)$ of the form

$$F(\gamma) = g_F(\langle \gamma, \varphi_1 \rangle, \dots, \langle \gamma, \varphi_N \rangle), \ \gamma \in \Gamma_X,$$

where $\varphi_1, \dots, \varphi_N \in \mathcal{D}$ and $g_F \in C^{\infty}_b(\mathbb{R}^N)$, we have

$$(\nabla^{\Gamma} F)(\gamma; x) = \sum_{j=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \gamma, \varphi_1 \rangle, \dots, \langle \gamma, \varphi_N \rangle) \nabla^X \varphi_i(x), \ \gamma \in \Gamma_X, x \in X.$$

The above equality follows from an analogue of (3.13), i.e.,

$$(\nabla^{\Gamma}_v F)(\gamma) = \left\langle (\nabla^{\Gamma} F)(\gamma), v \right\rangle_{T_\gamma \Gamma},$$

where $v \in V_0(X)$ and the directional derivative ∇^{Γ}_v is as in Definition 3.1 with ω replaced by γ. Moreover for the the adjoint operator to the gradient ∇^{Γ} on $L^2(\Gamma, \pi_\sigma)$ equation (3.19) becomes equal to

$$\int_\Gamma (\nabla^{\Gamma}_v F)(\gamma)G(\gamma)d\pi_\sigma(\gamma)$$
$$= - \int_\Gamma F(\gamma)(\nabla^{\Gamma}_v G)(\gamma)d\pi_\sigma(\gamma) - \int_\Gamma F(\gamma)G(\gamma)B^{\pi_\sigma}_v(\gamma)d\pi_\sigma(\gamma),$$

or

$$\nabla^{\Gamma*}_v = -\nabla^{\Gamma}_v - B^{\pi_\sigma}_v$$

as an operator equality on the domain $\mathcal{F}C^{\infty}_b(\mathcal{D}, \Gamma)$ in $L^2(\Gamma, \pi_\sigma)$ and $B^{\pi_\sigma}_v$ stands for the logarithmic derivative of the Poisson measure π_σ along v, i.e.,

$$B^{\pi_\sigma}_v(\gamma) := \langle \gamma, \beta^\sigma_v \rangle = \int_X [\langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \mathrm{div}^X v(x)] d\gamma(x).$$

3.3. A lifting of the geometry.

In the consideration above we have constructed some objects related to the differential geometry of the space Ω. Now we present an interpretation of all the above formulas via a simple "lifting rule".

Any function $\varphi \in \mathcal{D}$ generates a (cylinder) function on Ω by the formula

$$(3.24) \qquad\qquad L_\varphi(\omega) := \langle \omega, \varphi \rangle, \ \omega \in \Omega.$$

We will call L_φ the lifting of φ. As before any vector field $v \in V_0(X)$ can be considered as a vector field on Ω (the lifting of v) which we denote by L_v, see Definition 3.2. For $v, w \in V_0(X)$ formula (3.12) can be written as

$$(3.25) \qquad \langle L_v, L_w \rangle_{T_\omega \Omega} = L_{\langle v, w \rangle_{TX}}(\omega),$$

i.e., the scalar product of lifting vector fields is computed as the lifting of the scalar product $\langle v(x), w(x) \rangle_{T_x X} = \varphi(x)$. This rule can be used as a definition of the tangent space $T_\omega \Omega$.

Formula (3.11) has now the following interpretation:

$$(\nabla_v^\Omega L_\varphi)(\omega) = L_{\nabla_v^X \varphi}(\omega), \ \omega \in \Omega,$$

and the gradient of L_φ is nothing but the lifting of the corresponding underlying gradient:

$$(\nabla^\Omega L_\varphi)(\omega) = L_{\nabla^X \varphi}(\omega).$$

As follows from (3.18) the logarithmic derivative $B_v^{\pi_\sigma^\tau} : \Omega \to \mathbb{R}$ is obtained via the same lifting procedure of the corresponding logarithmic derivative $\beta_v^\sigma : X \to \mathbb{R}$, namely,

$$B_v^{\pi_\sigma^\tau}(\omega) = L_{\beta_v^\sigma}(\gamma_\omega).$$

Or, equivalently, one has for the divergence of a lifted vector field:

$$\text{div}_{\pi_\sigma^\tau}^\Omega(L_v) = L_{\text{div}_\sigma^X v}.$$

3.4. Representations of the Lie algebra of vector fields.

Using the property of quasi-invariance of the compound Poisson measure π_σ^τ we can define a unitary representation of the diffeomorphism group $\text{Diff}_0(X)$ in the space $L^2(\pi_\sigma^\tau)$, see [GGV75]. Namely, for $\phi \in \text{Diff}_0(X)$ we define a unitary operator

$$(V_{\pi_\sigma^\tau}(\phi)F)(\omega) := F(\phi(\omega))\sqrt{\frac{d\pi_\sigma^\tau(\phi(\omega))}{d\pi_\sigma^\tau(\omega)}}, \ F \in L^2(\pi_\sigma^\tau).$$

Then we have

$$V_{\pi_\sigma^\tau}(\phi_1)V_{\pi_\sigma^\tau}(\phi_2) = V_{\pi_\sigma^\tau}(\phi_1 \circ \phi_2), \ \phi_1, \phi_2 \in \text{Diff}_0(X).$$

as in Subsection 3.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms $\phi_t^v, t \in \mathbb{R}$. It generates a one-parameter unitary group

$$(3.26) \qquad V_{\pi_\sigma^\tau}(\phi_t^v) := \exp[itJ_{\pi_\sigma^\tau}(v)], \ t \in \mathbb{R},$$

where $J_{\pi_\sigma^\tau}(v)$ denotes the self-adjoint generator of this group.

Proposition 3.8. *For any $v \in V_0(X)$ the following operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ holds:*

$$(3.27) \qquad J_{\pi_\sigma^\tau}(v) = \frac{1}{i}\nabla_v^\Omega + \frac{1}{2i}B_v^{\pi_\sigma^\tau}.$$

Proof. Let $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ be given. Then differentiating the left hand side of (3.26) at $t = 0$ we get

$$\frac{d}{dt}(V_{\pi_\sigma^\tau}(\phi_t^v)F)(\omega)|_{t=0} = \frac{d}{dt}F(\phi_t^v(\omega))|_{t=0} + F(\omega)\frac{1}{2}\frac{d}{dt}\left.\frac{d\pi_\sigma^\tau(\phi_t^v(\omega))}{d\pi_\sigma^\tau(\omega)}\right|_{t=0}$$

$$= (\nabla_v^\Omega F)(\omega) + \frac{1}{2}F(\omega)B_v^{\pi_\sigma^\tau}(\omega),$$

where we have used the form of the operator $V_{\pi_\sigma^\tau}(\phi_t^v)$, the definition of the directional derivative ∇_v^Ω and Theorem 3.5. On the other hand the same procedure on the right hand side of (3.26) produce $i(J_{\pi_\sigma^\tau}(v)F)(\omega)$. Hence the result of the proposition follows. $\square$

Remark 3.9. More generally, one can study a family of self-adjoint operators $J(v)$, $v \in V_0(X)$, in a Hilbert space $\mathcal{H}$ which gives a representation of the Lie algebra $V_0(X)$ in the sense of the following commutation relation:

$$(3.28) \qquad [J(v_1), J(v_2)] = -iJ([v_1, v_2])$$

(on a dense domain in $\mathcal{H}$), where $[v_1, v_2] = \langle v_1, \nabla v_2 \rangle_{TX} - \langle v_2, \nabla v_1 \rangle_{TX}$ is the Lie-bracket of the vector fields $v_1, v_2 \in V_0(X)$. In the case discussed, this relation is a direct consequence of (3.27). Thus, we have constructed a compound Poisson space representation of the Lie algebra $V_0(X)$.

Let us define, in addition, a unitary representation of the additive group $\mathcal{D}$ given by the formula

$$(U_{\pi_\sigma^\tau}(f)F)(\omega) := \exp(i\langle\omega, f\rangle)F(\omega), \ F \in L^2(\pi_\sigma^\tau), \ \omega \in \Omega,$$

for any $f \in \mathcal{D}$. As usual, the semi-direct product $\mathcal{G} := \mathcal{D} \wedge \mathrm{Diff}_0(X)$ of the groups $\mathcal{D}$ and $\mathrm{Diff}_0(X)$ is defined as the set of pairs (f, ϕ) with multiplication operation

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1),$$

see e.g. [GGV75]. Let us introduce for any element $(f, \phi) \in \mathcal{G}$ the following operator on $L^2(\pi_\sigma^\tau)$:

$$W_{\pi_\sigma^\tau}(f, \phi) := U_{\pi_\sigma^\tau}(f)V_{\pi_\sigma^\tau}(\phi).$$

These operators are unitary and form a representation of the group $\mathcal{G}$. If we introduce multiplication operators $\rho_{\pi_\sigma^\tau}(f)$, $f \in \mathcal{D}$, as self-adjoint operators on $L^2(\pi_\sigma^\tau)$ which are defined for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ by the formula

$$(\rho_{\pi_\sigma^\tau}(f)F)(\omega) := \langle\omega, f\rangle F(\omega), \ \omega \in \Omega,$$

then $U_{\pi_\sigma^\tau}(f) = \exp[i\rho_{\pi_\sigma^\tau}(f)]$ and the form of the multiplication in $\mathcal{G}$ implies

$$[\rho_{\pi_\sigma^\tau}(f), J_{\pi_\sigma^\tau}(v)] = i\rho_{\pi_\sigma^\tau}(\nabla_v^X f)$$

(on a dense domain in $L^2(\pi_\sigma^\tau)$) for all $f \in \mathcal{D}$, $v \in \mathrm{Diff}_0(X)$. We also have the relation $[\rho_{\pi_\sigma^\tau}(f_1), \rho_{\pi_\sigma^\tau}(f_2)] = 0$. The family of operators $J_{\pi_\sigma^\tau}(v), \rho_{\pi_\sigma^\tau}(f)$, $v \in V_0(X)$, $f \in \mathcal{D}$, thus forms a compound Poisson representation of an infinite-dimensional Lie algebra. In the particular case when $\tau = \varepsilon_1$ this representation is known as Lie algebra of currents in non relativistic quantum field theory, e.g. [GGPS74].

4. INTRINSIC DIRICHLET FORMS ON COMPOUND POISSON SPACE

4.1. Definition of the intrinsic Dirichlet form.

We start with introducing some useful spaces of cylinder functions on Ω in addition to $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$. By $\mathcal{F}\mathcal{P}(\mathcal{D}, \Omega)$ we denote the set of all cylinder functions of the form (3.10) in which the generating function g_F is a polynomial on $\mathbb{R}^N$, i.e., $g_F \in \mathcal{P}(\mathbb{R}^N)$. Analogously we define $\mathcal{F}C_p^\infty(\mathcal{D}, \Omega)$, where now $g_F \in C_p^\infty(\mathbb{R}^N)$ (the set of all C^∞-functions f on $\mathbb{R}^N$ such that f and all its partial derivatives of any order are polynomially bounded).

We have obviously

$$\mathcal{F}C_b^\infty(\mathcal{D}, \Omega) \subset \mathcal{F}C_p^\infty(\mathcal{D}, \Omega),$$
$$\mathcal{F}\mathcal{P}(\mathcal{D}, \Omega) \subset \mathcal{F}C_p^\infty(\mathcal{D}, \Omega),$$

and these spaces are algebras with respect to the usual operations. The existence of the Laplace transform $l_{\pi_\sigma^\tau}(f)$, $f \in \mathcal{D}$, implies $\mathcal{F}C_p^\infty(\mathcal{D}, \Omega) \subset L^2(\pi_\sigma^\tau)$.

Note that after the embedding $\Omega \hookrightarrow \mathcal{D}'$ (see Subsection 2.3) and a natural extension to $\mathcal{D}'$ the space $\mathcal{F}\mathcal{P}(\mathcal{D}, \mathcal{D}')$ is nothing but the well-known space of cylinder polynomials on $\mathcal{D}'$, see [BK88, Chap. 2].

Definition 4.1. For $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Omega)$ we introduce a pre-Dirichlet form as

$$(4.29) \qquad \mathcal{E}_{\pi_\sigma^\tau}^\Omega(F, G) = \int_\Omega \langle (\nabla^\Omega F)(\omega), (\nabla^\Omega G)(\omega) \rangle_{T_\omega \Omega} d\pi_\sigma^\tau(\omega).$$

Note that for $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Omega)$ formula (3.14) is still valid and therefore

$$\left\langle \nabla^\Omega F, \nabla^\Omega G \right\rangle_{T\Omega} \in \mathcal{F}C_p^\infty(\mathcal{D}, \Omega),$$

such that (4.29) is well-defined.

We will call $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$ the intrinsic pre-Dirichlet form corresponding to the compound Poisson measure π_σ^τ on Ω. The name "intrinsic" means that $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$ is associated with the geometry of Ω generated by the original Riemannian structure of X, in particular, by the intrinsic gradient ∇^Ω. In the next subsection we shall prove the closability of $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$.

4.2. Intrinsic Dirichlet operators.

Let us introduce a differential operator $H_{\pi_\sigma^\tau}^\Omega$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ which is given on any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ of the form

$$(4.30) \quad F(\omega) = g_F(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle), \ \omega \in \Omega, g_F \in C_b^\infty(\mathbb{R}^N), \varphi_1, \dots, \varphi_N \in \mathcal{D},$$

by the formula

$$(H_{\pi_\sigma^\tau}^\Omega F)(\omega)$$

$$:= \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \int_X \langle \nabla^X \varphi_i(x), \nabla^X \varphi_j(x) \rangle_{T_x X} d\omega(x)$$

$$(4.31) \quad - \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \int_X \triangle^X \varphi_i(x) d\omega(x)$$

$$- \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \int_X \langle \nabla^X \varphi_i(x), \beta^\sigma(x) \rangle_{T_x X} d\omega(x),$$

where $\triangle^X$ denotes the Laplace-Beltrami operator on X. In this formula all expressions are from $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ or have the form $\langle \omega, \psi \rangle$, $\omega \in \Omega$, $\psi \in \mathcal{D}$, except for the functions $\langle \omega, h_i \rangle$, $\omega \in \Omega$, with

$$h_i(x) = \left\langle (\nabla^X \varphi_i)(x), \beta^\sigma(x) \right\rangle_{T_x X}, \; x \in X, i = 1, \dots, N.$$

To clarify the situation with these functions note that due to the assumption on σ we have $\rho^{1/2} \in H^{1,2}_{loc}(X)$ which gives $h_i \in L^1(\sigma)$ and these functions have compact supports. Therefore $h_i \in L^1(\sigma)$, $j = 1, \dots, N$. On the other hand we know that a function $\langle \omega, f \rangle$, $\omega \in \Omega$, is from $L^2(\pi_\sigma^\tau)$ if $f \in L^1(\sigma) \cap L^2(\sigma)$. The latter follows from the formula for the second moment of the measure π_σ^τ, namely

$$(4.32) \quad \int_\Omega \langle \omega, f \rangle^2 \, d\pi_\sigma^\tau(\omega) = m_2(\tau) \int_X f^2(x) d\sigma(x) + (m_1(\tau))^2 \left(\int_X f(x) d\sigma(x) \right)^2,$$

where $m_1(\tau)$ and $m_2(\tau)$ are the first and second moment of the measure τ on $\mathbb{R}^+$, respectively. Equation (4.32) is a direct consequence of (2.4). As a result the right hand side of (4.31) is well-defined. To show that the operator $H_{\pi_\sigma^\tau}^\Omega$ is well-defined we still have to show that its definition does not depend on the representation of F in (4.30) which will be done below.

Remark 4.2. In the applications to the study of unitary representations of the group $\mathrm{Diff}_0(X)$ given by compound Poisson measures, there is usually an additional assumption on the smoothness of the density $\rho := d\sigma/dm$, namely $\rho \in C^\infty(X)$, $\rho(x) > 0$, $x \in X$, see e.g. [GGV75]. In this case it is obvious that the operator $H_{\pi_\sigma^\tau}^\Omega$ preserves the spaces $\mathcal{F}C_p^\infty(\mathcal{D}, \Omega)$ and $\mathcal{F}\mathcal{P}(\mathcal{D}, \Omega)$.

Let us also consider the classical pre-Dirichlet form corresponding to the measure σ on X :

$$(4.33) \quad \mathcal{E}_\sigma^X(\varphi, \psi) := \int_X \langle \nabla^X \varphi(x), \nabla^X \psi(x) \rangle_{T_x X} d\sigma(x),$$

where $\varphi, \psi \in \mathcal{D}$. This form is associated with the Dirichlet operator H_σ^X which is given on $\mathcal{D}$ by

$$(4.34) \quad (H_\sigma^X \varphi)(x) := -\triangle^X \varphi(x) - \left\langle \beta^\sigma(x), \nabla^X \varphi(x) \right\rangle_{T_x X},$$

and which satisfies

$$\mathcal{E}_\sigma^X(\varphi, \psi) = (H_\sigma^X \varphi, \psi)_{L^2(\sigma)}, \; \varphi, \psi \in \mathcal{D}.$$

The closure of this form on $L^2(\sigma)$ is defined by $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$. Note that $D(\mathcal{E}_\sigma^X)$ is nothing but the Sobolev space of order 1 in $L^2(\sigma)$ (sometimes also denoted by $H_0^{1,2}(X, \sigma)$). $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ generates a positive self-adjoint operator in $L^2(\sigma)$ (the so-called Friedrich's extension of H_σ^X, see e.g. [BKR97] and [RS75]).

For this extension we preserve the previous notation H_σ^X and denote the domain by $D(H_\sigma^X)$. Using the underlying Dirichlet operator we obtain the representation

$$(H_{\pi_\sigma^\tau}^\Omega F)(\omega)$$

$$= \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \langle \nabla^X \varphi_i, \nabla^X \varphi_j \rangle_{T_\omega \Omega}$$

$$+ \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \langle \omega, H_\sigma^X \varphi_i \rangle.$$

Let us define for any $\omega \in \Omega$, $x, y \in X$

$$
(\nabla^\Omega \nabla^\Omega F)(\omega, x, y)
$$
$$
:= \sum_{i,j=1}^{N} \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \nabla^X \varphi_i(x) \otimes \nabla^X \varphi_j(y) \in T_x X \otimes T_y X.
$$

Then

$$
\triangle^\Omega F(\omega) := \mathrm{Tr}(\nabla^\Omega \nabla^\Omega F)(\omega)
$$
$$
= \sum_{i,j=1}^{N} \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \langle \nabla^X \varphi_i, \nabla^X \varphi_j \rangle_{T_\omega \Omega}.
$$

Hence the operator $H^\Omega_{\pi^\tau_\sigma}$ can be written as

$$
(H^\Omega_{\pi^\tau_\sigma} F)(\omega) = -(\triangle^\Omega F)(\omega) - \langle \omega, \mathrm{div}^X_\sigma (\nabla^\Omega F)(\omega; \cdot) \rangle.
$$

The following theorem implies that both $H^\Omega_{\pi^\tau_\sigma}$ and $\triangle^\Omega$ are well-defined as linear operators on $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$, i.e., independently of the representation of F on (4.30).

Theorem 4.3. *The operator $H^\Omega_{\pi^\tau_\sigma}$ is associated with the intrinsic Dirichlet form $\mathcal{E}^\Omega_{\pi^\tau_\sigma}$, i.e., for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ we have*

$$
\mathcal{E}^\Omega_{\pi^\tau_\sigma}(F, G) = (H^\Omega_{\pi^\tau_\sigma} F, G)_{L^2(\pi^\tau_\sigma)},
$$

or

$$
H^\Omega_{\pi^\tau_\sigma} = -\mathrm{div}^\Omega_{\pi^\tau_\sigma} \nabla^\Omega \quad \text{on} \ \mathcal{F}C_b^\infty(\mathcal{D}, \Omega).
$$

We call $H^\Omega_{\pi^\tau_\sigma}$ the intrinsic Dirichlet operator of the measure π^τ_σ.

Proof. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ of the form (4.30) we have

$$
(\nabla^\Omega F)(\omega; x) = \sum_{j=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \nabla^X \varphi_i(x).
$$

By (3.23) we conclude that

$$
\mathrm{div}^\Omega_{\pi^\tau_\sigma}(\nabla^\Omega F) = -H^\Omega_{\pi^\tau_\sigma} F
$$

which by (3.21) for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ gives

$$
(H^\Omega_{\pi^\tau_\sigma} F, G)_{L^2(\pi^\tau_\sigma)} = -\int_\Omega \mathrm{div}^\Omega_{\pi^\tau_\sigma}(\nabla^\Omega F)(\omega) G(\omega) d\pi^\tau_\sigma(\omega)
$$
$$
= \int_\Omega \langle (\nabla^\Omega F)(\omega), (\nabla^\Omega G)(\omega) \rangle_{T_\omega \Omega} d\pi^\tau_\sigma(\omega).
$$

$\square$

Remark 4.4. 1. In the case $\sigma = m$ and $\tau = \varepsilon_1$ we call the Dirichlet form $\mathcal{E}_{\pi_\sigma}^\Gamma$ the canonical Dirichlet form on Γ. The canonical Dirichlet form and canonical Dirichlet operator $H_{\pi_\sigma}^\Gamma$ are defined directly in terms of the Riemannian geometry of X.

2. The operator $H_{\pi_\sigma}^\Gamma$ can be naturally extended to cylinder functions of the form

$$F(\omega) := \exp(\langle \omega, \varphi \rangle), \varphi \in \mathcal{D}, \omega \in \Omega,$$

since such F belongs to $L^2(\Gamma, \pi_\sigma)$. We then have

$$(H_{\pi_\sigma}^\Gamma \exp(\langle \cdot, \varphi \rangle))(\gamma) = \langle \omega, H_\sigma^X \varphi - \left| \nabla^X \varphi \right|_{TX}^2 \rangle \exp(\langle \gamma, \varphi \rangle)$$

and for $\sigma = m$ and $\tau = \varepsilon_1$

$$(H_{\pi_m}^\Gamma \exp(\langle \cdot, \varphi \rangle))(\gamma) = -\langle \triangle^X \varphi + \left| \nabla^X \varphi \right|_{TX}^2 \rangle \exp(\langle \gamma, \varphi \rangle).$$

As an immediate consequence of Theorem 4.3 we obtain

Corollary 4.5. *$(\mathcal{E}_{\pi_\sigma^\tau}^\Omega, \mathcal{F}C_b^\infty(\mathcal{D}, \Omega))$ is closable on $L^2(\pi_\sigma^\tau)$. Its closure $(\mathcal{E}_{\pi_\sigma^\tau}^\Omega, D(\mathcal{E}_{\pi_\sigma^\tau}^\Omega))$ is associated with a positive definite self-adjoint operator, the Friedrichs extension of $H_{\pi_\sigma^\tau}^\Omega$ which we also denote by $H_{\pi_\sigma^\tau}^\Omega$ (and its domain by $D(H_{\pi_\sigma^\tau}^\Omega)$).*

Clearly, ∇^Ω also extends to $D(\mathcal{E}_{\pi_\sigma^\tau}^\Omega)$. We denote this extension again by ∇^Ω.

Corollary 4.6. *Let*

$$F(\omega) = g_F(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle), \ \omega \in \Omega, g_F \in C_b^\infty(\mathbb{R}^N),$$
$$\varphi_1, \dots, \varphi_N \in D(\mathcal{E}_\sigma^\Omega).$$

Then $F \in D(\mathcal{E}_{\pi_\sigma^\tau}^\Omega)$ and

$$(\nabla^\Omega F)(\cdot) = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \cdot, \varphi_1 \rangle, \dots, \langle \cdot, \varphi_N \rangle) \nabla^X \varphi_i.$$

Proof. By approximation this is an immediate consequence of (3.14) and the fact that for all $1 \le i \le N$

$$\int_\Omega \langle \omega, |\nabla^X \varphi_i|_{TX}^2 \rangle d\pi_\sigma^\tau(\omega) = (m_1(\tau))^{-1} \mathcal{E}_\sigma^X(\varphi_i, \varphi_i).$$

$\square$

Remark 4.7. Of course the domain of $\mathcal{E}_{\pi_\sigma^\tau}^\Omega$ $D(\mathcal{E}_{\pi_\sigma^\tau}^\Omega)$ is nothing but the Sobolev space $H_0^{1,2}(\Omega, \pi_\sigma^\tau)$ on Ω of order 1 in $L^2(\Omega, \pi_\sigma^\tau)$.

5. Identification of the process on compound Poisson space

In this section we will prove the existence of a diffusion process corresponding to our Dirichlet form $(\mathcal{E}^\Omega_{\pi_\tau}, \mathcal{D}(\mathcal{E}^\Omega_{\pi_\tau}))$. For a general theory of processes corresponding to Dirichlet forms we refer to [MR92, Chap. IV], see also [F80].

After all our preparation and taking into account the general description of compound Poisson space given in Section 6 we will see that this process is nothing but a direct consequence ("lifting") of the corresponding process on the space of simple configurations Γ_X, see [AKR97, Sect. 6] for a detailed description. Let us clarify this in more detail. After Section 6 there is an obvious identification between compound configurations $\omega \in \Omega$ and a marked configurations $(\gamma_\omega, m_\omega) \in \Omega_X^{\mathbb{R}_+}$ which gives the possibility to obtain an embedding from $L^2(\Gamma_X, \pi_\sigma)$ into $L^2(\Omega_X, \pi_\sigma^\tau)$, i.e.,

$$L^2(\Omega_X, \pi_\sigma^\tau) \ni F(\omega) = G(\gamma_\omega) \in L^2(\Gamma_X, \pi_\sigma).$$

Hence all operators acting in $L^2(\Gamma_X, \pi_\sigma)$, e.g., ∇^Γ, $\nabla^{\Gamma*}$, $H^\Gamma_{\pi_\sigma}$ etc. are applicable on $L^2(\Omega_X, \pi_\sigma^\tau)$ w.r.t. part of the variables. Moreover we have the following relation

$$(\nabla^\Omega F)(\omega) = (\nabla^\Gamma F)((\gamma_\omega, m_\omega)), \ \omega = (\gamma_\omega, m_\omega) \in \Omega,$$

for the intrinsic gradient, from which everything else follows, see below.

Let us consider a probability measure τ on $\mathbb{R}_+$ (or more general a probability measure on the space of marks M, cf. Section 6). In what follows we always identify any compound configuration $\omega \in \Omega$ (or in general marked configuration) with $(\gamma_\omega, m_\omega)$, i.e.,

$$\Omega \ni \omega \rightsquigarrow (\gamma_\omega, m_\omega)$$

and by Subsection 2.5 we have for any diffeomorphism $\phi \in \mathrm{Diff}_0(X)$ its action on $(\gamma_\omega, m_\omega)$ is given by

$$\phi(\gamma_\omega, m_\omega) = (\phi(\gamma_\omega), m_\omega).$$

It follows from Proposition 2.8 and the assumption on τ that the Radon-Nikodym density of π_σ^τ and π_σ with respect to the group $\mathrm{Diff}_0(X)$ are equal, i.e., $p_\phi^{\pi_\tau}(\omega) = p_\phi^{\pi_\sigma}(\gamma_\omega)$, where γ_ω corresponds to ω.

Let us compute the action of the gradient ∇^Γ on cylinder functions $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$. To this end let $v \in V_0(X)$ be a vector field on X with compact support and ϕ_t^v the corresponding flow. Then by definition we have

$$(\nabla_v^\Gamma F)((\gamma_\omega, m_\omega)) := \frac{d}{dt} F((\phi_t^v(\gamma_\omega), m_\omega))|_{t=0}.$$

On cylinder functions of the form

$$F((\gamma_\omega, m_\omega)) = g_F(\langle(\gamma_\omega, m_\omega), \varphi_1\rangle, \ldots, \langle(\gamma_\omega, m_\omega), \varphi_N\rangle),$$

where $\omega = (\gamma_\omega, m_\omega) \in \Omega$, $g_F \in C_b^\infty(\mathbb{R}^N)$, and $\varphi_1, \ldots \varphi_N \in \mathcal{D}$, the above definition gives

$$\sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle(\gamma_\omega, m_\omega), \varphi_1\rangle, \ldots, \langle(\gamma_\omega, m_\omega), \varphi_N\rangle)\langle(\gamma_\omega, m_\omega), \nabla_v^X \varphi_i\rangle.$$

Therefore the following equality on $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$ (dense in $L^2(\pi_\sigma^\tau)$) of the directional derivatives holds

$$(\nabla_v^\Omega F)(\omega) = (\nabla_v^\Gamma F)((\gamma_\omega, m_\omega))$$

which implies the equality between the intrinsic gradients, i.e.,

$$(\nabla^\Omega F)(\omega) = (\nabla^\Gamma F)((\gamma_\omega, m_\omega)).$$

From these considerations on the intrinsic gradient we get a relation between the Dirichlet forms, namely

$$
\begin{aligned}
\mathcal{E}_{\pi_\sigma^\tau}^\Omega(F, G) &= \int_\Omega \langle (\nabla^\Omega F)(\omega), (\nabla^\Omega G)(\omega) \rangle_{T_\omega \Omega} \, d\pi_\sigma^\tau(\omega) \\
&= \int_\Omega \langle (\nabla^\Gamma F)((\gamma_\omega, m_\omega)), (\nabla^\Gamma G)((\gamma_\omega, m_\omega)) \rangle_{T_{\gamma_\omega} \Gamma} \, d\pi_\sigma^\tau(\omega) \\
&= \mathcal{E}_{\pi_\sigma^\tau}^\Gamma(F, G).
\end{aligned}
$$

On the other hand the above relation between Dirichlet forms allowed us to derive easily the following relation for the intrinsic Dirichlet operators

$$(H_{\pi_\sigma^\tau}^\Omega F)(\omega) = (H_{\pi_\sigma, \gamma_\omega}^\Gamma F)(\gamma_\omega, m_\omega), \ F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Omega),$$

where $H_{\pi_\sigma, \gamma_\omega}^\Gamma$ acts w.r.t. the variable γ_ω. Of course the corresponding semigroups (whose generators are $H_{\pi_\sigma^\tau}^\Omega$ and $H_{\pi_\sigma, \gamma_\omega}^\Gamma$) are related by

$$e^{-tH_{\pi_\sigma^\tau}^\Omega} = e^{-tH_{\pi_\sigma, \gamma_\omega}^\Gamma} \otimes \mathbf{1}_{m_\cdot}, \ t > 0.$$

From this it follows that the process, Ξ_t, $t \geq 0$, on compound Poisson space (or in marked Poisson space) is nothing but the equilibrium process $X_t^{\gamma_\omega}$ (distorted Brownian motion on Ω) together with marks of the corresponding configuration, i.e.,

$$\Xi_t = \{X_t^{\gamma_\omega}, m_\omega\}.$$

For more detailed description of properties of the process $X_t^{\gamma_\omega}$ we refer to [AKR97, Sect. 6].

6. Marked Poisson measures

In this section we present some general results on marked Poisson measures which generalizes the compound Poisson measures introduced in Subsection 2.3. For more detailed information on marked Poisson processes we refer to [BL95, Chap. 6], [K93, Chap. 5], [MM91], and references therein.

6.1. Definition and measurable structure.

In this subsection we define and describe the space of marked configurations as well as its associated measurable structure. Before we recall the definition of the configuration space over the Cartesian space $X \times M$ between a Riemannian manifold X and a complete separable metric space M.

The configuration space $\Gamma_{X \times M}$ over the Cartesian product $X \times M$ is defined as the set of all locally finite subsets (configurations) in $X \times M$:

$$\Gamma_{X \times M} := \{\hat\gamma \subset X \times M \, | \, |\hat\gamma \cap K| < \infty \text{ for any compact } K \subset X \times M\}.$$

For any $\Lambda \subset X$ we sometimes use the shorthand γ_Λ for $\gamma \cap \Lambda$, for any $\gamma \in \Gamma_X$ and define

$$\Gamma_\Lambda := \{\gamma \in \Gamma_X | \gamma \cap (X \backslash \Lambda) = \emptyset\}.$$

For any $n \in \mathbb{N}_0$ and $\Lambda \in \mathcal{O}_c(X)$ we introduce the space of n points configurations as

$$\Gamma_\Lambda^{(n)} := \{\gamma \in \Gamma_\Lambda | |\gamma| = n\}, \ \Gamma_\Lambda^{(0)} := \{\emptyset\}.$$

Let us now introduce the space of marked configurations which will plays the same role as Γ_X played for Poisson measure but now for the marked Poisson measure, see Subsection 6.3 below. It is defined as

$$\Omega_X^M := \{\omega = (\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_X, m_\omega \in M^\omega\}.$$

Here M^ω stands for the set of all maps $\gamma_\omega \ni x \mapsto m_x \in M$. We may also write the marked configuration space Ω_X^M as a subspace of $\Gamma_{X \times M}$ as follows

$$\Omega_X^M := \{\omega = \{(x, m_x)\} \subset \Gamma_{X \times M} | \{x\} = \gamma_\omega \in \Gamma_X, m_x \in M\}.$$

For any $\Lambda \subset X$ we define in an analogous way the set Ω_Λ^M, i.e.,

$$\Omega_\Lambda^M := \{\omega = (\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_\Lambda, m_\omega \in M^\omega\}$$

and

$$\Omega_\Lambda^M := \{\omega = \{(x, m_x)\} \subset \Gamma_{\Lambda \times M} | \{x\} = \gamma_\omega \in \Gamma_\Lambda, m_x \in M\}.$$

In order to describe the σ-algebra $\mathcal{B}(\Omega_X^M)$ we proceed as follows. Let $\Lambda \in \mathcal{O}_c(X)$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be given. We define an equivalent relation $\sim$ on $(\Lambda \times M)^n$ setting

$$((x_1, m_{x_1}), (x_2, m_{x_2}), \dots, (x_n, m_{x_n})) \sim ((y_1, m_{y_1}), (y_2, m_{y_2}), \dots, (y_n, m_{y_n}))$$

iff there exists a permutation $\pi \in \mathfrak{S}_n$ (the group of permutations of n elements) such that

$$(x_i, m_{x_i}) = (y_{\pi(i)}, m_{y_{\pi(i)}}), \ \forall i = 1, \dots, n.$$

Hence we obtain the quotient space $(\Lambda \times M)^n / \mathfrak{S}_n$ by means of $\sim$. Then we introduce the subset of $(\Lambda \times M)^n / \mathfrak{S}_n$, $\Omega_\Lambda^M(n)$, defined as follows

$$\Omega_\Lambda^M(n) := \{((x_1, m_{x_1}), \dots, (x_1, m_{x_1})) | x_i \in \Lambda, x_i \neq x_j, i \neq j, m_{x_i} \in M\},$$

or equivalently

$$\Omega_\Lambda^M(n) := \{(\gamma_\omega, m_\omega) | \gamma_\omega \in \Gamma_\Lambda^{(n)}, m_\omega \in M^\omega\}, \ \Omega_\Lambda^M(0) := \{\emptyset\}$$

The space $\Omega_\Lambda^M(n)$ is endowed with the relative metric from $(\Lambda \times M)^n / \mathfrak{S}_n$, i.e.,

$$\delta([x], [y]) = \inf_{x' \in [x], y' \in [y]} d^n(x', y'),$$

where d^n is the metric defined on $(\Lambda \times M)^n$ driven from the original metrics on X and M. Therefore $\Omega_\Lambda^M(n)$ becomes a metrizable topological space.

It is obvious that

$$\Omega_\Lambda^M = \bigsqcup_{n=0}^{\infty} \Omega_\Lambda^M(n).$$

This space can be equipped with the topology of disjoint union of topological spaces, namely, the strongest topology on Ω_Λ^M such that all the embeddings

$$i_n : \Omega_\Lambda^M(n) \to \Omega_\Lambda^M, \ n \in \mathbb{N}_0$$

are continuous. $\mathcal{B}(\Omega_\Lambda^M)$ stands for the corresponding Borel σ-algebra.

For any $\Lambda \in \mathcal{O}_c(X)$ there are natural restriction maps

$$p_\Lambda : \Omega_X^M \to \Omega_\Lambda^M$$

defined by

(6.35) $$p_\Lambda(\gamma_\omega, m_\omega) = (\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda}) \in \Omega_\Lambda^M, \ (\gamma_\omega, m_\omega) \in \Omega_X^M.$$

The topology on Ω_X^M is defined as the weakest topology making all the mappings p_Λ continuous. The associated Borel σ-algebra is denoted by $\mathcal{B}(\Omega_X^M)$.

6.2. The projective limit.

Finally we want to show that Ω_X^M coincides with the projective limit of the family of topological spaces $\{\Omega_\Lambda^M | \Lambda \in \mathcal{O}_c(X)\}$. First we recall the definition of projective limit of topological spaces, see e.g. [BD68, Chap. 3] and [S71, Chap. 2].

Definition 6.1. Let $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ be given with $\Lambda_1 \subset \Lambda_2$. There are natural maps

$$p_{\Lambda_2, \Lambda_1} : \Omega_{\Lambda_2}^M \to \Omega_{\Lambda_1}^M$$

defined by

$$p_{\Lambda_2, \Lambda_1}(\gamma_\omega, m_\omega) = (\gamma_\omega \cap \Lambda_1, m_{\omega|\gamma_\omega \cap \Lambda_1}) \in \Omega_{\Lambda_1}^M, \ (\gamma_\omega, m_\omega) \in \Omega_{\Lambda_2}^M.$$

The projective limit of the family $\{\Omega_\Lambda^M | \Lambda \in \mathcal{O}_c(X)\}$ denoted by

$$\mathop{\mathrm{prlim}}_{\Lambda \in \mathcal{O}_c(X)} \Omega_\Lambda^M$$

is a topological space Ω and a family of continuous projections

$$P_\Lambda : \Omega \to \Omega_\Lambda^M, \ \Lambda \in \mathcal{O}_c(X),$$

such that the following two conditions are satisfied:

1. If $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ with $\Lambda_1 \subset \Lambda_2$, then

$$P_{\Lambda_1} = p_{\Lambda_2, \Lambda_1} \circ P_{\Lambda_2}.$$

2. If Ω' is a topological space and

$$P_\Lambda' : \Omega' \to \Omega_\Lambda^M, \ \Lambda \in \mathcal{O}_c(X),$$

a family of continuous projections which fulfills Condition 1 above, then there exists a unique continuous map $u : \Omega' \to \Omega$ such that $P_\Lambda' = P_\Lambda \circ u$, for all $\Lambda \in \mathcal{O}_c(X)$.

Remark 6.2. The projective limit of the family $\{\Omega_\Lambda^M | \Lambda \in \mathcal{O}_c(X)\}$ exists and is unique in the following sense: let Ω and Ω' be projective limits, then there exists a map $u : \Omega \to \Omega'$ such that u and u^{-1} are continuous, see e.g. [P67, Chap. 4].

Theorem 6.3. *The space of marked configurations Ω_X^M is the projective limit of the family $\{\Omega_\Lambda^M | \Lambda \in \mathcal{O}_c(X)\}$ together with the family of projections $\{p_\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ (defined in (6.35)) and*

$$\mathcal{B}(\Omega_X^M) = \sigma(p_\Lambda^{-1}(\mathcal{B}(\Omega_\Lambda^M)); \Lambda \in \mathcal{O}_c(X)).$$

In other words there exists a bicontinuous bijective mapping between Ω_X^M and the projective limit. This will be denoted by $\Omega_X^M \simeq \mathrm{prlim}_{\Lambda \in \mathcal{O}_c(X)} \Omega_\Lambda^M$.

Proof. We will use always in the proof the convention that $\Lambda, \Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ with $\Lambda_1 \subset \Lambda_2$. First we verify Condition 1 of Definition 6.1. This can easily be done as follows

$$
\begin{aligned}
p_{\Lambda_2, \Lambda_1} \circ p_{\Lambda_2}(\gamma_\omega, m_\omega) &= p_{\Lambda_2, \Lambda_1}(\gamma_\omega \cap \Lambda_2, m_{\omega | \gamma_\omega \cap \Lambda_2}) \\
&= (\gamma \cap \Lambda_1, m_{\omega | \gamma_\omega \cap \Lambda_1}) \\
&= p_{\Lambda_1}(\gamma_\omega, m_\omega),
\end{aligned}
$$

for $(\gamma_\omega, m_\omega) \in \Omega_X^M$ which is the desired result.

Let us now construct a version of the projective limit of the family $\{\Omega_\Lambda^M, \Lambda \in \mathcal{O}_c(X)\}$, see e.g. [P67]. As Ω we take

$$\Omega := \{\omega \in \times_{\Lambda \in \mathcal{O}_c(X)} \Omega_\Lambda^M | p_{\Lambda_2, \Lambda_1}((\omega)_{\Lambda_2}) = (\omega)_{\Lambda_1}\},$$

where $(\omega)_\Lambda$ denotes the Λ-component of ω. As projections we choose $P_\Lambda(\omega) := (\omega)_\Lambda$ and define the σ-algebra as $\mathcal{B}(\Omega) := \sigma(\{P_\Lambda | \Lambda \in \mathcal{O}_c(X)\})$.

Now we define a (bijective) mapping $I : \Omega_X^M \to \Omega$ by

$$(I(\gamma_\omega, m_\omega))_\Lambda := p_\Lambda(\gamma_\omega, m_\omega) \in \Omega_\Lambda^M, \ (\gamma_\omega, m_\omega) \in \Omega_X^M, \Lambda \in \mathcal{O}_c(X).$$

We first show that I is well defined, this means that

$$p_{\Lambda_2, \Lambda_1}((I(\gamma_\omega, m_\omega))_{\Lambda_2}) = (\gamma_\omega \cap \Lambda_1, m_{\omega | \gamma_\omega \cap \Lambda_1}).$$

Indeed we have

$$
\begin{aligned}
p_{\Lambda_2, \Lambda_1}((I(\gamma_\omega, m_\omega))_{\Lambda_2}) &= p_{\Lambda_2, \Lambda_1} \circ p_{\Lambda_2}(\gamma_\omega, m_\omega) \\
&= p_{\Lambda_1}(\gamma_\omega, m_\omega) \\
&= (\gamma_\omega \cap \Lambda_1, m_{\omega | \gamma_\omega \cap \Lambda_1})
\end{aligned}
$$

which proves that I is well defined. Let us prove in addition that I is a bijective mapping between Ω_X^M and Ω.

Injectivity. Let $(\gamma_\omega, m_\omega), (\gamma_{\omega'}, m_{\omega'}) \in \Omega_X^M$ such that $I(\gamma_\omega, m_\omega) = I(\gamma_{\omega'}, m_{\omega'})$, that means by definition of I that $(\gamma_\omega \cap \Lambda, m_{\omega | \gamma_\omega \cap \Lambda}) = (\gamma_{\omega'} \cap \Lambda, m_{\omega' | \gamma_{\omega'} \cap \Lambda})$ for all $\Lambda \in \mathcal{O}_c(X)$. Since the manifold X can be written as countable union of sets from $\mathcal{O}_c(X)$, i.e.,

$$X = \bigcup_{n \in \mathbb{N}_0} \Lambda_n, \ \Lambda_n \in \mathcal{O}_c(X), n \in \mathbb{N}_0$$

this implies that $(\gamma_\omega, m_\omega) = (\gamma_{\omega'}, m_{\omega'})$ and therefore the injectivity of I is proved.

Surjectivity. Let $\omega = ((\omega)_\Lambda)_{\Lambda \in \mathcal{O}_c(X)} = ((\gamma_\omega^*, m_\omega^*)_\Lambda)_{\Lambda \in \mathcal{O}_c(X)} \in \Omega$ be given and take a family of pairewise disjoint subsets from $\mathcal{O}_c(X)$, $\{\Lambda_n, n \in \mathbb{N}\}$ such that

$$X = \bigsqcup_{n \in \mathbb{N}} \Lambda_n;$$

moreover we may assume that for any $\Lambda \in \mathcal{O}_c(X)$ $\exists m \in \mathbb{N}$ such that

(6.36) $$\Lambda \subset \bigsqcup_{n=1}^{m} \Lambda_n =: \Lambda_n^m.$$

Let us define an element (γ, m_γ) from Ω_X^M as follows:

$$\gamma_\omega := \bigcup_{n \in \mathbb{N}} (\gamma_\omega^* \cap \Lambda_n)$$

and

$$m_\omega : \bigcup_{n \in \mathbb{N}} (\gamma_\omega^* \cap \Lambda_n) \ni x \mapsto m_x \in M, \ x \in \gamma_\omega^* \cap \Lambda_n, n_0 \in \mathbb{N}.$$

First we note that the assumption (6.36) gives

$$\gamma_\omega \cap \Lambda = \bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda), \ \Lambda \in \mathcal{O}_c(X).$$

Secondly we must prove that $(I(\gamma_\omega, m_\omega))_\Lambda = (\omega)_\Lambda$ for any $\Lambda \in \mathcal{O}_c(X)$. From the definition of I and p_Λ we have

$$(I(\gamma_\omega, m_\omega))_\Lambda := p_\Lambda(\gamma_\omega, m_\omega) := (\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda})$$

and from the above representation for $\gamma_\omega \cap \Lambda$ we obtain

$$
\begin{aligned}
(\gamma_\omega \cap \Lambda, m_{\omega|\gamma_\omega \cap \Lambda}) &= \left(\bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda), m_{\omega|\gamma_\omega \cap \Lambda} \right) \\
&= \bigcup_{n=1}^{m} (\gamma_\omega^* \cap \Lambda_n \cap \Lambda, m_{\omega|\gamma_\omega^* \cap \Lambda_n \cap \Lambda}) \\
&= \bigcup_{n=1}^{m} p_{\Lambda_n \cap \Lambda}(\gamma_\omega^*, m_\omega) = \bigcup_{n=1}^{m} p_\Lambda(\gamma_\omega^* \cap \Lambda_n, m_{\omega|\gamma_\omega^* \cap \Lambda_N}) \\
&= p_\Lambda(\gamma_\omega^* \cap \Lambda_n^m, m_{\omega|\gamma_\omega^* \cap \Lambda_n^m}) = (\gamma_\omega^* \cap \Lambda, m_{\omega|\gamma_\omega^* \cap \Lambda}) \\
&= (\omega)_\Lambda
\end{aligned}
$$

which proves the surjectivity of I.

Taking into account Condition 1 of Definition 6.1 it follows that I is continuous. Hence only remained to proof that $\mathcal{B}(\Omega_X^M)$ and $\mathcal{B}(\Omega)$ coincide. This is an immediate consequence of the definition of the σ-algebras and the continuity of I. $\square$

6.3. Marked Poisson measure.

The underlying manifold X is endowed with a non-atomic Radon measure σ, (cf. Section 2). Let a probability measure τ be given on the space M. The space $X \times M$ is endowed with the product measure between σ and τ denoted by $\hat\sigma$, i.e., $\hat\sigma := \sigma \otimes \tau$. The measure $\hat\sigma^n$ can be considered as a finite measure on $(\Lambda \times M)^n$ for any $\Lambda \in \mathcal{O}_c(X)$ which induces on $\Omega_\Lambda^M(n)$ the measure

$$\hat\sigma_{\Lambda,n} = \frac{1}{n!}(\sigma \otimes \tau)^n, \ n \geq 0, \ \hat\sigma_{\Lambda,0}(\emptyset) = 1.$$

Then we consider a measure $\lambda_{\hat\sigma}^\Lambda$ on Ω_Λ^M which coincides on each $\Omega_\Lambda^M(n)$ with the measure $\hat\sigma_{\Lambda,n}$ as follows

$$\lambda_{\hat\sigma}^\Lambda = \sum_{n=0}^\infty \hat\sigma_{\Lambda,n} = \sum_{n=0}^\infty \frac{1}{n!}(\sigma \otimes \tau)^n.$$

Measure $\lambda_{\hat\sigma}^\Lambda$ is a finite measure on Ω_Λ^M and $\lambda_{\hat\sigma}^\Lambda(\Omega_\Lambda^M) = e^{\sigma(\Lambda)}$, therefore we define a probability measure $\mu_{\hat\sigma}^\Lambda$ on Ω_Λ^M setting

(6.37)
$$\mu_{\hat\sigma}^\Lambda = e^{-\sigma(\Lambda)}\lambda_{\hat\sigma}^\Lambda.$$

The measure $\mu_{\hat\sigma}^\Lambda$ has the following property

$$\mu_{\hat\sigma}^\Lambda(\Omega_\Lambda^M(n)) = \frac{1}{n!}\sigma^n(\Lambda)e^{-\sigma(\Lambda)}$$

which gives the probability of the occurrence of exactly n points of the marked Poisson process (with arbitrary values of marks) inside the volume Λ.

In order to obtain the existence of a unique probability measure $\mu_{\hat\sigma}$ on $\mathcal{B}(\Omega_X^M)$ such that

$$\mu_{\hat\sigma}^\Lambda = p_\Lambda^* \mu_{\hat\sigma}, \ \Lambda \in \mathcal{O}_c(X),$$

on should check the consistency property of the family $\{\mu_{\hat\sigma}^\Lambda | \Lambda \in \mathcal{O}_c(X)\}$. In other words one should verify the following equality of measures,

(6.38)
$$\mu_{\hat\sigma}^{\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1} = \mu_{\hat\sigma}^{\Lambda_1}, \ \Lambda_1, \Lambda_2 \in \mathcal{O}_c(X), \Lambda_1 \subset \Lambda_2.$$

It is known, see e.g. [GGV75], that the σ-algebra $\mathcal{B}(\Omega_\Lambda^M)$ coincides with the σ-algebra generated by the cylinder sets from Ω_Λ^M, $C_{B,n}^\Lambda$, $B \in \mathcal{O}_c(\Lambda)$, $n \in \mathbb{N}_0$. Here $C_{B,n}^\Lambda$ has the following representation

$$C_{B,n}^\Lambda := \{\omega = (\gamma_\omega, m_\omega) \in \Omega_\Lambda^M \,||\gamma \cap B| = n\}.$$

Hence for a given $B \in \mathcal{O}_c(\Lambda_1)$, $n \in \mathbb{N}_0$ the pre-image under p_{Λ_2,Λ_1} of the cylinder set $C_{B,n}^{\Lambda_1}$ from $\Omega_{\Lambda_1}^M$ is a cylinder set from $\Omega_{\Lambda_2}^M$, i.e.,

$$p_{\Lambda_2,\Lambda_1}^{-1}(C_{B,n}^{\Lambda_1}) = \{\omega = (\gamma_\omega, m_\omega) \in \Omega_{\Lambda_2}^M \,||\gamma_\omega \cap B| = n\} = C_{B,n}^{\Lambda_2}.$$

On the other hand it is well known, see e.g. [AKR97] and [S94] that

$$\mu_{\hat\sigma}^{\Lambda_2}(C_{B,n}^{\Lambda_2}) = \frac{1}{n!}\sigma^n(B)e^{-\sigma(B)}.$$

which is the same as $\mu_{\hat{\sigma}}^{\Lambda_1}(C_{B,n}^{\Lambda_1})$. Therefore the consistency property (6.38) is proved.

It is possible to compute in closed form the Laplace transform of the measure $\mu_{\hat{\sigma}}$ if we make an additional assumption for the marks associated to each configuration, namely that they are independent and identically τ-distributed.

Let f be a continuous function on $X \times M$ such that the supp$f \subset \Lambda \times M$ for some $\Lambda \in \mathcal{O}_c(X)$. Let $\omega = (\gamma_\omega, m_\omega)$ be an element of Ω_X^M and define the pairing between f and ω by

$$\langle f, \omega \rangle := \sum_{x \in \gamma_\omega} f(x, m_x).$$

Then we have

$$\int_{\Omega_X^M} e^{\langle f,\omega \rangle} d\mu_{\hat{\sigma}}(\omega) = \int_{\Omega_\Lambda^M} e^{\langle f,\omega \rangle} d\mu_{\hat{\sigma}} \circ p_\Lambda(\omega) = \int_{\Omega_\Lambda^M} e^{\langle f,\omega \rangle} d\mu_{\hat{\sigma}}^\Lambda(\omega).$$

Using (6.37) the last integral is equal to

$$e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \int_{(\Lambda \times M)^n} \exp\left(\sum_{k=0}^n f(x_k, m_{x_k}) \right) d\hat{\sigma}(x_1, m_{x_1}) \ldots d\hat{\sigma}(x_n, m_{x_n})$$

$$= e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \left(\int_{\Lambda \times M} e^{f(x,m_x)} d\hat{\sigma}(x, m_x) \right)^n$$

$$= \exp\left(\int_{X \times M} (e^{f(x,m_x)} - 1) d\hat{\sigma}(x, m_x) \right).$$

That is, for any f in the above conditions the following formula holds:

$$l_{\mu_{\hat{\sigma}}}(f) = \int_{\Omega_X^M} e^{\langle f,\omega \rangle} d\mu_{\hat{\sigma}}(\omega) = \exp\left(\int_{X \times M} (e^{f(x,m_x)} - 1) d\hat{\sigma}(x, m_x) \right).$$

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