Feynman Integrals and White Noise Analysis

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Abstract

We review some basic notions and results of white noise analysis that are used in the construction of the Feynman integrand as a generalized White Noise functional for different classes of interactions. After sketching this construction for a large class of potentials we show that the resulting Feynman integrals solve the Schrdinger equation.

1 Introduction

As an alternative approach to quantum mechanics Feynman introduced the concept of path integrals, [11] which developed into an extremely useful tool in many branches of theoretical physics. In this work we review certain realizations of Feynman integrals in white noise analysis for different classes of interactions. As our basic example we think of a quantum mechanical particle.

On one hand it is possible to represent solutions of the heat equation by a path integral representation, based on the Wiener measure in a mathematically rigorous way. This is stated by the famous Feynman-Kac formula. On the other hand there have been a lot of attempts to write solutions of the Schrdinger equation as a Feynman (path) integral in a useful mathematical way. The methods are always more involved and less direct than in the Euclidean (i.e. Feynman-Kac) case. Among them are analytic continuation, limits of finite dimensional approximations and Fourier transform. Instead of enumerating a comprehensive list of publications on theories concerned with Feynman integrals we refer to the method proposed by [2] and the references therein. Here we have chosen a white noise approach.

White noise analysis is a mathematical framework which offers various generalizations of concepts known from finite dimensional analysis, among them are differential operators and Fourier transform. Although we will give a brief introduction to white noise calculus in Section 2 the reader unfamiliar with this topic is recommended to the monographs [5], [14], [22], [25], [13] and the articles [18], [21], [26], [28].

The idea of realizing Feynman integrals within the white noise framework goes back to [15]. The "'integral" is understood as the dual pairing of a distribution with a test function, so that the Feynman integrand itself has meaning as a distribution in a suitable distribution space depending on the class of potentials we deal with. This allows us to calculate not only the propagator but, more generally, time ordered expectation values. Important for the usefulness of any approach to Feynman integrals is the class of potentials we are able to handle. In white noise analysis Feynman integrals have been constructed for different classes of potentials. The first were proposed in [10] and by Khandekar and Streit [16]. This latter construction was generalized in [23], [8] and [12] to a wider class, which we give an account in this work. Potentials there were given as superpositions of δ-functions. Unfortunately this is restricted to one space dimension. In [2, Chap. 5] the path integral of the harmonic oscillator is defined within the theory of Fresnel integrals. Compared to our ansatz this procedure has the advantage of being manifestly independent of the space dimension. Despite the lack of a generalization to higher dimensional quantum systems our construction has some interesting features:

1. The admissible potentials may be very singular.
2. We are not restricted to smooth initial wave functions and may thus study the propagator directly.
3. Instead of giving a meaning to the Feynman integral we define the Feynman integrand as a Hida distribution. By taking expectation we get the propagator.

In [20] another class of potentials was considered, namely potentials which are Laplace transforms of finite complex measures on $\mathbb{R}^d$, $d \in \mathbb{N}$. These potentials are locally bounded and without singularities, however they grow in general exponentially at $\pm\infty$. They are too singular to be handled by Kato-Rellich perturbation theory. Nevertheless the propagator is analytic in the coupling constant g and we write it as a Dyson series. For the special case of Morse potentials $V(x) := e^{2ax} - be^{ax}$, $a, b \in \mathbb{R}$ the problem is solvable (see e.g., [17]) and it is known that quantities like the Green function, the spectrum and the eigenfunctions are not analytic in the coupling constant g. If we change from positive to negative g we also lose the essential self-adjointness of the corresponding Hamilton operator. This dramatic change however does not destroy the analyticity of the propagator. We give the details for that class in Section 6. We would like to emphasize that for this class of potentials the Feynman integrand belongs to a bigger distribution space, so-called Kondratiev distribution space $(S_d)^{-1}$ see [19]. The Hida space form a subspace of $(S_d)^{-1}$. We also would like to mention the work of Asai et al. [3] concerning Feynman integrals for the Albeverio-Høegh-Krohn potentials as a generalized white noise functional.

Most recently, white noise techniques were used to describe Feynman paths with boundaries [6], [7] and [9] for all the above classes of potentials.

2 Review of white noise analysis

In this section we give a sketch of white noise analysis which is suited for applications in later sections. For a general account on white noise analysis.

We start with the fundamental real Gel'fand triple:

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}), \tag{1}$$

where $S(\mathbb{R})$ and $S'(\mathbb{R})$ denotes the real Schwartz space of test functions and tempered distributions, respectively. Via Minlos' theorem we construct a measure space $(S'(\mathbb{R}), \mathcal{B}, \mu)$ called the white noise space by fixing the characteristic functional in the following way:

$$C_\mu(f) = \int_{S'(\mathbb{R})} e^{i\langle w,\varphi\rangle} d\mu(w) = e^{-\frac{1}{2}|\varphi|^2},\ \varphi \in S(\mathbb{R}).$$

Here the dual pairing $\langle \cdot, \cdot \rangle$ of $S'(\mathbb{R})$ and $S(\mathbb{R})$ is realized as an extension of the inner product in $L^2(\mathbb{R})$, $\langle f, \varphi\rangle := (f, \varphi)$, $f \in L^2(\mathbb{R})$, $\varphi \in S(\mathbb{R})$, and $|\cdot|$ denotes the norm in $L^2(\mathbb{R})$.

Within this formalism a version of Wiener's Brownian motion is given by:

$$B(t, w) := \langle w, 1\!\!1_{[0,t)}\rangle = \int_0^t w(s)ds.$$

We now consider the space $L^2(\mu)$ which is defined to be the complex Hilbert space $L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$. For applications the space $L^2(\mu)$ is often too small. A convenient way to solve this problem is to introduce a space of test functionals in $L^2(\mu)$ and to use its larger dual space.

Take a system of Hilbert norms $\{|\cdot|_p\}$ topologizing $S(\mathbb{R})$ which grows sufficiently fast. Then $S(\mathbb{R})$ is realized as a projective limit of Hilbert spaces $S_p(\mathbb{R})$, where $S_p(\mathbb{R})$ denotes the completion of $S(\mathbb{R})$ with respect to $|\cdot|_p$. Then the space of tempered distributions $S'(\mathbb{R})$ is the inductive limit of the Hilbert spaces $S_{-p}(\mathbb{R})$, where the dual norm $|\cdot|_{-p}$ topologizes the space $S_{-p}(\mathbb{R})$.

One convenient choice is

$$|\varphi|_p := |A^p\varphi|, \quad \varphi \in S(\mathbb{R}),$$

where $(A\varphi)(t) = -\varphi''(t) + (t^2 + 1)\varphi(t)$ is the Hamiltonian of a harmonic oscillator with ground state eigenvalue 2. Since $L^2(\mu)$ is Segal isomorphic to the symmetric Fock space $\Gamma(L^2(\mathbb{R}))$ of $L^2_{\mathbb{C}}(\mathbb{R}) := L^2(\mathbb{R}) + iL^2(\mathbb{R})$, we can identify the Fock space $\Gamma(S_p(\mathbb{R}))$ with a subspace $(S)_p$ of $L^2(\mu)$ and define the nuclear space

$$(S) = \bigcap_{p\geq 0} (S)_p.$$

Thus we arrived at the Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)'.$$

Elements of the space $(S)'$ are called Hida distributions (or generalized Brownian functionals). It is possible to characterize the spaces (S) and $(S)'$ by their S- or T-transforms. For $\Phi \in (S)'$ and $\varphi \in S(\mathbb{R})$ these transforms are defined as

$$\begin{aligned}(T\Phi)(\varphi) &:= \langle\langle \Phi, e^{i\langle\cdot,\varphi\rangle}\rangle\rangle = \int_{S'(\mathbb{R})} e^{i\langle w,\varphi\rangle}\Phi(w)d\mu(w)\\ (S\Phi)(\varphi) &:= \langle\langle \Phi, e_\mu(\varphi,\cdot)\rangle\rangle.\end{aligned}$$

Here $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the bilinear dual pairing between $(S)'$ and (S) and $e_\mu(\varphi,\cdot) := \exp(\langle\cdot,\varphi\rangle - \frac{1}{2}|\varphi|^2)$, $\varphi \in S(\mathbb{R})$. We denote by $\mathbb{E}(\Phi) := \langle\langle \Phi, 1\rangle\rangle$ the expectation of a Hida distribution Φ. S- and T-transform have extensions to the complex Schwartz space $S_{\mathbb{C}}(\mathbb{R})$ and are related by the following formula: $(T\Phi)(\varphi) = C_\mu(\varphi)(S\Phi)(\varphi)$ for any $\varphi \in S_{\mathbb{C}}(\mathbb{R})$.

Now we give the characterization theorem, which is due to Potthoff and Streit [27] and has been generalized in several ways, see e.g., [18] and references therein.

Theorem 2.1 *The following statements are equivalent:*

1. $F : S(\mathbb{R}) \to \mathbb{C}$ *has*
   (a) *"ray-analyticity": for all* $\varphi, \psi \in S(\mathbb{R})$ *the mapping* $\mathbb{C} \ni z \mapsto F(z\varphi + \psi)$ *is entire, and*
   (b) *"bound"': F is uniformly of order two, i.e., there exist constants* $K_1, K_2 > 0$ *and* $p \in \mathbb{N}_0$ *such that for all* $z \in \mathbb{C}$, $\varphi \in S(\mathbb{R})$,
   $$|F(z\varphi)| \leq K_1 \exp(K_2|z|^2|\varphi|_p^2).$$
2. *F is the S-transform of a unique Hida distribution* $\Phi \in (S)'$.
3. *F is the T-transform of a unique Hida distribution* $\hat{\Phi} \in (S)'$.

A functional satisfying 1 is usually called a U-functional. As an application of this theorem we give the following example.

Example 2.2 *Consider the composition* $\delta_a \circ B(t)$ *of the Dirac distribution* δ_a *at* $a \in \mathbb{R}$ *with Brownian motion* $B(t)$, $t > 0$,

$$\Phi = \delta(B(t) - a) = \delta(\langle\cdot, 1\!\!1_{[0,t)}\rangle - a), \quad a \in \mathbb{R}.$$

The S-transform of Φ is calculated to be (see e.g., [14])

$$(S\Phi)(\varphi) = (2\pi t)^{-1/2} \exp\left(-\frac{1}{2t}\left(\int_0^t \varphi(s)ds - a\right)^2\right)$$

and Theorem 2.1 gives immediately that Φ is well defined element in $(S)'$.

Theorem 2.1 enables us to discuss convergence of a sequence of generalized functionals.

Theorem 2.3 *Let $(F_n)_{n\in\mathbb{N}}$ denote a sequence of U-functionals with the following properties:*

1. *for all $\varphi \in S(\mathbb{R})$, $(F_n(\varphi))_{n\in\mathbb{N}}$ is a Cauchy sequence, and*
2. *there exist $K_1, K_2 > 0$ and $p \in \mathbb{N}_0$ such that the bound*

$$|F_n(z\varphi)| \leq K_1 \exp(K_2|z|^2|\varphi|_p^2), \quad \varphi \in S(\mathbb{R}), \quad z \in \mathbb{C},$$

holds for almost all $n \in \mathbb{N}$.

Then there is a unique $\Phi \in (S)'$ such that $T^{-1}F_n$ converges strongly to Φ.

As a second application we consider a theorem which concerns the integration of a family of generalized functionals.

Theorem 2.4 *Let (Ω, B, ν) denote a measure space and $\omega \mapsto \Phi_\omega$ a mapping from Ω to $(S)'$. We assume that the T-transform $F_\omega = T\Phi_\omega$ satisfies the following conditions for all $\omega \in \Omega$:*

1. *for every $\varphi \in S(\mathbb{R})$ the mapping $\omega \mapsto F_\omega(\varphi)$ is measurable, and*
2. *there exists $p \in \mathbb{N}_0$ such that*

$$|F_\omega(z\varphi)| \leq K_1(\omega) \exp(K_2(\omega)|z|^2|\varphi|_p^2), \quad \varphi \in S(\mathbb{R}), \quad z \in \mathbb{C},$$

with $K_1 \in L^1(\nu)$ and $K_2 \in L^\infty(\nu)$.

Then Φ is Bochner integrable in some $(S)_{-q}$ and thus

$$\int_\Omega \Phi_\omega d\nu(\omega) \in (S)'$$

and the T-transform and integration commute:

$$T\left(\int_\Omega \Phi_\omega d\nu(\omega)\right)(\varphi) = \int_\Omega (T\Phi_\omega)(\varphi) d\nu(\omega), \quad \varphi \in S(\mathbb{R}).$$

The last two theorems are also valid for the S-transform.

Example 2.5 *The Donsker's delta function from Example 2.2 is given by*

$$\delta(B(t) - a) = \frac{1}{2\pi} \int_\mathbb{R} \exp(i\lambda(B(t) - a)) d\lambda. \tag{2}$$

in the sense of Bochner integration, see e.g., [14].

Remark 2.6 *For later use we have to define pointwise products of a Hida distribution* Φ *with a Donsker delta function* $\delta(\langle w, \varphi \rangle - a)$*, i.e.,*

$$\Phi \cdot \delta(\langle w, \varphi \rangle - a). \tag{3}$$

If the mapping $\lambda \mapsto (T\Phi)(\varphi + \lambda\psi)$ *is integrable on* $\mathbb{R}$ *the following formula may be used to define the product in (3) as*

$$(T\Phi \cdot \delta(\langle w, \varphi \rangle - a))(\varphi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda a} (T\Phi)(\varphi + \lambda\psi) d\lambda \tag{4}$$

in the case that the right hand integral is indeed a U*-functional.*

Let us give one more example of a Hida distribution which is a first approximation of the Feynman integrand that we will introduce in the next section.

Example 2.7 *Let us consider the following formal expression for complex* c*:*

$$\exp\left(c \int_a^b w^2(s) ds\right), \qquad c \neq \frac{1}{2}.$$

Its S*-transform produces a* U*-functional "up to an infinite constant" (for details see [14]). So, as a renormalization, we omit this factor and get a well-defined* U*-functional:*

$$F(\varphi) = \exp\left(\frac{c}{1-2c} \int_a^b \varphi^2(s) ds\right), \quad \varphi \in S(\mathbb{R}).$$

Hence, we may define

$$\text{Nexp}\left(c \int_a^b w^2(s) ds\right) = S^{-1}F,$$

or, formally,

$$\text{Nexp}\left(c \int_a^b w^2(s) ds\right) = \frac{\exp(c \int_a^b w^2(s) ds)}{\mathbb{E}(c \int_a^b w^2(s) ds)}.$$

In Section 6 we need a d-dimensional version of the above white noise analysis for a bigger space of generalized functions. Hence, instead of the triple (1) we use

$$S_d \subset L_d^2 \subset S_d',$$

where $L_d^2 := L^2(\mathbb{R}) \otimes \mathbb{R}^d$, $d \in \mathbb{N}$ with norm given by $|f|^2 = \sum_{i=1}^d \int_{\mathbb{R}} f_j^2(s) ds$, $f \in L_d^2$. The space $S_d := S(\mathbb{R}) \otimes \mathbb{R}^d$ is a densely embedded nuclear space in L_d^2 and $S_d' := S'(\mathbb{R}) \otimes \mathbb{R}^d$ is its dual. A typical element $\varphi \in S_d$ is a d-dimensional vector where each component is a Schwartz test functions. The Gaussian measure on $(S_d', \mathcal{B})$ is given in terms of its characteristic functional by $C_\mu(\varphi) := \exp(-\frac{1}{2}|\varphi|^2)$, $\varphi \in S_d$ and a version of a d-dimensional Wiener's Brownian motion is given by

$$B(t, \vec{w}) := (\langle \vec{w}, 1\!\!1_{[0,t)} \otimes e_1 \rangle, \ldots, \langle \vec{w}, 1\!\!1_{[0,t)} \otimes e_d \rangle), \quad \vec{w} \in S_d',$$

where $\{e_1, \dots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$.

We proceed by choosing first a special subspace $(S_d)^1$ of test functionals and then construct the corresponding Gel'fand triple

$$(S_d)^1 \subset L^2(\mu) \subset (S_d)^{-1}.$$

Elements of the space $(S_d)^{-1}$ are called Kondratiev distributions, the space $(S)'$ of Hida distributions form a subspace. As before we will characterize the space $(S_d)^{-1}$ by its T-transform, see [19] for details. Let $\Phi \in (S_d)^{-1}$ be given, then there exist $p, q \in \mathbb{N}_0$ such that we can define, for every $\varphi \in U_{p,q} := \{\psi \in S_d | \; |\psi|_p^2 < 2^{-q}\}$ the T-transform by

$$(T\Phi)(\varphi) := \langle\langle \Phi, e^{i\langle \cdot, \varphi \rangle} \rangle\rangle$$

and its extension via analytic continuation to the complexifications of S_d denoted by $S_{d,\mathbb{C}}$. We also need the definition of holomorphy in a nuclear space, see [4].

Definition 2.8 *A function $F : U \to \mathbb{C}$ is holomorphic on an open set $U \subset S_{d,\mathbb{C}}$ iff for all $\eta_0 \in U$*

1. *for any $\varphi \in S_{d,\mathbb{C}}$ the mapping $\lambda \mapsto F(\eta_0 + \lambda\varphi)$ is holomorphic in some neighborhood of 0 in $\mathbb{C}$,*
2. *there exists an open neighborhood U' of η_0 such that F is bounded on U'.*

F is holomorphic at 0 iff F is holomorphic in a neighborhood of 0.

We now give the characterization of $(S_d)^{-1}$, see [19].

Theorem 2.9 *Let $U \subset S_{d,\mathbb{C}}$ be open and $F : U \to \mathbb{C}$ be holomorphic at zero, then there exists a unique $\Phi \in (S_d)^{-1}$ such that $T\Phi = F$. Conversely, let $\Phi \in (S_d)^{-1}$ then $T\Phi$ is holomorphic at zero. The correspondence between F and Φ is a bijection if we identify holomorphic functions which coincide on an open neighborhood of zero.*

As a consequence of the characterization theorem we have the analogous of Theorem 2.3 and 2.4.

Theorem 2.10 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(S_d)^{-1}$ such that there exists $U_{p,q}$, $p, q \in \mathbb{N}_0$ with*

1. *all $T\Phi_n$ are holomorphic on $U_{p,q}$,*
2. *there exists $C > 0$ such that $|(T\Phi_n)(\varphi)| \leq C$ for all $\varphi \in U_{p,q}$ and all $n \in \mathbb{N}$,*
3. *$((T\Phi_n)(\varphi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\varphi \in U_{p,q}$.*

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(S_d)^{-1}$.

Theorem 2.11 *Let (Ω, B, ν) be a measure space and $\Omega \ni \omega \mapsto \Phi_\omega \in (S_d)^{-1}$ a mapping. We assume that there exists $U_{p,q}$, $p, q \in \mathbb{N}_0$, such that*

1. *$T\Phi_\omega$ is holomorphic on $U_{p,q}$ for every $\omega \in \Omega$. The mapping $\omega \mapsto (T\Phi_\omega)(\varphi)$ is measurable for every $\varphi \in U_{p,q}$,*

2. *there exists $C \in L^1(\nu)$ such that $|(T\Phi_\omega)(\varphi)| \leq C(\omega)$ for all $\varphi \in U_{p,q}$ and for ν-almost all $\omega \in \Omega$.*

Then there exist $p', q' \in \mathbb{N}_0$ which only depend on p, q such that Φ_ω is Bochner integrable. In particular, $\int_\Omega \Phi_\omega d\nu(\omega) \in (S_d)^{-1}$ and $T\int_\Omega \Phi_\omega d\nu(\omega)$ is holomorphic on $U_{p',q'}$. We may interchange dual pairing and integration

$$\left\langle\!\!\left\langle \int_\Omega \Phi_\omega d\nu(\omega), \varphi \right\rangle\!\!\right\rangle = \int_\Omega \langle\!\langle \Phi_\omega, \varphi \rangle\!\rangle \, d\nu(\omega), \quad \varphi \in (S_d)^1.$$

3 The free Feynman integrand

We follow [15] and [10] in viewing the Feynman integral as a weighted average over Brownian paths which are modeled here by

$$x(t) := x_0 + \sqrt{\frac{\hbar}{m}} \langle w, 1\!\!1_{[0,t)} \rangle, \quad w \in S'(\mathbb{R}),$$

in the sequel we set $\hbar = m = 1$. The corresponding Feynman integrand for the free motion is

$$I_0 = \mathrm{N}\exp\left(\frac{i+1}{2}\int_0^t w^2(\tau)d\tau\right)\delta(x(t)-x_0),$$

where N is the normalizing factor introduced in Example 2.7. The delta distribution $\delta(x(t)-x_0)$ is used to fix the starting point and plays the role of an initial distribution. The T-transform of I_0 is

$$(TI_0)(\varphi) = (2\pi i|\Delta|)^{-1/2}\exp\left(-\frac{i}{2}|\varphi_\Delta|^2 - \frac{1}{2}|\varphi_{\Delta^c}|^2 + \frac{i}{2|\Delta|}\left(\int_{t_0}^t \varphi(\tau)d\tau + x - x_0\right)^2\right),$$

where $\Delta := [t_0, t]$ and φ_Δ is the restriction of φ to Δ, etc. Furthermore the Feynman integral $\mathbb{E}(I_0) = (TI_0)(0)$ is indeed the free particle propagator $\frac{1}{\sqrt{2\pi i|t-t_0|}}\exp(\frac{i}{2|t-t_0|}(x-x_0)^2)$. Not only the expectation but also the T-transform has a physical meaning. By a formal integration by parts

$$(TI_0)(\varphi) = \exp(-\frac{1}{2}|\varphi_{\Delta^c}|^2 + ix\varphi(t) - ix_0\varphi(t_0))\mathbb{E}\left(I_0\exp\left(-i\int_{t_0}^t x(\tau)\dot\varphi(\tau)d\tau\right)\right).$$

The term $\exp(-i\int_{t_0}^t x(\tau)\dot\varphi(\tau)d\tau)$ would thus arise from a time-dependent potential $W(x,t) = \dot\varphi(\tau)x$. And indeed it is straightforward to verify that

$$(TI_0)(\varphi) = \exp\left(-\frac{1}{2}|\varphi_{\Delta^c}|^2 + ix\varphi(t) - ix_0\varphi(t_0)\right)K_0^{(\dot\varphi)}(x_0, t_0|x, t), \qquad (5)$$

where

$$K_0^{(\dot\varphi)}(x_0,t_0|x,t) = \frac{1}{\sqrt{2\pi i|\Delta|}}\exp(ix_0\varphi(t_0)-ix\varphi(t)) \times\exp\left(-\frac{i}{2}|\varphi_\Delta|^2+\frac{i}{2|\Delta|}\left(\int_{t_0}^t\varphi(\tau)d\tau+x-x_0\right)^2\right)$$

is the Green's function corresponding to the potential W, i.e., $K_0^{(\dot\varphi)}$ obeys the Schrdinger equation

$$(i\partial_t+\frac{1}{2}\partial_x^2-\dot\varphi(t)x)K_0^{(\dot\varphi)}(x_0,t_0|x,t)=i\delta(t-t_0)\delta(x-x_0).$$

More generally one calculates

$$T\left(I_0\prod_{j=1}^n\delta(x(t_j)-x_j)\right)(\varphi) = \exp\left(-\frac{1}{2}|\varphi_{\Delta^c}|^2+ix\varphi(t)-ix_0\varphi(t_0)\right) \times\prod_{j=1}^n K_0^{(\dot\varphi)}(x_{j-1},t_{j-1}|x_j,t_j),$$

where $t_0<t_1<\ldots<t_{n+1}:=t$ and $x_{n+1}=x$.

4 The perturbed Feynman integrand

In order to pass from the free motion to more general situations, one has to give a rigorous definition of the heuristic expression

$$I_V=I_0\exp\left(-i\int_{t_0}^t V(x(\tau))d\tau\right).$$

In [16] the authors accomplished this by perturbative methods in case V is a finite signed Borel measure with compact support. This construction was generalized in [23] to a wider class of potentials by allowing time-dependent potentials and a Gaussian fall-off instead of a bounded support. The starting point is a power series expansion of $\exp(-i\int_{t_0}^t V(x(\tau))d\tau)$ using $V(x(\tau),\tau)=\int_{-\infty}^{\infty}V(x,\tau)\delta(x(\tau)-x)dx$:

$$\exp\left(-i\int_{t_0}^t V(x(\tau))d\tau\right)=\sum_{n=0}^\infty(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\prod_{j=1}^n V(x_j,t_j)\delta(x(t_j)-x_j)dt_jdx_j,$$

where $\Lambda_n=\{(t_1,\ldots t_n)|\ t_0<t_1<\ldots<t_n<t\}$. In order to consider singular potentials V is no longer taken to be a function V but a measure ν. Under suitable conditions on ν it is proven in [16] and [23] that

$$I_V=I_0+\sum_{n=1}^\infty(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\left(I_0\prod_{j=1}^n\delta(x(t_j)-x_j)\right)\prod_{j=1}^n\nu(dx_j,dt_j)$$

exists as a well-defined element of $(S)'$.

5 The Feynman integrand for the harmonic oscillator

Later in [8] and [12] it was shown that the Feynman integrand for the time-dependent harmonic oscillator in an external potential is a well defined element in $(S)'$, i.e., a Hida distribution. In both cases the class of potentials are time-dependent with a Gaussian fall-off. Here we give the main steps from [8] and the time-dependent case in [12] uses the same arguments with longer expression.

To define the Feynman integrand

$$I_h = I_0 \exp\left(-i\int_{t_0}^{t} U(x(\tau))d\tau\right), \qquad U(x) = \frac{1}{2}k^2x^2$$

of the harmonic oscillator, at least two things have to be done. First we have to justify the pointwise multiplication of I_0 with the interaction term and secondly it has to be shown that $\mathbb{E}(I_h)$ solves the Schrdinger equation for the harmonic oscillator. Both has been done in [10]. There the T-transform of I_h has been calculated and shown to be a U-functional, hence $I_h \in (S)'$. Here we use the following modified version of the T-transform

$$\begin{aligned}(TI_h)(\varphi) \quad = \quad & \frac{\sqrt{k}}{\sqrt{2\pi i \sin(k|\Delta|)}} \exp\left(-\frac{1}{2}|\varphi_{\Delta^c}|^2 - -\frac{i}{2}|\varphi_{\Delta}|^2\right) \exp\left(\frac{ik}{2\sin(k|\Delta|)}\Bigg((x_0^2+x^2)\right. \\ & \times \cos(k|\Delta|) - 2x_0x + 2x\int_{t_0}^{t}\varphi(t')\cos(k(t'-t_0))dt' - 2x_0\int_{t_0}^{t}\varphi(t')\cos(k(t-t'))dt' \\ & \left. +2\int_{t_0}^{t}\int_{t_0}^{s_1}\varphi(s_1)\varphi(s_2)\cos(k(t-s_1))\cos(k(s_2-t_0))ds_2ds_1\Bigg)\right),\end{aligned}$$

with $0 < k|\Delta| < \frac{\pi}{2}$ which is easily seen to be a U-functional. If we introduce the propagator

$$K_h^{(\dot{\varphi})}(x,t|x_0,t_0) = (TI_h)(\varphi)\exp\left(\frac{1}{2}|\varphi_{\Delta^c}|^2 - ix\varphi(t) + ix_0\varphi(t_0)\right)$$

of a particle in a time-dependent potential $\frac{1}{2}k^2x^2 + x\dot{\varphi}(t)$ and define the product

$$I_h \prod_{j=1}^{n} \delta(B(t_j) - x_j)$$

as an element in $(S)'$ using repeatedly (2) in Remark 2.6. Then we arrive at

$$T\left(I_h \prod_{j=1}^{n} \delta(B(t_j)-x_j)\right)(\varphi) = \exp\left(-\frac{1}{2}|\varphi_{\Delta^c}|^2 + ix\varphi(t) - ix_0\varphi(t_0)\right)\prod_{j=1}^{n+1} K_h^{(\dot{\varphi})}(x_{j-1},t_{j-1}|x_j,t_j).$$

This is achieved by induction and the fact that for $[t_0,t] \subset [t_0',t']$ we have $K_h^{(\dot{\psi})}(x_0,t_0|x,t) = K_h^{(\dot{\varphi})}(x_0,t_0|x,t)$ with $\psi = \varphi + \lambda 1\!\!1_{[t_0,t]}$ and $\lambda \in \mathbb{R}$.

The Feynman integrand for the harmonic oscillator in an external potential is defined as

$$I_{h,V} = I_h \exp\left(-i\int_{t_0}^{t} V(x(\tau),\tau)d\tau\right)$$

such that the perturbation V is introduced via the series expansion of the exponential. Hence we have to find conditions for V such that the following object exists in $(S)'$

$$I_{h,V} = I_h + \sum_{n=1}^{\infty}(-i)^n \int_{\mathbb{R}^n}\int_{\Lambda_n} I_h \left(\prod_{j=1}^{n} V(x_j,t_j)\delta(x(t_j)-x_j)\right).$$

Since we want to study singular time-dependent potentials, we consider ν a finite signed Borel measure on $\mathbb{R}\times\Delta$. Let ν_x denote the marginal measure $\nu_x(A) = \nu(A\times\Delta)$ for any $A\in\mathcal{B}(\mathbb{R})$ and similarly $\nu_t(B) = \nu(\mathbb{R}\times B)$ for any $B\in\mathcal{B}(\Delta)$. The following theorem gives sufficient conditions under which $I_{h,V}$ is a Hida distribution.

Theorem 5.1 *Let $\nu = \nu_x - \nu_-$ be a finite signed Borel measure on $\mathbb{R}\times\Delta$ where the marginal measures $|\nu_x| := (\nu_+ - \nu_-)_x$ and $|\nu_t|$ satisfy:*

1. *$\exists R>0, \forall r>R$: $|\nu|_x(\{x : |x|>r\}) < \exp(-\beta r^2)$ for some $\beta>0$,*
2. *$|\nu|_t$ has a L^∞ density.*

Then $I_{h,V}$ given by the following expression is a Hida distribution

$$I_{h,V} = I_h + \sum_{n=1}^{\infty}(-i)^n \int_{\mathbb{R}^n}\int_{\Lambda_n} \left(I_h \prod_{j=1}^{n} \delta(x(t_j)-x_j)\right)\prod_{j=1}^{n}\nu(dx_j,dt_j).$$

Condition 1 is satisfied for very singular potentials, e.g. $V(x) = \sum_{n=1}^{\infty} e^{-n^2}\delta_n(x)$, $x\in\mathbb{R}$. For cutoff interaction, i.e., compactly supported ν_x, condition 1 is of course valid. Furthermore all potentials V for which there exists $R', A, B>0$ and $C\in\mathbb{R}$ such that $|V(x)|\le A|x-C|e^{-B(x-C)^2}$ for all $|x|\ge R'$ are in the class of admissible potentials. Note also that ν is not supposed to be a product measure, hence the time-dependence can be more intricate than simple multiplication by a function of time.

Prova. We have to perform a central estimate for $T\left(I_h\prod_{j=1}^{n}\delta(x(t_j)-x_j)\right)(z\varphi)$ such that it survives to n-fold integration and summation. These technicalities which are necessary to establish the estimate can be found in e.g., [8] (or [12] for time-dependent mass and frequency) and is given by

$$\begin{aligned}
&\left|T\left(I_h\prod_{j=1}^{n}\delta(x(t_j)-x_j)\right)(z\varphi)\right| \\
\le\ & \left(\prod_{j=1}^{n+1}\frac{1}{2\sqrt{|\Delta_j|}}\right)\exp(X^2\gamma)\exp\left(|z|^2\|\varphi\|^2\left(\frac{1}{2}+\frac{\pi}{2}|\Delta|+\frac{L^2}{2\gamma}\right)\right),
\end{aligned}$$

where $\Delta_j := [t_{j-1}, t_j]$, $X^2 = \sup_{0\leq j\leq n+1} |x_j|$, $\gamma > 0$, L is a constant and $\|\cdot\|$ is a continuous norm in $S(\mathbb{R})$.
In order to apply Theorem 2.4 to perform the integration we need to show that $F := (\prod_{j=1}^{n+1} \frac{1}{2\sqrt{|\Delta_j|}}) \exp(X^2\gamma)$ is integrable with respect to ν. First we prove that $\exp(X^2\gamma) \in L^q(\mathbb{R}^n \times \Delta, |\nu|)$ with

$$|\exp(X^2\gamma)|_q \leq \exp\exp(\gamma(x_0^2 + x^2))Q^n < \infty,$$

where $q > 2$ is such that $0 < \gamma < \frac{\beta}{q}$ and $Q = |\exp(x^2\gamma)|_q$. For the factor $\prod_{j=1}^{n+1} \frac{1}{2\sqrt{|\Delta_j|}}$ we have the following estimate

$$\left| \prod_{j=1}^{n+1} \frac{1}{2\sqrt{|\Delta_j|}} \right|_p \leq |\nu|_{t\infty}^{n/p} \frac{\Gamma\left(\frac{2-p}{2}\right)^{(n+1)/p} |\Delta|^{n/p-(n+1)/2}}{2^{n+1}\Gamma\left((n+1)\frac{2-p}{2}\right)^{1/p}},$$

where $|\nu|_{t\infty}$ is shorthand notation for the essential supremum of the L^∞-density of $|\nu|_t$ which exists due to condition 2 and p is the conjugate exponent of q. finally an application of Hlder's inequality gives

$$|F|_1 \leq \exp(\gamma(x_0^2 + x^2))Q^n |\nu|_{t\infty}^{n/p} \frac{\Gamma\left(\frac{2-p}{2}\right)^{(n+1)/p} |\Delta|^{n/p-(n+1)/2}}{2^{n+1}\Gamma\left((n+1)\frac{2-p}{2}\right)^{1/p}} =: C_n < \infty.$$

Hence Theorem2.4 yields

$$I_n := \int_{\mathbb{R}^n} \int_{\Lambda_n} \left(I_h \prod_{j=1}^{n} \delta(x(t_j) - x_j) \right) \prod_{j=1}^{n} \nu(dx_j, dt_j) \in (S)'.$$

As the C_n are rapidly decreasing in n the hypotheses of Theorem 2.3 are fulfilled and hence $I_{h,V} \in (S)'$. ■

6 The Feynman integrand for a class of unbounded potentials

Now we construct the Feynman integrand for a new class of unbounded potentials and calculate the propagators. We show that the propagators solve the corresponding Schrdinger equation. We start we the definition of the class of interactions.

Definition 6.1 *Let m be a complex measure on the Borel sets on $\mathbb{R}^d$, $d \geq 1$ fulfilling the following condition*

$$\int_{\mathbb{R}^d} e^{C|\alpha|} d|m|(\alpha) < \infty, \qquad \forall C > 0. \tag{6}$$

We define a potential V on $\mathbb{R}^d$ by

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha\cdot x} dm(x).$$

Remark 6.2 *Condition (6) implies that the measure m is finite. By Lebesgue's dominated convergence theorem we obtain that the potentials are restrictions to the real line of entire functions. In particular they are locally bounded and without singularities. However they are in general unbounded at $\pm\infty$. It is possible to consider time-dependent potentials with the same type of condition as in (6), see [20, Subsection 4.5].*

Example 6.3

1. *Every finite measure with compact support fulfills the above condition (6),*
2. *The simplest example is the Dirac measure in one dimension $m(\alpha) = g\delta_a(\alpha)$ for $a > 0$ and $g \in \mathbb{R}$. The corresponding potential is $V(x) = ge^{ax}$. All polynomials of exponential functions of the above kind are also in that class too, e.g., $\sinh(sx)$, $\cosh(ax)$. In particular the well known Morse potential $V(x) = g(e^{-2ax} - 2\gamma e^{-ax})$ with $g, a, x \in \mathbb{R}$ and $\gamma > 0$ is included in this class.*
3. *If we choose a Gaussian density, we get potentials of the form $V(x) = ge^{bx^2}$ with $b, x \in \mathbb{R}$. Entire functions of arbitrary high order of growth are inside of this class. More explicitly, the measures $m(\alpha) = 1\!\!1_{[0,\infty)}(\alpha)\exp(-k\alpha^{1+b})$, $b, k > 0$ fulfill the condition (6). The associated potentials are entire functions of order $1 - \frac{1}{b}$, see e.g., [24, Lemma 7.2.1].*

Following the same procedure of Section 3 adapted for this case we arrive at the following expression for the Feynman integrand

$$I = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)} \prod_{j=1}^{n} dm(\alpha_j) d^n\tau.$$

First of all we have to give a meaning to the product $I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)}$.

Proposition 6.4 *Let $\tau_j \in [t_0, t]$ for $j = 1, \ldots, n$, $t_0 < t$ and $\alpha_j \in \mathbb{R}^d$. The pointwise product*

$$\Phi_n = I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)}$$

is a Kondratiev distribution and its T-transform is given by

$$\begin{aligned}(T\Phi_n)(\varphi) &= \frac{1}{\sqrt{(2\pi i|\Delta|)^d}} \exp\left(-\frac{i}{2} \int_{\mathbb{R}} \left(\varphi(s) + i \sum_{j=1}^{n} \alpha_j 1\!\!1_{(\tau_j,t]}(s) \right)^2 ds \right) \\ &\quad \times \exp\left(-\frac{1}{2i|\Delta|} \left(\int_{t_0}^{t} \varphi(s) ds + i \sum_{j=1}^{n} \alpha_j (t - \tau_j) + (x - x_0) \right)^2 \right) \\ &\quad \times \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x \right).\end{aligned}$$

Prova. The arguments of the proof goes as follows. First the T-transform have an extension in $\varphi \in S_d$ to all $\psi \in S_{d,\mathbb{C}}$ and the first part of Definition 2.8 is fulfilled. To apply Theorem 2.9 we need a bound

$$\begin{aligned}(T\Phi_n)(\varphi) &\leq \frac{1}{\sqrt{(2\pi|\Delta|)^d}} \exp\left(\sum_{j=1}^{n} |\alpha_j||x-x_0| + \sum_{j=1}^{n} |\alpha_j||x_0|\right) \\ &\quad \times \exp\left(|\varphi|^2 + \left(2\sqrt{|\Delta|}\sum_{j=1}^{n}|\alpha_j| + \frac{|x-x_0|}{\sqrt{\Delta}}\right)|\varphi|\right) \\ &=: C_n(\alpha_1,\ldots,\alpha_n,\varphi). \end{aligned} \tag{7}$$

Thus Φ_n is Kondratiev distribution. ∎

Theorem 6.5 *Let V be as in Definition 6.1. Then*

$$I = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)} \prod_{j=1}^{n} dm(\alpha_j) d^n\tau$$

exists as a generalized white noise functional. The series converges in the strong topology of $(S_d)^{-1}$. The integrals exist in the sense of Bochner integrals. Therefore we can express the T-transform by

$$(TI)(\varphi) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)}\right)(\varphi) \prod_{j=1}^{n} dm(\alpha_j) d^n\tau$$

for all φ in a neighborhood of zero $U_{p,q} := \{\psi \in S_{d,\mathbb{C}}|\ 2^q|\psi|_p < 1\}$ for some $p, q \in \mathbb{N}_0$.

Prova. We have already shown in Proposition 6.4 that the product $\Phi_n = I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)}$ is a Kondratiev distribution, moreover, we derived the estimate (7). In order to see that the integrals exist in the Bochner sense we want to apply Theorem 2.11. As the T-transform of Φ_n is entire in $\varphi \in S_{d,\mathbb{C}}$ and measurable, it remains only to derive a suitable bound

$$\begin{aligned} &\int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} C_n(\alpha_1,\ldots,\alpha_n,\varphi) \prod_{j=1}^{n} d|m|(\alpha_j) d^n\tau \\ \leq\ & \frac{|\Delta|^n}{\sqrt{(2\pi|\Delta|)^d}} \exp\left(|\varphi|^2 + \frac{|x-x_0|}{\sqrt{|\Delta|}}|\varphi|\right) \\ &\times \left(\int_{\mathbb{R}^d} \exp((|x-x_0| + |x_0| + 2\sqrt{|\Delta|}|\varphi|)|\alpha|) d|m|(\alpha)\right)^n \end{aligned}$$

and this is finite since the measure satisfies condition (6). Due to Theorem 2.11 there exists an open neighborhood U independent of n and

$$I_n := \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} \Phi_n \prod_{j=1}^{n} dm(\alpha_j) d^n\tau \in (S_d)^{-1}, \quad \forall n \in \mathbb{N}.$$

To finish the proof we must show that the series converges in $(S_d)^{-1}$ in the strong sense. First we observe that TI_n is holomorphic on U, then in order to apply Theorem 2.10 we derive the following bound for $\varphi \in U$

$$\begin{aligned} |(TI)(\varphi)| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} |(TI_n)(\varphi)| \\ &\leq \frac{1}{\sqrt{(2\pi|\Delta|)^d}} \exp\left(|\varphi|^2 + \frac{|x-x_0|}{\sqrt{|\Delta|}} |\varphi| \right) \\ &\quad \times \exp\left(|\Delta| \int_{\mathbb{R}^d} \exp((|x| + 2|x_0| + 2\sqrt{|\Delta|}|\varphi|)|\alpha|) d|m|(\alpha) \right) \\ &\leq \infty. \end{aligned}$$

Therefore $I \in (S_d)^{-1}$. ■

Remark 6.6 *The bound established above has rather surprising consequence. For the forthcoming discussion it is convenient to show the dependence on the coupling constant explicitly, so that we get*

$$(TI)(\varphi) = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \prod_{j=0}^{n} e^{\alpha_j \cdot x(\tau_j)} \right)(\varphi) \prod_{j=1}^{n} dm(\alpha_j) d^n\tau$$

which is a perturbation series in the coupling constant. The bound we have already calculated guarantees that $(TI)(\varphi)$ is an entire in the coupling constant g for all fixed $x, x_0, t_0 < t$ and $\varphi \in U_{p,q}$. This is surprising, since the corresponding Hamilton operators, even if they are essentially self-adjoint for $g > 0$, lose this property for $g < 0$ in general. Quantities such as eigenvalues and eigenvectors will not be analytic in the coupling constant. On the other hand under a stronger condition than (6) Albeverio et al. [1] have shown that the solution of the Schrdinger equation $\Psi_t(x)$ is analytic in the coupling constant if the initial wave function Ψ_0 as a function of x is from a certain class of analytic functions.

Finally we show that for $t_0 < t$ the propagator does solve the Schrdinger equation not only in the sense of distributions, but also in the sense of ordinary functions. We can also give the to test function in the T-transform a physical meaning corresponding to a time dependent homogeneous external small force. To this end we proceed as in (5) and consider

$$(TI)(\varphi) \exp\left(\frac{i}{2} \int_{\Delta^c} \varphi^2(s) ds + ix_0 \cdot \varphi(t_0) - ix \cdot \varphi(t) \right).$$

This produces the Schrdinger propagator $K^{(\dot{\varphi})}$ which solves the Schrdinger equation for all $x, x_0, t_0 < t$

$$(i\partial_t + \frac{1}{2}\triangle_d - gV - x \cdot \dot{\varphi}(t)) K^{(\dot{\varphi})}(x_0, t_0 | x, t) = 0,$$

with initial condition $\delta(x - x_0)$.

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