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Syphon dynamics —a soluble model of multi-agents cooperative behavior

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Abstract. – We consider the dynamics of a "one queue one server" feedback queueing system where the decision of an agent to use the feedback loop is based upon its waiting time in the system. We investigate the dynamics for very patient agents and quantify the emerging stable and almost deterministic oscillations of the queue length. The resulting delay dynamics can be viewed as an auto-siphoning hydrodynamic device. Using this picture, we can further discuss the transport phenomenon in serial networks of feedback queues. Global purging of the network and noise-induced transport are emphasized.

Introduction. – Models able to capture the global behavior of interacting agents attract a strong attention in science, technology and in areas where humans decisions are important such as finance and management sciences. In this context, Multi-Agents Systems (MAS), *i.e.* systems composed of an assembly of interacting items responding to individual decision mechanisms, are rightfully regarded as prototypes for understanding the global behavior of elementary societies formed by "intelligent" members. Computer simulations are widely used to study such systems involving a high number of degrees of freedom with strong-coupling nonlinearities and fluctuations. In parallel, relatively few models offer a balanced compromise between the representativeness of the salient features of the MAS mechanisms together with a sufficient simplicity to allow for analytical treatment. To focus on such a solvable model is the goal of the present paper. Our model involves an elementary decision mechanism based on the past experience (*i.e.* the *"history"*) lived by each agent. As time evolves, this individual experience is, due to his/her mutual interactions with other members of the MAS, modified and so are the decisions he/she takes. The present model, partly inspired from [1], describes the dynamics of the population level of customers in a queue line waiting for service. The arrival of new customers and the required service times are i.i.d. random variables. Once a customer leaves the service, his *decision* to return or to quit the system is based on the waiting time he has spent into the line before service. This type of dynamical system belongs to the class of *queueing system with a state-dependent feedback* [2]. Despite its simplicity, this model already offers several salient features of the general class of MAS described in [3]. Indeed the

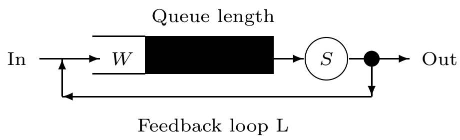

Fig. 1 – Sketch of the feedback queue. The routing of an agent i leaving the server S depends on the time spent in W plus the service time. If this time exceeds a patience threshold P_i, the agent leaves the system. Otherwise, he uses the loop L.

waiting time of each agent (*i.e.* each customer) is the internal degree of freedom on which the customers decisions —return or leave the system— are taken. The return mechanism (*i.e.* the feedback feeding of the queue) is intrinsically nonlinear. The customers perceive their environment by measuring their individual waiting times in the system and then, based on this measure, act autonomously. They exhibit a proactive behavior as they permanently auto-regulate the queue length to avoid explosion while trying to permanently keep the server in the busy state. Our system exhibits an *emergent dynamical behavior* materialized here by stable oscillations of population level in the waiting room. In the limiting regime, characterized by an assembly composed of very patient customers, an analytical discussion of the dynamics is possible. In this limit, the law of large numbers applies and implies that the dynamics obeys a deterministic, nonlinear evolution. A similar reduction to deterministic evolution has also been identified in [4], where a MAS composed of "bullish" and "bearish" financial agents interact. For the model to be presented here, the resulting deterministic dynamical system can further be identified with an auto-siphoning hydrodynamic device. We shall refer to this dynamics by speaking of the *siphon dynamic model* (SDM). This elementary representation enables us to intuitively guess most relevant features of the evolution of the underlying highly nonlinear system. In addition, the simplicity of the basic model directly suggests to study the collective behavior of several SDM coupled via a network. Here, we shall focus on the flows travelling in a cascade of coupled SDM.

Basic model. – We consider a queuing system with feedback composed of a waiting room W with infinite capacity, a server facility S and a feedback loop (fig. 1). The inter-arrival times of external agents (*i.e.* agents coming from outside and not from the feedback loop) into W are the outcomes of independent drawings of a positive random variable A distributed according to a given probability density $a(t)$. We suppose A to have finite mean and set:

$$\frac{1}{\lambda} := \int_0^\infty t a(t) \mathrm{d}t. \tag{1}$$

The parameter λ can be interpreted as the mean frequency of external arrivals into W. Similarly, the inter-departure times from S, $\{d_j\}_{j\in\mathbb{N}}$, are the independent realizations of a positive random variable D. We suppose D to be independent of A, to be distributed according to a given probability density $d(t)$ with finite mean and set:

$$\frac{1}{\mu} := \int_0^\infty t d(t) \mathrm{d}t. \tag{2}$$

The parameter μ can be interpreted as the mean frequency of service completions at S. We further assume the relation $\lambda < \mu$ to hold which ensures stability of the queue. The queueing discipline is "first in first out" such that the i-th agent (coming from outside or from the loop) lining up in the waiting room is also the i outgoing agent. Immediately after an agent

i receives service at S, he compares his system time W_i (*i.e.* the time spent in the waiting room plus the service time) with his patience factor P_i, the i-th outcome of a positive random variable P with density $p(t)$. We suppose $p(t)$ to be supported on an interval $[P_{\min}, P_{\max}]$, where $0 < P_{\min} \leq P_{\max} < \infty$ are two control parameters of the dynamics.

Let us now fix the routing of agents in the feedback queue and hence, the dynamics of the system. Immediately upon finishing service, agent i adopts one of the two following alternatives:

a) leave the system forever if $W_i > P_i$,

b) line up once again and without delay in the queue (by using the feedback loop) if $W_i \leq P_i$.

We say that agent i is satisfied by the service if he adopts decision b) and is unsatisfied if he chooses a). Note that if agent i opts for decision b), he joins the queue as agent number j for some $j > i$ (*i.e.* the agents memory is cleared after every decision).

Despite its apparent simplicity, this feedback dynamic offers a rich variety of behaviors including self-organization of the traffic intensity and stable queue-length oscillations. The origin of these behaviors is due to the facts that 1) the feedback routing induces correlations between the input and output process of the queue and 2) the routing decision (a) or b)) at time t of an agent i depends on the systems history up to time $t - W_i$. The key feature is that satisfied agents drive the system into states generating unsatisfied agents and vice versa. Accordingly, clusters of unsatisfied agents decreasing the waiting line are followed by clusters of satisfied agents crowding the waiting line. There results a remarkably stable self-organized behavior where the traffic intensity $\rho = \lambda/\mu$ is oscillating around its critical value 1, provided

$$P_{\min} \gg 1/\mu\,. \tag{3}$$

This relation is supposed to be satisfied from now on. The stability of the oscillations results from the law of large numbers which reduces the relative fluctuations of the waiting time W_i. To see this, let $N(t)$ be the number of agents in the system at time t (*i.e.* the agents in the waiting room plus the one in service at time t). Observe indeed that the waiting time W_i of agent i entering at time t into the waiting room satisfies

$$\frac{1}{N(t)} \sum_{j=1}^{N(t)-1} d_j \leq \frac{W_i}{N(t)} \leq \frac{1}{N(t)} \sum_{j=1}^{N(t)} d_j \tag{4}$$

which, due to the law of large numbers, converges for $N(t) \to \infty$ almost surely to $\frac{1}{\mu}$. Therefore, for large $N(t)$ we have

$$W_i \approx \frac{N(t)}{\mu} \tag{5}$$

with high probability. Hence, for $N(t) \gg P_{\max}\mu$, eq. (5) applies and we have $W_i > P_{\max}$ *i.e.*, incoming agents will, with high probability, take decision a). Suppose now that an agent i joins at time t a relatively short waiting line (*i.e.* such that $N(t) \ll P_{\min}\mu$). To show that this agent i will —with high probability— take decision b), we introduce the notation $N(t) = n$ and remark that

$$W_i \leq \sum_{j=1}^{n} d_j \leq n \max_{1 \leq j \leq n} \{d_j\}. \tag{6}$$

Thanks to this relation, we deduce the inequality

$$\mathrm{Prob}\left(n \max_{1\leq j\leq n} \{d_j\} \leq P_{\min}\right) \leq \mathrm{Prob}\left(W_i \leq P_{\min}\right). \tag{7}$$

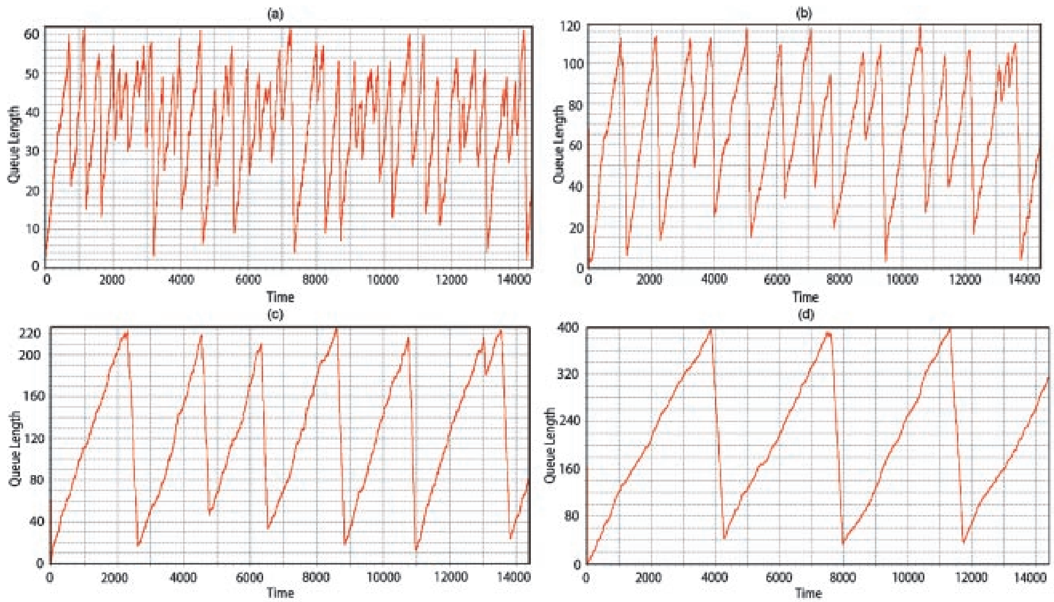

Fig. 2 – Simulations of the queue length $N(t)$ over 14000 outflow events. A and D are exponentially distributed with parameter $\lambda = 0.1$ and $\mu = 1$. The patience parameter P is uniformly distributed over the interval $[P_{\min}, P_{\min} + 10]$. Parameters: (a) $P_{\min} = 50$, (b) $P_{\min} = 100$, (c) $P_{\min} = 200$, (d) $P_{\min} = 350$. Note the stabilization of the oscillations with increasing $P_{\min}$.

The RHS of (7) is the probability of i to take decision b). Thanks to the independence of the d_j's, the LHS of (7) equals $F_d(P_{\min}/n)^n$ *i.e.*, the n-th power of the cumulative distribution function of d evaluated at $P_{\min}/n$. Hence, for $P_{\min}\mu \gg n$, the LHS will be close to one and therefore the agent i will, with high probability, take decision b). The simulations presented in fig. 2 indeed exhibit for large $P_{\min}$ stable oscillations (*i.e.* a limit-cycle type of dynamics).

A) *The syphon dynamics with deterministic patience parameter.* Here we focus on the case where all the agents do have the same patience factor $P := P_{\min} = P_{\max}$. The following deterministic dynamics captures the essence of the original queueing system (see fig. 3A): starting with $N(0) = 0$, the queue length grows linearly with speed λ up to $N = P\mu + P\lambda$ and decreases linearly with speed $\mu - \lambda > 0$ until $N(t)$ reaches the value $P\lambda$. The amplitude Δ of the oscillator is $\Delta = P\mu$ and the period is $\pi = P(2 + \frac{\lambda}{\mu-\lambda} + \frac{\mu-\lambda}{\lambda})$. The switches between increasing and decreasing states are triggered by the agents leaving S which, at the moment of arrival, have seen exactly $P\mu$ agents ahead waiting in the queue. The resulting *delay dynamics* admits a simple and enlightening physical realization as an auto-syphoning system *without feedback* sketched in fig. 3B. We call this system the syphon dynamic model (SDM).

B) *The syphon dynamics with individualized patience parameter.* Let us now focus on the more realistic case where the agents patience factors P_i are individualized by fixing them at random according to the outcomes of i.i.d. random variables with values in $[P_{\min}, P_{\max}]$. We restrict however the individual character by imposing $\eta := \frac{P_{\max}-P_{\min}}{P_{\min}} < 1/2$, a restriction to become clear in eq. (9). The syphon model is still adequate and the value $P_{\max} - P_{\min}$ plays the role of maximal water level fluctuations (see fig. 4B). It differs from the former case, as now the amplitudes Δ of the oscillations (*i.e.* the cluster length of unsatisfied agents) depend not only on μ but also on λ (see fig. 4A). To see this, let i denote the first agent which upon arrival

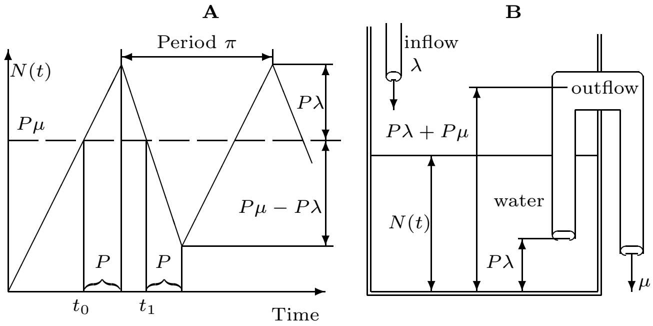

Fig. 3 – A. The agent entering at t_0 is the first one of a whole cluster C of unsatisfied customers and triggers the switch of $N(t)$ from the increasing to the decreasing state at $t_0 + P$. The last agent belonging to the cluster of unsatisfied customers C is the one entering just before t_1 and triggers the switch of $N(t)$ from the decreasing to the increasing state at $t_1 + P$. This simple delay dynamic repeats and creates stable oscillations. B. The syphon model. The queue length corresponds to the water level $N(t)$. The inflow and outflow rates are λ, respectively μ. The syphon leaves a water residue of hight $P\lambda$ due to the constant inflow during P. The effective syphon length is $P\mu$.

into W sees exactly $P_{\min}\mu$ agents ahead. Consider the two following regimes of the external traffic parameter $\rho := \lambda/\mu$: 1) the low regime quantified by $\rho < 1/(P_{\min}\mu)$ and 2) the high regime quantified by $\eta < \rho < 1$. In case 1), the queue length perceived upon arrival changes slowly and when i leaves S the queue length is, with high probability, still equal to $P_{\min}\mu + 1$. Therefore, in the case 1) the mean number of unsatisfied agents is well approximated by

$$\Delta_{\min} = \left(P_{\min}\mu + 1\right) \text{Prob}\left(P < P_{\min} + \frac{1}{\mu}\right). \tag{8}$$

This corresponds to the minimal cluster length Δ because a single unsatisfied agent decreases the line to $P_{\min}\mu$ excluding the creation of new unsatisfied agents (provided meanwhile no external entrances occurred).

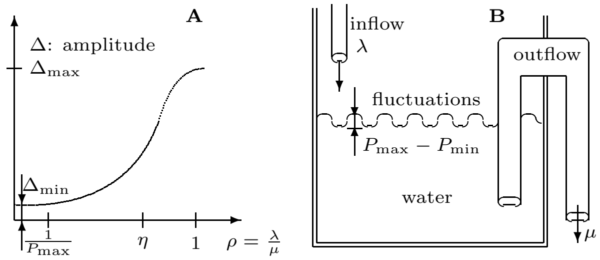

Fig. 4 – A. In contrast to the deterministic case, the amplitude Δ now depends on λ. Δ increases monotonically with the external traffic intensity $\frac{\lambda}{\mu}$ from a mean minimal value $\Delta_{\min}$ to a mean maximal value $\Delta_{\max} = \mathbb{E}(P)\mu$, a behavior observed by simulations. B. The syphon model is still adequate for $P_{\min}$ large and $(P_{\max} - P_{\min})/P_{\min}$ small. The value $P_{\max} - P_{\min}$ plays the role of maximal water level fluctuations and the condition $(P_{\max} - P_{\min})/P_{\min} \ll 1$ says that these fluctuations are small with respect to the syphon depth.

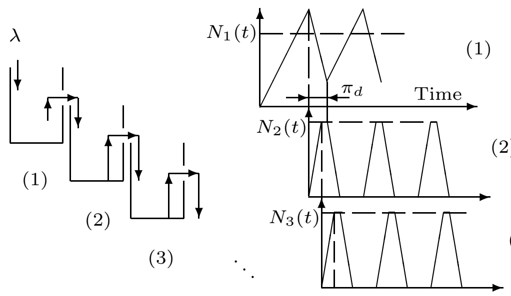

Fig. 5 – The serial coupling of identical feedback queues. For large $P = P_{\min} = P_{\max}$, the analogy with the syphon model applies and the linear network is seen to create a periodic pumping of water with speed 1 [syphon/P].

If the external traffic is high, the queue length changes rapidly and grows beyond $P_{\max}\mu$. Already after $(P_{\max} - P_{\min})\frac{\mu}{\lambda}$ units of time measured from the moment agent i entered into the system, the entering agents will see more than $P_{\max}\mu$ agents in the line. After another time interval $T := P_{\min} - (P_{\max} - P_{\min})\frac{\mu}{\lambda}$ agent i reaches the server. During this time interval only satisfied agents left the server and $\Delta_T := T(\lambda+\mu)$ agents entered into W. All of these Δ_T agents will certainly quit the system upon arrival at the server. A direct calculation shows that

$$\rho > \frac{\eta + \sqrt{\eta^2 + 4\eta}}{2} \Longrightarrow \Delta_T > P_{\min}\mu. \tag{9}$$

Therefore, in the high-traffic case $\rho > \eta$, the cluster length $\Delta(\rho)$ of unsatisfied agents is larger than $P_{\min}\mu$.

Networks of syphon stages. – The simple syphon model allows to discuss the transport phenomenon in networks of feedback queues. Here we restrict ourselves to feedback queues where the outflow from system n is the inflow to system $n+1$ (fig. 5).

A) *Serial coupling of syphons with deterministic patience parameter.* We suppose $P := P_{\min} = P_{\max}$. The first syphon is filled at a rate λ and is periodically discharged during $P\frac{\mu}{\mu-\lambda}$ at rate $\mu - \lambda$ according to fig. 3A. During such a period, say π_d, syphon 2 is filled at rate μ and starts to overflow at time P —thereby stabilizing the water level and wetting syphon 3. At the end of π_d, syphon 2 is not feeded anymore by syphon 1 during time $\pi - \pi_d$ and starts to discharge *all* the water into syphon 3 at rate μ. This scenario repeats itself at the next discharge of syphon 1. The so-created water-clusters propagate with speed 1 [syphon/P] (fig. 5).

B) *Serial coupling of syphons with individualized patience parameter.* The random patience parameter P takes now values in $[P_{\min}, P_{\max}]$. Syphon 1 is periodically discharged during $\frac{\Delta(\rho)}{\mu-\lambda}$ at rate $\mu - \lambda$. Depending on the cluster length $\Delta(\rho)$ (fig. 4A) two repeating outflow scenarios occur:

1) For $\Delta(\rho) \le P_{\min}\mu$, water will remain in syphon 2 until another avalanche triggers the overflow of syphon 2. When this happens, the length of the *coalesced* clusters is big enough to trigger overflows in every downstream syphon. Hence a global purge of the system occurs (*i.e.* residual water trapped in any downstream syphon(s) is collected by the propagating cluster which leaves behind nothing but dry syphons).

2) For $\Delta(\rho) > P_{\min}\mu$, syphon 2 is likely to overflow. The cluster length $\Delta \in [P_{\min}\mu, P_{\max}\mu]$ however is critical as water may get trapped in syphon 2 and syphon 3 cannot overflow

anymore. Generically for $\Delta(\rho) > P_{\min}\mu$, the first cluster leaves droplets of water (corresponding to patient agents) in the first few syphons and is trapped as soon as its length drops below $P_{\min}\mu$. As $\rho > 0$, succeeding travelling clusters form. They produce, together with the trapped water, an overflow of syphon 2. This triggers the avalanche of a big cluster which purges the whole system.

In both cases, sufficiently large clusters which can propagate downstream are formed. The propagation speed is random between 1 syphon/$P_{\max}$ and 1 syphon/$P_{\min}$ with mean $1/\mathbb{E}(P)$ and succeeding clusters can coalesce to form bigger ones. As big clusters purge the system, the following long-run scenario is independent of any initial distribution of water clusters in the network: Away from syphon 1, syphons always change between wet and dry states and the wetting periods, which increase with the cluster size, increase with the distance to syphon 1. Besides this clearing effect of the initial condition let us mention the possibility of noise-induced transport. For $\rho = 0$ and for a given quantity of water in syphon 1, the fluctuations can indeed induce transport whenever the noisy water level is able to trigger the syphoning effect.

Conclusion. – While feedback loops in queuing systems (QS) are extensively studied in the available literature, multi-agent queueing systems (MAQS) —*i.e.*, QS in which "intelligent" items (*i.e.*, items able to process individual information) circulate— are far less investigated. A single server QS with a feedback loop feeded by a Markov decision process cannot possibly give rise to stationary oscillations of the queue length. As our simple model clearly shows, this is not the case in MAQS. Even an elementary form of individual intelligence (here the recording of the time spent in the queue), when coupled with nonlinearities (the feedback loop), is able to generate rich and very stable time structures (here the oscillations in the queue length).

As far as analytical discussions are concerned, our model also shows the determinant role played by the law of large numbers. This feature will undoubtedly be further exploited to cope with randomness in MAQS.

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