EVOLUTION EQUATION RELATED TO THE GROSS LAPLACIAN

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Abstract. In this paper we study a Cauchy problem associated to Δ_K^*, the adjoint of Δ_K which is related to the Gross Laplacian for certain choice of the operator K. We show that the solution is a well defined generalized function in an appropriate space. Finally, using infinite dimensional stochastic calculus we give a probabilistic representation of the solution in terms of K-Wiener process W.

1. Introduction

The Gross Laplacian Δ_G was introduced by L. Gross in [Gro67] in order to study the heat equation in infinite dimensional spaces. It has been shown that the solution is represented as an integral with respect to Gaussian measure, see [Gro67] and [Pie70]. There exists a huge literature dedicated to the Gross Laplacian with different points of view. We would like to mention the white noise analysis approach, see [CJ99], [HKPS93], [HOS92], [Kuo86] and references therein.

In this paper we study the Cauchy problem

$$\frac{\partial}{\partial t}U(u)=\frac{1}{2}\Delta_K^*U(t), \qquad U(0)=\Phi, \tag{1.1}$$

where Φ is a generalized functions and Δ_K^* is the adjoint operator of Δ_K which is related to the Gross Laplacian for certain choice of the operator K, see (2.6) for more details. As the main tool we use the Laplace transform and the fact that Δ_K^* is a convolution operator. It is straightforward to show that the solution of (1.1) is a well defined element in an appropriate space of generalized functions, see Section 2 for details. Thus, the main result of the paper is to give a probabilistic representation for that solution. This entails, between others things, a stochastic calculus in infinite dimensions such as the Itô formula, see Theorem 3.4 below.

The paper is organized as follows. In Section 2 we provide the mathematical background needed to solve the Cauchy problem (1.1); namely we construct the appropriate test functions space $\mathcal{F}_\theta(N')$ and the associated generalized functions $\mathcal{F}'_\theta(N')$. The elements in $\mathcal{F}_\theta(N')$ are entire functions on the co-nuclear space N' with exponential growth of order θ (a Young function) and of minimal type. For special choices of θ the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{F}'_\theta(N')$ are well studied in the literature, see Remark 2.1.

The main tools we use are the Laplace transform which characterizes the space $\mathcal{F}'_\theta(N')$ in terms of holomorphic functions with certain growth conditions. We introduce the convolution $\Phi*\varphi$ between a generalized function $\Phi\in\mathcal{F}'_\theta(N')$ and

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a test function $\varphi \in \mathcal{F}_\theta(N')$ which generalizes the convolution of a measure and a function. We then introduce the convolution of two generalized functions as an extension of the convolution of two measures. It turns out that, indeed, the operator Δ_K^* is given as a convolution. Finally in Section 3 we prove the existence of the solution of the Cauchy problem (1.1) as a well defined element in $\mathcal{F}'_\theta(N')$ and give a probabilistic representation of it. We use the stochastic integration in Hilbert spaces, as developed in [DPZ92] and [Mét82] with respect to K-Wiener process W and the Itô formula for $\mathcal{F}'_\theta(N')$-valued processes $t_{W(u)}\Phi$, where $t_x\Phi$ is the translation of Φ by x.

2. Preliminaries

In this section we will introduce the framework which is necessary later on. Let X be a real nuclear Fréchet space with topology given by an increasing family $\{|\cdot|_p;\, p \in \mathbb{N}_0\}$ of Hilbertian norms, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Then X is represented as

$$X = \bigcap_{p\in\mathbb{N}_0} X_p,$$

where X_p is the completion of X with respect to the norm $|\cdot|_p$. We use X_{-p} to denote the dual space of X_p. Then the dual space X' of X can be represented as

$$X' = \bigcup_{p\in\mathbb{N}_0} X_{-p}$$

which is equipped with the inductive limit topology.

Let $N = X + iX$ and $N_p = X_p + iX_p$, $p \in \mathbb{Z}$, be the complexifications of X and X_p, respectively. For $n \in \mathbb{N}_0$, we denote by $N^{\hat{\otimes} n}$ the n-fold symmetric tensor product of N equipped with the π-topology and by $N_p^{\hat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product of N_p. We will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $N_p^{\hat{\otimes} n}$ and $N_{-p}^{\hat{\otimes} n}$, respectively.

Functional spaces. Let $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty \qquad \text{and} \quad \theta(0) = 0.$$

Such a function is called a Young function. For a Young function θ we define

$$\theta^*(x) := \sup_{t\geq 0}\{tx - \theta(t)\}, \qquad x \geq 0.$$

This is called the polar function associated to θ. It is known that θ^* is again a Young function and $(\theta^*)^* = \theta$, see [KR61] for more details and general results.

For a Young function θ, we denote by $\mathcal{F}_\theta(N')$ the space of holomorphic functions on N' with exponential growth of order θ and of minimal type. Similarly, let $\mathcal{G}_\theta(N)$ denote the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type. Moreover, for each $k \in \mathbb{Z}$ and $m > 0$, define $\mathcal{F}_{\theta,m}(N_p)$ to be the Banach space of entire functions f on N_p satisfying the condition

$$|f|_{\theta,p,m} := \sup_{x\in N_p} |f(x)|e^{-\theta(m|x|_p)} < \infty.$$

Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ may be represented as

$$\mathcal{F}_\theta(N') = \bigcap_{p\in\mathbb{N}_0, m>0} \mathcal{F}_{\theta,m}(N_{-p}),$$

$$\mathcal{G}_\theta(N) = \bigcup_{p\in\mathbb{N}_0, m>0} \mathcal{F}_{\theta,m}(N_p)$$

which are equipped with the projective limit topology and the inductive limit topology, respectively. The space $\mathcal{F}_\theta(N')$ is called the space of test functions on N'. Its dual space $\mathcal{F}'_\theta(N')$, equipped with the strong topology, is called the space of generalized functions. The dual pairing between $\mathcal{F}'_\theta(N')$ and $\mathcal{F}_\theta(N')$ is denoted by $\langle\!\langle \cdot,\cdot \rangle\!\rangle$.

For $k \in \mathbb{N}_0$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_p) = \left\{ \vec{\varphi} = (\varphi_n)_{n=0}^\infty;\ \varphi_n \in N_p^{\hat{\otimes} n},\ \sum_{n=0}^\infty \theta_n^{-2} m^{-n} |\varphi_n|_p^2 < \infty \right\},$$

$$G_{\theta,m}(N_{-p}) = \left\{ \vec{\Phi} = (\Phi_n)_{n=0}^\infty\ ;\ \Phi_n \in N_{-p}^{\hat{\otimes} n},\ \sum_{n=0}^\infty (n!\theta_n)^2 m^n |\Phi_n|_{-p}^2 < \infty \right\},$$

where

$$\theta_n = \inf_{x>0} \frac{e^{\theta(x)}}{x^n}, \quad n \in \mathbb{N}_0.$$

We define

$$F_\theta(N) := \bigcap_{p\in\mathbb{N}_0, m>0} F_{\theta,m}(N_p)$$

$$G_\theta(N') := \bigcup_{p\in\mathbb{N}_0, m>0} G_{\theta,m}(N_{-p}).$$

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Fréchet space, see [GHOR00, Proposition 2]. The space $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the bilinear pairing given by

$$\langle\!\langle \vec{\Phi}, \vec{\varphi} \rangle\!\rangle = \sum_{n=0}^\infty n! \langle \Phi_n, \varphi_n \rangle, \tag{2.1}$$

where $\vec{\Phi} = (\Phi_n)_{n=0}^\infty \in G_\theta(N')$ and $\vec{\varphi} = (\varphi_n)_{n=0}^\infty \in F_\theta(N)$.

The Taylor map defined by

$$\mathfrak{T} : \mathcal{F}_\theta(N') \longrightarrow F_\theta(N),\ \varphi \mapsto \left(\frac{1}{n!} \varphi^{(n)}(0) \right)_{n=0}^\infty$$

is a topological isomorphism. The same is true between $\mathcal{G}_{\theta^*}(N)$ and $G_\theta(N')$. The action of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ on a test function $\varphi \in \mathcal{F}_\theta(N')$ can be expressed in terms of the Taylor map as follows:

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \langle\!\langle \vec{\Phi}, \vec{\varphi} \rangle\!\rangle, \tag{2.2}$$

where $\vec{\Phi} = (\mathfrak{T}^*)^{-1}\Phi$ and $\vec{\varphi} = \mathfrak{T}\varphi$.

Remark 2.1. Let us identify the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{F}'_\theta(N')$ for particular Young function θ. We notice that these spaces are constructed without reference to any measure.

White noise: 1. Suppose $\theta(t) = t^2/2$, then $\theta^*(x) = x^2/2$. Then the space $\mathcal{F}_\theta(N')$ coincides with the space of Hida test functions and $\mathcal{F}'_\theta(N')$ is the Hida distribution space, see [HKPS93].
2. Let $\theta(t) = t^{2/(1+\beta)}$, $\beta \in [0,1)$, then $\theta^*(x) = x^{2/(1-\beta)}$ and the resulting spaces are the Kondratiev-Streit spaces, see [KS93].

Poisson noise: For $\theta(t) = \lambda(e^t - 1)$, $\lambda > 0$ we have $\theta^*(x) = -x + \lambda + x\log(x/\lambda)$. The corresponding spaces are the Poisson noise test and generalized functions, see [IK88].

Laplace transform. It is easy to see that for each $\xi \in N$, the exponential function

$$e_\xi(z) = e^{\langle z,\xi\rangle}, \quad z \in N',$$

is a test function in the space $\mathcal{F}_\theta(N')$ for any Young function θ, cf. [GHOR00, Lemme 2]. Thus the Laplace transform of a generalized function $\Phi \in \mathcal{F}'_\theta(N')$

$$\mathcal{L}\Phi(\xi) := \langle\!\langle \Phi, e_\xi \rangle\!\rangle, \quad \xi \in N, \tag{2.3}$$

is well defined. The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(N')$ and $\mathcal{G}_{\theta^*}(N)$ (cf. [GHOR00, Théorème 1]).

Remark 2.2. Let $\theta(t) = t^2/2$ and γ be the Gaussian measure on X'. Moreover consider the injection

$$I : \mathcal{F}_\theta(N') \hookrightarrow L^2(X',\gamma),\ \varphi \mapsto (I\varphi)(x) = \sum_{n=0}^\infty \langle :x^{\otimes n}:, \varphi_n\rangle,$$

where $\vec{\varphi} = (\varphi_n)_{n=0}^\infty \in F_\theta(N)$ and $\langle :x^{\otimes n}:, \varphi_n\rangle$ are the Hermite polynomials. Using an obvious inequality $n! \le \sqrt{2\pi} e^{-n} n^{n+1/2}$ it is easy to see that for each $m \in (0,1)$ there exists a constant $C(m) > 0$ such for any $p \in \mathbb{N}_0$ we have

$$|I\varphi|^2_{L^2(X',\gamma)} = \sum_{n=0}^\infty n! |\varphi_n|_0^2 \le C \sum_{n=0}^\infty \theta_n^{-2} m^{-n} |\varphi_n|_p^2,$$

where $\theta_n = e^{n/2} n^{-n/2}$. Hence I is a continuous injection. As an example, for $\varphi = e_\xi$, $\xi \in X$ we have

$$(Ie_\xi)(x) = \sum_{n=0}^\infty \frac{1}{n!}\langle :x^{\otimes n}:, \xi^{\otimes n}\rangle = e^{\langle x,\xi\rangle - |\xi|^2/2}, \qquad x \in X'.$$

Then the duality $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ given in (2.2) between $\mathcal{F}'_\theta(N')$ and $\mathcal{F}_\theta(N')$ is an extension of scalar product in $L^2(X',\gamma)$. Namely for any $\Phi \in L^2(X',\gamma)$ and $\varphi \in \mathcal{F}_\theta(N')$

$$\langle\!\langle \Phi, \varphi\rangle\!\rangle = \sum_{n=0}^\infty n! \langle \Phi_n, \varphi_n\rangle = \int_{X'} \Phi(x)(I\varphi)(x)\, d\gamma(x).$$

As a consequence, it turns out that the Laplace transform $\mathcal{L}$ coincides with the S-transform in white noise analysis.

Convolution. For $\varphi \in \mathcal{F}_\theta(N')$, the translation $t_x\varphi$ of φ by $x \in N'$ is defined by

$$(t_x\varphi)(y) = \varphi(y-x), \quad y \in N'.$$

It is easy to check that, for any $x \in N'$, t_x is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself , cf. [GHKO00, Proposition 2.1]. We may define the translation

t_x on $\mathcal{F}'_\theta(N')$ as follows: Let $\Phi \in \mathcal{F}'_\theta(N')$ be given, then $t_x\Phi \in \mathcal{F}'_\theta(N')$ is defined by

$$\langle\!\langle t_x\Phi, \varphi\rangle\!\rangle := \langle\!\langle \Phi, t_{-x}\varphi\rangle\!\rangle, \quad \forall\varphi \in \mathcal{F}_\theta(N').$$

We define the convolution $\Phi * \varphi$ of a generalized function $\Phi \in \mathcal{F}'_\theta(N')$ and a test function $\varphi \in \mathcal{F}_\theta(N')$ to be the test function

$$(\Phi * \varphi)(x) := \langle\!\langle \Phi, t_{-x}\varphi\rangle\!\rangle, \quad x \in N'.$$

For the proof, see [GHKO00, Lemme 2.1].

Remark 2.3. The definition of $\Phi * \varphi$ generalize the notion of convolution between a measure and a function.

For any $\Phi, \Psi \in \mathcal{F}'_\theta(N')$ we define the convolution $\Phi * \Psi \in \mathcal{F}'_\theta(N')$ by

$$\langle\!\langle \Phi * \Psi, \varphi\rangle\!\rangle := \langle\!\langle \Phi, \Psi * \varphi\rangle\!\rangle, \quad \varphi \in \mathcal{F}_\theta(N').$$

Remark 2.4. We notice that the above definition does generalize the notion of convolution of measures. It is not surprising that the commutative and associative laws holds because it does for measures.

As a consequence we have the following

Lemma 2.5. *Let $\Phi, \Psi \in \mathcal{F}'_\theta(N')$ be given, then we have*

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi. \tag{2.4}$$

Operator Δ_K and Δ_K^*. Let $K \in L(N, N')$ be given, where $L(N, N')$ is the set of continuous linear operators from N to N'. We denote by $\tau(K)$ the kernel associated to K in $(N \otimes N)'$ (which is isomorphic to $L(N, N')$, see [BK95]) which is defined by

$$\langle \tau(K), \xi \otimes \eta\rangle = \langle K\xi, \eta\rangle.$$

For $\varphi \in \mathcal{F}_\theta(N')$ of the form

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi^{(n)}\rangle, \tag{2.5}$$

we define the operator Δ_K of φ at $x \in N'$ by

$$(\Delta_K\varphi)(x) := \sum_{n=0}^{\infty} (n+2)(n+1)\langle x^{\otimes n}, \langle \tau(K), \varphi^{(n+2)}\rangle\rangle, \tag{2.6}$$

where the contraction $\langle \tau(K), \varphi^{(n+2)}\rangle$ is defined by

$$\langle x^{\otimes n}, \langle \tau(K), \varphi^{(n+2)}\rangle\rangle := \langle x^{\otimes n} \hat{\otimes} \tau(K), \varphi^{(n+2)}\rangle.$$

In particular, for $K = I$ (embedding of N in N'), $\tau(I)$ is the trace operator and Δ_K is the Gross Laplacian.

We state the following useful

Lemma 2.6. *Let $\varphi \in \mathcal{F}_\theta(N')$ be given and let K be an operator as described above. Then Δ_K is a convolution operator, namely*

$$\Delta_K\varphi = \mathcal{T}(\tau(K)) * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'). \tag{2.7}$$

where $\mathcal{T}(\tau(K)) \in \mathcal{F}'_\theta(N')$ is associated with $\overrightarrow{\mathcal{T}(\tau(K))} = (0, 0, \tau(K), 0, \dots) \in G_\theta(N')$.

Proof. We take φ of the form: $\varphi(x) = e^{\langle x,\xi\rangle} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle x^{\otimes n}, \varphi^{(n)}\rangle$, $\varphi^{(n)} = \xi^{\otimes n}$. Then we have

$$\begin{aligned}(\Delta_K\varphi)(x) &= \sum_{n=0}^{\infty}(n+2)(n+1)\langle x^{\otimes n}, \langle \tau(K), \frac{1}{(n+2)!}\xi^{\otimes(n+2)}\rangle\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\langle x^{\otimes n}, \langle \tau(K), \xi^{\otimes(n+2)}\rangle\rangle \\ &= \langle K\xi,\xi\rangle\varphi(x).\end{aligned}$$

Noting that $\varphi^{(2)}(\cdot + x) = \frac{1}{2}\xi^{\otimes 2}e^{\langle x,\xi\rangle}$ then we have

$$(\mathcal{T}(\tau(K)) * \varphi)(x) = \langle\!\langle \mathcal{T}(\tau(K)), t_{-x}\varphi\rangle\!\rangle = 2\langle \tau(K), \varphi^{(2)}(\cdot + x)\rangle = \langle K\xi,\xi\rangle\varphi(x).$$

The result follows by density of the exponential functions on $\mathcal{F}_\theta(N')$. □

Let A be the operator defined for any $\Phi \in \mathcal{F}'_\theta(N')$ by

$$A\Phi := \mathcal{T}(\tau(K)) * \Phi.$$

It follows that for all $\varphi \in \mathcal{F}_\theta(N')$ we have

$$\begin{aligned}\langle\!\langle A\Phi, \varphi\rangle\!\rangle &= \langle\!\langle \mathcal{T}(\tau(K)) * \Phi, \varphi\rangle\!\rangle \\ &= \langle\!\langle \Phi, \mathcal{T}(\tau(K)) * \varphi\rangle\!\rangle \\ &= \langle\!\langle \Phi, \Delta_K\varphi\rangle\!\rangle \\ &=: \langle\!\langle \Delta_K^*\Phi, \varphi\rangle\!\rangle\end{aligned}$$

which proves that A is the adjoint operator Δ_K^*.

It is clear, using (2.4), that

$$(\mathcal{L}(\Delta_K^*\Phi))(\xi) = \langle K\xi,\xi\rangle(\mathcal{L}\Phi)(\xi). \tag{2.8}$$

3. Convolution equation: existence and probabilistic representation

Now we are able to investigate the following Cauchy problem

$$\frac{\partial}{\partial u}U(u) = \frac{1}{2}\Delta_K^* U(u), \quad u \in [0,T], \quad U(0) = \Phi \in \mathcal{F}'_\theta(N'). \tag{3.1}$$

Applying the Laplace transform to (3.1) we obtain

$$\frac{\partial}{\partial u}\mathcal{L}U(u) = \frac{1}{2}\mathcal{L}\mathcal{T}(\tau(K))\mathcal{L}U(u), \quad u \in [0,T], \quad \mathcal{L}U(0) = \mathcal{L}\Phi \in \mathcal{G}_{\theta^*}(N). \tag{3.2}$$

Therefore the unique solution of (3.2) is given by

$$\mathcal{L}U(u) = (\mathcal{L}\Phi)\exp\left(\frac{u}{2}\mathcal{L}\mathcal{T}(\tau(K))\right). \tag{3.3}$$

Finally, the solution of (3.1) is obtained using the characterization theorem as

$$U(u) = \Phi * e^{*u\mathcal{T}(\tau(K))/2}. \tag{3.4}$$

We proceed in order to give a probabilistic representation of the solution (3.4). First we keep the notation K for its extension to X_{-p} ($p \in \mathbb{N}_0$ fixed) into itself. Moreover we assume that K is a symmetric, nonnegative linear operator with finite trace. We follow closely the ideas from [EOOS05] and [RT03]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\in[0,T]}, P)$ be a filtered probability space with a filtration $(\mathcal{F}_u)_{u\in[0,T]}$ satisfying the usual conditions. By a K-Wiener process $W = (W(u))_{u\in[0,T]}$ we mean a X_{-p}-valued process on $(\Omega, \mathcal{F}, P)$ such that

- $W(0) = 0$,
- W has P-a.s. continuous trajectories,
- the increments of W are independent,
- the increments $W(u) - W(v)$, $0 < v \leq u$ have the following Gaussian law:
$$P \circ (W(u) - W(v))^{-1} = N(0, (u-v)K),$$
where $N(0, (u-v)K)$ denotes the Gaussian distribution with zero mean and covariance operator $(u-v)K$.

A K-Wiener process with respect to the filtration $(\mathcal{F}_u)_{u\in[0,T]}$ is a K-Wiener process such that

- $W(u)$ is $\mathcal{F}_u$-adapted,
- $W(u) - W(v)$ is independent of $\mathcal{F}_u$ for all $v \in [0, u]$.

Later on we need to define stochastic integrals of $\mathcal{F}'_\theta(N')$-valued process. We use the theory of stochastic integration in Hilbert spaces developed in [DPZ92] and [Mét82]. Before we introduce some notations. Given two separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $L(\mathcal{H}_1, \mathcal{H}_2)$ (resp. $L_2(\mathcal{H}_1, \mathcal{H}_2)$) the space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (resp. Hilbert-Schmidt operators). The Hilbert-Schmidt norm of an element $S \in L_2(\mathcal{H}_1, \mathcal{H}_2)$ is denoted by $\|S\|_{HS}$.

Definition 3.1. Let $(\Phi(u))_{0\leq u\leq T}$ be a given $L(X_{-p}, \mathcal{F}'_\theta(N'))$-valued, $\mathcal{F}_u$-adapted continuous stochastic process. Assume that there exist $m > 0$ and $p \in \mathbb{N}_0$ such that $\mathfrak{T} \circ \mathcal{L}\Phi(u) \in L(X_{-p}, G_{\theta,m}(N_{-p}))$ and
$$P\left(\int_0^T \left\|(\mathfrak{T} \circ \mathcal{L}\Phi(u)) \circ K^{1/2}\right\|_{HS}^2 du < \infty\right) = 1. \tag{3.5}$$
Then, for $u \in [0, T]$, we define the generalized stochastic integral
$$\int_0^u \Phi(v) dW(v) \in \mathcal{F}'_\theta(N')$$
by
$$\mathfrak{T}\left(\mathcal{L}\left(\int_0^u \Phi(v) dW(v)\right)(\xi)\right) := \int_0^u \mathfrak{T}\left((\mathcal{L}\Phi(v))(\xi)\right) dW(v). \tag{3.6}$$
Notice that the right hand side of (3.6) is a well defined stochastic integral in a Hilbert space by the condition (3.5), see [DPZ92].
We are going to derive the Itô formula for $t_{W(u)}\Phi$, $\Phi \in \mathcal{F}'_\theta(N')$. Before we give a technical lemma. By a $\mathcal{F}'_\theta(N')$-valued continuous $\mathcal{F}_u$-semimartingale $(Z(u))_{u\in[0,T]}$, we mean processes of the form
$$Z(u) = Z(0) + \int_0^u \Phi(v) dW(v) + \int_0^u \Psi(v) dv,$$
when all terms in the right hand side are well defined. We state to following lemma which the simple proof is left to the interested reader.

Lemma 3.2. *Let $\Phi \in \mathcal{F}'_\theta(N')$, $\xi \in N$ and $g : X_{-p} \longrightarrow \mathbb{C}$, $g(x) := \langle\langle t_x\Phi, e_\xi\rangle\rangle = (\mathcal{L}(t_x\Phi))(\xi)$ be given. Then g is twice continuously differentiable and*
$$(Dg)(x)(y) = \langle\langle \Phi, t_{-x}e_\xi\rangle\rangle\langle y, \xi\rangle, \qquad x, y \in X_{-p}. \tag{3.7}$$
$$(D^2g)(x)(y_1, y_2) = \langle\langle \Phi, t_{-x}e_\xi\rangle\rangle\langle y_1, \xi\rangle\langle y_2, \xi\rangle, \qquad y_1, y_2 \in X_{-p}. \tag{3.8}$$
Moreover g, Dg and D^2g are uniformly continuous on bounded sets of X_{-p}.

Remark 3.3.

(1) In the conditions of Lemma 3.2 we may rewrite (3.7) and (3.8) as
$$(Dg)(x) = -\langle\!\langle D(t_x\Phi), e_\xi\rangle\!\rangle$$
$$(D^2g)(x) = \langle\!\langle D^2(t_x\Phi), e_\xi\rangle\!\rangle.$$
(2) Since the family of exponential functions $\{e_\xi,\ \xi \in N\}$ is dense in $\mathcal{F}_\theta(N')$ the same result holds for $g(x) = \langle\!\langle t_x\Phi, \varphi\rangle\!\rangle$ with $\varphi \in \mathcal{F}_\theta(N')$.

Now we are able to prove the announced Itô formula.

Theorem 3.4. *Let $(W(u))_{u\in[0,T]}$ be a K-Wiener process with respect to the filtration $(\mathcal{F}_u)_{u\in[0,T]}$ and let $\Phi \in \mathcal{F}'_\theta(N')$ be given. Then $t_{W(u)}\Phi$ is a $\mathcal{F}'_\theta(N')$-valued continuous $\mathcal{F}_u$-semimartingale which has the following decomposition*
$$t_{W(u)}\Phi = t_{W(0)}\Phi - \int_0^u D(t_{W(v)}\Phi)dW(v) + \frac{1}{2}\int_0^u \Delta_K^*(t_{W(v)}\Phi)dv.$$

Proof. By Lemma 3.2 the function $g : X_{-p} \longrightarrow \mathbb{C}$, $g(x) := \langle\!\langle t_x\Phi, e_\xi\rangle\!\rangle$ for $\xi \in N$ is twice continuously differentiable. Then applying Itô's formula we get
$$g(W(u)) = g(W(0)) + \int_0^u Dg(W(v))dW(v) + \frac{1}{2}\int_0^u \mathrm{tr}[D^2g(W(v))K]dv.$$
The explicit representation for g gives
$$\begin{aligned}\langle\!\langle t_{W(u)}\Phi, e_\xi\rangle\!\rangle = \langle\!\langle t_{W(0)}\Phi, e_\xi\rangle\!\rangle &- \int_0^u \langle\!\langle D(t_{W(v)}\Phi), e_\xi\rangle\!\rangle dW(v)\\ &+ \frac{1}{2}\int_0^u \mathrm{tr}[\langle\!\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\!\rangle K]dv.\end{aligned}$$
The trace in the last integral above may be written as
$$\begin{aligned}\mathrm{tr}[\langle\!\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\!\rangle K] &= \sum_{i\geq 1}\langle\!\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\!\rangle\langle Kf_i, \xi\rangle\langle f_i, \xi\rangle\\ &= \langle\!\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\!\rangle\langle K\xi, \xi\rangle,\end{aligned}$$
where we have used the symmetry of K and the fact $\sum_{i\geq 1}\langle f_i, K\xi\rangle\langle f_i, \xi\rangle = \langle K\xi, \xi\rangle$ for one (hence every) orthonormal basis $\{f_i;\ i \geq 1\}$ in X_{-p}. The result follows by (2.8). ∎

In order to show that the solution of (3.1) is given in terms of an expectation, first we prove the following

Lemma 3.5. *Let $\Phi \in \mathcal{F}'_\theta(N')$, $\xi \in N$ be given and $g : X_{-p} \longrightarrow \mathbb{C}$, $g(x) = \langle\!\langle t_x\Phi, e_\xi\rangle\!\rangle$. Then we have*
$$\mathbb{E}\left(\int_0^T \left\|Dg(W(u)) \circ K^{1/2}\right\|_{HS}^2 du\right) < \infty.$$
Thus, $\{\int_0^u Dg(W(v))dW(v),\ u \in [0,T]\}$ is a $\mathcal{F}'_\theta(N')$-valued continuous, square integrable martingale.

Proof. For every orthonormal basis $\{f_i,\ i \geq 1\}$ in X_{-p} we have
$$\left\|(Dg)(W(u)) \circ K^{1/2}\right\|_{HS}^2 \leq |\langle\!\langle t_{W(u)}\Phi, e_\xi\rangle\!\rangle|^2|\xi|_p^2\sum_{i\geq 1}|K^{1/2}f_i|_{-p}^2.$$

Notice that $\langle\langle t_{W(u)}\Phi, e_\xi\rangle\rangle = \langle\langle \Phi, e_\xi\rangle\rangle e^{\langle W(u),\xi\rangle}$ and $\sum_{i\geq 1}|K^{1/2}f_i|^2_{-p} = ||K^{1/2}||^2_{HS}$. Therefore we obtain

$$\left\|(Dg)(W(u))\circ K^{1/2}\right\|^2_{HS} \leq C(\xi,\Phi,K,\alpha)e^{\alpha|W(u)|^2_{-p}},$$

where the constant $C(\xi,\Phi,K,\alpha)$ is given by

$$C(\xi,\Phi,K,\alpha) := |\xi|^2_p|e^{|\xi|^2_p/\alpha}|\langle\langle\Phi,e_\xi\rangle\rangle|^2||K^{1/2}||^2_{HS}$$

and we have used, for any $\alpha > 0$,

$$\begin{cases} 2|\langle W(u),\xi\rangle| \leq |W(u)|_{-p}|\xi|_p \leq \dfrac{|\xi|^2_p}{\alpha} + \alpha|W(u)|^2_{-p}\,, \\ |e^{\langle W(u),\xi\rangle}| \leq e^{|\langle W(u),\xi\rangle|} \leq e^{|\xi|^2_p/\alpha}e^{\alpha|W(u)|^2_{-p}}. \end{cases}$$

Since $W(u)$ has Gaussian law $N(0,uK)$ it follows that

$$\mathbb{E}\left(\int_0^T \left\|(Dg)\left(W(u)\right)\circ K^{1/2}\right\|^2_{HS} du\right) \leq C\left(\xi,\Phi,K,\alpha\right)\int_0^T\int_{X_{-p}} e^{\alpha u|x|^2_{-p}}dN(0,K)du.$$

For $\alpha \in [0, \frac{1}{2T\mathrm{tr}[K]}[$ the stochastic integral above admits a Fernique estimation of Gaussian measure (cf. [DPZ92, Proposition 2.16, pag. 56]) so that

$$\int_{X_{-p}} e^{\alpha u|x|^2_{-p}}dN(0,K) \leq \frac{1}{\sqrt{1-2u\alpha\mathrm{tr}[K]}}.$$

So, we obtain

$$\mathbb{E}\left(\int_0^T \left\|(Dg)\left(W(u)\right)\circ K^{1/2}\right\|^2_{HS} du\right) \leq C\left(\xi,\Phi,K,\alpha\right)\int_0^T \frac{du}{\sqrt{1-2u\alpha\mathrm{tr}[K]}} < +\infty.$$

□

As consequence of the above lemma for each $\xi \in N$ the process

$$\left\{\int_0^u \langle\langle D(t_{W(v)}\Phi), e_\xi\rangle\rangle dW(v),\ u\in[0,T]\right\},$$

is a $L^2(P)$-bounded martingale. Therefore we have

Corollary 3.6. *The following stochastic integral*

$$\left\{\int_0^v \mathfrak{T}\circ\mathcal{L}(D(t_{W(u)}\Phi))dW(u),\ v\in[0,T]\right\},$$

is a $L^2(P)$-bounded martingale.

Now we give a probabilistic representation formula of the solution of the Cauchy problem (3.1).

Theorem 3.7. *The solution of the Cauchy problem (3.1) is given by*

$$U(u) = \mathbb{E}_{P^x}(t_{W(u)}\Phi). \tag{3.9}$$

where P^x is the probability law of W starting at $x \in X_{-p}$.

Proof. To check that $U(u) = \mathbb{E}_{P^x}(t_{W(u)}\Phi)$ is the solution of the Cauchy problem (3.1), it suffices to show that its Laplace transform $\mathcal{L}U(u)$ satisfies the Cauchy problem (3.2). It follows from Itô's formula, with $\xi \in N$, that

$$\begin{aligned}\langle\langle t_{W(u)}\Phi, e_\xi\rangle\rangle &= \langle\langle t_{W(0)}\Phi, e_\xi\rangle\rangle - \int_0^u \langle\langle D(t_{W(v)}\Phi), e_\xi\rangle\rangle dW(v)\\ &+ \frac{1}{2}\int_0^u \mathrm{tr}[\langle\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\rangle K]dv.\end{aligned}$$

Taking expectation and using the fact that $\left(\int_0^u \langle\langle D(t_{W(v)}\Phi), e_\xi\rangle\rangle dW(v)\right)$ is a $L^2(P)$-bounded martingale yields

$$\begin{aligned}\mathbb{E}_{P^x}\langle\langle t_{W(u)}\Phi, e_\xi\rangle\rangle &= \mathbb{E}_{P^x}\langle\langle t_{W(0)}\Phi, e_\xi\rangle\rangle + \frac{1}{2}\int_0^u \mathbb{E}_{P^x}\mathrm{tr}[\langle\langle D^2(t_{W(v)}\Phi), e_\xi\rangle\rangle K]dv.\\ &= \mathbb{E}_{P^x}\langle\langle t_{W(0)}\Phi, e_\xi\rangle\rangle + \frac{1}{2}\int_0^u \mathbb{E}_{P^x}\Delta_K^*(t_{W(v)}\Phi)(\xi)dv.\end{aligned}$$

Using the definition of Laplace transform (2.3), the last equation can be written as

$$\mathcal{L}U(u)(\xi) = \mathcal{L}U(0)(\xi) + \frac{1}{2}\int_0^u \mathcal{L}\Delta_K^* U(v)(\xi)dv$$

or making use of the explicit form (2.8) as

$$\mathcal{L}U(u)(\xi) = \mathcal{L}U(0)(\xi) + \frac{1}{2}\int_0^u \mathcal{L}U(v)(\xi)\langle K\xi, \xi\rangle dv$$

which implies that $\mathcal{L}U(u)(\xi)$ solves the Cauchy problem (3.2). □

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