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The Square of Self Intersection Local Time of Brownian Motion

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We dedicate this paper to Sergio. M. F. and L. S. are among the many whose professional progress would have been a much lesser one without Sergio's helping hand; beyond this, his friendship has enriched the decades we shared.

1. Introduction

Self intersection local time of Brownian motion B have been the subject of numerous studies over half a century [**2**], [**4**]-[**11**], [**13**], [**17**]-[**25**], [**27**]-[**41**]. An informal definition, in terms of an integral over Donsker's δ-function would be

$$L = \int d^2t\, \delta(B(t_2) - B(t_1)).$$

To make sense of this integral one can invoke a regularization such as

$$L_\varepsilon(t) \equiv \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta_\varepsilon\left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right)\right),$$

with

$$\delta_\varepsilon\left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right)\right) \equiv (2\pi\varepsilon)^{-d/2} e^{-\frac{|\mathbf{B}(t_2)-\mathbf{B}(t_1)|^2}{2\varepsilon}}. \tag{1}$$

(Alternatively, one might suppress the short time singularities by considering near passages instead of self crossings, via the substitution

$$\delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \to \delta(\mathbf{B}(t_2) - \mathbf{B}(t_1) + \varepsilon) \tag{2}$$

as e.g. in [**41**]).

The central problem is then the removal of the regularization: $\varepsilon \to 0$. As is well known, for $d > 1$, we need for this to center the regularized local time

$$L_{\varepsilon,c} = L_\varepsilon - E\left(L_\varepsilon\right)$$

(Varadhan [**35**]). For $d > 2$, a further multiplicative renormalization $r(\varepsilon)$ is required for the existence of a limiting process. M. Yor shows, using the regularization (2) for $d = 3$, that

$$r(\varepsilon)\left(L_\varepsilon - E\left(L_\varepsilon\right)\right) \xrightarrow{\mathcal{L}} c\beta$$

with β a Brownian motion independent of $\mathbf{B}$.

This can be understood in the light of the fact that each term in the (renormalized) chaos expansion for the local time converges in law to a Brownian motion, for any $d > 2$ as shown by us in [4].

2. Tools from White Noise Analysis

We quote some White Noise Analysis concepts as introduced in [4], referring to [16] for a systematic presentation.

A vector of Brownian motions $B_i, i = 1, ..., d$, has a version in terms of independent white noises ω_i via

$$B_i(t) =< \omega_i, 1_{[0,t]} >= \int_0^t \omega_i(s)ds.$$

Hence we consider independent d-tuples of Gaussian white noise $\omega = (\omega_1, ..., \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, ..., f_d) \in S(R, R^d)$, and use the following multi-index notation:

$$\vec{n} = (n_1, \ldots, n_d), \;\; n = \sum_1^d n_i, \;\; \vec{n}! = \prod_1^d n_i!$$

$$< \mathbf{f}, \mathbf{f} > = \sum_{i=1}^d \int dt \; f_i^2(t)$$

$$< F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} >= \int d^n t \; F_{\vec{n}}(t_1, ..., t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, ..., t_n)$$

and similarly for $<: \omega^{\otimes \vec{n}} :, F_{\vec{n}} >$ where for d-tuples of white noise the Wick product $: \ldots :$ [16] generalizes to

$$: \omega^{\otimes \vec{n}} := \bigotimes_{i=1}^d : \omega_i^{\otimes n_i} : .$$

The vector valued white noise ω has the characteristic function

$$C(\mathbf{f}) = E(e^{i<\omega, \mathbf{f}>}) = \int_{S^*(R,R^d)} d\mu [\omega] \, e^{i<\omega, \mathbf{f}>} = e^{-\frac{1}{2}<\mathbf{f},\mathbf{f}>}, \tag{3}$$

where $< \omega, \mathbf{f} > = \sum_{i=1}^d < \omega_i, f_i>$ and $f_i \in S(R, R)$.

The Hilbert space

$$(L^2) \equiv L^2(d\mu)$$

is canonically isomorphic to the d–fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq (\bigoplus_{k=0}^{\infty} Sym L^2(R^k, k! d^k t))^{\otimes d} \equiv \mathfrak{F}, \tag{4}$$

for the general element of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} <: \omega^{\otimes \vec{n}} :, F_{\vec{n}} > \tag{5}$$

with kernel functions F in $\mathfrak{F}$.

For suitable functionals Φ of Brownian motion, expressed in terms of Itô integrals

$$\Phi = E(\Phi) + \int \varphi(\tau)dB(\tau) \tag{6}$$

the Clark-Ocone formula [3], [26] provides us with an explicit formula for the integrand $\varphi(\cdot)$. Here we cite a rather general version of this result in the context of regular generalized functions [15] of white noise, see also [1].

THEOREM 2.1. [12] *Let Φ be a regular generalized function of white noise, $\Phi \in \mathcal{G}^{-1}$. Then it can be written as a generalized Itô integral*

$$\Phi = E(\Phi) + I(\varphi)$$

with

$$\varphi(t) = \Gamma(1_{[0,t]})\partial_t\Phi \tag{7}$$

Here ∂_t is the Hida derivative

$$\partial_t\Phi(\omega) = \lim_{\varepsilon\to 0} \frac{\Phi(\omega+\varepsilon\delta_t) - \Phi(\omega)}{\varepsilon}.$$

and $\Gamma(A)$ is the "2nd quantization" of the operator A, transforming kernel functions as follows: if $\Phi \in (L^2)$ has kernel functions F_n then $\Gamma(A)\Phi$ has kernel functions $A^{\otimes n}F_n$, so that

$$S(\Gamma(A)\Phi)(f) = S(\Phi)(A^{\dagger}f).$$

3. Local Times in terms of White Noise

We set, for $\varepsilon > 0$,

$$L_\varepsilon(t) \equiv \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1)),$$

with

$$\delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \equiv (2\pi\varepsilon)^{-d/2} e^{-\frac{|\mathbf{B}(t_2)-\mathbf{B}(t_1)|^2}{2\varepsilon}}. \tag{8}$$

It has the following chaos expansion, for $d \geq 3$.

THEOREM 3.1. [11] *For any $\varepsilon > 0$, $L_\varepsilon - E(L_\varepsilon)$ has kernel functions $F \in \mathfrak{F}$ given by*

$$F_{\varepsilon,\vec{n}}(s_1,\dots,s_n) = (-1)^{\frac{n}{2}} \left(\varkappa(\varkappa+1)(2\pi)^{d/2}2^{\frac{n}{2}}\,\tfrac{n}{2}!\right)^{-1}\cdot \tag{9}$$
$$\Theta(u)\Theta(t-v)\cdot\left((v-u+\varepsilon)^{-\varkappa} + (t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}\right)$$

if all n_i are even, and zero otherwise, with $v(s_1,\dots,s_n) \equiv max(s_1,\dots,s_n)$, $u(s_1,\dots,s_n) \equiv min(s_1,\dots,s_n)$, and $\varkappa \equiv (n+d)/2 - 2$. Θ is the Heaviside function.

THEOREM 3.2. [4] *Let $d \geq 3$, all n_i even. The terms*

$$L_{\varepsilon,\vec{n}}(\omega) = \langle :\omega^{\otimes\vec{n}}:, F_{\varepsilon,\vec{n}}\rangle \tag{10}$$

in the chaos expansion converge in law, as $\varepsilon \to +0$:

$$r(\varepsilon, d)L_{\varepsilon,\vec{n}} \xrightarrow{\mathcal{L}} c_{d,\vec{n}}\beta_{\vec{n}},$$

where the $\beta_{2\vec{n}}$ are independent Brownian motions and

$$r(\varepsilon) = \begin{cases} |ln\varepsilon|^{-1/2} & \text{for } d = 3 \\ \varepsilon^{(d-3)/2} & \text{for } d > 3 \end{cases}$$

$$c^2_{d,\vec{n}} = (2\pi)^{-d} 2^{-n} \left(((n+d)/2 - 2)((n+d)/2 - 1) \frac{\vec{n}}{2}! \right)^{-2} \vec{n}! \cdot \begin{cases} n(n-1) \text{ if } d = 3 \\ \frac{n!(d-4)!}{(n+d-5)!} \text{ if } d > 3 \end{cases}$$

One would thus expect the variances of these Brownian motions to sum up to that of the renormalized local time.

In this case we should verify

$$t \sum_{\substack{m_1=0 \\ \vec{m}\neq 0}}^{\infty} \cdots \sum_{m_d=0}^{\infty} \sum_{m_d=0}^{\infty} c^2_{d,2\vec{m}} = \lim_{\varepsilon \to +0} E\left((r(\varepsilon)L_{\varepsilon,c})^2 \right) \tag{11}$$

On the lhs we can calculate the sum over all $\vec{m}$ with fixed $m = m_1 + \ldots + m_d$ so that

$$\sum_{\substack{m_1=0 \\ \vec{m}\neq 0}}^{\infty} \cdots \sum_{m_d=0}^{\infty} c^2_{d,2\vec{m}} = (2\pi)^{-d} \sum_{m=0}^{\infty} ((m + d/2 - 2)(m + d/2 - 1))^{-2} a_m(d) \cdot \begin{cases} 2m(2m-1) \text{ if } d = 3 \\ \frac{(2m)!(d-4)!}{(2m+d-5)!} \text{ if } d > 3 \end{cases} \tag{12}$$

with

$$a_m(d) \equiv 2^{-2m} \sum_{\substack{m_1=0 \\ m_1+\ldots+m_d=m>0}} \cdots \sum_{m_d=0} \frac{\left(2\,\vec{m}\right)!}{\left(\vec{m}!\right)^2} \tag{13}$$

Lemma 4.3.

$$\begin{aligned} a_m(d) &= \frac{1}{m!} \frac{d^m}{dx^m} (1-x)^{-d/2} \Big|_{x=0} \\ &= \frac{\Gamma(m + \frac{d}{2})}{\Gamma(m+1)\Gamma(\frac{d}{2})} \end{aligned}$$

Proof: For $d = 1$ (13) reads

$$\begin{aligned} a_m(1) &= 2^{-2m} \frac{(2m)!}{(m!)^2} = \frac{1}{m!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2m-1}{2} \\ &= \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}. \end{aligned}$$

Also note that

$$(1-x)^{-1/2} = \sum_{m=0}^{\infty} a_m(1)x^m.$$

Furthermore we have from (13)

$$a_n(d) = \sum_{m=0}^{n} a_m(d-1)a_{n-m}(1).$$

Hence, by induction,

$$(1-x)^{-d/2} = \sum_{m=0}^{\infty} a_m(d)x^m. \tag{14}$$

The sum (11)

$$E(t,d) = \frac{t}{(2\pi)^d} \sum_{m=1}^{\infty} a_m(d) \frac{1}{\left((m+d/2-2)(m+d/2-1)\right)^2} \cdot \begin{cases} 2m(2m-1) \text{ if } d=3 \\ \frac{(2m)!(d-4)!}{(2m+d-5)!} \text{ if } d>3 \end{cases}$$

can then be evaluated in closed form for any d, by suitable integrations and/or differentiations of (14) to produce the extra m-dependent factors in the denominator and/or numerator.

This is to be compared to

$$\lim_{\varepsilon\to+0} E\left((r(\varepsilon)L_{\varepsilon,c})^2\right) = \frac{2}{(2\pi)^d} \lim_{\varepsilon\to 0} \int_0^{\frac{t}{\varepsilon}} dt_2 \int_{s_1}^{\frac{t_2}{\varepsilon}} dt_1 \int_0^{\frac{t}{\varepsilon}} ds_2 \int_0^{\frac{s_2}{\varepsilon}} ds_1 \cdot \left(\left(1+\tau+\sigma+\tau\sigma-\delta^2\right)^{-\frac{d}{2}} - \left((1+\tau)(1+\sigma)\right)^{-\frac{d}{2}}\right) \tag{15}$$

where we use the abbreviations

$$\sigma = s_2 - s_1, \tau = t_2 - t_1, \delta = \left|[s_1, s_2] \cap [t_1, t_2]\right| \tag{16}$$

and have assumed

$$s_1 < t_1, \tag{17}$$

compensating this restriction by a factor 2.

Both the sums and the integral have been calculated for specific values of d (and agree), in particular e.g.

$$\begin{pmatrix} E(t,3) \\ E(t,4) \\ E(t,5) \\ E(t,6) \\ E(t,7) \\ E(t,8) \\ E(t,9) \\ E(t,10) \end{pmatrix} = (2\pi)^{-d} t \begin{pmatrix} 16\pi \\ 2 \\ -\frac{16}{9} + \frac{2}{3}\pi \\ -\frac{5}{6} + \frac{4}{3}\ln 2 \\ \frac{48}{25} - \frac{3}{5}\pi \\ \frac{34}{45} - \frac{16}{15}\ln 2 \\ -\frac{656}{441} + \frac{10}{21}\pi \\ -\frac{33}{56} + \frac{6}{7}\ln 2 \end{pmatrix} \tag{18}$$

The calculation of the integrals in (15) involves the cancelation of divergences due to the substraction of the second term (squared expectation). They are further complicated by the different analytic form of the integral depending on whether the interval $[s_1, s_2] \cap [t_1, t_2]$ is partially or totally contained in $[s_1, s_2]$.

It would of course be desirable to have a closed form expression for general values of the dimension d. We do indeed obtain this by a different route.

Proposition 4.4.

$$\lim_{\varepsilon \to 0} E\left(\left(r(\varepsilon) L_{\varepsilon,c} \right)^2 \right) = \frac{8t}{(2\pi)^d (d-2)} \begin{cases} \left(\frac{2(d-4)}{(d-3)(d-2)} \sum_{k=1}^{\frac{d}{2}-2} \frac{(-1)^k k}{d-2k-2} - \frac{2}{d-3} \sum_{k=1}^{\frac{d}{2}-3} \frac{(-1)^k k}{d-2k-4} + \frac{(-1)^{\frac{d}{2}}}{d-2} - (-1)^{\frac{d}{2}} \frac{d-4}{d-3} \ln 2 \right) \\ \text{if } d \geq 4, \text{even} \\ \\ \frac{d-4}{d-3} \left(2 \sum_{k=0}^{\frac{d-7}{2}} \frac{(-1)^k}{d-2k-6} - \frac{d-1}{(d-2)(d-4)} + (-1)^{\frac{d+3}{2}} \frac{\pi}{2} \right) \\ \text{if } d \geq 5, \text{odd} \end{cases}$$

To prove this we use the well-known formula for the L^2–norm of Wiener integrals:

$$E\left(\left(\int \varphi dB \right)^2 \right) = E\left(\int \varphi^2 ds \right)$$

To apply this to $E\left((L_{\varepsilon,c})^2 \right)$ we need to determine φ, given

$$L_{\varepsilon,c} = \int \varphi dB.$$

This is done by the Clark-Ocone formula (7) and one finds for the regularized and centered local time

$$L_{\varepsilon,c}(t) = \int_0^t \overrightarrow{\varphi(\tau)} \cdot d\overrightarrow{B(\tau)}$$

with

$$\begin{aligned} \overrightarrow{\varphi}(\tau) &= -(2\pi)^{-d/2} \int_\tau^t dt_2 \int_0^\tau dt_1 \\ &\quad \cdot (\varepsilon + t_2 - \tau)^{-d/2-1} \left\langle \overrightarrow{\omega}, 1_{[t_1,\tau]} \right\rangle e^{-\frac{|B(\tau)-B(t_1)|^2}{2(\varepsilon+t_2-\tau)}}. \end{aligned}$$

This in turn permits the explicit calculation of

$$E\left((L_{\varepsilon,c})^2 \right) = E\left(\int \overrightarrow{\varphi}^2 (\tau) \, d\tau \right) =$$

$$\frac{2d}{(2\pi)^{d/2}} \int_0^t d\tau \int_\tau^t dt_2 \int_0^\tau dt_1 \int_\tau^t ds_2 \int_0^{t_1} ds_1$$

$$\frac{\tau - t_1}{\left((t_1 - \tau + s_2 - s_1 + \varepsilon)(t_2 - t_1 + \varepsilon) + (\tau - t_1)(t_2 - \tau + \varepsilon) \right)^{1+d/2}}$$

Renormalizing this by a factor of $r^2(\varepsilon)$ and taking the limit, one obtains the desired results.

4. Conclusion

In [4] we had established, for any dimension $d > 2$, the weak convergence to a Brownian motion of each term $r(\varepsilon, d)L_{\varepsilon,\vec{n}}$ in the multiple Wiener integral expansion of the renormalized, regularized, centered local time $r(\varepsilon, d)L_{\varepsilon,c}$. Here we calculate for a number of dimensions d, the value of the sums

(19) $$\sum_{\vec{n}\neq 0} \lim_{\varepsilon\to+0} E\left(\left(r(\varepsilon)L_{\varepsilon,\vec{n}}\right)^2\right)$$

We also give a closed form expression for $\lim_{\varepsilon\to+0} E\left((r(\varepsilon)L_{\varepsilon,c})^2\right)$ and find that

(20) $$\lim_{\varepsilon\to+0} E\left((r(\varepsilon)L_{\varepsilon,c})^2\right) = \sum_{\vec{n}\neq 0} \lim_{\varepsilon\to+0} E\left(\left(r(\varepsilon)L_{\varepsilon,\vec{n}}\right)^2\right)$$

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