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Hankel Operators over the Complex Wiener Space

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Abstract. This work introduces and investigates (small) Hankel operators H_b on the Hilbert space of holomorphic, square integrable Wiener functionals. A regularity condition on the symbol b, which guarantees the boundedness of H_b, is provided. The symbols b for which H_b is of Hilbert–Schmidt type are characterized, and a representation of H_b by an integral operator is given. The proofs employ the hypercontractivity of the Ornstein–Uhlenbeck semigroup, together with approximations by finitely many variables. These results extend known results from a finite-dimensional context.

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1. Introduction

Denote by γ_c the Gauss measure on $\mathbb{C}^d$ with Lebesgue density

$$p_c(z) = \frac{1}{(2\pi c)^d} e^{-\frac{|z|^2}{2c}}, \quad z \in \mathbb{C}^d,$$

and by $\mathcal{H}L^p(\mathbb{C}^d, \gamma_c)$ the space of holomorphic functions on $\mathbb{C}^d$ which are p-times integrable with respect to γ_c. The investigation of Hankel operators on $\mathcal{H}L^p(\mathbb{C}^d, \gamma_c)$ was initiated in [11] and continued in [15, 25]; related topics are studied in [9, 10, 12, 16, 18]. The authors of [11] pointed out that some of their results are independent of the dimension d, and therefore should remain valid in some infinite-dimensional version. In [2] the hypercontractivity of the Ornstein–Uhlenbeck semigroup e^{-tN} in $\mathcal{H}L^p(\mathbb{C}^d, \gamma_c)$ was used to investigate continuity properties of Hankel operators. It turns out that essentially the same approach also works in the infinite-dimensional context of holomorphic Wiener functionals. This paper investigates some details of such an approach. We will see that some results, known for $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$, naturally extend to infinite dimensions but others do not. For general background on Hankel operators we refer to the discussion in [2, Remark 1.1] and to the literature given there.

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Let us introduce some notation. The complex Wiener space $(W_{\mathbb{C}}, \mathcal{B}, \mu_c)$ over the fixed time interval $[0, T]$ consists of the following objects:

$$W_{\mathbb{C}} := W_{\mathbb{C}}([0, T]) := \{w \in C([0, T], \mathbb{C}) \mid w(0) = 0\}. \tag{1.1}$$

$\mathcal{B}$ is the Borel σ-algebra on $W_{\mathbb{C}}$ induced by the $\|\cdot\|_\infty$-topology on $W_{\mathbb{C}}$.

$$Z_t(w) := X_t(w) + iY_t(w) := w(t), \quad w \in W_{\mathbb{C}}, \tag{1.2}$$

is the canonical process, and μ_c the Wiener measure with variance parameter $c > 0$. I.e. μ_c is the unique probability measure on $\mathcal{B}$ which is such that

(Z1) X and Y defined in (1.2) are independent, real processes.
(Z2) X and Y have independent, centered Gaussian increments.
(Z3) $E[(X_t - X_s)^2] = E[(Y_t - Y_s)^2] = c(t - s)$ for all $0 \leqslant s < t \leqslant T$.

REMARK. In the finite-dimensional context of Gauss measure γ_c on $\mathbb{C}^d$ one has

$$L^p(\gamma_{\tilde{c}}) \subset L^p(\gamma_c) \quad \text{for all } \tilde{c} \geqslant c. \tag{1.3}$$

In the infinite-dimensional context the measures μ_c and $\mu_{\tilde{c}}$ with $c \neq \tilde{c}$ are mutually singular, $\mu_c \perp \mu_{\tilde{c}}$. In particular there is no natural relation such as (1.3) between the spaces $L^p(\mu_c)$ and $L^p(\mu_{\tilde{c}})$. This significant difference from the $\mathbb{C}^d$-case requires some care when one extends results for Hankel operators from finite to infinite dimensions. This is why we will sometimes distinguish between an equivalence class of functions $[f]_c \in L^2(\mu_c)$ and corresponding pointwise defined representatives $f \in [f]_c$.

We denote by $\mathcal{P}(Z)$ the *algebra of holomorphic polynomials* generated by the complex Gaussian variables Z_t, $t \in [0, T]$. Since the point evaluation $Z_t(w) = w(t)$ is a continuous linear functional on the complex Banach space $(W_{\mathbb{C}}, \|\cdot\|_\infty)$ the space $\mathcal{P}(Z)$ consists of holomorphic functions on $(W_{\mathbb{C}}, \|\cdot\|_\infty)$. For background on holomorphic functions in Banach spaces see, e.g., [8].

One can uniquely identify $Q \in \mathcal{P}(Z)$ with its equivalence class $[Q]_c \in L^p(\mu_c)$ because $[Q]_c$ contains exactly one representative in $\mathcal{P}(Z)$. To see this assume $Q_1, Q_2 \in \mathcal{P}(Z)$ are in $[Q]_c$. Then $R := Q_1 - Q_2$ is a polynomial in variables $Z_{t_1}, \ldots, Z_{t_n}$, $R = R(Z_{t_1}, \ldots, Z_{t_n})$, for suitable $0 < t_1 < \cdots < t_n$. The equality $[R]_c = [0]_c$ gives

$$0 = \int_{W_{\mathbb{C}}} |R(Z_{t_1}, \ldots, Z_{t_n})|^p \, \mathrm{d}\mu_c = \int_{\mathbb{C}^n} |R(z_1, \ldots, z_n)|^p \, \mathrm{d}\gamma(z),$$

where γ is the (regular) induced Gaussian measure $\gamma = (Z_{t_1}, \ldots, Z_{t_n})_* \mu_c$ on $\mathbb{C}^n$. This gives $R = 0$. Subsequently we simply identify a polynomial Q with its class $[Q]_c$, and thereby $\mathcal{P}(Z)$ with a subspace in $L^p(\mu_c)$. Notice that this allows to identify the classes $[Q]_c$ and $[Q]_{\tilde{c}}$, which a priori are not comparable.

NOTATION. For $p \in [1, \infty)$ define the space of *holomorphic L^p-Wiener functionals* to be

$$\mathcal{H}L^p(\mu_c) := L^p(\mu_c)\text{-closure of } \mathcal{P}(Z). \tag{1.4}$$

REMARKS. (1) Holomorphic L^p-Wiener functionals can be introduced and characterized in different ways, and they have been studied to some extent in the literature [3, 4, 13, 14, 17, 19–21, 23, 24]. Our choice (1.4) requires the least amount of terminology.

(2) There is almost no extra effort in replacing $W_{\mathbb{C}}([0, T])$ by $W_{\mathbb{C}}(\mathbb{R}_+)$, but some minor technicalities are easier to handle for $W_{\mathbb{C}}([0, T])$. In fact the main results in this paper can be formulated and proved in the context of an abstract complex Wiener space, at the cost of introducing more terminology.

(3) An arbitrary class $[f]_c \in \mathcal{H}L^2(\mu_c)$ can in general not be identified in a unique way with a pointwise defined representative f (as for polynomials). In particular $[f]_c$ need not contain a continuous representative f on $(W_{\mathbb{C}}, \|\cdot\|_\infty)$, see [22]. So in the strict sense $\mathcal{H}L^p(\mu_c)$ is not a space of holomorphic functions on $(W_{\mathbb{C}}, \|\cdot\|_\infty)$. However, $\mathcal{H}L^p(\mu_c)$ is naturally isomorphic to a space of genuine holomorphic functions on the complex Cameron–Martin subspace of $W_{\mathbb{C}}$. For details see [7, 20]. In the present work we do not use this identification in order to be as self-contained as possible. But in the forthcoming paper [1] this identification will be crucial for the characterization of finite rank Hankel operators.

Subsequently we study bilinear forms Γ_b on $\mathcal{H}L^2(\mu_c)$ represented by

$$\Gamma_b(f, g) = \langle fg, b\rangle = \int_{W_{\mathbb{C}}} \bar{b} f g \, \mathrm{d}\mu_c \quad \forall f, g \in \mathcal{P}(Z), \tag{1.5}$$

with suitable symbol functions $b \in \mathcal{H}L^2(\mu_c)$. Notice that the integral in (1.5) is well-defined for all $f, g \in \mathcal{P}(Z)$, but in general not for all $f, g \in \mathcal{H}L^2(\mu_c)$. The (by definition!) dense inclusion $\mathcal{P}(Z) \subset \mathcal{H}L^2(\mu_c)$ shows that Γ_b extends to a unique continuous bilinear form on $\mathcal{H}L^2(\mu_c)$ if and only if an estimate of the following type holds:

$$|\Gamma_b(f, g)| \leqslant \text{const.} \|f\|_{L^2(\mu_c)} \|g\|_{L^2(\mu_c)} \quad \forall f, g \in \mathcal{P}(Z). \tag{1.6}$$

There is then associated a continuous, anti-linear operator H_b on $\mathcal{H}L^2(\mu_c)$ satisfying

$$\Gamma_b(f, g) = \langle g, H_b f\rangle \quad \forall f, g \in \mathcal{H}L^2(\mu_c).$$

Γ_b and H_b are called *Hankel form* respectively *Hankel operator with symbol b*. The equality $\langle b, fg\rangle = \langle H_b f, g\rangle$, which holds for all $f, g \in \mathcal{P}(Z)$, implies that

$$H_b f = P(b\bar{f}) \tag{1.7}$$

provided $b\bar{f} \in L^2(\mu_c)$, and where $P : L^2(\mu_c) \to \mathcal{H}L^2(\mu_c)$ is the orthogonal projection. So if we denote by M_b the multiplication by b and the complex conjugation by C we obtain $H_b = P \circ M_b \circ C$, a more conventional form for a Hankel operator.

Among the classical questions about Hankel operators are characterizations of properties of H_b in terms of properties of b. In this work we investigate the continuity and the Hilbert–Schmidtness of H_b in terms of $b \in \mathcal{H}L^2(W_{\mathbb{C}}, \mu_c)$.

The content of this paper can be summarized as follows: Section 2 prepares some facts about $\mathcal{H}L^2(\mu_c)$ and about the Ornstein–Uhlenbeck semigroup on that space. The main results are Theorem 3.2 (on boundedness) and Theorem 4.2 (on Hilbert–Schmidtness). Their proofs consist in a reduction to the finite-dimensional case. We also extend the integral representation for Hilbert–Schmidt Hankel operators (Theorem 4.5). In contrast to $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ such a representation seems not possible for general continuous H_b. This deviation from the finite-dimensional context arises from the singularity of measures μ_c and μ_{2c}. Another (obvious) deviation is the absence of a reproducing kernel in $\mathcal{H}L^2(W_{\mathbb{C}}, \mu_c)$. Because of it we cannot prove a general integral representation using the reproducing kernel (as in finite dimensions). We use "finite variable approximations" instead.

2. Preparations about Holomorphic Wiener Functionals

The finite-dimensional background to be generalized subsequently reads as follows: Let $\varphi \in \mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ be given by its Taylor series

$$\varphi(z) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha z^\alpha, \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. Then (2.1) is also an orthogonal series in $\mathcal{H}L^2(\gamma_c)$ having the norm

$$\|\varphi\|^2_{L^2(\gamma_c)} = \sum_{\alpha \in \mathbb{N}_0^d} (2c)^{|\alpha|} \alpha! |a_\alpha|^2, \tag{2.2}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and $\alpha! := \alpha_1! \cdots \alpha_d!$. The Ornstein–Uhlenbeck semigroup (OU) e^{-tN} on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ can be defined by

$$\mathrm{e}^{-tN} \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha z^\alpha = \sum_{n=0}^{\infty} \mathrm{e}^{-tn/c} \left(\sum_{|\alpha|=n} a_\alpha z^\alpha \right). \tag{2.3}$$

This has the immediate consequences that $\mathrm{e}^{-tN}\varphi(z) = \varphi(\mathrm{e}^{-t/c}z)$ and

$$\|\mathrm{e}^{-tN}\varphi\|_{L^2(\gamma_c)} = \|\varphi\|_{L^2(\gamma_{c\mathrm{e}^{-2t/c}})}. \tag{2.4}$$

If $\varphi \in D(\mathrm{e}^{tN})$ then these two properties remain valid if one skips all minus signs in the exponents. In particular $\varphi \in D(\mathrm{e}^{tN})$ is simply characterized by the growth requirement $\varphi \in \mathcal{H}L^2(\gamma_{c\mathrm{e}^{2t/c}})$, a fact which is false in the absence of holomorphy.

The semigroup e^{-tN} has the following *hypercontractivity property* (see, e.g., [6]): Let $0 < q \leqslant p < \infty$. Then

$$\|e^{-tN}\|_{\mathcal{H}L^q(\gamma_c)\to\mathcal{H}L^p(\gamma_c)} \leqslant 1, \quad \text{if } t \geqslant \frac{c}{2}\ln\frac{p}{q}. \tag{2.5}$$

We now construct an orthogonal basis in $\mathcal{H}L^2(W_{\mathbb{C}}, \mu_c)$ which allows to represent every $\varphi \in \mathcal{H}L^2(\mu_c)$ by a power series w.r.t. complex "variables" $Z_1, Z_2, \ldots$, quite analogous to (2.1). This has the advantage that one can restrict every such φ to finitely many variables $Z_1, \ldots, Z_d$ and thereby to obtain a natural link to the spaces $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$, as explained in Lemma 2.2. The basic variables Z_k are complex Gaussian random variables constructed by the elementary Wiener integral as follows: Let f be a step function on $[0, T]$, i.e. there are time points $0 = t_0 < t_1 < \cdots < t_n = T$ and constants $f_i \in \mathbb{C}$ such that $f = \sum_{i=1}^n f_i 1_{[t_{i-1},t_i)}$. Denote by $S[0, T]$ the vector space of such step functions. Then the stochastic integral of $f \in S[0, T]$ w.r.t. complex Brownian motion is defined as

$$\int_0^T f(t)\,\mathrm{d}Z_t(w) := \sum_{i=1}^n f_i(Z_{t_i}(w) - Z_{t_{i-1}}(w)), \quad \forall w \in \Omega. \tag{2.6}$$

The properties (Z1), (Z2), (Z3) from Section 1 satisfied by the Brownian motion Z imply

$$\left\|\int_0^T f(t)\,\mathrm{d}Z_t\right\|^2_{L^2(\mu_c)} = 2c\|f\|^2_{L^2(\mathrm{d}x)}, \tag{2.7}$$

so the linear map $f \mapsto \int_0^T f(t)\,\mathrm{d}Z_t$ is isometric from $(S[0, T], \|\cdot\|_{L^2(\mathrm{d}x)})$ to $L^2(\mu_c)$, up to the normalization constant $2c$. For general $f \in L^2([0, T], \mathrm{d}x)$ the Wiener integral $\int_0^T f(t)\,\mathrm{d}Z_t$ is simply obtained by continuous extension of this map, and it defines a complex Gaussian random variable. Clearly the Itô-isometry (2.7) remains valid under this extension, and by definition (1.4) we have $\int_0^T f(t)\,\mathrm{d}Z_t \in \mathcal{H}L^2(\mu_c)$.

The following fact is an immediate consequence of the Segal–Bargmann isomorphism [7] applied to the well-known Hermite-basis over the real Wiener space $W_{\mathbb{R}}([0, T])$, and it generalizes the standard orthogonal basis (OGB) in $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ in a natural way:

THEOREM 2.1. *Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis in $L^2([0, T], \mathrm{d}x)$. Define*

$$Z_k := \int_0^T e_k(t)\,\mathrm{d}Z_t, \quad k = 1, 2, \ldots. \tag{2.8}$$

Put $\mathbb{N}_c^\infty := \{(\alpha_1, \alpha_2, \ldots) \mid$ only finitely many $\alpha_i \in \mathbb{N}_0$ are non-zero$\}$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and $\alpha! := \alpha_1!\cdots\alpha_d!$, where d is the largest index such that $\alpha_d \neq 0$. Define

$$Z^0 := 1, \qquad Z^\alpha := Z_1^{\alpha_1}\cdots Z_d^{\alpha_d}.$$

Then $\{Z^\alpha, \alpha \in \mathbb{N}_c^\infty\}$ *is an OGB in* $\mathcal{H}L^2(\mu_c)$ *with normalization* $\|Z^\alpha\|^2_{L^2(\mu_c)} = (2c)^{|\alpha|}\alpha!$.

By the previous theorem any $\varphi \in \mathcal{H}L^2(\mu_c)$ admits an orthogonal expansion

$$\varphi = \sum_{n=0}^\infty \varphi^{(n)} = \sum_{n=0}^\infty \sum_{|\alpha|=n} a_\alpha Z^\alpha = \sum_{\alpha\in\mathbb{N}_c^\infty} a_\alpha Z^\alpha, \tag{2.9}$$

with

$$\|\varphi\|^2_{L^2(\mu_c)} = \sum_{\alpha\in\mathbb{N}_c^\infty} (2c)^{|\alpha|}\alpha!|a_\alpha|^2 < \infty. \tag{2.10}$$

This generalizes (2.1) and (2.2) in a most natural way. In particular observe that the function $\tilde{Z}_k(z_1, \dots, z_d) := z_k$ defines a complex Gaussian random variable on $(\mathbb{C}^d, \gamma_c)$, so (2.1) can also be viewed as a series of Gaussian random variables.

REMARK. Although the functions Z_k given in (2.8) depend on the choice of the orthonormal basis $\{e_1, e_2, \dots\}$, the components $\varphi^{(n)}$ given in (2.9) do not. This follows from the corresponding property of the Hermite decomposition over $W_\mathbb{R}([0, T])$ together with the Segal–Bargmann transformation. We call $\varphi = \sum_{n=0}^\infty \varphi^{(n)}$ the *complex chaos decomposition of* φ, and $\varphi^{(n)}$ the *chaos components* of φ.

From now on we fix an arbitrary ONB $\{e_1, e_2, \dots\}$ in $L^2([0, T], \mathrm{d}x)$ and thereby the corresponding Z_k given in (2.8). Non of the results in this paper depends on the choice of such a basis. The OU-semigroup (2.3) now generalizes in the obvious (and basis independent) way to $\mathcal{H}L^2(\mu_c)$ as follows.

NOTATION. For $\varphi = \sum_{n=0}^\infty \varphi^{(n)} \in \mathcal{H}L^2(\mu_c)$ we denote by

$$\mathrm{e}^{-tN} \sum_{n=0}^\infty \varphi^{(n)} := \sum_{n=0}^\infty \mathrm{e}^{-tn/c}\varphi^{(n)}$$

the *Ornstein–Uhlenbeck semigroup* e^{-tN} on $\mathcal{H}L^2(W_\mathbb{C}, \mu_c)$. To distinguish the finite-dimensional case we will write $\mathrm{e}^{-t\tilde{N}}$ for the OU-semigroup on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$.

We now investigate how e^{-tN} relates to the "finite variable restriction" of $\varphi \in \mathcal{H}L^2(\mu_c)$. So let φ be given by the expansion (2.9), put $F_d := \overline{\mathcal{P}(Z_1, \dots, Z_d)}$ (the closure in $L^2(\mu_c)$), and let $\pi_d : \mathcal{H}L^2(\mu_c) \to F_d$ be the orthogonal projection. We claim that this projection is given by

$$\pi_d\varphi = \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha Z^\alpha, \tag{2.11}$$

which justifies the name “finite variable restriction” for $\pi_d\varphi$. To verify (2.11) note that

$$\overline{\mathcal{P}(Z_1,\ldots,Z_d)} = \left\{\varphi \in \mathcal{H}L^2(\mu_c) \mid \varphi = \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha Z^\alpha\right\}. \tag{2.12}$$

Since the inclusion “$\supset$” in (2.12) is clear choose $\varphi \in \overline{\mathcal{P}(Z_1,\ldots,Z_d)}$ and expand φ as in (2.9). For indices $\alpha \neq (\alpha_1,\ldots,\alpha_d,0,0,\ldots)$ we have $Z^\alpha \perp \mathcal{P}(Z_1,\ldots,Z_d)$ by Theorem 2.1. Thus $Z^\alpha \perp \overline{\mathcal{P}(Z_1,\ldots,Z_d)}$, i.e. the coefficient a_α in (2.9) vanishes. So “$\subset$” in (2.12) follows. Since the set $\{Z^\alpha \mid \alpha = (\alpha_1,\ldots,\alpha_d,0,0,\ldots)\}$ is an OGB in F_d we see that (2.11) holds.

The following result can be viewed as a “holomorphic factorization lemma”. It is basic for our transition from finite to infinite dimensions given in Sections 3 and 4.

LEMMA 2.2. *To $f \in F_d$ there exists a unique $\tilde{f} \in \mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$ such that*

$$f = \tilde{f}(Z_1,\ldots,Z_d) \quad \mu_c\text{-}a.s. \tag{2.13}$$

The map $J : f \mapsto \tilde{f}$ is isometric from $(F_d, \|\cdot\|_{L^2(\mu_c)})$ onto $\mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$. Moreover

$$J \circ \mathrm{e}^{-tN} \circ J^{-1} = \mathrm{e}^{-t\tilde{N}}, \tag{2.14}$$

i.e. we have $\mathrm{e}^{-tN} f = (\mathrm{e}^{-t\tilde{N}}\tilde{f})(Z_1,\ldots,Z_d)$ μ_c-a.s.

Proof. Represent f as in (2.12). Then $\|f\|^2_{L^2(\mu_c)} = \sum_{\alpha\in\mathbb{N}_0^d}(2c)^{|\alpha|}|a_\alpha|^2\alpha! < \infty$. The completeness of $\mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$ thus implies that

$$Jf(z_1,\ldots,z_d) := \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha z^\alpha \tag{2.15}$$

defines a function $\tilde{f} \in \mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$ and the series (2.15) converges for *every* $z \in \mathbb{C}^d$. This implies that the series for f does not only converge in $\mathcal{H}L^2(\mu_c)$, but also μ_c-a.s. Thus (2.13) follows for every choice of pointwise defined representatives $Z_1,\ldots,Z_d$. To verify that $J : f \mapsto \tilde{f}$ is isometric (and thus injective) note that

$$\int_{W_\mathbb{C}} |f|^2\,\mathrm{d}\mu_c = \int_{W_\mathbb{C}} |\tilde{f}(Z_1,\ldots,Z_d)|^2\,\mathrm{d}\mu_c = \int_{\mathbb{C}^d} |\tilde{f}(z)|^2\,\mathrm{d}\gamma_c(z) \tag{2.16}$$

because $Z_1,\ldots,Z_d$ are complex Gaussian random variables with $\langle Z_k, Z_l\rangle_{L^2(\mu_c)} = 2c\delta_{kl}$, i.e. $(Z_1,\ldots,Z_d)_*\mu_c = \gamma_c$. Clearly (2.16) implies that there is only one $\tilde{f} \in \mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$ such that (2.13) holds. Now given $\tilde{f} \in \mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$ define $f := \tilde{f}(Z_1,\ldots,Z_d)$. By (2.16) $f \in \mathcal{H}L^2(W_\mathbb{C},\mu_c)$ and by definition (2.15) $Jf = \tilde{f}$. Thus J maps onto $\mathcal{H}L^2(\mathbb{C}^d,\gamma_c)$. Finally we verify (2.14):

$$\begin{aligned}
e^{-tN} f(w) &= \sum_{n=0}^{\infty} e^{-tn/c} \sum_{\alpha_1+\cdots+\alpha_d=n} a_\alpha Z^\alpha(w) \quad (\text{for } \mu_c\text{-a.e. } w) \\
&= \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha (e^{-t/c} Z(w))^\alpha \\
&= \tilde{f}(e^{-t/c} Z_1(w), \ldots, e^{-t/c} Z_d(w)) \\
&= (e^{-t\tilde{N}} \tilde{f})(Z_1(w), \ldots, Z_d(w)).
\end{aligned}$$

□

3. Bounded Hankel Operators

In this section we continue the notation $F_d = \overline{\mathcal{P}(Z_1, \ldots, Z_d)}$, and we let $\pi_d : \mathcal{H}L^2(\mu_c) \to F_d$ be the orthogonal projection. The following key lemma holds:

LEMMA 3.1. *Let $b \in \mathcal{H}L^2(\mu_c)$ be such that H_b is a continuous Hankel operator on $\mathcal{H}L^2(\mu_c)$. Then*

$$H_{\pi_d b} = \pi_d \circ H_b \circ \pi_d. \tag{3.1}$$

Let $J : f \mapsto \tilde{f}$ be the isometry defined in Lemma 2.2, *and put $b_d := \pi_d b$. Then*

$$H_{J(b_d)} = J \circ H_{b_d} \circ J^{-1}. \tag{3.2}$$

Proof. The Hankel form $\Gamma_{\pi_d b}$ evaluated at $f, g \in \mathcal{P}(Z_n, n \in \mathbb{N})$ reads

$$\begin{aligned}
\Gamma_{\pi_d b}(f, g) &= \langle fg, \pi_d b\rangle = \langle \pi_d(fg), b\rangle \\
&= \langle \pi_d f \pi_d g, b\rangle = \langle \pi_d g, H_b(\pi_d f)\rangle \\
&= \langle g, (\pi_d \circ H_b \circ \pi_d) f\rangle.
\end{aligned}$$

Since the right side of this equation defines a continuous bilinear form on $\mathcal{H}L^2(\mu_c)$, and $\mathcal{P}(Z_n, n \in \mathbb{N})$ is dense in $\mathcal{H}L^2(\mu_c)$ (Theorem 2.1) we obtain (3.1).

Now let $f, g \in \mathcal{P}(Z_1, \ldots, Z_d)$. Then

$$\langle g, H_{\pi_d b} f\rangle = \int_{W_\mathbb{C}} \overline{(\pi_d b)} fg \, d\mu_c = \int_{\mathbb{C}^d} \overline{\tilde{b}_d} \tilde{f} \tilde{g} \, d\gamma_c = \langle \tilde{g}, H_{\tilde{b}_d} \tilde{f}\rangle. \tag{3.3}$$

Since $H_{\pi_d b} f \in F_d$ and $g \in F_d$ the left side of (3.3) can also be written as

$$\langle g, H_{b_d} f\rangle = \langle Jg, J(H_{b_d} f)\rangle = \langle \tilde{g}, (J \circ H_{b_d} \circ J^{-1}) \tilde{f}\rangle. \tag{3.4}$$

Since (3.3) and (3.4) hold for all $\tilde{f}, \tilde{g} \in \mathcal{P}(z_1, \ldots, z_d)$ we conclude $H_{\tilde{b}_d} = J \circ H_{b_d} \circ J^{-1}$. □

REMARK. Equation (3.1) shows that the "finite variable restriction" $\pi_d \circ H_b \circ \pi_d$ of H_b to the subspace F_d is again a Hankel operator, H_{b_d}. (This statement is false for general orthogonal projections in $\mathcal{H}L^2(\mu_c)$.) Moreover (3.2) shows that H_{b_d} is unitary equivalent to the Hankel operator $H_{\tilde{b}_d}$ on the space $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$. These two properties provide a tight relation between Hankel operators on $\mathcal{H}L^2(W_{\mathbb{C}}, \mu_c)$ and Hankel operators on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$.

We now give a condition on the symbol b that guarantees the continuity of H_b. In its proof we do not directly use the hypercontractivity of the semigroup e^{-tN} on $\mathcal{H}L^2(\mu_c)$, we only use the hypercontractivity (2.5) on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$.

THEOREM 3.2. *Let* e^{-tN} *be the OU-semigroup on* $\mathcal{H}L^2(\mu_c)$ *and let* $\varphi \in \mathcal{H}L^p(\mu_c)$ *with* $p \geqslant 2$. *Put*

$$b := \mathrm{e}^{-tN}\varphi, \quad \textit{with} \quad t \geqslant t_J := \frac{c}{2}\ln p', \tag{3.5}$$

where p' *denotes the conjugate index to* p, *and define* $\Gamma_b(f, g) := \int_{W_{\mathbb{C}}} \bar{b} f g \, \mathrm{d}\mu_c$ *on* $\mathcal{P}(Z)$. *Then* Γ_b *extends by continuity to* $\mathcal{H}L^2(\mu_c)$: *For all* $f, g \in \mathcal{P}(Z_n, n \in \mathbb{N})$

$$\left| \int_{W_{\mathbb{C}}} \bar{b} f g \mathrm{d}\mu_c \right| \leqslant \|\varphi\|_{L^p(\mu_c)} \|f\|_{L^2(\mu_c)} \|g\|_{L^2(\mu_c)}. \tag{3.6}$$

REMARKS. (1) (3.6) implies the continuity of Γ_b because $\mathcal{P}(Z_n, n \in \mathbb{N})$ is dense in $\mathcal{H}L^2(\mu_c)$.

(2) One may interpret (3.5) as a regularization of φ. Notice that $p \to \infty$ implies $p' \to 1$ and therefore $t_J \to 0$. So the larger p is the less φ needs to be regularized.

Proof of Theorem 3.2. Let $f, g \in \mathcal{P}(Z_n, n \in \mathbb{N})$, so $f, g \in F_d$ for a suitable d. As before let $\pi_d : \mathcal{H}L^2(\mu_c) \to F_d$ be the orthogonal projection. Since e^{-tN} is a diagonal operator on the basis $\{Z^\alpha, \alpha \in \mathbb{N}_c^\infty\}$ the formula (2.11) implies

$$\mathrm{e}^{-tN} \circ \pi_d = \pi_d \circ \mathrm{e}^{-tN}. \tag{3.7}$$

In view of this and the symmetry of the projection π_d we have

$$\begin{aligned} \Gamma_{\mathrm{e}^{-tN}\varphi}(f, g) &= \int_{W_{\mathbb{C}}} \overline{\mathrm{e}^{-tN}\varphi} \pi_d(fg) \, \mathrm{d}\mu_c \\ &= \int_{W_{\mathbb{C}}} \overline{\mathrm{e}^{-tN}(\pi_d\varphi)} f g \, \mathrm{d}\mu_c \\ &= \int_{\mathbb{C}^d} \overline{\mathrm{e}^{-t\tilde{N}}\tilde{\varphi}_d} \tilde{f} \tilde{g} \, \mathrm{d}\gamma_c \quad \text{(by Lemma 2.2)}, \end{aligned}$$

where $\tilde{\varphi}_d = J(\pi_d\varphi)$, $\tilde{f} = J(f)$ and $\tilde{g} = J(g)$. We can now apply the finite-dimensional estimate from Theorem 4.5 in [2] to the previous integral on the right side to obtain

$$\begin{aligned}|\Gamma_{e^{-tN}\varphi}(f,g)| &\leqslant \|\tilde{\varphi}_d\|_{L^p(\gamma_c)}\|\tilde{f}\|_{L^2(\gamma_c)}\|\tilde{g}\|_{L^2(\gamma_c)}\\ &\leqslant \|\varphi\|_{L^p(\mu_c)}\|f\|_{L^2(\mu_c)}\|g\|_{L^2(\mu_c)}. \qquad \square\end{aligned}$$

REMARK. Assume that Γ_b is continuous and abbreviate $b_d := \pi_d b$. Equation (3.2) implies $\|H_{b_d}\| = \|H_{\tilde{b}_d}\|$, and (3.1) gives $\|H_{b_d}\| \leqslant \|H_b\|$. In [2, Remark 6.11] the following estimate we derived for the symbol $\tilde{b}$ of a Hankel operator on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c) : (\frac{u}{s})^d \|\tilde{b}\|^2_{L^2(\gamma_{c+u})} \leqslant \|H_{\tilde{b}}\|^2$ for all $s \in (0, \infty)$, with $u^{-1} := c^{-1} + s^{-1}$. In view of (2.4) and with $\tilde{b} = \tilde{b}_d$ this converts to

$$\left(\frac{u}{s}\right)^d \|e^{\frac{c}{2}\ln(1+\frac{u}{s})\tilde{N}}\tilde{b}_d\|^2_{L^2(\gamma_c)} \leqslant \|H_{\tilde{b}_d}\|^2.$$

With Lemma 2.2 and $\|H_{\tilde{b}_d}\| \leqslant \|H_b\|$ this yields

$$\left(\frac{u}{s}\right)^d \|e^{\frac{c}{2}\ln(1+\frac{u}{s})N} b_d\|^2_{L^2(\mu_c)} \leqslant \|H_b\|^2, \quad \forall s \in (0, \infty).$$

Since $u/s < 1$ we cannot derive the boundedness of $\|e^{\frac{c}{2}\ln(1+\frac{u}{s})N} b\|_{L^2(\mu_c)}$ in the limit $d \to \infty$. Therefore the necessary regularity condition which holds in the d-dimensional case does not carry over to infinite dimensions by letting d go to infinity.

4. Hilbert Schmidtness and Integral Representation

In this section we investigate the Hilbert–Schmidt (HS) property of H_b. Let us first recall the finite-dimensional situation, as discussed in [11, Theorem 10.1] and [2, Example 6.9]: *Assume* $b \in \mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$. *Then*

$$H_b \text{ is HS on } \mathcal{H}L^2(\mathbb{C}^d, \gamma_c) \Longleftrightarrow b \in \mathcal{H}L^2(\mathbb{C}^d, \gamma_{2c}). \tag{4.1}$$

Moreover,

$$\|H_b\|_{HS(\gamma_c)} = \|b\|_{L^2(\gamma_{2c})}. \tag{4.2}$$

Subsequently we generalize (4.1) and (4.2) to the Wiener space context. Since $\mu_c \perp \mu_{2c}$ the spaces $L^2(\mu_c)$ and $L^2(\mu_{2c})$ contain fundamentally different function classes. So (4.2) does not make sense if we replace γ by μ. However, Theorem 2.1 has the important consequence that for $c \leqslant \tilde{c}$ we can naturally identify $\mathcal{H}L^2(\mu_{\tilde{c}})$ with a subspace in $\mathcal{H}L^2(\mu_c)$:

LEMMA 4.1. *Let* $\tilde{c} \geqslant c > 0$. *Then the identity map* $I : \mathcal{P}(Z) \to \mathcal{P}(Z)$ *extends by continuity to a continuous, injective map* $\tilde{I} : \mathcal{H}L^2(\mu_{\tilde{c}}) \to \mathcal{H}L^2(\mu_c)$. *Moreover, the classes* $[f]_{\tilde{c}} \in \mathcal{H}L^2(\mu_{\tilde{c}})$ *and* $\tilde{I}[f]_{\tilde{c}} \in \mathcal{H}L^2(\mu_c)$ *have a common representative* g.

Proof. The step functions $S[0, T]$ are dense in $L^2([0, T], \mathrm{d}x)$. So we can choose an ONB $\{e_1, e_2, \ldots\}$ in $L^2([0, T], \mathrm{d}x)$ with elements $e_n \in S[0, T]$. The w-wise well-defined integrals given in (2.8) generate a subspace of polynomials $\mathcal{P}(Z_n, n \in \mathbb{N}) \subset \mathcal{P}(Z)$ which is dense in $\mathcal{H}L^2(\mu_c)$ by Theorem 2.1, for every $c > 0$. For $Q \in \mathcal{P}(Z_n, n \in \mathbb{N})$ we obtain with (2.10) the estimate

$$\|Q\|_{L^2(\mu_c)} \leqslant \|Q\|_{L^2(\mu_{\tilde{c}})}, \quad \forall c \leqslant \tilde{c}. \tag{4.3}$$

So the identity $I : (\mathcal{P}(Z_n, n \in \mathbb{N}), \|\cdot\|_{L^2(\mu_{\tilde{c}})}) \to (\mathcal{P}(Z_n, n \in \mathbb{N}), \|\cdot\|_{L^2(\mu_c)})$ is continuous and thus has a unique continuous extension $\tilde{I}$. Write $[f]_{\tilde{c}} \in \mathcal{H}L^2(\mu_{\tilde{c}})$ as $[f]_{\tilde{c}} = \sum_{\alpha \in \mathbb{N}_c^\infty} a_\alpha Z^\alpha$ and define the polynomial approximation $f_m(w) := \sum_{n=0}^m \sum_{\alpha_1 + \cdots + \alpha_m = n} a_\alpha Z^\alpha(w)$. By (4.3) f_m converges both in $\mathcal{H}L^2(\mu_{\tilde{c}})$ and in $\mathcal{H}L^2(\mu_{\tilde{c}})$. So $\tilde{I}$ is given by

$$\tilde{I} : \sum_{\alpha \in \mathbb{N}_c^\infty} a_\alpha Z^\alpha \mapsto \sum_{\alpha \in \mathbb{N}_c^\infty} a_\alpha Z^\alpha. \tag{4.4}$$

(Notice that the left side is considered as an orthogonal series in $\mathcal{H}L^2(\mu_{\tilde{c}})$, while the right side is an orthogonal series in $\mathcal{H}L^2(\mu_c)$.) This implies that $\tilde{I}$ is injective, and it is straightforward to verify that the restriction $\tilde{I}|_{\mathcal{P}(Z)}$ is the identity on $\mathcal{P}(Z)$. A suitable subsequence of (f_m), denoted $(f_{m'})$, converges both $\mu_{\tilde{c}}$-a.s. and μ_c-a.s. Thus the set of divergence points w of $(f_{m'})$ is contained in $N := N_c \cap N_{\tilde{c}}$, where $\mu_c(N_c) = 0$ and $\mu_{\tilde{c}}(N_{\tilde{c}}) = 0$. We conclude that the w-wise limit $g := \lim_{m\to\infty}(f_{m'} 1_{N^c})$ is in $[f]_c \cap [f]_{\tilde{c}}$. □

REMARKS. (1) We will identify $\mathcal{H}L^2(\mu_{\tilde{c}})$ with its image in $\mathcal{H}L^2(\mu_c)$ under the map $\tilde{I}$ whenever $\tilde{c} \geqslant c$. (4.4) shows that this identification is most natural.

(2) For the full space $L^2(\mu_c)$ the identity map on polynomials $\mathcal{P}(X_t, Y_t, t \in [0, T])$ is not continuous. The proof of Lemma 4.1 breaks down in that case, because Hermite-polynomials (replacing the Z^α) with respect to variance $\tilde{c}$ are not mutual orthogonal in $L^2(\mu_c)$ if $\tilde{c} \neq c$.

(3) We derived (4.4) for the special (w-wise everywhere defined) basis elements Z^α based on step functions e_n. Clearly $\tilde{I}$ is also given by (4.4) for any choice of orthogonal vectors (2.8).

THEOREM 4.2. *Let $b \in \mathcal{H}L^2(\mu_c)$. Then*

$$H_b \text{ is HS in } \mathcal{H}L^2(\mu_c) \iff b \in \mathcal{H}L^2(\mu_{2c}). \tag{4.5}$$

Moreover, in case H_b is HS we have

$$\|H_b\|^2_{HS(\mu_c)} = \|b\|^2_{L^2(\mu_{2c})}. \tag{4.6}$$

Proof. (⇒) Let H_b be HS. Define $b_n := \pi_n b$, where π_n projects on $F_n = \overline{\mathcal{P}(Z_1, \ldots, Z_n)}$ (closure in $\mathcal{H}L^2(\mu_c)$). Then $H_{b_n} = \pi_n \circ H_b \circ \pi_n$ by Lemma 3.1. Thus H_{b_n} is HS on $\mathcal{H}L^2(\mu_c)$ and thus on F_n. By (3.2) also $H_{\tilde{b}_n} : \mathcal{H}L^2(\gamma_c) \to \mathcal{H}L^2(\gamma_c)$ is HS. In view of (4.2) this implies

$$\|H_{b_n}\|_{HS(\mu_c)} = \|b_n\|_{L^2(\mu_{2c})}. \tag{4.7}$$

In this derivation we may replace b by $b - b_m$. For $n \geqslant m$ we have $\pi_n(b - b_m) = b_n - b_m$ and $H_{b_n - b_m} = H_{b_n} - H_{b_m}$. So instead of (4.7) we arrive at

$$\|H_{b_n} - H_{b_m}\|_{HS(\mu_c)} = \|b_n - b_m\|_{L^2(\mu_{2c})}. \tag{4.8}$$

It is simple to check that H_{b_n} $(= \pi_n \circ H_b \circ \pi_n)$ converges to H_b in HS(μ_c)-norm. So (4.8) shows that the sequence (b_n), which converges to b in $\mathcal{H}L^2(\mu_c)$, in fact converges in $\mathcal{H}L^2(\mu_{2c})$. Clearly its $\mathcal{H}L^2(\mu_{2c})$-limit is again b, so "⇒" in (4.5) holds. Taking n to infinity in (4.7) yields (4.6).

(⇐) Let $b \in \mathcal{H}L^2(\mu_{2c}) \subset \mathcal{H}L^2(\mu_c)$ and $f, g \in \mathcal{P}(Z_k, k \in \mathbb{N})$. As before define $b_n := \pi_n b$, $f_n := \pi_n f$ and $g_n := \pi_n g$. Then $b_n \in \mathcal{H}L^2(\mu_{2c})$, $b_n \to b$ in $\mathcal{H}L^2(\mu_{2c})$, and

$$\langle b_n, fg\rangle = \langle b_n, f_n g_n\rangle = \langle \tilde{b}_n, \tilde{f}_n \tilde{g}_n\rangle. \tag{4.9}$$

Since $\tilde{b}_n \in \mathcal{H}L^2(\gamma_{2c})$ we know that $\Gamma_{\tilde{b}_n}$ is a HS bilinear form on $\mathcal{H}L^2(\gamma_c)$. From (4.9) we conclude that Γ_{b_n} is HS on $\mathcal{H}L^2(\mu_c)$ and has the same HS-norm as $\Gamma_{\tilde{b}_n}$. Thus

$$\begin{aligned} \|H_{b_n} - H_{b_m}\|_{HS} &= \|H_{b_n - b_m}\|_{HS} \\ &= \|H_{\tilde{b}_n - \tilde{b}_m}\|_{HS} \\ &= \|\tilde{b}_n - \tilde{b}_m\|_{L^2(\gamma_{2c})} \to 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

Thus H_{b_n} is a Cauchy sequence of HS-operators. Denote by H the HS-limit. It remains to verify $H = H_b$. For $f, g \in \mathcal{P}(Z_k, k \in \mathbb{N})$ we have

$$\begin{aligned} \langle g, Hf\rangle &= \lim_{n\to\infty} \langle g, H_{b_n} f\rangle = \lim_{n\to\infty} \langle fg, b_n\rangle = \langle fg, b\rangle \\ &= \Gamma_b(f, g). \end{aligned}$$

Since the left-hand side of this equation defines a continuous bilinear form on the whole space $\mathcal{H}L^2(\mu_c)$ we conclude $H = H_b$. □

REMARK. In Theorem 4.2 we cannot just write $\|H_b\|^2_{HS(\mu_c)} = \|b\|^2_{L^2(\mu_{2c})}$ for all $b \in \mathcal{H}L^2(\mu_c)$ because if $b \notin \mathcal{H}L^2(\mu_{2c})$ we have no canonical identification of b with a function modulo μ_{2c}-zero sets.

We next generalize the integral representation known for $\mathcal{H}L^2(\gamma_c)$, i.e. the assertion that every continuous Hankel operator H_b on $\mathcal{H}L^2(\gamma_c)$ with $b \in \mathcal{H}L^2(\gamma_c)$ is given by

$$H_b f(z) = \int_{\mathbb{C}^d} b(z + w)\overline{f(w)}\, \mathrm{d}\gamma_c(w), \quad \forall z \in \mathbb{C}^d,\ \forall f \in \mathcal{H}L^2(\gamma_c). \tag{4.10}$$

In contrast to (4.10) it makes no sense to choose $b \in [b]_c \in \mathcal{H}L^2(\mu_c)$ and to consider $b(w + \cdot)$ with fixed w because this function depends significantly on the choice of representative b (see Lemma 4.5). So in general the integral kernel $b(w + w')$ corresponding to the one in (4.10) is not well-defined in the Wiener space context. However, if $b \in \mathcal{H}L^2(\mu_{2c}) \subset \mathcal{H}L^2(\mu_c)$ the following exception holds:

LEMMA 4.3. *Let $[b]_{2c} \in \mathcal{H}L^2(\mu_{2c})$. Then $\hat{b}(w, w') := b(w + w')$ is well-defined as an element in $L^2(\mu_c \otimes \mu_c)$. In particular, $\hat{b}$ does not depend on the specific representative $b \in [b]_{2c}$.*

Proof. Choose $b \in [b]_{2c}$. By convolution $\mu_c * \mu_c = \mu_{2c}$ we obtain

$$\begin{aligned} \int |\hat{b}(w, w')|^2 \, \mathrm{d}(\mu_c \otimes \mu_c)(w, w') \\ = \int |b(w + w')|^2 \, \mathrm{d}(\mu_c \otimes \mu_c)(w, w') \\ = \int |b(u)|^2 \, \mathrm{d}\mu_{2c}(u) < \infty. \end{aligned} \tag{4.11}$$

Now choose $b_1, b_2 \in [b]_{2c}$. If we replace in the previous calculation b by $b_1 - b_2$ we obtain

$$\begin{aligned} \int |\hat{b}_1(w, w') - \hat{b}_2(w, w')|^2 \, \mathrm{d}(\mu_c \otimes \mu_c)(w, w') \\ = \int |b_1(u) - b_2(u)|^2 \, \mathrm{d}\mu_{2c}(u) = 0. \end{aligned} \tag{4.12}$$

(4.11) and (4.12) yield the assertion. □

THEOREM 4.4. *Let $b \in \mathcal{H}L^2(\mu_{2c})$. Then $H_b f$ is given by*

$$H_b f(w) = \int_{W_\mathbb{C}} b(w + w') \overline{f(w')} \, \mathrm{d}\mu_c(w') \;\; \mu_c\text{-}a.s. \tag{4.13}$$

Proof. We continue with the notation in the proof of Theorem 4.2. Let $f \in \overline{\mathcal{P}(Z_1, \ldots, Z_n)}$. In view of [2, Remark 6.8] we have $\tilde{b}_n(z + \cdot) \in \mathcal{H}L^2(\gamma_c)$ for every $z \in \mathbb{C}^n$, and

$$H_{\tilde{b}_n} \tilde{f}(z) = \int_{\mathbb{C}^n} \tilde{b}_n(z + u) \overline{\tilde{f}(u)} \, \mathrm{d}\gamma_c(u).$$

With $Z = (Z_1, \ldots, Z_n)$ this representation and (3.2) imply

$$\begin{aligned} H_{b_n} f(w) &= H_{\tilde{b}_n} \tilde{f}(Z(w)) \quad \mu_c\text{-a.s.} \\ &= \int_{\mathbb{C}^n} \tilde{b}_n(Z(w) + u) \overline{\tilde{f}(u)} \, \mathrm{d}\gamma_c(u) \end{aligned}$$

$$= \int_{W_{\mathbb{C}}} \tilde{b}_n(Z(w) + Z(w'))\overline{\tilde{f}(Z(w'))}\,\mathrm{d}\mu_c(w')$$
$$= \int_{W_{\mathbb{C}}} b_n(w + w')\overline{f(w')}\,\mathrm{d}\mu_c(w').$$

This holds for all $f \in \overline{\mathcal{P}(Z_1, \dots, Z_n)}$. For general $f \in \mathcal{H}L^2(\mu_c)$ (3.1) gives

$$H_{b_n} f(w) = H_{b_n}\pi_n f(w) = \int_{W_{\mathbb{C}}} b_n(w + w')\overline{\pi_n f(w')}\,\mathrm{d}\mu_c(w') \quad \mu_c\text{-a.s.}$$
$$= \int_{W_{\mathbb{C}}} b_n(w + w')\overline{f(w')}\,\mathrm{d}\mu_c(w'). \tag{4.14}$$

The functions $\hat{b}_n(w, w') := b_n(w+w')$ and $\hat{b}(w, w') := b(w+w')$ are in $L^2(\mu_c \otimes \mu_c)$ by Lemma 4.3. Since $H_{b_n} \to H_b$ in HS(μ_c)-norm the isometry (4.8) and $\|\hat{b}_n\|_{\mu_c\otimes\mu_c} = \|b_n\|_{\mu_{2c}}$ imply that $\hat{b}_n \to \hat{b}$ in $\mathcal{H}L^2(\mu_c \otimes \mu_c)$. Thus the right-hand side in (4.14) converges in $L^2(\mu_c)$ to $\int_{W_{\mathbb{C}}} b(w + w')\overline{f(w')}\,\mathrm{d}\mu_c(w')$. On the other hand $H_{b_n} \to H_b$ in HS-norm implies $H_{b_n} f \to H_b f$ in $\mathcal{H}L^2(\mu_c)$. These two arguments show that we can pass to the limit in (4.14) which yields (4.13). □

REMARK. (4.1) and (4.2) are simple consequences of the integral representation for Hankel operators on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ because this integral representation holds for all continuous H_b on $\mathcal{H}L^2(\mathbb{C}^d, \gamma_c)$ (see the proof in [2]). In the Wiener space context that proof does not generalize, which is why we used the finite variable approximation.

We finally discuss the integral kernel $b(w + w')$ if b is not in $\mathcal{H}L^2(\mu_{2c})$. This illustrates one of the problems arising from $\mu_c \perp \mu_{2c}$.

LEMMA 4.5. *Let $[b]_c \in \mathcal{H}L^2(\mu_c)$ and $b_1 \in [b]_c$. Then there is a $b_2 \in [b]_c$ and a μ_c zero set N such that*

$$\int_{W_{\mathbb{C}}} |b_1(w + w') - b_2(w + w')|^2\,\mathrm{d}\mu_c(w') \neq 0, \quad \forall w \in N^c. \tag{4.15}$$

Proof. (4.15) is equivalent to the assertion that there exists $b_0 \in [0]_c$ such that

$$\int_{W_{\mathbb{C}}} |b_0(w + w')|^2\,\mathrm{d}\mu_c(w') \neq 0, \quad \forall w \in N^c. \tag{4.16}$$

Since $\mu_c \perp \mu_{2c}$ there is a μ_c zero set N_0 such that $\mu_{2c}(N_0) = 1$. Put $b_0 = 1_{N_0}$, so $b_0 \in [0]_c$. With $\mu_c * \mu_c = \mu_{2c}$ we have

$$\int |b_0(w + w')|^2\,\mathrm{d}(\mu_c \otimes \mu_c)(w, w') = \int_{W_{\mathbb{C}}} |1_{N_0}(u)|^2\,\mathrm{d}\mu_{2c}(u) = 1.$$

So Fubini applied to the left side gives

$$\int \left(\int |1_{N_0}(w + \cdot)|^2 \, \mathrm{d}\mu_c \right) \mathrm{d}\mu_c(w) = 1. \tag{4.17}$$

Since the term in brackets is between 0 and 1 it must in fact equal 1 μ_c-a.s. in order to satisfy (4.17). But this implies the assertion. □

Notice that Lemma 4.5 does not contradict Lemma 4.3: $b(w + w')$ is not well-defined for representatives $b \in [b]_c$ but it is well-defined for representatives $b \in [b]_{2c}$.

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