

A note on the σ -algebra of cylinder sets and all
that

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Abstract

In this note we introduced and describe the different kinds of σ -algebras in infinite dimensional spaces. We emphasize the case of Hilbert spaces and nuclear spaces. We prove that the σ -algebra generated by cylinder sets and the Borel σ -algebra coincides.

Contents

1 Introduction

In many situations of infinite dimensions appear measures defined on different kind of σ -algebras. The most common one being the Borel σ -algebra and the σ -algebra generated by cylinder sets. Also many times appear the σ -algebra associated to the weak topology and as well the strong topology. Here we will be concern mostly with the first two, i.e., the σ -algebra of cylinder sets and the Borel σ -algebra. In applications we find also not unique situations where one should have the above mentioned σ -algebras. Therefore we will treat the most common described in the literature. Namely the case of Hilbert riggings and nuclear triples.

Rigged Hilbert spaces is an abstract version of the theory of distributions or generalized functions, therefore this case is very important it self. As it is well known nuclear triples appear often in mathematical physics, hence we will also consider it. Booths cases are well known in the literature but for the convenience of the reader and completeness of this note we include on the appendix its definitions. Nevertheless we recommended the interested reader to look the classical book of (?) for more details and historical remarks. Thus in Section 2 we describe the σ -algebra generated by cylinder sets on rigged Hilbert spaces. Moreover we prove that, in fact, that this σ -algebra coincides with the Borel σ -algebra. In Section 3 we obtain analogous results for nuclear triples.

2 Case of rigged Hilbert spaces

Let $(\mathcal{H}, (\cdot, \cdot))$ be a Hilbert space and consider $(\mathcal{H}_+, (\cdot, \cdot))$ another Hilbert space which is such that $\mathcal{H}_+ \subset \mathcal{H}$ densely. Then we can consider the triple

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \quad (1)$$

which is called rigged Hilbert space, see Appendix A.1 for more details or (?), (?), (?), and (?).

Now we would like to introduce a σ -algebra in \mathcal{H}_- . To this end we proceed as follows: consider the following collection of finite dimensional subsets from \mathcal{H}

$$\mathcal{L} = \{L | L \subset \mathcal{H}, \dim(L) < \infty\} \quad (2)$$

This family is such that the following conditions holds:

1. if $L \in \mathcal{L}$, then an arbitrary subspace $\tilde{L} \subset L$ belongs to \mathcal{L} ,
2. if $L_1, L_2 \in \mathcal{L}$, $l.h.\{L_1, L_2\} \in \mathcal{L}$, where $l.h.\{L_1, L_2\} = \{\alpha l_1 + \beta l_2 | \alpha, \beta \in \mathbb{R}, l_1 \in L_1, l_2 \in L_2\}$,
3. $\bigcup_{L \in \mathcal{L}} L$ is dense in \mathcal{H} .

For each $L \in \mathcal{L}$ and any $A \in \mathcal{B}(L)$ (the Borel σ -algebra on L) we define a cylinder set with coordinate L and base A by

$$C(L, A) := \{\xi \in \mathcal{H}_- | P_L(\xi) \in A\}. \quad (3)$$

Here P_L is the projection on \mathcal{H} which can be extended to \mathcal{H}_- . More precisely, let $\{e_j, j \in \mathbb{N}\}$ be an orthonormal bases in \mathcal{H}_+ which is also orthonormal in \mathcal{H} . Then any element $h \in \mathcal{H}$ has the following decomposition:

$$h = \sum_{j=1}^{\infty} (h, e_j) e_j,$$

this implies with $\dim(L) = N$

$$P_L(h) = \sum_{j=1}^N (h, e_j) e_j.$$

On the other hand, any $\xi \in \mathcal{H}_-$ is the limit in the norm $|\cdot|_-$ of a Cauchy sequence of elements in \mathcal{H} i.e., $\exists (h_n)_{n=1}^{\infty} \subset \mathcal{H}$ such that $|\xi - h_n|_- \rightarrow 0, n \rightarrow \infty$. Therefore any element $\xi \in \mathcal{H}_-$ also admits a decomposition in the basis $(e_j)_{j=1}^{\infty}$ as

$$\xi = \sum_{j=1}^{\infty} \langle \xi, e_j \rangle e_j. \quad (4)$$

Hence we easily find that

$$P_L(\xi) = \sum_{j=1}^N \langle \xi, e_j \rangle e_j, \quad N = \dim(L). \quad (5)$$

This implies that, indeed, the cylinder set (??) is well defined. We denote by $\mathcal{C}(L)$ the σ -algebra of cylinder sets with a fixed coordinate L , i.e.,

$$\mathcal{C}(L) = \{C(L, A) | A \in \mathcal{B}(L)\}.$$

Then we can define the algebra of cylinder sets by

$$\mathcal{C}_{\mathcal{L}}(\mathcal{H}_-) = \bigcup_{L \in \mathcal{L}} \mathcal{C}(L).$$

And finally the σ -algebra of cylinder sets is given by

$$\mathcal{C}_{\sigma}(\mathcal{H}_-) = \sigma(\mathcal{C}_{\mathcal{L}}(\mathcal{H}_-)),$$

that means, the σ -algebra generated by the algebra of cylinder sets. We would like to stress that in fact $\mathcal{C}(\mathcal{H}_-)$ is only an algebra and not a σ -algebra. This can be seen with the following example: consider the following family of cylinder sets

$$\{C(L_i, A_i), i = 1, 2, \dots\},$$

where each L_i is finite dimensional and $A_i \in \mathcal{B}(L_i)$. Then the set

$$K = \bigcap_{i=1}^{\infty} C(L_i, A_i)$$

is not a cylinder set because the coordinate is not finite dimensional.

Proposition 2.1 *For any class \mathcal{L} we have $\mathcal{C}_\sigma(\mathcal{H}_-) = \mathcal{B}(\mathcal{H}_-)$, i.e., $\mathcal{C}_\sigma(\mathcal{H}_-)$ is independent of \mathcal{L} .*

Proof. First we prove that $\mathcal{C}_\sigma(\mathcal{H}_-) \subset \mathcal{B}(\mathcal{H}_-)$: in fact to see that every cylinder set belongs to $\mathcal{B}(\mathcal{H}_-)$ we take into account the following representation for $C(L, A)$ which is a consequence of (??):

$$C(L, A) = \{\xi \in \mathcal{H}_- \mid (\langle \xi, e_1 \rangle, \dots, \langle \xi, e_n \rangle) \in A\}.$$

To prove the other inclusion we proceed as follows: let $\{e_j^-, j \in \mathbb{N}\}$ be an orthonormal basis in \mathcal{H}_- . Then consider the following sets in \mathcal{H}_- :

$$\bar{B}^-(0, R) = \bigcap_{N=1}^{\infty} \left\{ \xi \in \mathcal{H}_- \mid \sum_{n=1}^N (\xi, e_n)_- \leq R \right\}, \quad R > 0$$

which is the closed ball in \mathcal{H}_- with radius R . The open ball is obtained as

$$B^-(0, R) = \bigcup_{n=1}^{\infty} \bar{B}^-(0, R - \frac{1}{n})$$

which belongs to $\mathcal{C}_\sigma(\mathcal{H}_-)$ because each $\bar{B}^-(0, R)$ is an element of $\mathcal{C}_\sigma(\mathcal{H}_-)$. Since any open set in \mathcal{H}_- is the union of open balls, then it belongs to $\mathcal{C}_\sigma(\mathcal{H}_-)$. This implies that $\mathcal{B}(\mathcal{H}_-) \subset \mathcal{C}_\sigma(\mathcal{H}_-)$. ■

3 Case of nuclear triples

The case of nuclear triples is the most common in applications, e.g., the Schwartz triple, tempered distributions, etc. Hence we suppose given a nuclear triple

$$\mathcal{N}' \supset \mathcal{H} \supset \mathcal{N}, \tag{6}$$

where \mathcal{H} is a Hilbert space, \mathcal{N} is a dense subset in \mathcal{H} which is nuclear, see Appendix ?? for a detailed description of (??) or (?), (?), and (?).

Hence we will take a family of subsets from \mathcal{N} each being of finite dimension:

$$\mathcal{L} = \{L \subset \mathcal{N} \mid \dim(L) < \infty\}. \tag{7}$$

Given $L \in \mathcal{L}$ we define a cylinder set with coordinate L and base $A \in \mathcal{B}(L)$ by

$$C(L, A) = \{\Phi \in \mathcal{N}' \mid P_L(\Phi) \in A\}. \quad (8)$$

As before P_L is an orthogonal projection onto L defined in \mathcal{H} which extends to \mathcal{N}' by continuity as in the previous case. Sometimes it is useful to have also cylinder sets of the form (??) in coordinates. Thus we choose a basis $(e_j)_{j=1}^n$ in L which is also orthogonal in \mathcal{N} . Then for any $\Phi \in \mathcal{N}'$ we have

$$P_L(\Phi) = \sum_{j=1}^n \langle \Phi, e_j \rangle e_j.$$

This implies the following representations for $C(L, A)$:

$$C(L, A) = \{\Phi \in \mathcal{N}' \mid (\langle \Phi, e_1 \rangle, \dots, \langle \Phi, e_n \rangle) \in A\}.$$

We proceed introducing the σ -algebra of cylinder sets with a fixed coordinate $L \in \mathcal{L}$ by

$$\mathcal{C}(L) = \{C(L, A), A \in \mathcal{B}(L)\}.$$

Then the algebra of cylinder sets in \mathcal{N}' we denote by $\mathcal{C}(\mathcal{N}')$ and is defined by

$$\mathcal{C}(\mathcal{N}') := \bigcup_{L \in \mathcal{L}} \mathcal{C}(L).$$

Finally the σ -algebra of cylinder sets is the σ -algebra generated by $\mathcal{C}(\mathcal{N}')$, i.e.,

$$\mathcal{C}_\sigma(\mathcal{N}') = \sigma(\mathcal{C}(\mathcal{N}')).$$

Proposition 3.1 *If the space \mathcal{N} is a countable Hilbert space, then we have*

$$\mathcal{C}_\sigma(\mathcal{N}') = \mathcal{B}_w(\mathcal{N}') = \mathcal{B}_s(\mathcal{N}),$$

where $\mathcal{B}_w(\mathcal{N}')$ (resp. $\mathcal{B}_s(\mathcal{N})$) is the Borel σ -algebra degenerated by the weak (resp. strong) topology.

Proof. The prove of these facts can be founded in (2, Appendix 5) or (?). ■

A Rigged Hilbert spaces

The concept of rigged Hilbert spaces was introduced as an abstraction of the theory of generalized functions of the type of Sobolev-Schwartz, i.e.,

$$\mathcal{D}'(\mathbb{R}^d) \supset L^2(\mathbb{R}^d) \supset \mathcal{D}(\mathbb{R}^d),$$

or

$$S'(\mathbb{R}^d) \supset L^2(\mathbb{R}^d) \supset S(\mathbb{R}^d).$$

Let $(\mathcal{H}_0, (\cdot, \cdot)_0)$ be a complex Hilbert space and assume additionally that $(\mathcal{H}_+, (\cdot, \cdot)_+)$ is another complex Hilbert space such that $\mathcal{H}_+ \hookrightarrow \mathcal{H}_0$ is dense and

$$|\varphi|_0 \leq c|\varphi|_+, \quad \forall \varphi \in \mathcal{H}_+, \quad c > 0. \quad (\text{A.1})$$

Remark A.1 The inequality (??) may be reduced to the following inequality

$$|\varphi|_0 \leq |\varphi|'_+, \quad \forall \varphi \in \mathcal{H}_+, \quad (\text{A.2})$$

where $|\varphi|'_+$ is an equivalent norm in \mathcal{H}_+ . Hence, in this appendix we suppose without loss of generality that the inequality (??) verifies with $c = 1$.

Remark A.2 The elements from \mathcal{H}_0 will be denoted by f, g, h, \dots and we call them ordinary functions, while the elements from \mathcal{H}_+ play the role of test function. We denote the elements from \mathcal{H}_+ by $\varphi, \psi, \eta, \dots$

Let $f \in \mathcal{H}_0$ be a fixed element. Define a functional l_f associated to f on \mathcal{H}_+ by

$$l_f : \mathcal{H}_+ \rightarrow \mathbb{C}, \quad \varphi \mapsto l_f(\varphi) := (\varphi, f)_0. \quad (\text{A.3})$$

Proposition A.3 The functional l_f has the following properties.

1. Is linear, i.e., $\forall \alpha, \beta \in \mathbb{C}, l_f(\alpha\varphi + \beta\psi) = \alpha l_f(\varphi) + \beta l_f(\psi)$.
2. Is continuous, i.e., $|l_f(\varphi)| \leq |f|_0 |\varphi|_+$.
3. $\forall \alpha, \beta \in \mathbb{C}, \forall f, g \in \mathcal{H}_0, l_{\alpha f + \beta g}(\varphi) = \bar{\alpha} l_f + \bar{\beta} l_g$.

Proof. These properties follows immediately from the definition (??). ■

Let us define the following mapping in \mathcal{H}_0

$$|\cdot|_- : \mathcal{H}_0 \rightarrow \mathbb{R}_+, \quad f \mapsto |f|_- := \|l_f\| := \sup\{|l_f(\varphi)|, \varphi \in \mathcal{H}_+, |\varphi|_+ = 1\}. \quad (\text{A.4})$$

Proposition A.4 The mapping $|\cdot|_-$ verifies the properties of a norm.

1. $|f|_- \geq 0, \forall f \in \mathcal{H}_0$.
2. $|\lambda f|_- = |\lambda| |f|_-, \forall \lambda \in \mathbb{C}, \forall f \in \mathcal{H}_0$.
3. $|f + g|_- \leq |f|_- + |g|_-, \forall f, g \in \mathcal{H}_0$.
4. In $|f|_- = 0$, then $f = 0$.

Proof. Again the proof of the above proposition follows by applying the definition of $|\cdot|_-$. ■

Definition A.5 We define \mathcal{H}_- as the completion of \mathcal{H}_0 with respect to $|\cdot|_-$, i.e., $\mathcal{H}_- := \overline{\mathcal{H}_0}^{|\cdot|_-}$. The space \mathcal{H}_- plays the role of generalized functions and its elements will be denoted by $\Phi, \Psi, \Theta, \dots$.

Remark A.6 Consider the sesquilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{H}_+ \times \mathcal{H}_+ \rightarrow \mathbb{C}, \quad (\varphi, \psi) \mapsto \langle \varphi, \psi \rangle := (\varphi, \psi)_0.$$

1. Then $\langle \cdot, \cdot \rangle$ can be extended to $\mathcal{H}_+ \times \mathcal{H}_-$, i.e., we can define $\langle \varphi, \Phi \rangle := (\varphi, \Phi)_0$, $\forall \varphi \in \mathcal{H}_+$, $\forall \Phi \in \mathcal{H}_-$. Note that $\langle \varphi, \Phi \rangle$ is well defined and independent of the sequence $(f_n)_{n=1}^\infty \subset \mathcal{H}$ which converges to Φ .

2. The following generalization of the Cauchy-Schwarz inequality is valid

$$|(\varphi, \Phi)_0| \leq |\varphi|_+ |\Phi|_{\mathcal{H}_-}.$$

Theorem A.7 The space \mathcal{H}_- is a Hilbert space.

Proof. It is enough to show that the norm $|\cdot|_-$ is given by a scalar product, i.e., $\forall f \in \mathcal{H}_0$, we have $|f|_- = \sqrt{(f, f)_-}$. Denote by O the embedding operator from \mathcal{H}_+ into \mathcal{H} and its dual by I , i.e.,

$$O : \mathcal{H}_+ \hookrightarrow \mathcal{H}_0, \quad \varphi \mapsto O\varphi := \varphi, \quad I \equiv O^* : \mathcal{H}_0 \rightarrow \mathcal{H}_+, \quad f \mapsto If,$$

such that

$$(f, \varphi)_0 = (f, O\varphi)_0 = (If, \varphi)_+, \quad f \in \mathcal{H}_0, \quad \varphi \in \mathcal{H}_+.$$

Then, we have

$$\begin{aligned} |f|_- &= \sup\{|(f, \varphi)_0|, \varphi \in \mathcal{H}_+, |\varphi|_+ = 1\} \\ &= \sup\{|(If, \varphi)_+|, \varphi \in \mathcal{H}_+, |\varphi|_+ = 1\} \\ &= |If|_+. \end{aligned}$$

Hence, if we define the scalar product $(\cdot, \cdot)_-$ in \mathcal{H}_0 by

$$(f, g)_- := (If, Ig)_+, \quad f, g \in \mathcal{H}_0,$$

we obtain $|f|_-^2 = (f, f)_-$. Therefore the pre-Hilbert space $(\mathcal{H}_0, (\cdot, \cdot)_-)$ after completion turns into a Hilbert space which coincides with \mathcal{H}_- . \blacksquare

Remark A.8 1. The scalar product $(\cdot, \cdot)_-$ can be written as

$$(f, g)_- = (If, g)_0 = (f, Ig), \quad f, g \in \mathcal{H}_0.$$

2. Since $\|O\| = \|I\| = 1$, then

$$|f|_- \leq |f|_0, \quad f \in \mathcal{H}_0.$$

3. The scalar product in \mathcal{H}_- is given by

$$(\Phi, \Psi)_{\mathcal{H}_-} := \lim_{n \rightarrow \infty} (f_n, g_n)_-,$$

where $(f_n)_{n=0}^\infty, (g_n)_{n=0}^\infty \subset \mathcal{H}_0$ are Cauchy sequences in the norm $|\cdot|_-$ converging to Φ and Ψ , respectively.

4. Since $\forall f, g \in \mathcal{H}_0$ $(f, g)_{\mathcal{H}_-} = (f, g)_- = (If, Ig)_+$, then $I : D(I) = \mathcal{H}_0 \subset \mathcal{H}_- \rightarrow \mathcal{H}_+$ is an isomorphism between \mathcal{H}_- and \mathcal{H}_+ defined in a dense set. We denote its extension by continuity by \tilde{I} . Hence \tilde{I} is a unitary operator from \mathcal{H}_- onto \mathcal{H}_+ .
5. The operator \tilde{I} is continuous.
6. We have the following equality

$$(\varphi, \Phi)_0 = (\varphi, \tilde{I}\Phi)_+, \quad \forall \varphi \in \mathcal{H}_+, \quad \forall \Phi \in \mathcal{H}_-.$$

Theorem A.9 1. The space \mathcal{H}_- coincides with the dual of \mathcal{H}_+ , with respect to \mathcal{H}_0 , i.e.,

$$\mathcal{H}_- = (\mathcal{H}_+)' \tag{A.5}$$

2. The dual of \mathcal{H}_- coincides with \mathcal{H}_+ with respect to \mathcal{H}_0 .

Remark A.10 We have constructed the following chain

$$\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+ \tag{A.6}$$

called *Rigged Hilbert spaces* or *Gelfand triple*.

1. The fact that $\mathcal{H}_- = (\mathcal{H}_+)'$ means that $\forall l \in (\mathcal{H}_+)' \exists \Phi_l \in \mathcal{H}_-$ such that

$$l(\varphi) = (\varphi, \Phi_l)_0, \quad \forall \varphi \in \mathcal{H}_+,$$

and we say that \mathcal{H}_- is the dual of \mathcal{H}_+ in terms of \mathcal{H}_0 which is reflected in the representation of the functional $l \in (\mathcal{H}_+)'$ using $(\cdot, \cdot)_0$.

2. The classical identification $\mathcal{H}_+ = (\mathcal{H}_+)'$ says that the functional $\forall l \in (\mathcal{H}_+)'$ has representation in terms of the scalar product in \mathcal{H}_+ , i.e.,

$$l(\varphi) = (\varphi, \psi_l)_+, \quad \varphi, \psi_l \in \mathcal{H}_+.$$

Definition A.11 If $O : \mathcal{H}_+ \rightarrow \mathcal{H}_0$ is an Hilbert-Schmidt operator (or quasi nuclear), then the triple (??) is called nuclear.

Example A.12 Let $\mathcal{H}_0 = L^2(\mu) := L^2(X, \mathcal{B}(X), \mu)$ with $\mu(X) < \infty$ and $\mathcal{H}_+ = L^2(p\mu)$ with $p : X \rightarrow [0, \infty]$ a measurable finite μ -a.e. Then the corresponding triple is

$$L^2(p^{-1}\mu) \supset L^2(\mu) \supset L^2(p\mu). \tag{A.7}$$

B Nuclear triples

Nuclear spaces are essentially infinite dimensional. In infinite dimensions closed ball is not pre-compact (or in other words, from a sequence we can not take a convergent subsequence). In the case of nuclear spaces we have the following very important lemma.

Lemma B.1 *Let \mathcal{N} be a nuclear space. Then any bounded set $A \subset \mathcal{N}$ is pre-compact, i.e., after closure it is compact.*

Proof. For the proof see e.g., (? , pag. 55). ■

Let T be an arbitrary set of indices and $(\mathcal{H}_\tau, (\cdot, \cdot)_\tau)_{\tau \in T}$ a family of Hilbert spaces indexed in T . We assume that the family of Hilbert spaces satisfies the following conditions:

(DE) The family $(\mathcal{H}_\tau, (\cdot, \cdot)_\tau)_{\tau \in T}$ is directed by embedding, i.e., $\forall \tau, \tau' \in T, \exists \tau'' \in T$ such that the embedding

$$i_{\tau'', \tau} : \mathcal{H}_{\tau''} \hookrightarrow \mathcal{H}_\tau, \quad i_{\tau'', \tau'} : \mathcal{H}_{\tau''} \hookrightarrow \mathcal{H}_{\tau'}$$

are continuous.

(D) The linear space \mathcal{N} given by

$$\mathcal{N} := \bigcap_{\tau \in T} \mathcal{H}_\tau$$

is dense in each $\mathcal{H}_\tau, \tau \in T$.

In \mathcal{N} we introduce the projective limit topology, i.e., the weakest topology in \mathcal{N} such that the embedding $i_\tau : \mathcal{N} \hookrightarrow \mathcal{H}_\tau, \tau \in T$ are continuous. As a system of base neighborhoods we can take the following collection

$$\Sigma = \{U(\varphi, \varepsilon) | \varphi \in \mathcal{N}, \tau \in T, \varepsilon > 0\}, \quad (\text{B.1})$$

where

$$U(\varphi, \varepsilon) = \{\psi \in \mathcal{N} | |\varphi - \psi|_\tau < \varepsilon\},$$

i.e., the open ball in \mathcal{N} are intersections of \mathcal{N} with open balls in each \mathcal{H}_τ .

Definition B.2 *1. The space $\mathcal{N} = \bigcap_{\tau \in T} \mathcal{H}_\tau$ with the topology generated by the system of neighborhoods Σ in (??) is called projective limit of the family $(\mathcal{H}_\tau)_{\tau \in T}$ and is denoted by*

$$\mathcal{N} = \text{pr} \lim_{\tau \in T} \mathcal{H}_\tau.$$

2. The space $\mathcal{N} = \text{pr} \lim_{\tau \in T} \mathcal{H}_\tau$ is called nuclear iff $\forall \tau \in T \exists \tau' \in T$ such that the embedding operator

$$i_{\tau', \tau} : \mathcal{H}_{\tau'} \hookrightarrow \mathcal{H}_\tau$$

is of Hilbert-Schmidt type (or quasi nuclear).

Remark B.3 1. If the set T is countable, then $\mathcal{N} = \text{pr} \lim_{\tau \in T} \mathcal{H}_\tau$ is called countable Hilbert space.

2. The projective limit of Banach space is constructed in the same way where \mathcal{H}_τ is replaced by a Banach space B_τ .
3. If $T = \mathbb{N}$, then the system of base neighborhoods Σ in (??) is countable, therefore, \mathcal{N} turns into a Fréchet space, i.e., a metrizable locally convex space. In this case the topology in \mathcal{N} is generated by the following metric

$$\rho(\varphi, \psi) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|\varphi - \psi|_n}{1 + |\varphi - \psi|_n}.$$

4. The convergence in \mathcal{N} is equivalent to the convergence in each \mathcal{H}_n , $n \in \mathbb{N}_0$, $(\varphi_j)_{j=1}^{\infty} \subset \mathcal{N}$ converges to $\varphi \in \mathcal{N}$ iff $\forall n \in \mathbb{N}_0 |\varphi_j - \varphi|_n \rightarrow 0, j \rightarrow \infty$.
5. Without lost of generality we can suppose that $\forall \tau \in T \mathcal{H}_\tau \hookrightarrow \mathcal{H}_0$ is a continuous embedding and

$$|\varphi|_0 \leq |\varphi|_\tau, \varphi \in \mathcal{H}_\tau, \tau \in T, 0 \in T.$$

Example B.4 Let $T = \{\tau = (\tau_k)_{k=1}^{\infty}, \tau_k \geq 1\}$, $\mathcal{H}_0 := \ell^2(\mathbb{C})$ be given, i.e.,

$$\mathcal{H}_0 = \left\{ f = (f_k)_{k=1}^{\infty}, f_k \in \mathbb{C} \mid |f|_0^2 = \sum_{k=1}^{\infty} |f_k|^2 < \infty \right\}.$$

For each $\tau \in T$ define $\mathcal{H}_\tau := \ell^2(\tau)$ by

$$\mathcal{H}_\tau = \left\{ \varphi = (\varphi_k)_{k=1}^{\infty}, \varphi_k \in \mathbb{C} \mid |\varphi|_\tau^2 = \sum_{k=1}^{\infty} |\varphi_k|^2 \tau_k < \infty \right\},$$

with the following scalar product

$$(\varphi, \psi)_\tau := \sum_{k=1}^{\infty} \varphi_k \psi_k \tau_k, \quad \varphi, \psi \in \ell^2(\tau).$$

It is not hard to show that the family $(\ell^2(\tau))_{\tau \in T}$ satisfies (DE) and (D) above and, hence, we obtain a nuclear space $\mathcal{N} := \bigcap_{\tau \in T} \ell^2(\tau) = \mathbb{C}_0^\infty$, where \mathbb{C}_0^∞ is the space of infinite sequences with only a finite number of entries different from zero, e.g., $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n, 0, 0, \dots)$.

Remark B.5 *The projective limit of Hilbert spaces appear usually in the following form: given a linear space E and a family of scalar products $\{(\cdot, \cdot)_n | n \in \mathbb{N}_0\}$ compatibles, i.e., if $(f_n)_{n=1}^\infty \subset E$ is a convergent sequence to zero in the norm $|\cdot|_n$ and is Cauchy in $|\cdot|_m$, then also it converges to zero in $|\cdot|_m$. Denote by \mathcal{H}_n the completion of E with respect to $|\cdot|_n$ and $\mathcal{N} = \bigcap_{n=0}^\infty \mathcal{H}_n$. Since $E \subset \mathcal{N}$, then \mathcal{N} is dense in each \mathcal{H}_n , $n \in \mathbb{N}_0$. We can always assume that the family of norms are increasing, i.e.,*

$$|\cdot|_0 \leq |\cdot|_1 \leq \cdots |\cdot|_n \leq |\cdot|_{n+1} \leq \cdots. \quad (\text{B.2})$$

For each $\tau \in T$ we can construct the following rigging of Hilbert spaces

$$\mathcal{H}_{-\tau} \supset \mathcal{H}_0 \supset \mathcal{H}_\tau,$$

where $\mathcal{H}_{-\tau}$ is the dual of \mathcal{H}_τ with respect to \mathcal{H}_0 , cf. Appendix ??.

Theorem B.6 *Let $\mathcal{N} = \bigcap_{\tau \in T} \mathcal{H}_\tau$ be given. Then we have*

$$\mathcal{N}' = \bigcup_{\tau \in T} \mathcal{H}_{-\tau}.$$

Proof. The proof of this theorem can be founded in (?). ■

Remark B.7 *Hence we have constructed the following chain of spaces*

$$\mathcal{N}' = \bigcup_{\tau \in T} \mathcal{H}_{-\tau} \supset \mathcal{H}_{-\tau} \supset \mathcal{H}_0 \supset \mathcal{H}_\tau \supset \mathcal{N} = \bigcap_{\tau \in T} \mathcal{H}_\tau$$

which we abbreviate by

$$\mathcal{N}' \supset \mathcal{H}_0 \supset \mathcal{N}$$

and is called Gelfand triple.

Remark B.8 *The paring between \mathcal{N}' and \mathcal{N} is determined by the scalar product of \mathcal{H}_0 , i.e., if $\Phi \in \mathcal{N}'$ and $\varphi \in \mathcal{N}$, then $\langle \varphi, \Phi \rangle = (\varphi, \Phi)_0 \in \mathbb{C}$. In fact $\Phi \in \mathcal{H}_{-\tau}$, for some $\tau \in T$ and $\varphi \in \mathcal{H}_\tau$, $\forall \tau \in T$ which implies that the paring between \mathcal{N}' and \mathcal{N} is determined by the paring between $\mathcal{H}_{-\tau}$ and \mathcal{H}_τ .*

Example B.9 *In the same conditions as in Example ?? we obtain that $\ell^2(\tau)' = \ell^2(\tau^{-1})$ (here if $\tau = (\tau_k)_{k \in \mathbb{N}}$, then $\tau^{-1} = (\tau_k^{-1})_{k \in \mathbb{N}}$) and as a consequence we have $\mathcal{N}' = \bigcup_{\tau \in T} \ell^2(\tau^{-1}) = \mathbb{C}^\infty$.*

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