

# The Renormalization of Self Intersection Local Times

## I: The Chaos Expansion

Margarida de Faria\*      Custódia Drumond\*  
Ludwig Streit\*<sup>+</sup>

\*CCM, Universidade da Madeira, P 9000 Funchal  
<http://www.uma.pt/>

<sup>+</sup>BiBoS, Univ. Bielefeld, D 33615 Bielefeld

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## 1 Introduction

The intersections of Brownian motion paths have been investigated since the Forties [20]. One can consider intersections of sample paths with themselves or e.g. with other, independent Brownian motions [36], one can study simple [5] or n-fold intersections [6] and one can ask all of these questions for linear, planar, spatial or - in general - d-dimensional Brownian motion: evidently self-intersections become increasingly scarce as the dimension d increases.

Intersection local times of Brownian motion were studied by many authors, see e.g. [1], [3]-[11], [15], [18]-[38]. A more systematic review than we can give here of the subject will be found e.g. in the recent reference [15].

An informal but rather suggestive definition of self intersection local time of Brownian motion  $B$  is in terms of an integral over Dirac's - or Donsker's -  $\delta$ -function

$$L \equiv \int d^2t \delta(B(t_2) - B(t_1)),$$

intended to sum up the contributions from each pair of "times"  $t_1, t_2$  for which the Brownian motion  $B$  is at the same point.

In Edwards' modeling of polymer molecules by Brownian motion paths,  $L$  is used to model the "excluded volume" effect: different parts of the molecule should not be located at the same point in space. As another application, Symanzik introduced  $L$  as a tool for constructive quantum field theory in [31].

A rigorous definition, such as e.g. through a sequence of Gaussians approximating the  $\delta$ -function or in terms of generalized Brownian functionals [10] [30] [33], will lead to increasingly singular objects and will necessitate various "renormalizations" as the dimension  $d$  increases. For  $d > 1$  the expectation will diverge in the limit and must be subtracted [18] [32], clearly  $L$  will then no more be positive. For  $d > 3, 5, 7, \dots$  further subtractions have been proposed [33] that will make  $L$  into a well defined generalized function of Brownian motion.

For  $d = 3$  another renormalization has been constructed by Westwater to make the Gibbs factor  $e^{-g \cdot L}$  of the polymer model well-defined [35], for yet another, recent approach see [1].

Yor, in [38], first suppresses the short time accumulation of self-intersections by the regularization

$$\delta(\vec{B}(t_2) - \vec{B}(t_1)) \rightarrow \delta(\vec{B}(t_2) - \vec{B}(t_1) + \vec{\varepsilon})$$

and shows, again for  $d = 3$ , that a multiplicative renormalization

$$r(\varepsilon)(L_\varepsilon - E(L_\varepsilon))$$

gives rise to another, independent Brownian motion as the weak limit of regularized and subtracted approximations to  $L$ .

In the present note we study similar limits, for arbitrary  $d \geq 3$ , using a Gaussian regularization of the  $\delta$ -function for which the chaos expansion of the corresponding regularized  $L_\varepsilon$  is available [10]. For a suitably subtracted and renormalized local time, each term in this expansion converges in law to a Brownian motion.

We prepare and state these results in the following section 2, in section 3 we give their proofs. In a companion paper [3] we extend these results to the corresponding series.

## 2 Definitions and Main Results

### 2.1 White Noise Analysis and Local Times

We reproduce here some White Noise Analysis concepts as introduced in [10], referring to [14] for a systematic presentation.

Brownian motions  $B_i, i = 1, \dots, d$ , have version in terms of white noise  $\omega_i$  via

$$B_i(t) = \langle \omega_i, 1_{[0,t]} \rangle = \int_0^t \omega_i(s) ds.$$

Hence we consider independent d-tuples of Gaussian white noise  $\omega = (\omega_1, \dots, \omega_d)$  and correspondingly, d-tuples of test functions  $\mathbf{f} = (f_1, \dots, f_d) \in S(R, R^d)$ , and introduce the following notation:

$$\vec{n} = (n_1, \dots, n_d), \quad n = \sum_1^d n_i, \quad \vec{n}! = \prod_1^d n_i!$$

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt f_i^2(t)$$

$$\langle F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} \rangle = \int d^n t F_{\vec{n}}(t_1, \dots, t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \dots, t_n)$$

and similarly for  $\langle : \omega^{\otimes \vec{n}} :, F_{\vec{n}} \rangle$  where for d-tuples of white noise the Wick product  $: \dots :$  [14] generalizes to

$$: \omega^{\otimes \vec{n}} := \bigotimes_{i=1}^d : \omega_i^{\otimes n_i} : .$$

The vector valued white noise  $\omega$  has the characteristic function

$$C(\mathbf{f}) = E(e^{i\langle \omega, \mathbf{f} \rangle}) = \int_{S^*(R, R^d)} d\mu [\omega] e^{i\langle \omega, \mathbf{f} \rangle} = e^{-\frac{1}{2}\langle \mathbf{f}, \mathbf{f} \rangle}, \quad (2.1)$$

where  $\langle \omega, \mathbf{f} \rangle = \sum_{i=1}^d \langle \omega_i, f_i \rangle$  and  $f_i \in S(R, R)$ .

The Hilbert space

$$(L^2) \equiv L^2(d\mu)$$

is canonically isomorphic to the  $d$ -fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq \left( \bigoplus_{k=0}^{\infty} \text{Sym} L^2(R^k, k!d^k t) \right)^{\otimes d} \equiv \mathfrak{F}, \quad (2.2)$$

for the general element of  $(L^2)$  this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} \langle : \omega^{\otimes \vec{n}} : , F_{\vec{n}} \rangle \quad (2.3)$$

with kernel functions  $F$  in  $\mathfrak{F}$ .

It is desirable to introduce regularizations for the intersection local time, with a view towards the construction of well-defined, "renormalized" intersection local times in higher dimensions where the latter do not exist without subtractions. A computationally simple regularization is, for  $\varepsilon > 0$ ,

$$L_{\varepsilon} \equiv \int_0^t dt_2 \int_0^{t_2} dt_1 \delta_{\varepsilon}(\mathbf{B}(t_2) - \mathbf{B}(t_1)),$$

with

$$\delta_{\varepsilon}(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \equiv (2\pi\varepsilon)^{-d/2} e^{-\frac{|\mathbf{B}(t_2) - \mathbf{B}(t_1)|^2}{2\varepsilon}}. \quad (2.4)$$

It has the following chaos expansion, which we quote here only for  $d \geq 3$ :

**Theorem 2.1** [10] *For any  $\varepsilon > 0$ ,  $L_{\varepsilon} - E(L_{\varepsilon})$  has kernel functions  $F \in \mathfrak{F}$  given by*

$$F_{\varepsilon, \vec{n}}(s_1, \dots, s_n) = (-1)^{\frac{n}{2}} \left( \varkappa(\varkappa + 1)(2\pi)^{d/2} 2^{\frac{n}{2}} \frac{\vec{n}!}{2} \right)^{-1}.$$

$$\Theta(u)\Theta(t-v) \cdot ((v-u+\varepsilon)^{-\varkappa} + (t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}) \quad (2.5)$$

if all  $n_i$  are even, and zero otherwise, with  $v(s_1, \dots, s_n) \equiv \max(s_1, \dots, s_n)$ ,  $u(s_1, \dots, s_n) \equiv \min(s_1, \dots, s_n)$ , and  $\varkappa \equiv (n+d)/2 - 2$ .  $\Theta$  is the Heaviside function.

Each kernel function is thus the sum of four terms. The first one gives rise to

**Definition 2.2**

$$M_t(d, \vec{n}, \varepsilon) \equiv \int_{[0,t]^n} d^n s (v - u + \varepsilon)^{-\varkappa} : \omega^{\otimes \vec{n}}(s) : \quad (2.6)$$

$$\begin{aligned} &= \int_{[0,t]^n} d^n s (v - u + \varepsilon)^{-\varkappa} \bigotimes_{i=1}^d : \omega_i(s_1^i) \dots \omega_i(s_{n_i}^i) : \\ &= \sum_{i=1}^d \sum_{m=1}^{n_i} \int_0^t ds_m^i \int_0^{s_m^i} d^{n-1} s (v - u + \varepsilon)^{-\varkappa} \bigotimes_{i=1}^d : \omega_i(s_1^i) \dots \omega_i(s_{n_i}^i) : \\ &= \sum_{i=1}^d n_i \int_0^t dB_i(\tau) \int_0^\tau d^{n-1} s (\tau - u + \varepsilon)^{-\varkappa} : \omega^{\otimes \vec{n} - \vec{\delta}_i}(s) : \\ &= \sum_{k=1}^d \int_0^t dB_k(v) m_k(v) \end{aligned} \quad (2.7)$$

$$\equiv \sum_{k=1}^d M_{k,t} \quad (2.8)$$

The others give

$$N_t(d, \vec{n}, \varepsilon) \equiv \int_{[0,t]^n} d^n s ((t + \varepsilon)^{-\varkappa} - (v + \varepsilon)^{-\varkappa} - (t - u + \varepsilon)^{-\varkappa}) : \omega^{\otimes \vec{n}}(s) :$$

All of the above processes are continuous.

**Definition 2.3** We denote the  $\vec{n}$ <sup>th</sup> order contribution to the regularized local time  $L_\varepsilon$  by

$$K_t(d, \vec{n}, \varepsilon) \equiv (-1)^{\frac{n}{2}} \left( \varkappa(\varkappa + 1)(2\pi)^{d/2} 2^{\frac{n}{2}} \frac{\vec{n}!}{2} \right)^{-1} (M_t(d, \vec{n}, \varepsilon) + N_t(d, \vec{n}, \varepsilon))$$

**Remark 2.4** Our key observation is that, as  $\varepsilon$  goes to zero,  $M$  is more singular than  $N$ , and that it is a Brownian martingale.

## 2.2 The main theorems

**Theorem 2.5** For  $d \geq 3$ , the renormalized  $M$  converge in distribution to independent Brownian motions  $\beta_i$ :

$$(rM_i; i = 1, \dots, d) \xrightarrow[\varepsilon \rightarrow +0]{\mathfrak{L}} \left( \frac{n_i}{n} k_n \beta_i; i = 1, \dots, d \right)$$

with

$$k_n^2 = \vec{n}! \begin{cases} n(n-1) & \text{if } d = 3 \\ \frac{n!(d-4)!}{(n+d-5)!} & \text{if } d > 3 \end{cases} \quad (2.9)$$

if

$$r(\varepsilon) = \begin{cases} |ln\varepsilon|^{-1/2} \\ \varepsilon^{(d-3)/2} \end{cases}$$

**Theorem 2.6** For  $d \geq 3$ , the renormalized  $\vec{n}$  <sup>th</sup> order contributions to the regularized local time  $L_\varepsilon$  converge in distribution to Brownian motions  $\beta$

$$rK(d, \vec{n}, \varepsilon) \xrightarrow[\varepsilon \rightarrow +0]{\mathfrak{L}} c_{\vec{n}, d} \beta$$

with

$$c_{\vec{n}, d}^2 = k_n^2 \left( \varkappa(\varkappa + 1) (2\pi)^{d/2} 2^{\frac{n}{2}} \frac{\vec{n}}{2}! \right)^{-2}$$

### 3 Proofs

**Proposition 3.1**  $M_{k,t}$  are orthogonal Brownian martingales

Proof: Orthogonality is obvious. For the martingale property see [13] and [2], it is a consequence of the fact that the kernel functions of  $M_t$  in (2.6) do not depend on  $t$  (except through the limit of integrations). ■

Their limiting behavior, as  $\varepsilon \rightarrow +0$ , is studied in the following lemma (from now on we shall consider only the situations which require renormalisation, i.e.,  $d \geq 3$ .)

**Lemma 3.2** As  $\varepsilon \rightarrow +0$ ,

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} k_n^2 (t + o(1)) \begin{cases} |ln\varepsilon| & \text{for } d = 3 \\ \varepsilon^{-(d-3)} & \text{for } d > 3 \end{cases}$$

Proof:

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} \overrightarrow{n}! \|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2$$

For  $d > 3$

$$\begin{aligned} \|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2 &= \int_0^t d^n s (v-u+\varepsilon)^{-2\varkappa} \\ &= n(n-1) \int_0^t dv \int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{2\varkappa}} \\ &= n(n-1)\varepsilon^{3-d} \int_0^t dv \int_0^{v/\varepsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \\ &= \varepsilon^{3-d} t \left( \frac{n!(d-4)!}{(n+d-5)!} + o(1) \right) \end{aligned}$$

while for  $d = 3$

$$\|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2 = n(n-1)t |\ln \varepsilon| (1 + o(1))$$

■

To show convergence of these martingales by Theorem VIII.3.11 of [16] we must show convergence of characteristics. Since the processes  $M$  are continuous this reduces to showing convergence in probability of  $\langle rM_i, rM_k \rangle_t$  as  $\varepsilon \rightarrow +0$ . This will be taken care of by proposition 3.7 which we prepare now:

$$\langle rM_i, rM_k \rangle_t = r^2 \delta_{ik} \int_0^t dv (m_i(v))^2$$

and we need to estimate

$$\int_0^t dv (m_i(v))^2 \equiv \sum_{\overrightarrow{k}} \langle : \omega^{\otimes \overrightarrow{k}} : , G_{\overrightarrow{k}}^{(i)} \rangle$$

Let us note that

$$E(\langle rM_i, rM_k \rangle_t) = E(r^2 M_i M_k) = \delta_{ik} r^2 \frac{n_k}{n} \overrightarrow{n}! \|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2$$

$$E(\langle rM_i, rM_k \rangle_t) = \delta_{ik} \frac{n_k}{n} k_n^2 (t + o(1))$$

Next we intend to show that the rhs gives indeed that the rest of  $\langle rM_i, rM_k \rangle_t$  goes to zero. The kernel of the highest order is

$$G_{2\vec{n}-2\vec{\delta}_i}^{(i)}(s_1, s'_1; \dots; s_{n-1}, s'_{n-1}) = \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-\varkappa} (\tau - u' + \varepsilon)^{-\varkappa} \quad (3.1)$$

$v^{(l)}$  and  $u^{(l)}$  are the largest and the smallest of the  $s_i^{(l)}$ , and  $\left(\vec{\delta}_i\right)_k \equiv \delta_{ik}$ .

(For  $n = 2$ :  $u^{(l)} = v^{(l)} = s^{(l)}$ ).

**Remark 3.3** *The integral (3.1) can be calculated in closed form (using e.g. [12] nos. 2.15 and 2.263.4), but one gets a useful approximation by introducing the following auxiliary functions:*

$$\begin{aligned} & H_{2n-2}(s_1, \dots, s_{n-1}; s'_1, \dots, s'_{n-1}; \varepsilon; d) \\ & \equiv (v \vee v' - u \vee u' + \varepsilon)^{-(n+d)/2+3} (v \vee v' - u \wedge u' + \varepsilon)^{-(n+d)/2+2} \end{aligned}$$

where  $v \equiv \max(s_i)$ ,  $u \equiv \min(s_i)$  and  $v' \equiv \max(s'_i)$ ,  $u' \equiv \min(s'_i)$ . These functions majorize the kernel functions

**Lemma 3.4** *For  $n > 2$ , and  $\varkappa > 1$*

$$\begin{aligned} 0 & \leq G_{2\vec{n}-2\vec{\delta}_i}^{(i)}(s_1, s'_1; \dots; s_{n-1}, s'_{n-1}) \\ & \leq \frac{1}{\varkappa - 1} H_{2n-2}(s_1, \dots, s_{n-1}; s'_1, \dots, s'_{n-1}; \varepsilon; d) \end{aligned} \quad (3.2)$$

Proof:

$$\begin{aligned} 0 & \leq \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-\varkappa} (\tau - u' + \varepsilon)^{-\varkappa} \\ & \leq \int_{v \vee v'}^t d\tau (\tau - u \vee u' + \varepsilon)^{-\varkappa} (v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa} \\ & \leq \frac{1}{\varkappa - 1} (v \vee v' - u \vee u' + \varepsilon)^{-\varkappa+1} (v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa} \end{aligned}$$

This estimate suffices to show

**Lemma 3.5** *For  $2\vec{n} - 2\vec{\delta}_i \neq 0$*

$$r^2 G_{2\vec{n}-2\vec{\delta}_i}^{(i)} \rightarrow 0 \text{ in } L^2(R^{2n-2})$$

as  $\varepsilon$  goes to  $+0$ .



Proof: Consider first  $n \geq 3$ . By the above estimate it is sufficient to show that

$$\lim_{\varepsilon \rightarrow +0} r^4 \|H_{2n-2}\|_{L^2([0,t]^{2n-2})}^2 = 0.$$

$$\begin{aligned} \|H_{2n-2}\|_{L^2(\mathbb{R}^{2n-2})}^2 &= \int d^{n-1}s \int d^{n-1}s' (v \vee v' - u \vee u' + \varepsilon)^{2-2\kappa} (v \vee v' - u \wedge u' + \varepsilon)^{-2\kappa} \\ &= c_n \int_0^t dv \int_0^v dv' \int_0^v du \int_0^{v'} du' \\ &\quad \frac{(v-u)^{n-3} (v'-u')^{n-3}}{(v \vee v' - u \vee u' + \varepsilon)^{n+d-6} (v \vee v' - u \wedge u' + \varepsilon)^{n+d-4}} \\ &\leq c_n \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^y dx \int_0^{y'} dx' \\ &\quad \frac{1}{(y-x \vee x' + 1)^{d-3} (y-x \wedge x' + 1)^{d-1}} \end{aligned}$$

and then decompose the x-integration into the following two domains

$$x' < x \text{ and } x < x'$$

In the first case

$$\begin{aligned} I &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3} (y-x'+1)^{d-1}} \\ &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \frac{1}{(y-x'+1)^{d-1}} \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3}} \end{aligned}$$

We first estimate the last integral

$$\int_{x'}^y \frac{1}{(y-x+1)^{d-3}} dx \leq \begin{cases} y-x' & \text{if } d=3 \\ \ln(y-x'+1) & \text{if } d=4 \\ \frac{1}{d-4} & \text{if } d>4 \end{cases}$$

and one finds

$$I = \begin{cases} O(1) & \text{if } d=3 \\ O(\varepsilon^{7-2d}) & \text{if } d>3 \end{cases}$$

Hence, in both cases,  $r^4 I$  vanishes as  $\varepsilon \rightarrow 0$ .

The second case is

$$I = \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{1}{(y-x+1)^{d-1}} \int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \quad (3.3)$$

Estimating the last integrand by 1 we find

$$\int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \leq \frac{(y-x)^2}{2}$$

Substitution of these estimates into the integrals over  $x$  and  $y$  gives for  $d \geq 3$

$$I \leq \text{const.} \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{(y-x)^2}{(y-x+1)^{d-1}} = \begin{cases} O(1) & \text{if } d = 3 \\ O(\varepsilon^{-1} \ln \varepsilon) & \text{if } d = 4 \\ O(\varepsilon^{7-2d}) & \text{if } d > 4 \end{cases}$$

so in that  $r^4 I$  vanishes as  $\varepsilon \rightarrow 0$ . For  $n = 2$  it is sufficient to use the estimate

$$\begin{aligned} 0 &\leq G_2 = \int_0^t d\tau \Theta(\tau - u \vee u') (\tau - u + \varepsilon)^{1-d/2} (\tau - u' + \varepsilon)^{1-d/2} \\ &\leq \int_{u \vee u'}^t d\tau (\tau - u \vee u' + \varepsilon)^{1-d/2} (u \vee u' - u \wedge u' + \varepsilon)^{1-d/2} = H_2(u, u') \end{aligned}$$

and to verify that

$$\lim_{\varepsilon \rightarrow +0} r^4 \|H_2\|_{L^2([0,t]^2)}^2 = 0.$$

■

With this lemma we have established that the highest order term of the (renormalized) quadratic variation goes to zero in quadratic mean for any  $t > 0$ . The kernels  $G_k$  of the other terms are obtained by integrating over pairs of  $s, s'$  such as e.g. in

$$\text{Sym} \int_0^t ds G_{2\left(\vec{n} - \vec{\delta}_i\right)}^{(i)}(s, s; s_2, s'_2; \dots; s_{n-1}, s'_{n-1}),$$

these new functions are in fact also bounded by an expression like (3.1), and hence by (3.2), so that for all  $\varepsilon > 0$ ,

$$\|G_k\| \leq \text{const.} \|H_k\|$$

With this goal we show

**Lemma 3.6** Let  $n \geq 2$  and  $m > 1$ , and let  $F_{n,m}$  be a function symmetric in the variables  $s$  and in the variables  $s'$ , with

$$\begin{aligned} 0 &\leq F_{n,m}(s_1, \dots, s_{n-1}, s_n, s'_1, \dots, s'_{n-1}, s'_n) \\ &\leq c \int_0^t d\tau \Theta \left( \tau - \max_{i \leq n} (s, s') \right) (\tau - \min_{i \leq n} (s_i) + \varepsilon)^{-m} (\tau - \min_{i \leq n} (s'_i) + \varepsilon)^{-m} \end{aligned} \quad (3.4)$$

Then  $\exists c_m < \infty$  such that

$$\begin{aligned} 0 &\leq \int_0^t ds F_{n,m}(s_1, \dots, s_{n-1}, s, s'_1, \dots, s'_{n-1}, s) \\ &\leq c_m \int_0^t d\tau \Theta \left( \tau - \max_{i < n} (s, s') \right) (\tau - \min_{i < n} (s_i) + \varepsilon)^{-m+1/2} (\tau - \min_{i < n} (s'_i) + \varepsilon)^{-m+1/2} \end{aligned} \quad (3.5)$$

Proof: Under the assumption of the lemma

$$\begin{aligned} &\int_0^t ds F_{n,m}(s_1, \dots, s_{n-1}, s, s'_1, \dots, s'_{n-1}, s) \leq \\ &c \int_0^t ds \int_0^t d\tau \Theta \left( \tau - \max_{0 \leq i \leq n-1} (s_i, s'_i, s) \right) \left( \tau - \min_{0 \leq i \leq n-1} (s_i, s) + \varepsilon \right)^{-m} \left( \tau - \min_{0 \leq i \leq n-1} (s'_i, s) + \varepsilon \right)^{-m}. \end{aligned}$$

Using

$$\begin{aligned} u &= \min_{0 \leq i \leq n-1} s_i, & u' &= \min_{0 \leq i \leq n-1} s'_i \\ v &= \max_{0 \leq i \leq n-1} s_i, & v' &= \max_{0 \leq i \leq n-1} s'_i \end{aligned}$$

(For  $n = 2$ :  $u^{(l)} = v^{(l)} = s_1^{(l)}$ ). Assuming without loss of generality that  $u < u'$  we can decompose the  $s$ -integration of our estimate as follows

$$\begin{aligned} &c \int_0^t ds \int_0^t d\tau \Theta (\tau - \max(v, v', s)) (\tau - \min(u, s) + \varepsilon)^{-m} (\tau - \min(u', s) + \varepsilon)^{-m} \\ &= c \int_0^t d\tau \left( \int_0^u + \int_u^{u'} + \int_{u'}^{v \vee v'} + \int_{v \vee v'}^t \right) ds \end{aligned}$$

$$\begin{aligned}
& \cdot \Theta(\tau - \max(v, v', s))(\tau - \min(u, s) + \varepsilon)^{-m} (\tau - \min(u', s) + \varepsilon)^{-m} \\
= & c \int_{v \vee v'}^t d\tau \left( \int_0^u ds (\tau - s + \varepsilon)^{-2m} + \int_u^{u'} ds (\tau - u + \varepsilon)^{-m} (\tau - s + \varepsilon)^{-m} \right. \\
& \left. + \int_{v \vee v'}^{v \vee v'} ds (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} + \int_{v \vee v'}^\tau ds (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} \right).
\end{aligned}$$

We now show that each of the four terms obeys the postulated estimate (3.5)

$$\begin{aligned}
& \int_{v \vee v'}^t d\tau \int_0^u ds (\tau - s + \varepsilon)^{-2m} \leq \frac{1}{2m-1} \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-2m+1} \\
& \leq \frac{1}{2m-1} \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}
\end{aligned}$$

since  $u < u'$  and  $m > 1/2$ .

The second term

$$\begin{aligned}
& \int_{v \vee v'}^t d\tau \int_u^{u'} ds (\tau - u + \varepsilon)^{-m} (\tau - s + \varepsilon)^{-m} \\
& \leq \frac{1}{m-1} \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m+1} \\
& \leq \frac{1}{m-1} \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}.
\end{aligned}$$

using again  $u < u'$ . The third term

$$\begin{aligned}
& \int_{v \vee v'}^t d\tau \int_{u'}^{v \vee v'} ds (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} \\
= & \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (v \vee v' - u')
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - u') \\
&\leq \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}.
\end{aligned}$$

Finally

$$\begin{aligned}
&\int_{v \vee v'}^t d\tau \int_{v \vee v'}^{\tau} ds (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} \\
&= \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - v \vee v') \\
&\leq \int_{v \vee v'}^t d\tau (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - u') \\
&\leq \int_0^t d\tau \Theta(\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}
\end{aligned}$$

■

Combining this Lemma with the previous one we conclude that for all kernel functions  $G_k$  with  $k \geq 2$  arguments

$$\lim_{\varepsilon \rightarrow +0} r^4 \|\text{Sym}G_k\|^2 \leq \lim_{\varepsilon \rightarrow +0} r^4 \|G_k\|^2 \leq \lim_{\varepsilon \rightarrow +0} r^4 \|H_k\|^2 = 0$$

i.e. we have shown

**Proposition 3.7**

$$ms - \lim_{\varepsilon \rightarrow +0} \langle rM_i, rM_k \rangle_t = \delta_{ik} \frac{n_k}{n} k_n^2 t$$

**Proof of Theorem 2.5:** The above limit is clearly (up to constants) the quadratic variation of a Brownian motion. Theorem 2.5 is then a consequence of e.g. Theorem VIII.3.11 in [16], which

in the present case of continuous martingales requires convergence in probability of quadratic variations for a dense set of  $t$ .

To control the remaining terms  $N_t(d, \vec{n}, \varepsilon;)$  in the chaos expansion we observe that

$$\begin{aligned} \|N_t(d, \vec{n}, \varepsilon)\|_{(L^2)}^2 &= \vec{n}! \|(t + \varepsilon)^{-\varkappa} - (v + \varepsilon)^{-\varkappa} - (t - u + \varepsilon)^{-\varkappa}\|_{L^2([0, t]^n)}^2 \\ &\leq \vec{n}! \left( \|(t + \varepsilon)^{-\varkappa}\|_{L^2}^2 + \|(v + \varepsilon)^{-\varkappa}\|_{L^2}^2 + \|(t - u + \varepsilon)^{-\varkappa}\|_{L^2}^2 \right) \end{aligned}$$

The first of these three norms is equal to  $t^n(t + \varepsilon)^{-2\varkappa}$ , i.e.  $O(1)$ . The second one is

$$\begin{aligned} \int_{[0, t]^n} d^n s (v + \varepsilon)^{-2\varkappa} &= n \int_0^t dv \frac{v^{n-1}}{(v + \varepsilon)^{n+d-4}} = n\varepsilon^{4-d} \int_0^{t/\varepsilon} dx \frac{x^{n-1}}{(x + 1)^{n+d-4}} \\ &= \begin{cases} O(1) & \text{for } d = 3 \\ O(\ln \varepsilon) & \text{for } d = 4 \\ O(\varepsilon^{4-d}) & \text{for } d > 4 \end{cases} \end{aligned}$$

which are suppressed by the renormalization

$$r^2(\varepsilon) = \begin{cases} |\ln \varepsilon|^{-1} & \text{for } d = 3 \\ \varepsilon^{d-3} & \text{for } d > 3 \end{cases} .$$

A similar estimate holds for the third term of  $N$ , so that we have shown

**Lemma 3.8**

$$ms - \lim_{\varepsilon \rightarrow +0} r(\varepsilon) N_t(d, \vec{n}, \varepsilon) = 0.$$

In fact the convergence is uniform in any finite t-interval. Next we show

**Lemma 3.9** *The processes  $\{r(\varepsilon)N.(d, \vec{n}, \varepsilon; ) : \varepsilon > 0\}$ ,  $\{r(\varepsilon)M.(d, \vec{n}, \varepsilon; ) : \varepsilon > 0\}$  and their linear combinations are tight.*

Proof: A criterion for tightness of  $M$  (following [17], p.64) is

$$\sup_{\varepsilon > 0} E |rM_t - rM_s|^\alpha \leq C_T (t - s)^{1+\beta}, \quad (3.6)$$

$\forall T > 0$  and  $0 \leq s < t \leq T$  and for some positive constants  $\alpha, \beta$  and  $C_T$ .

As a first step we show

$$\sup_{\varepsilon > 0} E |rM_t - rM_s|^2 \leq C_T (t - s) \quad (3.7)$$

by direct calculation:

$$E |M_t - M_s|^2 = \frac{t-s}{n} \int_s^t dv \int_0^v d^{n-1} s(v-u+\varepsilon)^{-2\alpha}$$

The second integral may be estimated as follows

$$\begin{aligned} \int_0^v d^{n-1} s(v-u+\varepsilon)^{-2\alpha} &= (n-1) \int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{n+d-4}} \\ &\leq (n-1) \varepsilon^{3-d} \int_0^{v/\varepsilon} dx \frac{1}{(x+1)^{d-2}} \\ &= (n-1) \begin{cases} \ln(v+\varepsilon) - \ln \varepsilon & \text{for } d=3 \\ O(\varepsilon^{3-d}) & \text{for } d>3 \end{cases} \end{aligned}$$

Renormalization of this estimate by the factor  $r^2$  makes it bounded on  $[0, T]$ , and the integral over  $v$  gives the desired estimate (3.7), i.e.

$$\|rM_t - rM_s\|_2^2 \leq c_T (t-s) \quad (3.8)$$

Note that the kernel functions of  $K$  are all dominated by those of  $M$ . Hence we have also, possibly with a larger constant  $c_T$ , the estimate

$$\|rK_t - rK_s\|_2^2 \leq c_T (t-s) \quad (3.9)$$

By the hypercontractivity of the Ornstein-Uhlenbeck semigroup (see e.g. [14], p.235), one has for  $n^{\text{th}}$  order white noise monomials  $\varphi \in (L^2)$ , and any  $\alpha > 2$

$$\|\varphi\|_\alpha^\alpha \leq c_{n,\alpha} \|\varphi\|_2^\alpha$$

For  $\varphi = rK_t - rK_s$ , and using the above estimate for the 2-norm, we get

$$E |rK_t - rK_s|^\alpha \leq C_T (t-s)^{\alpha/2}$$

as required to ensure tightness. The estimates for  $rN$  etc. are of the same kind. ■

**Proof of Theorem 2.6:** We need to consider

$$rK = rM + rN$$

knowing that, as  $\varepsilon \rightarrow +0$ , the  $rK$  are tight by Lemma 3.9, the  $rM$  converge in law, and the  $rN_t$  go to zero in mean square. The latter two facts are sufficient, via the Cramér-Wold device (see e.g. [17] p.61), for finite dimensional convergence of  $rK$ ; tightness then implies convergence in law.

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