# Complex Gaussian Analysis and the Bargmann-Segal Space

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**Abstract**

A definition of the Bargmann-Segal space in complex Gaussian analysis is given and it is proved that there exists a canonical isomorphism to a complex Gaussian space. In this representation the Bargmann-Segal space is used in order to characterize certain spaces of test and generalized functions.

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# 1 Introduction

Gaussian analysis and in particular white noise analysis have been developed into a useful tool in infinite dimensional analysis. Their applications are found in quantum physics and in the theory of stochastic systems.

In finite dimensional analysis the Lebesgue measure is the basic measure in integration theory. It is a well-known fact that no measure exists in infinite dimensions which has the properties of the Lebesgue measure (first of all the translations-invariance property). However, there exists an important and useful class of measures on infinite dimensional spaces, the so called Gaussian measures. Gaussian measures have many properties similar to those of the Lebesgue measure except for translations-invariance. In addition, stochastic systems are often considered to be Gaussian or can be described by using Gaussian ones, hence Gaussian measures are a natural choice. For a detailed exposition of the theory and for examples of applications we refer to recent monographs, see [BeKo88], [HiHi76], [Hi75], [Hi80], [HKPS93], [Kuo96] and [Ob94].

In this paper we present constructions in complex Gaussian analysis, i.e., in the theory of Gaussian measures on infinite dimensional complex spaces. This theory has already been introduced and discussed in [Hi71], [Itô52], [Jo81], [Ko80a] and [Ko80b].

The transition from real to complex Gaussian analysis shows that certain structures of complex Gaussian analysis are simpler than the corresponding structures of real Gaussian analysis. This simplification is based on the following fact: consider a Gaussian measure on the complex plane and the corresponding space of complex valued square-integrable functions. It is well-known that in this case the monomials of a different order are orthogonal with respect to the usual scalar product. In the real case, however, the monomials of a different order are only linear independent. To obtain an orthogonal system an additional orthogonalization procedure is needed. The result of this procedure are the Hermite polynomials. In a generalization to infinite dimensions we have the Wick monomials in the real and the holomorphic monomials in the complex case. Handling smooth holomorphic monomials is easier than handling Wick monomials. Moreover, smooth holomorphic monomials are defined independently of a Gaussian measure. Consequently, there are connections and formulas in complex Gaussian analysis which are absent from real Gaussian analysis.

Now we give a short survey of this paper. In Section 3 we present a re-

view on the results from real and complex Gaussian analysis. This review is as detailed as necessary to define the Bargmann-Segal space within the framework of Gaussian analysis. The Bargmann-Segal space has originally been introduced by V. Bargmann, see [Ba61], [Ba62a] and [Ba62b], and I.E. Segal, see [Se60], [Se62] and [Se78]. Our definition is different in the sense that we use a measure on an infinite dimensional complex co-nuclear space instead of a sequence of measures on finite dimensional subspaces of a complex separable Hilbert space. The main result of the first part which at the same time is the underlying structure for the second part of this paper is given in Theorem 4.3. There we prove that there is a canonical isomorphism between the Bargmann-Segal space and a strict subspace of $L^2(\nu)$, the space of complex valued functions which are square-integrable with respect to the Gaussian measure $\nu$. Thus, this representation of the Bargmann-Segal space is one as a space of square-integrable functions with respect to a Gaussian measure on an infinite dimensional complex space. Furthermore, this implies a natural unitary isomorphism to the symmetric Fock space and to $L^2(\mu)$, the canonical space of complex valued square-integrable functions in real Gaussian analysis, see the diagram on page 18. The latter isomorphism is the Segal-Bargmann transform or coherent state transform. In this context we wish to point out the work by P. Krée [Kr74], [Kr75], [Kr79] and [Kr88]. For a recent paper on the Segal-Bargmann transform which contains some of the history, an introductionary appendix, and a large body of pertinent references, we refer the interested reader also to [Ha94]. Moreover, we wish to mention the forthcoming paper [GrMa] which contains a detailed account of the history of this transform. In Gaussian analysis and in particular white noise analysis this transform is called $S$-transform, for a detailed discussion of the connection between the Segal-Bargmann transform and the $S$-transform we refer to [KLPSW96].

The aim of the second part of this note is to characterize certain spaces of test and generalized functions which can be constructed in Gaussian analysis, see Section 5.

Y.-L. Lee has used the Bargmann-Segal space in order to characterize so called U-functionals of generalized functions, see [Lee91]. U-functionals are entire functions which, in addition, satisfy a certain growth property. The spaces of generalized functions which we consider cannot in general be characterized by a growth property of their $S$-transform in a satisfying way. One of these spaces is the generalized function space $\mathcal{G}'$ which has been introduced by J. Potthoff and M. Timpel in [PoTi94]. There the authors

have worked out a growth condition which is *sufficient* for functions to be the $S$-transform of generalized functions from $\mathcal{G}'$.

Our representation of the Bargmann-Segal space, however, enables us to develop a new kind of characterization. We also use the analyticity of the $S$-transform of generalized functions as in the characterizations given before. But instead of a growth property in our characterization we have to check integrability of the $S$-transform of generalized functions.

In Section 6 we prepare the proofs of the characterization theorems in Section 7. The main idea is to generalize the concept of the Bargmann-Segal space and to define families of them. The connections of these spaces provide us with tools for the characterization Theorems 7.1 and 7.3.

In a next step we examine the test and generalized functions from $\mathcal{G}$ and $\mathcal{G}'$, respectively, see [PoTi94]. Since in [BePo95] F.E. Benth and J. Potthoff have generalized basic concepts from the stochastic processes such as conditional expectation, martingales, sub- (super-) martingales etc. to mappings from the real line into $\mathcal{G}'$, the space $\mathcal{G}'$ is of interest from the probabilistic point of view. Generalized stochastic processes occur in the theory of stochastic partial differential equations, see [HØUZ96]. Solving such equations often leads to solutions in spaces of generalized stochastic processes.

A characterization of $\mathcal{G}$ and $\mathcal{G}'$ has not been given before. They can be found in Corollary 7.2 and 7.4.

Finally, we show how to apply the characterization theorems. As examples we consider Donsker's delta function and the Feynman integrand of the free particle propagator.

# 2 Preliminaries

## 2.1 Some facts on nuclear triples

We start with a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. For a given separable nuclear space $\mathcal{N}$ (in the sense of Grothendieck) densely topologically embedded in $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

The dual pairing $\langle\cdot,\cdot\rangle$ of $\mathcal{N}$ and $\mathcal{N}'$ is realized then as an extension of the

inner product in $\mathcal{H}$:

$$\langle f, \xi \rangle \;\; = \;\; (f, \xi), \quad f \in \mathcal{H}, \xi \in \mathcal{N}.$$

Instead of reproducing the abstract definition of nuclear spaces, see e.g. [Pi69] and [Sc71], we give a complete and convenient characterization in terms of projective limits of Hilbert spaces which has also been proved in [Pi69] and in [Sc71].

**Theorem 2.1** *The nuclear Fréchet space can be represented as*

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p,$$

*where* $\{(\mathcal{H}_p, (\cdot, \cdot)_p), p \in \mathbb{N}\}$ *is a family of Hilbert spaces such that for all* $p_1, p_2 \in \mathbb{N}$ *there exists* $p \in \mathbb{N}$ *such that the embeddings* $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ *and* $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ *are Hilbert-Schmidt. The topology of* $\mathcal{N}$ *is given by the projective limit topology, i.e., the coarsest topology on* $\mathcal{N}$ *such that the canonical embeddings* $\mathcal{N} \hookrightarrow \mathcal{H}_p$ *are continuous for all* $p \in \mathbb{N}$.

The Hilbertian norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$. Without loss of generality we always suppose that $\forall p \in \mathbb{N}$, $\forall \xi \in \mathcal{N}$: $|\xi| \leq |\xi|_p$, and that the system of norms is ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ if $p \leq q$. Applying the general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p}$$

which is provided with the inductive limit topology $\tau_{ind}$ by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}_p', p \in \mathbb{N}\}$. The inductive limit topology (with respect to this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$.

Additionally, we introduce the notion of symmetric tensor power of a nuclear space. The simplest way to do this is to start from usual symmetric tensor powers $\mathcal{H}_p^{\hat{\otimes} n}, n \in \mathbb{N}$, of Hilbert spaces. Since there is no danger of confusion we preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_p^{\hat{\otimes} n}$ and $\mathcal{H}_{-p}^{\hat{\otimes} n}$ respectively. Using the definition

$$\mathcal{N}^{\hat{\otimes} n} := \operatorname*{prlim}_{p \in \mathbb{N}} \mathcal{H}_p^{\hat{\otimes} n}$$

one can prove, see e.g [Pi69] and [Sc71], that $\mathcal{N}^{\hat{\otimes}n}$ is a nuclear space which is called the $n^{th}$ symmetric tensor power of $\mathcal{N}$. The dual space $\mathcal{N}'^{\hat{\otimes}n}$ can be written as

$$\mathcal{N}'^{\hat{\otimes}n} = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\hat{\otimes}n}.$$

All the results quoted above also hold for complex spaces, in particular for the complexified space $\mathcal{N}_{\mathbb{C}}$ where, by definition, an element of $\theta \in \mathcal{N}_{\mathbb{C}}$ decomposes into $\theta = \xi + i\eta$, $\xi, \eta \in \mathcal{N}$. The corresponding complexified Hilbert spaces $\mathcal{H}_{p,\mathbb{C}}$ have the inner product:

$$(f_1, f_2)_{p,\mathbb{C}} := (\overline{f_1}, f_2)_p := (g_1, g_2)_p + (h_1, h_2)_p - i(h_1, g_2)_p + i(g_1, h_2)_p,$$

with $f_1, f_2 \in \mathcal{H}_{p,\mathbb{C}}$, $f_1 = g_1 + ih_1$, $f_2 = g_2 + ih_2$, $g_1, g_2, h_1, h_2 \in \mathcal{H}_p$. Thus, we have introduced the nuclear triple

$$\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n} \subset \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n} \subset \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}.$$

In addition, we introduce the symmetric (or Boson) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$$

with the convention $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}0} := \mathbb{C}$ and the Hilbertian norm

$$\| \vec{f} \|_{\Gamma(\mathcal{H})} := \sum_{n=0}^{\infty} n! |f^{(n)}|^2, \quad \vec{f} = (f^{(n)}, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \in \Gamma(\mathcal{H}).$$

## 2.2 Holomorphy on locally convex spaces

Let $\mathcal{E}$ be a locally convex topological vector space (over the complex field $\mathbb{C}$), $\mathcal{L}(\mathcal{E}^n)$ the space of n-linear forms from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric n-linear forms. Also let $\mathcal{P}^n(\mathcal{E})$ denote the n-homogeneous polynomials on $\mathcal{E}$. Obviously, there is a linear bijection $\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \hat{A} \in \mathcal{P}^n(\mathcal{E})$.

Now let $\mathcal{U} \subset \mathcal{E}$ be open and consider a function $F : \mathcal{U} \mapsto \mathbb{C}$. $F$ is said to be G-holomorphic (Gâteaux-holomorphic) if for all $\theta_0 \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C}$: $\lambda \mapsto F(\theta_0 + \lambda\theta)$ is holomorphic in some

neighbourhood of zero in $\mathbb{C}$. If $F$ is G-holomorphic then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{d^n F}(\eta)$ such that

$$F(\eta + \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F}(\eta)(\theta)$$

for all $\theta$ from some open neighbourhood $\mathcal{V}$ of zero. $F$ is said to be holomorphic, if for all $\eta \in \mathcal{U}$ there exists an open neighbourhood $\mathcal{V}$ of zero such that $\sum_{n=0}^{\infty} \frac{1}{n!}\widehat{d^n F}(\eta)(\theta)$ converges uniformly in $\theta \in \mathcal{V}$ to a continuous function. We say that $F$ is holomorphic at $\theta_0$ if there is an open set $\mathcal{U}$ containing $\theta_0$ such that $F$ is holomorphic on $\mathcal{U}$. Furthermore, we say that $F$ is entire if it is holomorphic on $\mathcal{E}$.

The conjunction of both phenomena leads us to the following proposition which can be found in e.g. [Di81].

**Proposition 2.2** *$F$ is holomorphic if and only if it is G-holomorphic and locally bounded.*

# 3 Gaussian analysis

## 3.1 Gaussian analysis on real spaces

In order to introduce a probability measure on the vector space $\mathcal{N}'$, we consider the $\sigma$-algebra $\mathcal{C}_\sigma(\mathcal{N}')$ generated by cylinder sets

$$C_{F_1,\dots,F_n}^{\xi_1,\dots,\xi_n} = \Big\{ x \in \mathcal{N}' \Big| \langle x, \xi_1 \rangle \in F_1, \dots, \langle x, \xi_n \rangle \in F_n \Big\}, \quad \xi_i \in \mathcal{N}, F_i \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma$-algebra on $\mathbb{R}$.

The canonical Gaussian measure $\mu$ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ is given by its characteristic function

$$\int_{\mathcal{N}'} \exp(i\langle x, \xi \rangle)\, d\mu(x) = \exp(-\tfrac{1}{2}|\xi|^2), \quad \xi \in \mathcal{N},$$

via Minlos' theorem, see e.g. [BeKo88], [Hi80] and [HKPS93]. $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ is the basic probability space throughout this paper.

The integral $\int_{\mathcal{N}'} f(x)\, d\mu(x)$ of a measurable function $f$ defined on $\mathcal{N}'$ is called the expectation of $f$ if $f$ is integrable, i.e., the integral $\int_{\mathcal{N}'} |f(x)|\, d\mu(x)$

is finite. The space of the integrable functions is denoted by $L^1(\mu) := L^1(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$ and the expectation of a function $f$ is denoted by $\mathbb{E}(f)$.

The central space in our setup is the space of complex valued functions which are square-integrable with respect to this measure:

$$L^2(\mu) := L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu).$$

Firstly, we construct an orthogonal system in $L^2(\mu)$. For this we need the definition of Wick powers. For any $\xi \in \mathcal{N}$ the function

$$\begin{aligned} :\exp(\langle x, \xi\rangle): &:= \frac{\exp(\langle x, \xi\rangle)}{\mathbb{E}(\exp(\langle \cdot, \xi\rangle))} \\ &= \exp(\langle x, \xi\rangle - \tfrac{1}{2}(\xi, \xi)), \quad x \in \mathcal{N}', \end{aligned}$$

is called Wick exponential. In addition, we consider its Taylor expansion, see e.g. [BeKo88] or [HKPS93],

$$:\exp(\langle x, \xi\rangle): \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \langle h_n(x), \xi^{\otimes n}\rangle, \quad \xi \in \mathcal{N}.$$

The map $h_n : \mathcal{N}' \mapsto \mathcal{N}'^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, is called the n-th Wick power of $x \in \mathcal{N}'$ and is usually denoted as

$$h_n(x) \;:=\; :x^{\otimes n}:$$

($\mathcal{N}^{\hat{\otimes} 0} := \mathbb{R}$, $\langle :x^{\otimes 0}:, \xi^{\otimes 0}\rangle := 1$), see e.g. [BeKo88] or [HKPS93].

For any $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, we define the smooth Wick monomial of order $n$ corresponding to the kernel $\varphi^{(n)}$ as follows:

$$I(\varphi^{(n)})(x) \;:=\; \langle :x^{\otimes n}:, \varphi^{(n)}\rangle, \quad x \in \mathcal{N}', n \in \mathbb{N}_0.$$

Smooth Wick monomials of different order are orthogonal with respect to the inner product

$$(f, h)_{L^2(\mu)} \;:=\; \int_{\mathcal{N}'} \overline{f(x)} g(x)\, d\mu(x), \quad f, g \in L^2(\mu).$$

Furthermore, we can construct Wick monomials $I(f^{(n)}) = \langle :x^{\otimes n}:, f^{(n)}\rangle$ with kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ in the sense of measurable functions by using an approximation. More precisely, for any sequence $(\varphi_j^{(n)})_{j\in\mathbb{N}} \subset \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ which

converges to $f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ we have the convergence of $I(\varphi^{(n)})$ to $I(f^{(n)})$ in any $L^p(\mu), p \geq 1$, see [BeKo88] (note that $I(\varphi^{(n)}) = \langle :x^{\otimes n}:, \varphi^{(n)} \rangle$ is only a formal notation for the monomial introduced above). For Wick monomials associated to the kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ and $h^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}m}$, $n, m \in \mathbb{N}_0$, we have the following orthogonality property:

$$\begin{aligned} \Big(I(f^{(n)}), I(h^{(m)})\Big)_{L^2(\mu)} &= \int_{\mathcal{N}'} \overline{\langle :x^{\otimes n}:, f^{(n)} \rangle} \langle :x^{\otimes m}:, h^{(m)} \rangle \, d\mu(x) \\ &= \delta_{n,m} \, n! \, (\overline{f^{(n)}}, h^{(n)}) \end{aligned}$$

($\delta_{n,m}$ is the Kronecker delta), see e.g. [BeKo88] or [HKPS93].

This means that the Wick monomials $I(f^{(n)}), f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$, are orthogonal in $L^2(\mu)$ for different $n \in \mathbb{N}_0$ and, consequently, the spaces $\mathcal{L}_n(\mu)$ generated by $I(f^{(n)}), n \in \mathbb{N}_0$, are orthogonal, too. Obviously, we have

$$\mathcal{L}_n(\mu) = \{\langle h_n(\cdot), f^{(n)} \rangle | f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}\}.$$

Consider the space $\mathcal{P}(\mathcal{N}')$ of smooth continuous polynomials on $\mathcal{N}'$:

$$\mathcal{P}(\mathcal{N}') := \left\{ \varphi \middle| \varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \tilde{\varphi}^{(n)} \rangle, \tilde{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}.$$

It is well-known that any $\varphi \in \mathcal{P}(\mathcal{N}')$ can be written as a smooth Wick polynomial, i.e.,

$$\mathcal{P}(\mathcal{N}') = \left\{ \varphi \middle| \varphi(x) = \sum_{n=0}^{N} \langle :x^{\otimes n}:, \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\},$$

see e.g. [BeKo88] or [HKPS93], and that $\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu)$. From this we can infer that

$$L^2(\mu) = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{L}_n(\mu).$$

Hence, for any $f \in L^2(\mu)$ we have the Itô-Segal-Wiener chaos decomposition, see e.g. [BeKo88] and [HKPS93]:

$$f(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle, \quad f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}.$$

Its norm can be presented as

$$\| f \|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \, (\overline{f^{(n)}}, f^{(n)}).$$

## 3.2 Gaussian analysis on complex spaces

In this subsection we introduce a Gaussian measure on the complexification $\mathcal{N}'_{\mathbb{C}}$. Let $K : \mathcal{H} \mapsto \mathcal{H}$ be an operator such that $C_K(\xi) := \exp(-\frac{1}{2}(K\xi, \xi)$, $\xi \in \mathcal{N}$ is a characteristic function. Then we denote by $\mu_K$ the corresponding Gaussian measure. This notation reminds us that the measure $\mu_K$ has the covariance operator $K$ in the space $\mathcal{H}$. In the following definition we use the measure $\mu_{\frac{1}{2}} := \mu_{\frac{1}{2}\mathbb{1}}$ ($\mathbb{1}$ denotes the identity operator on $\mathcal{H}$) to define a Gaussian measure on the measurable space $(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}))$, see e.g. [Hi71] and [Ko80a].

**Definition 3.1** *Let $z = x + iy \in \mathcal{N}'_{\mathbb{C}}$, $x, y \in \mathcal{N}'$, then we define the measure*

$$d\nu(z) := d\mu_{\frac{1}{2}}(x) \times d\mu_{\frac{1}{2}}(y)$$

*on the measurable space $(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}))$.*

For $\theta \in \mathcal{N}_{\mathbb{C}}$ it is easy to calculate that

$$\int_{\mathcal{N}'_{\mathbb{C}}} \exp(iRe\langle\overline{z}, \theta\rangle)\, d\nu(z) = \exp(-\tfrac{1}{4}(\overline{\theta}, \theta)).$$

Thus, our definition is equivalent to the definition given in [Sh91].

Again we can consider the space of square-integrable complex valued functions $L^2(\nu) := L^2(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}), \nu)$.

For any $\varphi^{(n)} \in {\mathcal{N}_{\mathbb{C}}}^{\hat{\otimes} n}$ the bilinear pairing with $z^{\otimes n} \in {\mathcal{N}'_{\mathbb{C}}}^{\hat{\otimes} n}$ is defined correctly, hence we are able to introduce the smooth holomorphic monomial of order $n \in \mathbb{N}_0$ corresponding to the kernel $\varphi^{(n)} \in {\mathcal{N}_{\mathbb{C}}}^{\hat{\otimes} n}$ as follows:

$$M(\varphi^{(n)})(z) := \langle z^{\otimes n}, \varphi^{(n)}\rangle, \quad n \in \mathbb{N}_0, z \in \mathcal{N}'_{\mathbb{C}},$$

see e.g. [Ko80a] and [BeKo88]. As in the real case these monomials can also be constructed for kernels $g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ in the sense of measurable functions. Furthermore, for holomorphic monomials associated to kernels $g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ and $h^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} m}$, $n, m \in \mathbb{N}_0$, we have the following orthogonality property:

$$\begin{aligned}\Big(M(g^{(n)}), M(h^{(m)})\Big)_{L^2(\nu)} &:= \int_{\mathcal{N}'_{\mathbb{C}}} \overline{\langle z^{\otimes n}, g^{(n)}\rangle}\langle z^{\otimes m}, h^{(m)}\rangle\, d\nu(z) \\ &= \delta_{n,m}\, n!\, (\overline{g^{(n)}}, h^{(n)}), \qquad (1)\end{aligned}$$

see [Ko80a] or [BeKo88].

Finally, we give a class of positive exponential functions of $L^1(\nu)$. In applications they are useful as dominating functions in order to show that a function is square-integrable. Consider a symmetric trace class operator $A$ on $\mathcal{H}$ such that $0 \leq A < 1$. These properties are sufficient for the measurability of the function $\langle \overline{z}, Az \rangle$, $z \in \mathcal{N}'_{\mathbb{C}}$, with respect to the $\sigma$-algebra $\mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}})$, see e.g. [BeKo88]. Furthermore, we obtain by direct computation the following lemma.

**Lemma 3.2**

$$\int_{\mathcal{N}'_{\mathbb{C}}} \exp(\langle \overline{z}, Az \rangle)\, d\nu(z) = (det(\mathbb{1} - A))^{-1} < \infty.$$

# 4 The Bargmann-Segal space

Let $\mathbb{P}$ be the set of all orthogonal projections on $\mathcal{H}$ such that for any $P \in \mathbb{P}$: $\mathcal{R}(P) \subset \mathcal{N}$ ($\mathcal{R}(P)$ is the range of the operator $P$) and $d_P := \dim(\mathcal{R}(P)) < \infty$. For $h \in \mathcal{H}$ any projection $P \in \mathbb{P}$ can be represented as

$$Ph = \sum_{i=1}^{d_P} \langle h, e_i \rangle e_i,$$

where $\{e_i, 1 \leq i \leq d_P\} \subset \mathcal{N}$ is an orthonormal basis (with respect to the scalar product given on $\mathcal{H}$) of $\mathcal{R}(P)$. Now it is easy to see that for any $P \in \mathbb{P}$ the following extension to $\mathcal{N}'$ is well-defined:

$$\mathcal{N}' \ni x \mapsto Px = \sum_{i=1}^{d_P} \langle x, e_i \rangle e_i \quad \in \mathcal{N}.$$

And, obviously, such $P$ can be extended to the complexification $\mathcal{N}'_{\mathbb{C}}$.

Now we are prepared to define the Bargmann-Segal space, see [Ba61], [Ba62a], [Ba62b], [Se60], [Se62] and [Se78].

**Definition 4.1 (Bargmann-Segal space)** *A complex valued function $g$ defined on $\mathcal{H}_{\mathbb{C}}$ is an element of the Bargmann-Segal space $E^2(\nu)$ if it satisfies the following conditions:*
*1. $g$ is entire on $\mathcal{H}_{\mathbb{C}}$, and*
*2.*

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |g(Pz)|^2 \, d\nu(z) < \infty.$$

**Remark 4.2** *This concept essentially goes back to I.E. Segal, see [Se60], [Se62] and [Se78]. The difference between his and our definition is that he used a sequence of measures on finite dimensional subspaces of a complex separable Hilbert space whereas we use a measure on an infinite dimensional complex co-nuclear space. If $\mathcal{H} = \mathbb{R}^n$, $E^2(\nu)$ is reduced to the Bargmann space of analytic functions introduced in [Ba61], see also [Ba62a] and [Ba62b].*

Now we will show that there is a canonical isomorphism between $E^2(\nu)$ and a strict subspace of $L^2(\nu)$. In order to construct this space we define the set of smooth holomorphic polynomials:

$$\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}}) = \left\{ \varphi \middle| \varphi(z) = \sum_{n=0}^{N} \langle z^{\otimes n}, \varphi^{(n)} \rangle, \varphi^{(n)} \in {\mathcal{N}_{\mathbb{C}}}^{\hat{\otimes} n}, z \in \mathcal{N}'_{\mathbb{C}}, N \in \mathbb{N}_0 \right\}.$$

In accordance with (1) it is obvious that $\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}}) \subset L^2(\nu)$. We denote the closure of $\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})$ in $L^2(\nu)$ as $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$. Since the anti-holomorphic polynomials are not in $\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})$ we know that $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ it is a strict subspace of $L^2(\nu)$. Before we prove the isomorphism between $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ and the Bargmann-Segal space we describe how the isomorphic mapping acts between these spaces.

The functions $g \in E^2(\nu)$ and $f \in \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ are defined on different spaces. So we choose a mapping which restrict $f \in \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ to a function $\hat{f} : \mathcal{H}_{\mathbb{C}} \to \mathbb{C}$ such that $\hat{f} \in E^2(\nu)$. But the elements of $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ are equivalence classes of functions and $\mathcal{H}_{\mathbb{C}}$ is a space of measure zero with respect to $\nu$. That means that we need more structure in order to have a unique restriction. We get this additional structure from the following explicit expression of $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$:

$$\begin{aligned}\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)} \;=\; \Big\{ g \in L^2(\nu) \Big| g(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, g^{(n)} \rangle, \qquad (2)\\ \| g \|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)}) < \infty, \, g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n} \Big\}.\end{aligned}$$

To obtain the expression of $\| g \|^2_{L^2(\nu)}$ we used (1) and the Parseval relation. This expression allows a natural restriction of $f \in \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ to $\mathcal{H}_{\mathbb{C}}$. Although $\mathcal{H}_{\mathbb{C}}$ is a set of measure zero with respect to $\nu$, there exists for $g \in E^2(\nu)$ a "unique" (unique with respect to the norm in $L^2(\nu)$) extension $\hat{g} : \mathcal{N}'_{\mathbb{C}} \to \mathbb{C}$ such that $\hat{g} \in \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$. For we get the necessary additional structure from the holomorphy and integrability of the functions $g \in E^2(\nu)$ and the explicit expression of elements from $\overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ given in (2).

**Theorem 4.3** *There is a canonical isomorphism between the Bargmann-Segal space $E^2(\nu)$ and the closure of $\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})$ in the space $L^2(\nu)$, i.e.,*

$$E^2(\nu) \;\simeq\; \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}.$$

**Remark 4.4** *This theorem implies a natural unitary isomorphism of $E^2(\nu)$ to the symmetric Fock space and to $L^2(\mu)$, see the diagram on page 18. The latter isomorphism is the $S$-transform which we discussed in the introduction, see also the references given there.*

In the proof of Theorem 4.3 we need the following lemma which is easy to prove.

**Lemma 4.5** *If there exists $S \in \mathbb{R}$ such that for a given family of kernels $g^{(n)} \in {\mathcal{N}'_{\mathbb{C}}}^{\hat{\otimes} n}$ and for all $P \in \mathbb{P}$ we have the estimate $\sum_{n=0}^{\infty} n! \, |P^{\otimes n} g^{(n)}|^2 < S$, then*
*1. $g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$, and*
*2. $\sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)}) \leq S$.*

**Proof of Theorem 4.3:** $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)} \to E^2(\nu)$:

Property 1 of Definition 4.1: The expression (2) of $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ gives us an explicit representation of a function $g \in \overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$. Hence, we can consider their restriction to $z \in \mathcal{H}_{\mathbb{C}}$ given by

$$g(z) = \sum_{n=0}^{\infty} (z^{\otimes n}, g^{(n)}), \quad g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}.$$

This restriction is a pointwise defined function and we have for $z \in \mathcal{H}_{\mathbb{C}}$

$$\begin{aligned} |g(z)| &\leq \sum_{n=0}^{\infty} |(z^{\otimes n}, g^{(n)})| \\ &\leq \sum_{n=0}^{\infty} |z|^n |g^{(n)}| \\ &\leq \sqrt{\sum_{n=0}^{\infty} n! |g^{(n)}|^2} \sqrt{\sum_{m=0}^{\infty} \frac{1}{m!} |z|^{2m}} \\ &= \| \, g \, \|_{L^2(\nu)} \exp(\tfrac{1}{2}|z|^2). \end{aligned}$$

From this estimate we can conclude that the sum converges uniformly on any bounded set of $\mathcal{H}_{\mathbb{C}}$. Thus, $g$ is entire on $\mathcal{H}_{\mathbb{C}}$, see e.g. [Di81]. Hence, property 1 of Definition 4.1 is fulfilled.

Property 2 of Definition 4.1: For every $g \in \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ we have the representation

$$g(z) = \sum_{n=0}^{\infty} (z^{\otimes n}, g^{(n)})$$

and know that

$$\| g \|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)}) < \infty.$$

Therefore, we have the following estimate:

$$\begin{aligned}
& \sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |g(Pz)|^2 \, d\nu(z) \\
= & \sup_{P\in\mathbb{P}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\mathcal{N}'_{\mathbb{C}}} \overline{\langle (Pz)^{\otimes n}, g^{(n)} \rangle} \langle (Pz)^{\otimes m}, g^{(m)} \rangle \, d\nu(z) \\
= & \sup_{P\in\mathbb{P}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\mathcal{N}'_{\mathbb{C}}} \overline{\langle z^{\otimes n}, P^{\otimes n}(g^{(n)}) \rangle} \langle z^{\otimes m}, P^{\otimes m}(g^{(m)}) \rangle \, d\nu(z) \\
= & \sup_{P\in\mathbb{P}} \sum_{n=0}^{\infty} n! \, (\overline{P^{\otimes n}(g^{(n)})}, P^{\otimes n}(g^{(n)})) \\
\leq & \sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)}) < \infty.
\end{aligned}$$

This is the estimate required in property 2 of Definition 4.1.

$E^2(\nu) \to \overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$: Assuming that $g \in E^2(\nu)$ we know that $g$ is entire on $\mathcal{H}_{\mathbb{C}}$; hence, for $z \in \mathcal{H}_{\mathbb{C}}$ it has the Taylor expansion

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n g}(0)(z),$$

see Subsection 2.2. The polynomials $\widehat{d^n g}(0)$ are continuous functions on $\mathcal{H}_{\mathbb{C}}$ and, therefore, in particular continuous functions on $\mathcal{N}_{\mathbb{C}}$ with respect to the projective limit topology, see e.g. [BeKo88] and [HKPS93]. In this situation we can apply the kernel theorem, see e.g. [BeKo88] or [ReSi72], and find for all $z \in \mathcal{N}_{\mathbb{C}}$:

$$g(z) = \sum_{n=0}^{\infty} \langle g^{(n)}, z^{\otimes n} \rangle, \quad g^{(n)} \in {\mathcal{N}'_{\mathbb{C}}}^{\hat{\otimes} n}. \tag{3}$$

So we know that for all $P \in \mathbb{P}$, $z \in \mathcal{N}'_{\mathbb{C}}$:

$$g(Pz) = \sum_{n=0}^{\infty} \langle g^{(n)}, (Pz)^{\otimes n} \rangle$$

$$= \sum_{n=0}^{\infty} \langle z^{\otimes n}, P^{\otimes n} g^{(n)} \rangle.$$

From this representation of $g(Pz)$ and property 2 of Definition 4.1 we can deduce that there exists $S \in \mathbb{R}$ such that for all $P \in \mathbb{P}$:

$$\sum_{n=0}^{\infty} n! \, |P^{\otimes n} g^{(n)}|^2 < S.$$

Then we can apply Lemma 4.5 and obtain that

$$g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}, \quad n \in \mathbb{N}_0,$$

and

$$\sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)}) < \infty. \tag{4}$$

Now we have shown that for all $g \in E^2(\nu)$ we have the representation

$$g(z) = \sum_{n=0}^{\infty} (z^{\otimes n}, g^{(n)}), \quad g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}, \quad z \in \mathcal{N}_{\mathbb{C}}. \tag{5}$$

Since $g$ and $\sum_{n=0}^{\infty} (z^{\otimes n}, g^{(n)})$ are holomorphic on $\mathcal{H}_{\mathbb{C}}$ and since $\mathcal{N}_{\mathbb{C}}$ is a dense subspace of $\mathcal{H}_{\mathbb{C}}$, the representation (5) is valid for all $z \in \mathcal{H}_{\mathbb{C}}$. From (4) and the expression (2) of $\overline{\mathcal{P}(\mathcal{N}_{\mathbb{C}}')}^{L^2(\nu)}$ we can infer that the constructed extension of $g$ to $\mathcal{N}_{\mathbb{C}}'$ is correctly defined and that it is an element of $\overline{\mathcal{P}(\mathcal{N}_{\mathbb{C}}')}^{L^2(\nu)}$. $\square$

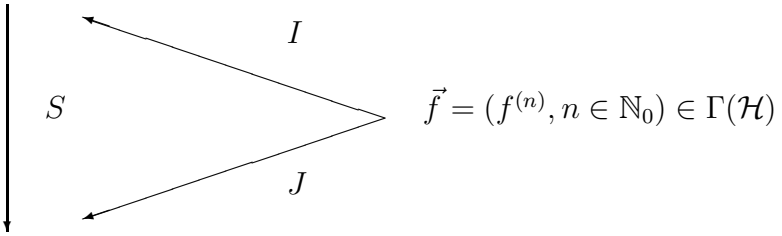
**Remark 4.6** *The proof of this theorem includes an additional statement which is interesting itself. Based on the assumption that $g$ is entire on $\mathcal{H}_{\mathbb{C}}$, the kernel theorem, see (3), allows the following expression of $g$:*

$$g(z) \;=\; \sum_{n=0}^{\infty} \langle g^{(n)}, z^{\otimes n} \rangle, \quad z \in \mathcal{N}_{\mathbb{C}}, \; g^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}. \tag{6}$$

*However, if we make the additional assumption $\sup_{P \in \mathbb{P}} \int_{\mathcal{N}_{\mathbb{C}}'} |g(Pz)|^2 \, d\nu(z) < \infty$ we can see according to the proof above that we do not need the restriction to $\mathcal{N}_{\mathbb{C}}$ and that (6) is valid for all $z \in \mathcal{H}_{\mathbb{C}}$.*

The following diagram shows the relations between the spaces $\Gamma(\mathcal{H})$, $L^2(\mu)$ and $E^2(\nu)$, see e.g. [BeKo88] and [HKPS93].

$$I\vec{f}(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle \in L^2(\mu)$$




$$J\vec{f}(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, f^{(n)} \rangle \in E^2(\nu)$$

The maps $I$, $J$ and $S$ are unitary isomorphisms. In Gaussian analysis $I$ is known as Itô-Segal-Wiener-isomorphism and $S$ as $S$-transform, see the introduction and Remark 4.4. In the terminology of [Se56] the map $J$ is called duality transform.

# 5 Test and generalized functions

Consider a selfadjoint (in generally unbounded) operator $(K, \mathcal{D}(K))$ ($\mathcal{D}(K)$ is the domain of the operator $K$) on $\mathcal{H}$ such that $K \geq 1$. Furthermore, suppose that $\mathcal{N} \subset \mathcal{D}(K)$, $K$ is an element of $\mathcal{L}(\mathcal{N}, \mathcal{N})$ (the space of continuous linear operators on $\mathcal{N}$) and $\mathcal{N}$ is a core of $K$. It is obvious that we have the usual extension of $K$ to $\mathcal{N}'$ and to the complexification $\mathcal{N}'_{\mathbb{C}}$. From the properties of $K$ we can conclude that $(K^s)^{\otimes n} \in \mathcal{L}(\mathcal{N}^{\hat{\otimes} n}, \mathcal{N}^{\hat{\otimes} n}), n \in \mathbb{N}_0, s \in \mathbb{Z}$. We denote the inner product $(\overline{(K^s)^{\otimes n}\varphi^{(n)}}, (K^s)^{\otimes n}\varphi^{(n)})$ by $(\overline{\varphi^{(n)}}, \varphi^{(n)})_{K^s}, \varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$, and the corresponding norm by $|\varphi^{(n)}|_{K^s}$. Furthermore, we denote the completion of $\mathcal{N}$ with respect to the norm $|\cdot|_{K^s}$ by $\mathcal{H}_{K^s,\mathbb{C}}$. Then it is easy to prove that the completion of $\mathcal{N}^{\hat{\otimes} n}$ with respect to the norm $|\cdot|_{K^s}$ is $\mathcal{H}_{K^s,\mathbb{C}}^{\hat{\otimes} n}$. Using these inner products we define the following Hilbertian norm for the smooth Wick polynomials $\varphi(x) = \sum_{n=0}^{N} \langle :x^{\otimes n}:, \varphi^{(n)} \rangle$, $x \in \mathcal{N}'$:

$$\| \varphi \|_{K^s}^2 := \sum_{n=0}^{\infty} n! \left( \overline{\varphi^{(n)}}, \varphi^{(n)} \right)_{K^s}, \quad \varphi \in \mathcal{P}(\mathcal{N}'), s \in \mathbb{N}_0.$$

Next we define $G_{K^s}$ as the completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{K^s}$. Or equivalently,

$$G_{K^s} = \left\{ f \in L^2(\mu) \middle| f(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle, \| f \|_{K^s}^2 < \infty, f^{(n)} \in \mathcal{H}_{K^s,\mathbb{C}}^{\hat{\otimes} n} \right\}. \quad (7)$$

Finally, the space of test functions $\mathcal{G}_K$ is defined as the projective limit of the spaces $G_{K^s}$:

$$\mathcal{G}_K = \bigcap_{s \geq 0} G_{K^s}. \quad (8)$$

We define $(G_{K^{-s}}, \|\cdot\|_{K^{-s}})$ as the dual of $(G_{K^s}, \|\cdot\|_{K^s})$ with respect to $(L^2(\mu), (\cdot,\cdot)_{L^2(\mu)})$, $s \in \mathbb{N}_0$. It is well-known that

$$\| \varphi \|_{K^{-s}}^2 = \sum_{n=0}^{\infty} n! \left( \overline{\varphi^{(n)}}, \varphi^{(n)} \right)_{K^{-s}}, \quad \varphi \in \mathcal{P}(\mathcal{N}'),$$

and that $G_{K^{-s}}$ is the completion of $\mathcal{P}(\mathcal{N}')$ with respect to the norm $\|\cdot\|_{K^{-s}}$, i.e.,

$$G_{K^{-s}} = \left\{ f \middle| f(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle, \| f \|_{K^{-s}}^2 < \infty, f^{(n)} \in \mathcal{H}_{K^{-s},\mathbb{C}}^{\hat{\otimes} n} \right\} \quad (9)$$

(note that $f \in G_{K^{-s}}$ is a generalized function and, therefore, not necessarily an element of $L^2(\mu)$). Let $\mathcal{G}_K'$ be the dual of $\mathcal{G}_K$ with respect to $(L^2(\mu), (\cdot,\cdot)_{L^2(\mu)})$. Then we know from general duality theory that $\mathcal{G}_K'$ is the inductive limit of $G_{K^{-s}}$:

$$\mathcal{G}_K' = \bigcup_{s \geq 0} G_{K^{-s}}. \quad (10)$$

We call the elements of $\mathcal{G}_K'$ generalized functions and have finally constructed the following triple:

$$\mathcal{G}_K \subset L^2(\mu) \subset \mathcal{G}_K'.$$

**Example 5.1** The nuclear space $\mathcal{N}$ can be chosen as the space of real-valued Schwartz test functions $S(\mathbb{R})$ for $\mathcal{H} = L^2_{\mathbb{R}}(\mathbb{R})$ the space of real valued square-integrable functions with respect to the Lebesque measure. This particular

choice is the usual one in white noise analysis, see e.g. [HKPS93]. A convenient choice of Hilbertian norms $\{|\cdot|_p, p \in \mathbb{N}\}$ topologizing $S(\mathbb{R})$ is

$$|f|_p := |H^p f|, \quad f \in S(\mathbb{R}),$$

where

$$Hf(t) := -\frac{d^2 f}{d^2 t}(t) + (t^2+1)f(t), \quad t \in \mathbb{R},$$

is the Hamiltonian of a harmonic oscillator with the ground state eigenvalue 2. $\mathcal{H}_p$ in this case is the completion of $S(\mathbb{R})$ with respect to $|\cdot|_p$. The operator $(H, \mathcal{D}(H))$ satisfies the assumptions required for the operator $(K, \mathcal{D}(K))$. Hence, we can consider the associated space of test function $\mathcal{G}_H$ which in white noise analysis is denoted by $(S)$. The elements of the corresponding dual space $(S)'$ are the well-known Hida distributions, see [KoSa78], [Ko80a], [Ko80b], [KuTa80], [HKPS93], [BeKo88] and [KLPSW96].

**Example 5.2** For $K = 2\mathbb{1}$ there appear the spaces $\mathcal{G} = \mathcal{G}_{2\mathbb{1}}$ and $\mathcal{G}' = \mathcal{G}'_{2\mathbb{1}}$ which have been introduced in [PoTi94].

# 6 A family of Bargmann-Segal spaces

In Section 4 we have defined the Bargmann-Segal space and in Theorem 4.3 we have shown that there is a canonical isomorphism to a strict subspace of $L^2(\nu)$, which is generated by the smooth holomorphic polynomials $\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})$. In the following subsections we generalize these spaces.

## 6.1 A family of complex Gaussian spaces

Let the operator $K$ be as in Section 5. For $\xi \in \mathcal{N}$ we consider the family of exponentials

$$C_{\frac{1}{2}K^{2s}}(\xi) := \exp(-\tfrac{1}{4}(\xi,\xi)_{K^s}), \quad s \in \mathbb{Z}.$$

These exponentials are characteristic functions with associated Gaussian measures $\mu_{\frac{1}{2}K^{2s}}$ on the measurable space $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$. Using these Gaussian measures we define the following Gaussian measures on the measurable space $(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}))$.

**Definition 6.1** *Let $z = x + iy \in \mathcal{N}'_{\mathbb{C}}$, $x, y \in \mathcal{N}'$, then we define the measures*

$$d\nu^{K,s}(z) := d\mu_{\frac{1}{2}K^{2s}}(x) \times d\mu_{\frac{1}{2}K^{2s}}(y), \quad s \in \mathbb{Z},$$

*on the measurable space $(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}))$.*

These measures enable us to consider the spaces of complex valued functions which are square-integrable with respect to one of those Gaussian measures: $L^2(\nu^{K,s}) := L^2(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}), \nu^{K,s})$, $s \in \mathbb{Z}$. For the smooth holomorphic monomials $M(\varphi^{(n)})$, $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, defined in Subsection 3.2, we again have orthogonality for different $n$. To be more precise we give the following proposition. The prove is essentially the same as in the case where $K$ is the identity, see Subsection 3.2.

**Proposition 6.2** *For any $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ and $\psi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$, $n, m \in \mathbb{N}_0$, we have:*

$$\begin{aligned} \Big(M(\varphi^{(n)}), M(\psi^{(m)})\Big)_{L^2(\nu^{K,s})} &:= \int_{\mathcal{N}'_{\mathbb{C}}} \overline{\langle z^{\otimes n}, \varphi^{(n)} \rangle} \langle z^{\otimes m}, \psi^{(m)} \rangle \, d\nu^{K,s}(z) \\ &= n! \, \delta_{n,m} (\overline{\varphi^{(n)}}, \psi^{(n)})_{K^s}, \quad s \in \mathbb{Z}. \end{aligned}$$

As a direct consequence of Proposition 6.2 we have the following corollary by using an approximation, for more details we refer to Subsection 3.1.

**Corollary 6.3** *For any $g^{(n)} \in \mathcal{H}_{K^s}^{\hat{\otimes} n}, n \in \mathbb{N}_0, s \in \mathbb{Z}$, there exists $M(g^{(n)}) \in L^2(\nu^{K,s})$ as a measurable monomial; the orthogonality property of Theorem 6.2 also holds for this extension.*

## 6.2 A family of Bargmann-Segal spaces: definition and properties

Now we are prepared to define a family of Bargmann-Segal spaces.

**Definition 6.4 (family of Bargmann-Segal spaces)** *Let the operator $K$ be as in Section 5. We define the spaces*

$$E^2(\nu^{K,s}) := \overline{\mathcal{P}_h(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu^{K,s})}, \quad s \in \mathbb{Z},$$

*and call them Bargmann-Segal spaces corresponding to the measures $\nu^{K,s}$.*

Next we will study the connections between these spaces.

**Theorem 6.5** *If $s \leq t$ then $E^2(\nu^{K,t}) \subseteq E^2(\nu^{K,s})$, $s,t \in \mathbb{Z}$.*

**Proof:** First, we give an explicit description of these spaces:

$$E^2(\nu^{K,s}) = \left\{ g \in L^2(\nu^{K,s}) \middle| g(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, g^{(n)} \rangle, \right.$$
$$\left. \| g \|^2_{L^2(\nu^{K,s})} < \infty,\ g^{(n)} \in \mathcal{H}^{\hat{\otimes} n}_{K^s,\mathbb{C}} \right\}, \quad s \in \mathbb{Z}. \qquad (11)$$

To obtain this explicit expression of the norm $\| \cdot \|_{L^2(\nu^{K,s})}$ we made use of Corollary 6.3. From this expression and the fact that $\mathcal{H}_{K^t,\mathbb{C}} \subseteq \mathcal{H}_{K^s,\mathbb{C}}$ the assertion follows immediately. □

In the following we give another connection between Bargmann-Segal spaces, which will be essential for the characterization of test and generalized functions in Section 7.

**Theorem 6.6** *The function $g \in E^2(\nu^{K,s})$ if and only if $g(K^s \cdot) \in E^2(\nu)$, $s \in \mathbb{Z}$.*

**Proof:** To begin with, we again have a look at the explicit description of these spaces in (11). Then it is plausible that it is sufficient to show that

$$\| g \|^2_{L^2(\nu^{K,s})} = \| g(K^s \cdot) \|^2_{L^2(\nu)} .$$

Indeed, we have that

$$\| g \|^2_{L^2(\nu^{K,s})} = \sum_{n=0}^{\infty} n! \, (\overline{g^{(n)}}, g^{(n)})_{K^s}$$
$$= \sum_{n=0}^{\infty} n! \, (\overline{(K^s)^{\otimes n} g^{(n)}}, (K^s)^{\otimes n} g^{(n)})$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\mathcal{N}'_{\mathbb{C}}} \overline{\langle z^{\otimes n}, (K^s)^{\otimes n} g^{(n)} \rangle} \langle z^{\otimes m}, (K^s)^{\otimes m} g^{(m)} \rangle \, d\nu(z)$$
$$= \int_{\mathcal{N}'_{\mathbb{C}}} |g(K^s z)|^2 \, d\nu(z) = \| g(K^s \cdot) \|^2_{L^2(\nu)} .$$

Here we made use of Corollary 6.3. □

# 7 Characterization of test and generalized functions via the Bargmann-Segal space

In this section we characterize the test and generalized functions which we introduced in Section 5 via Bargmann-Segal spaces. These characterizations are exceptional useful for the spaces $\mathcal{G}$ and $\mathcal{G}'$.

**Theorem 7.1** *The test function $\varphi \in \mathcal{G}_K$ if and only if for any $s \in \mathbb{N}_0$ the function $(S\varphi)(K^s\cdot) \in E^2(\nu)$.*

**Proof:** Let $\varphi \in \mathcal{G}_K$. We fix an arbitrary $s \in \mathbb{N}_0$. The definition of $\mathcal{G}_K$ in (8) shows that $\varphi \in G_{K^s}$. Using the description of the $S$-transform in the diagram on page 18, the definition of $G_{K^s}$ in (7) and the representation of $E^2(\nu^{K,s})$ in (11) we obtain

$$S\varphi \in S(G_{K^s}) = E^2(\nu^{K,s}).$$

Now we apply Theorem 6.6 and get

$$(S\varphi)(K^s\cdot) \in E^2(\nu).$$

Let us suppose that for any $s \in \mathbb{N}_0$ the function $(S\varphi)(K^s\cdot) \in E^2(\nu)$. An application of Theorem 6.6 and the same arguments as above show that for all $s \in \mathbb{N}_0$:

$$S\varphi \in E^2(\nu^{K,s}) = S(G_{K^s})$$

and, therefore, $\varphi \in G_{K^s}$ for all $s \in \mathbb{N}_0$. Then the definition of $\mathcal{G}_K$ in (8) shows that $\varphi \in \mathcal{G}_K$. □

A special case of the Theorem 7.1 is $K = 2\mathbb{1}$ with the corresponding test functions space $\mathcal{G}$, see Example 5.2. As a corollary we have the following characterization of this space.

**Corollary 7.2** *The test function $\varphi \in \mathcal{G}$ if and only if for any $\lambda > 0$ the function $(S\varphi)(\lambda\cdot) \in E^2(\nu)$.*

**Theorem 7.3** *The generalized function $\Phi \in \mathcal{G}'_K$ if and only if there exists $s \in \mathbb{N}_0$ such that $(S\Phi)(K^{-s}\cdot) \in E^2(\nu)$.*

**Proof:** Let us suppose that $\Phi \in \mathcal{G}'_K$. The description of $\mathcal{G}'_K$ in (10) shows that there exists $s \in \mathbb{N}_0$ such that $\Phi \in G_{K^{-s}}$. Using the obvious extension of the $S$-transform to $G_{K^{-s}}$, the description of $G_{K^{-s}}$ in (9) and the representation of $E^2(\nu^{K,-s})$ in (11) we obtain

$$S\Phi \in S(G_{K^{-s}}) = E^2(\nu^{K,-s}).$$

Now we apply Theorem 6.6 and get

$$(S\Phi)(K^{-s}\cdot) \in E^2(\nu).$$

Let for $s \in \mathbb{N}_0$ the function $(S\Phi)(K^{-s}\cdot) \in E^2(\nu)$. Then by using Theorem 6.6 and the same arguments as above we obtain that

$$S\Phi \in E^2(\nu^{K,-s}) = S(G_{K^{-s}})$$

and, hence, $\Phi \in G_{K^{-s}}$. Now the description of $\mathcal{G}'_K$ in (10) shows that $\Phi \in \mathcal{G}'_K$. □

Again we consider the special case $K = 2\mathbb{1}$ and obtain the following characterization of the distributions space $\mathcal{G}'$, see Example 5.2.

**Corollary 7.4** *The generalized function $\Phi \in \mathcal{G}'$ if and only if there exists $\epsilon > 0$ such that $(S\Phi)(\epsilon\cdot) \in E^2(\nu)$.*

Applying these characterizations to concrete problems one has to show that after a certain scaling a function is an element of the Bargmann-Segal space. The integrable exponentials considered in Lemma 3.2 can be useful as dominating functions in order to show integrability.

# 8 Applications

In this section we present two examples in which we show how to apply the characterization theorems to concrete problems.

**Example 8.1** As a first example of an application we consider Donsker's delta function which is given in the framework of white noise analysis, see Example 5.1. Let $f_t := \mathbb{1}_{[0,t)}$, $t > 0$, be the indicator function for $[0, t) \subset \mathbb{R}$. Then $f_t \in L^2_{\mathbb{R}}(\mathbb{R})$ and

$$B(t, \cdot) := \langle \cdot, f_t \rangle \in L^2(\mu)$$

is a realisation of Brownian motion at time $t$, see e.g. [HKPS93]. Let $\delta(\cdot - a)$ be the Dirac distribution at $a \in \mathbb{R}$ then the composition

$$\Phi := \delta(B(t, \cdot) - a)$$

is called Donsker's delta function, see e.g. [HKPS93]. There its $S$-transform is calculated to be

$$(S\Phi)(z) = \frac{1}{\sqrt{2\pi t}} \exp\big( - \frac{1}{2t}(\langle z, f_t\rangle - a)^2\big), \quad z \in S'_{\mathbb{C}}(\mathbb{R}). \tag{12}$$

Now we consider the restriction of $S\Phi$ to $L^2(\mathbb{R})$. If we can show that there exists an $\epsilon > 0$ such that $(S\Phi)(\epsilon\cdot)$ satisfies the properties 1 and 2 of Definition 4.1, then we can conclude by Theorem 4.3 and Corollary 7.4 that $\Phi \in \mathcal{G}'$.

It is obvious that $\langle \cdot, f_t\rangle$ is entire on $\mathcal{H}_{\mathbb{C}}$, hence, $(S\Phi)(\epsilon\cdot)$ is entire on $\mathcal{H}_{\mathbb{C}}$ as required in property 1 of Definition 4.1.

In order to show property 2 we calculate the following integral:

$$\begin{aligned} & \int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon z)|^2 \, d\nu(z) \\ = & \int_{S'_{\mathbb{C}}(\mathbb{R})} \frac{1}{2\pi t} \exp\Big( - \frac{1}{2t}\Big(\Big(\sqrt{t}\epsilon\Big\langle z, \frac{f_t}{\sqrt{t}}\Big\rangle - a\Big)^2 \\ & \qquad + \Big(\sqrt{t}\epsilon\overline{\Big\langle z, \frac{f_t}{\sqrt{t}}\Big\rangle} - a\Big)^2\Big)\Big) \, d\nu(z) \end{aligned}$$

For integrands whose dependence on $z \in S'_{\mathbb{C}}(\mathbb{R})$ is only the dependence on a normalized measurable monomial the infinite dimensional Gaussian integral reduces to a one-dimensional one, i.e.,

$$\int_{S'_{\mathbb{C}}(\mathbb{R})} F(\langle z, f\rangle) \, d\nu(z) = \int_{\mathbb{C}} F(u) d\gamma(u), \tag{13}$$

where $f \in L^2(\mathbb{R})$ with norm equal to 1 and

$$d\gamma(u) = \frac{1}{\pi} \exp(-(x^2 + y^2)) \, dx \, dy, \quad u = x + iy, \, x, y \in \mathbb{R}.$$

This can easily be shown by using the fact that in this case the characteristic function of the Gaussian measure $\mu_{\frac{1}{2}}$, see Subsection 3.2, can be considered

as the characteristic function of an one-dimensional Gaussian measure, for more details we refer to [BeKo88]. Therefore, we have:

$$\begin{aligned}
&\int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon z)|^2 \, d\nu(z) \\
= &\int_{\mathbb{C}} \frac{1}{2\pi t} \exp\Big( -\frac{\epsilon^2}{2}\Big(\Big(u-\frac{a}{\sqrt{t}}\Big)^2 + \Big(\overline{u}-\frac{a}{\sqrt{t}}\Big)^2\Big)\Big) \, d\gamma(u) \\
= &\frac{1}{2\pi t}\int_{\mathbb{C}} \exp\big( -\frac{\epsilon^2}{2}(u^2+\overline{u}^2)\big) \, d\gamma(u) \\
= &\frac{1}{2\pi^2 t}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left((3\epsilon^2-1)x^2 + (\epsilon^2-1)y^2\right) dxdy, \quad u = x+iy,\ x,y\in\mathbb{R}, \\
= &\frac{1}{2\pi^2 t}\int_{\mathbb{R}} \exp\left((3\epsilon^2-1)x^2\right) dx \int_{\mathbb{R}} \exp\left((\epsilon^2-1)y^2\right) dy.
\end{aligned}$$

Now it is obvious that the integral is finite for $0 < \epsilon < \frac{1}{\sqrt{3}}$. Using (12) and (13) we obtain that for any $P \in \mathbb{P}$

$$\int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon P z)|^2 \, d\nu(z) \leq \int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon z)|^2 \, d\nu(z) < \infty$$

and, consequently, that

$$\sup_{P\in\mathbb{P}} \int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon P z)|^2 \, d\nu(z) \leq \int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon z)|^2 \, d\nu(z) < \infty$$

for $0 < \epsilon < \frac{1}{\sqrt{3}}$. Therefore, property 1 of Definition 4.1 is fulfilled and we have proved that Donsker's delta is an element of $\mathcal{G}'$.

**Example 8.2** In this example we show that the Feynman integrand of the free particle propagator is not an element of $\mathcal{G}'$. The idea of realizing Feynman integrals within the white noise framework goes back to [HS83]. The "average over all path" is performed with a Hida distribution, see Example 5.1, as the weight (instead of a measure). The existence of such Hida distributions has been established in [FPS91]. There the $S$-Transform of the Feynman integrand of the free particle propagator $\Phi$ reads:

$$\begin{aligned}
(S\Phi)(f) \;=\; &\frac{1}{\sqrt{2\pi i(t-t_0)}} \exp\Big( -\frac{1}{2}\int_T f^2(s)\,ds + \frac{i}{2}\int_T f^2(s)\,ds \\
&+\frac{i}{2(t-t_0)}\Big( -i\int_T f(s)\,ds + k - k_0\Big)^2\Big),
\end{aligned}$$

where $f \in L^2(\mathbb{R})$, $T = [t_0, t]$, $t > t_0$, $t_0, t, k_0, k \in \mathbb{R}$. In our notation we have the following formula:

$$\begin{aligned}(S\Phi)(z) &= \frac{1}{\sqrt{2\pi i(t-t_0)}} \exp\Big( -\frac{1}{2}(z, A_T z) + \frac{i}{2}(z, A_T z) \\ &\quad + \frac{i}{2(t-t_0)}\Big( -i(z, \mathbb{1}_T) + k - k_0\Big)^2\Big),\ z \in L^2(\mathbb{R}),\end{aligned} \tag{14}$$

where the action of the operator $A_T$ is given by multiplication with the indicator function $\mathbb{1}_T$. It is easy to see that $A_T$ is a projection operator on $L^2(\mathbb{R})$. $A_T$ projects on a infinite dimensional subspace of $L^2(\mathbb{R})$, hence, $A_T$ is not a trace class operator. From (14) we obtain by straightforward computation:

$$|(S\Phi)(z)|^2 = \frac{1}{2\pi(t-t_0)} \exp\Big( -(x, A_T x) + (y, A_T y) + 2\frac{k-k_0}{(t-t_0)}(x, \mathbb{1}_T)\Big),$$

where $z = x + iy$, $x, y \in L^2_{\mathbb{R}}(\mathbb{R})$.

Assume that $\Phi \in \mathcal{G}'$. Then by Corollary 7.4 it follows that there exists an $\epsilon > 0$ such that $(S\Phi)(\epsilon\cdot) \in L^2(\nu)$. Hence, the integral

$$\begin{aligned}&\int_{S'_{\mathbb{C}}(\mathbb{R})} |(S\Phi)(\epsilon z)|^2\, d\nu(z) \\ = \ &\frac{1}{2\pi(t-t_0)} \int_{S'(\mathbb{R})} \exp\Big(\epsilon^2 \langle y, A_T y\rangle\Big)\, d\mu_{\frac{1}{2}}(y) \\ &\int_{S'(\mathbb{R})} \exp\Big( -\epsilon^2\langle x, A_T x\rangle + 2\frac{k-k_0}{(t-t_0)}\epsilon\langle x, \mathbb{1}_T\rangle\Big)\, d\mu_{\frac{1}{2}}(x)\end{aligned} \tag{15}$$

is finite. But $A_T$ is not a trace class operator, so $\langle y, A_T y\rangle$ does not exists as a measurable function, see e.g. [BeKo88]. This stands in contradiction to $(S\Phi)(\epsilon\cdot) \in L^2(\nu)$. Moreover, a similar computation as used in order to prove Lemma 3.2 shows that the integral (15) would be infinite, which also stands in contradiction to $(S\Phi)(\epsilon\cdot) \in L^2(\nu)$. Hence, the assumption $\Phi \in \mathcal{G}'$ is false, i.e., $\Phi \notin \mathcal{G}'$.

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