

Probability, White Noise Analysis, Infinite Particle Systems

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1 Introduction

These informal notes are intended as a supplement to lectures given at the Physics Department of MSU-IIT in January of 2009. Their main goal was twofold: to provide some additional mathematical underpinnings for applications of (Gaussian) White Noise Analysis in current and future research at that institute, and secondly to introduce recent developments in the theory of infinite particle systems in the continuum. These latter are based on Poisson White Noise, and we shall see that many structures of the Gaussian analysis have natural analogues in the Poissonian case.

To set the stage we first review concepts from finite dimensional analysis, more specifically from measure and integration, probability, and generalized functions.

Teaching this course was a great pleasure for the lecturer, and he would like to express his gratitude to his hosts at MSU-IIT for their generous hospitality: Chancellor Marcelo P. Salazar, Dean of the College of Science and Math Romulo C. Guerrero, Department Chair J. Bornales, and staff, and of course to the students, who kept the enthusiasm bubbling, for feedback and corrections. Drs. V. and Chr. Bernido of RCTP, Jagna, deserve thanks for establishing links and for helping me to choose and prepare the material. Material support came from CHED-ZRC Region X., DAAD, DOST-PCASTRD, and MSU-IIT, and is gratefully acknowledged.

2 Measure, Integration, and Probability

Refs. : [3][5][7][8][10][11], many others!

Measure theory is not only the basis for any modern theory of integration, but also for practical numerical methods such as Monte Carlo integration. It provides the foundation for probability theory, indispensable in virtually all branches of physics, and fundamental in quantum theory.

Consider a triplet $\{\Omega, \mathfrak{B}, m\}$, where

- Ω is an arbitrary set
- \mathfrak{B} is a collection of subsets of Ω : the "measurable sets"

- m attributes a non-negative number to each measurable set: its "measure".

Wanted: Some reasonable properties!

2.1 Measure spaces

Definition 1 Let Ω be an arbitrary set and \mathfrak{B} a collection of subsets of Ω , such that

$$A \in \mathfrak{B} \implies A^c = \Omega - A \in \mathfrak{B} \quad (1)$$

$$\emptyset \in \mathfrak{B} \quad (2)$$

$$A_k \in \mathfrak{B} \implies \bigcup_{k=1}^{\infty} A_k \in \mathfrak{B}$$

Then we say that \mathfrak{B} is a " σ -algebra over Ω ".

Problem 1 For an arbitrary set Ω , construct a couple of σ -algebras over Ω .

Problem 2 For an arbitrary σ -algebra \mathfrak{B} and $A, B \in \mathfrak{B}$ show that $A \cap B \in \mathfrak{B}$.

Definition 2 Let

$$m : \mathfrak{B} \rightarrow [0, \infty] \quad (3)$$

with

$$m(\emptyset) = 0 \quad (4)$$

and " σ -additivity"

$$A_k \in \mathfrak{B} \text{ disjoint} \implies m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum m(A_k). \quad (5)$$

Then m is called a measure, and the triplet $\{\Omega, \mathfrak{B}, m\}$ is called a "measure space".

Finite measures have $m(\Omega) < \infty$, σ -finite measures are such that Ω is a countable union of sets of finite measure.

Demanding only

$$A_k \in \mathfrak{B} \implies \bigcup_{k=1}^n A_k \in \mathfrak{B} \quad (6)$$

for finite unions, one would speak of an "algebra of sets".

Remark 1 Connection with logics:

\cap = "and",

\cup = "or"

$A \rightarrow A^c$ = "negation".

We usually reserve the label "algebra" for structures in which one can add and multiply elements with one another and multiply by "numbers". Let us just state here that algebras of sets are indeed algebras if we define multiplication as

$$A_1 \cdot A_2 = A_1 \cap A_2,$$

addition as

$$A_1 + A_2 = (A_1 \cap A_2^c) \cup (A_1^c \cap A_2)$$

and as "numbers", we permit the integers modulo 2 (where integers are identified if they differ by a multiple of 2), so that

$$\begin{aligned} 1 \cdot A &= A, \\ 0 \cdot A &= \emptyset \end{aligned}$$

Problem 3 1. Are addition and multiplication commutative?

2. Compute and interpret

$$A + A = ?$$

3. Check

$$\begin{aligned} (A + B) + C &= A + (B + C) \\ A(B + C) &= AB + AC \end{aligned}$$

2.1.1 Translation to Probability Theory

- Ω is called "Sample Space"
- \mathfrak{B} is the set of "Events"
- $m = p$ with $0 \leq p \leq 1$ is called a "Probability Measure" and $\{\Omega, \mathfrak{B}, p\}$ is a "Probability Space".

Problem 4 When you roll a die, the possible outcomes are represented by the numbers

$$\{1, 2, 3, 4, 5, 6\}.$$

Construct a suitable probability space $\{\Omega, \mathfrak{B}, p\}$.

Theorem 1 For any collection \mathfrak{M} of subsets of Ω there is a smallest σ -algebra over Ω containing \mathfrak{M} .

Proof. Take the intersection of all the σ -algebras containing \mathfrak{M} . ■

Theorem 2 (Carathéodory Extension Theorem) Any non-negative σ -additive set function f on an algebra of sets \mathfrak{A} can be extended to the smallest σ -algebra \mathfrak{B} containing \mathfrak{A} . For σ -finite f the extension is unique.

For proofs of this important fact consult books on measure and integration theory.

Example 1 "Counting Measure"

$$\mu(A) = \#(A) \text{ (= no. of elements in } A) \quad (7)$$

Example 2 "Dirac Measure"

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

2.2 Lebesgue Measure

Example 3 Lebesgue Measure on $\Omega = [a, b] \subset \mathbb{R}^1$.

$$\mathfrak{A} = \left\{ \bigcup_{i=1}^N [a_i, b_i] \subseteq \Omega \right\} \quad (9)$$

is a set algebra .

Definition 3 The σ -algebra \mathfrak{B}_0 generated by \mathfrak{A} is called "Borel algebra"

The interval length

$$\mu_0([a_i, b_i]) = |b_i - a_i| \quad (10)$$

extends on \mathfrak{A} to a σ -additive set function and hence to a measure, the "Lebesgue Measure" μ on $\mathfrak{B}_0(\mathbb{R}^n)$.

- Extension to \mathbb{R}^n is evident. Use volume instead of length:

$$\mu_0([\vec{a}, \vec{b})) = \prod_{i=1}^n |b_i - a_i| \quad (11)$$

- μ is translation- and for $n > 1$ rotation invariant:

$$\mu(A) = \mu(\{x + c : x \in A\}) \quad (12)$$

for $c \in \mathbb{R}^n$, and for rotations R

$$\mu(A) = \mu(\{Rx : x \in A\}) \quad (13)$$

- Countable subsets are Borel sets:

$$\{x_i \quad : \quad i = 1, 2, \dots\} = \bigcup_i \{x_i\} \quad (14)$$

$$= \bigcup_i \bigcap_m \left\{ [x_i, x_i + \frac{1}{m}) \right\} \in \mathfrak{B}_0. \quad (15)$$

The set \mathfrak{Q} of rational numbers is Borel-measurable:

$$\mu(\mathfrak{Q}) = 0. \quad (16)$$

Note:

$$\mu(\{x\}) \leq \mu(\{[x, x + \frac{1}{m}]\}) = \frac{1}{m}. \quad (17)$$

for all m , hence

$$\mu(\{x\}) = 0. \quad (18)$$

Use Sigma-Additivity to conclude: all countable sets have Lebesgue measure zero!

2.3 A Non-Measurable Set

Consider $\Omega = R$ and μ the Lebesgue measure.

Have translation invariance:

$$\mu(A + x) = \mu(A) \quad (19)$$

where

$$A + x \equiv \{a + x : a \in A\}. \quad (20)$$

Now construct a set $A \subset R$ as follows:

Call x, y equivalent if their difference is rational:

$$x \sim y \iff x - y \in Q. \quad (21)$$

Let now $A \subseteq [0, 1]$ consist of a maximal set of inequivalent numbers ("one from each equivalence class"). Then, for all rational $r_i \in Q$, with $r_i \neq r_k$, have

$$(A + r_i) \cap (A + r_k) = \emptyset$$

since otherwise

$$a + r_i = b + r_k \quad (22)$$

and hence

$$a - b = r_k - r_i \quad (23)$$

rational, which should not be the case for a and b in A . So

$$\bigcup_{r \in [-1, 1]} (A + r_i) \quad (24)$$

is a union of *disjoint* sets.

Using σ -additivity and translation invariance

$$\mu\left(\bigcup_{r \in [-1, 1]} (A + r_i)\right) = \sum_{r \in [-1, 1]} \mu(A + r_i) = \sum_{r \in [-1, 1]} \mu(A) = \begin{cases} 0 \\ \infty \end{cases} \quad (25)$$

depending on whether $\mu(A)$ is zero or positive. Note on the other hand

$$[0, 1] \subset \bigcup_{r \in [-1, 1]} (A + r_i) \subset [-1, 2]. \quad (26)$$

i.e.

$$1 \leq \mu \left(\bigcup_{r \in [-1, 1]} (A + r_i) \right) \leq 3, \quad (27)$$

in contradiction to the previous computation. The only way out: A is not measurable!

2.4 Distribution Functions

More measures on $\mathfrak{B}_0(R)$ can be obtained through "distribution functions":

Definition 4 *Functions*

$$F : R \rightarrow R \quad (28)$$

monotonely growing and continuous from the left are called distribution functions. The extension of

$$\mu_F([a, b)) \equiv F(b) - F(a) \quad (29)$$

to a measure on $\mathfrak{B}_0(R)$ is called a "Lebesgue-Stieltjes Measure with distribution F ".

Alternatively we could have defined

$$\mu_F((a, b]) = F_+(b) - F_+(a) \quad (30)$$

with

$$F_+(b) = F(b) + \mu_F(\{b\}) = \lim_{\varepsilon \rightarrow +0} F(b + \varepsilon). \quad (31)$$

1. F_+ is right continuous.

2. μ_F is a probability measure iff

$$F(\infty) - F(-\infty) = 1 \quad (32)$$

3. Example (Dirac Measure on R):

$$F(x) = \begin{cases} 1 & \text{for } x > \alpha \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

More generally:

$$F(b + \varepsilon) - F(b) \rightarrow m \Leftrightarrow \mu(\{b\}) = m \quad (34)$$

For discrete measures F is piecewise constant.

Definition 5 *Let*

$$F(x) = \int_{-\infty}^x \rho(y)dy \text{ with } \rho \geq 0. \quad (35)$$

Then ρ is called the "density".

Example 4 *(Cauchy distribution):*

$$-\frac{1}{2} < F(x) = \frac{1}{\pi} \arctan x < \frac{1}{2} \quad (36)$$

with

$$\rho(x) = F'(x) = \frac{1}{\pi(1+x^2)} : \quad (37)$$

Note: All distributions with density are continuous, and $F' = \rho$.

But there are continuous distributions without density as we shall see now.

2.4.1 Cantor Set and Devil's Staircase

Construct a set $T \subset [0, 1]$ as follows:

1. Take the open middle third $(1/3, 2/3)$ of the initial interval $[0, 1]$
2. Add the open middle thirds of the remaining intervals
3. Add the open middle thirds of the remaining intervals
4. ...etc....

$$T = (1/3, 2/3) \cup \quad (38)$$

$$(1/9, 2/9) \cup (7/9, 8/9) \cup \quad (39)$$

$$(1/27, 2/27) \cup \dots \quad (40)$$

Definition 6 *The remaining set*

$$C \equiv [0, 1] - T$$

is called the "Cantor Set".

To check it out write the $x \in [0, 1]$ as follows ("base three")

$$x = \sum a_n 3^{-n} \text{ with } a_n = 0, 1, 2. \quad (41)$$

Note that this is not always unique:

$$x = \sum_{n=2}^{\infty} 2 \cdot 3^{-n} = 2 \cdot \sum_{n=2}^{\infty} 3^{-n} = \frac{2}{9} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \quad (42)$$

$$= \frac{2}{9} \frac{1}{1 - \frac{1}{3}} = \frac{1}{3} = 1 \cdot 3^{-1}. \quad (43)$$

In these cases we always adopt the first, infinite series version. Then one finds that x is in T iff at least one a_n is equal to one. In other words $x \in C$ iff all the a_n are 2 or 0. For example $1/3 \in C$.

How big is C? Well, how big is T? We can easily calculate its Lebesgue measure:

$$\mu(T) = 1/3 + 2 \cdot 1/9 + 4 \cdot 1/27 + \dots \quad (44)$$

$$+ 2^{n-1} \cdot 3^{-n} + \dots \quad (45)$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1. \quad (46)$$

Hence *the Cantor set has measure zero! But it is not countable*: its elements are as many as there are sequences of zeroes and twos, i.e. as many as there are sequences of zeroes and ones, hence as many as there are real numbers in the interval $[0,1]$ (just write them in binary form!).

Now to the Devil's Staircase:

Consider the function

$$F(x) \equiv \frac{1}{2} \sum_{n=1}^{N-1} 2^{-n} a_n + 2^{-N} \quad (47)$$

where N is the smallest index for which $a_N = 1$. In particular on T

$$F = 1/2 \text{ resp. } 1/4, 3/4 \text{ resp. } 1/8, 3/8, 5/8, 7/8 \dots \text{etc.} \quad (48)$$

F is continuous, monotone hence a distribution function, but almost everywhere, i.e. on the intervals constituting T , F is in fact constant, with $F' = 0$, so that we cannot have

$$F = \int \rho dx. \quad (49)$$

The measure defined by F is "singular".

Generally there exists a "Lebesgue decomposition" of any measure into a discrete, a singular and an "absolutely continuous" part. While one encounters discrete and/or absolutely continuous measures most of the time in physics, singular ones cannot be disregarded: as an example we cite the occurrence of the devil's staircase in the context of the inviscid Burgers equation with (fractional) Brownian initial data [2][13][14][15].

2.4.2 Measurable Functions

Definition 7 Given $\{\Omega, \mathfrak{B}\}$. A function

$$f : \Omega \rightarrow \bar{R} \quad (50)$$

is called "measurable" if pre-images of Borel sets $A \in \mathfrak{B}_0$ are in the σ -algebra \mathfrak{B} :

$$f^{-1}(A) \equiv \{\omega \in \Omega : \} \in \mathfrak{B}. \quad (51)$$

In a probabilistic terminology they are called "random variables".

1. In the probabilistic context measurability is natural: we accept a function on the sample space as a random variable iff " $f(\omega) \in A$ " is an event, i.e. has a well-defined probability.
2. Measurability depends not on the measure, but only on the pair Ω, \mathfrak{B}

$$f \in \mathfrak{M}\{\Omega, \mathfrak{B}\}. \quad (52)$$

3. With Respect to. $\mathfrak{B} = \mathfrak{B}_0(R^n)$, all continuous functions are measurable. Reason: for continuous functions the pre-images of open sets are open, hence Borel measurable.

Problem 5 Let $\Omega = R^1$, and

$$f(\omega) = \sin \omega.$$

Compute $f^{-1}(A)$ for $A = (-2, 2)$, $A = (0, 1]$ and $A = (2, 3)$.

Problem 6 Consider the example of a die, and

$$f(\omega) = \begin{cases} 1 & \text{for even } \omega \\ -1 & \text{otherwise} \end{cases}$$

Compute $f^{-1}(A)$ for $A = \{x > 0\}$ and for $A = \{x > 4\}$.
Which functions are measurable?

Consider "Indicator Functions" of sets $M \in \Omega$:

$$\chi_M(\omega) = \begin{cases} 1 & \text{falls } \omega \in M \\ 0 & \text{otherwise} \end{cases}. \quad (53)$$

Now inspect the pre-images of sets $A \in \mathfrak{B}_0(R^1)$

$$\begin{aligned} 1 \in A \quad 0 \notin A &\implies \chi_M^{-1}(A) = M \\ 1 \in A \quad 0 \in A &\implies \chi_M^{-1}(A) = \Omega \\ 1 \notin A \quad 0 \in A &\implies \chi_M^{-1}(A) = \Omega - M \\ 1 \notin A \quad 0 \notin A &\implies \chi_M^{-1}(A) = \emptyset \end{aligned} \quad (54)$$

All possible pre-images are thus listed on the right, and are measurable if M is. Hence indicator functions χ_M are measurable iff M is.

Similarly for "simple" functions

$$f \equiv \sum_{i=1}^{\infty} a_i \chi_{M_i}. \quad (55)$$

Theorem 3 If f_i are $\{\Omega, \mathfrak{B}\}$ -measurable, then so are

- a) $g(\omega) = af_1(\omega) + bf_2(\omega)$ with real a, b
- b) $g(\omega) = \varphi(f_1(\omega))$ with $\varphi \in \mathfrak{B}_0(R^1)$
- c) $g(\omega) = f_1(\omega)f_2(\omega)$
- d) $g(\omega) = f_1(\omega)/f_2(\omega)$
- e) $g(\omega) = |f_1(\omega)|$
- f) $g(\omega) = \liminf_n f_n(\omega)$, $g(\omega) = \limsup_n f_n(\omega)$,
- $g(\omega) = \sup_n f_n(\omega)$, $g(\omega) = \inf_n f_n(\omega)$.

Approximation by Simple Functions:

Theorem 4 Let $f \geq 0$ and $\{\Omega, \mathfrak{B}\}$ -measurable. Then there is a monotonely growing sequence of simple functions with

$$f_n(\omega) \nearrow f(\omega) \quad (56)$$

for all $\omega \in \Omega$.

Proof. Set

$$f_n(\omega) = \begin{cases} n & \text{for } f(\omega) > n \\ [2^n f(\omega)]/2^n & \text{otherwise} \end{cases} . \quad (57)$$

■

2.5 Integrals, Expectations

Motivation: Let $\{\Omega, \mathfrak{B}, p\}$ be a probability space, and

$$f \equiv \sum_{i=1}^N a_i \chi_{M_i} \quad (58)$$

a simple random variable. A natural definition of expectation for this random variable would be the weighted average of its values a_i :

$$E(f) \equiv \sum_{i=1}^N a_i p(M_i) . \quad (59)$$

Definition 8 Let $\{\Omega, \mathfrak{B}, m\}$ be a measure space and

$$f \equiv \sum_{i=1}^{\infty} a_i \chi_{M_i} \geq 0 \quad (60)$$

$\{\Omega, \mathfrak{B}\}$ -measurable. Then we set

$$\int_{\Omega} f(\omega) dm(\omega) \equiv \sum_{i=1}^{\infty} a_i m(M_i) \quad (61)$$

as the "integral of f over Ω with respect to the measure m ".

Note:

1. For step functions

$$f|_{I_k} = f_k \quad (62)$$

and the Lebesgue measure, the integral is the same as the Riemann integral :

$$\int_{R^1} f(x) d\mu(x) \equiv \sum_{k=1}^{\infty} f_k \mu(I_k) = \sum_{k=1}^{\infty} f_k |I_k|. \quad (63)$$

2. But there are many more simple functions:

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ 0 & \text{otherwise} \end{cases} \quad (64)$$

This is a simple function, and

$$\int_{R^1} f(x) d\mu(x) = 1 \cdot \mu(Q) + 0 \cdot \mu(Q^c) = 0. \quad (65)$$

3. Notations

$$\int_{\Omega} f(\omega) dm(\omega) \equiv \int_{\Omega} f(\omega) m(d\omega) \equiv \int_{\Omega} f dm \quad (66)$$

and

$$\int_A f(\omega) dm(\omega) \equiv \int_{\Omega} f(\omega) \chi_A(\omega) dm(\omega). \quad (67)$$

Call $0 \leq f \in M(\Omega, \mathfrak{B})$ "integrable", if $\int_{\Omega} f dm$ is finite.

Theorem 5 Let $0 \leq f \in M(\Omega, \mathfrak{B})$ and f_n simple, with

$$f_n(\omega) \nearrow f(\omega). \quad (68)$$

then for any m on $\{\Omega, \mathfrak{B}\}$ there exists a unique

$$\int_{\Omega} f(\omega) dm(\omega) \equiv \lim \int_{\Omega} f_n(\omega) dm(\omega). \quad (69)$$

Special case: Lebesgue Integral, based on Borel measure, $\{\Omega, \mathfrak{B}, m\} = \{R^n, \mathfrak{B}_0, \mu\}$.
For continuous functions the Lebesgue integral coincides with that of Riemann:

$$\int_R f(\omega) d\mu(\omega) = \int_{-\infty}^{\infty} f(x) dx. \quad (70)$$

Properties of $\int_{\Omega} f(\omega) dm(\omega)$:

1. For $f \in M(\Omega, \mathfrak{B})$ set $f = f_+ - f_-$ with $f_{\pm} \geq 0$ and

$$\int_{\Omega} f(\omega) dm(\omega) \equiv \int_{\Omega} f_+ dm - \int_{\Omega} f_- dm. \quad (71)$$

(Must avoid $\infty - \infty$ on the rhs., if $\int_{\Omega} |f| dm$ finite, i.e.. $|f|$ integrable, we say that f is integrable).

2. Let $f \geq 0$. Then

$$\int_{\Omega} f(\omega) dm(\omega) \geq 0 \quad (72)$$

and

$$\int_{\Omega} f(\omega) dm(\omega) = 0 \implies m(\{f \neq 0\}) = 0 \quad (73)$$

3.

$$\int (af_1 + bf_2) dm = a \int f_1 dm + b \int f_2 dm \quad (74)$$

4.

$$f \geq g \implies \int f dm \geq \int g dm \quad (75)$$

5.

$$\int |f| dm \geq \left| \int f dm \right| \quad (76)$$

Consider the Lebesgue measure. The integral

$$\int_{R^2} f(x) d\mu^{(d=2)}(x)$$

based on the Lebesgue measure on the plane R^2 , and the two iterated one dimensional integrals

$$\int_{R^1} \left(\int_{R^1} f(x) d\mu^{(d=1)}(x_1) \right) d\mu^{(d=1)}(x_2) \quad (77)$$

$$\int_{R^1} \left(\int_{R^1} f(x) d\mu^{(d=1)}(x_2) \right) d\mu^{(d=1)}(x_1) \quad (78)$$

need not be equal! They are if one of them is finite for $|f|$ instead of f ("Fubini Theorem").

2.5.1 Convergence of Integrals

Often one has sequences $f_n \in M(\Omega, \mathfrak{B})$ and wants to control convergence of Integrals $\int_A f_n dm$. A simple criterion is

Theorem 6 *Let $m(A) < \infty$ and*

$$f_n \rightarrow f \quad (79)$$

***uniformly** on A . Then*

$$\int_A f_n dm \rightarrow \int_A f dm. \quad (80)$$

Proof. Let $\varepsilon > 0$. then there is N such that

$$|f_n(\omega) - f(\omega)| \leq \varepsilon \quad (81)$$

for all $n > N$ and all $\omega \in A$. Hence

$$\left| \int_A f_n dm - \int_A f dm \right| \leq \int_A |f_n - f| dm \leq \varepsilon \cdot m(A). \quad (82)$$

■

Some stronger results:

Let

$$f_n(\omega) \rightarrow f(\omega) \quad (83)$$

for m -almost-all ω . And further:

(M)

$$0 \leq f_n(\omega) \nearrow f(\omega) \quad (84)$$

"monotone convergence"

or

(D)

$$|f_n(\omega)| \leq g(\omega) \quad (85)$$

with integrable g ("dominated convergence")

Then

$$\int f_n dm \rightarrow \int f dm. \quad (86)$$

2.5.2 The Radon-Nikodym Theorem

Here we consider two sigma-finite measures m_i on $\{\Omega, \mathfrak{B}\}$, with

$$m_1(A) = 0 \implies m_2(A) = 0. \quad (87)$$

Definition 9 One calls " m_2 absolutely continuous with respect to m_1 " and writes

$$m_2 \ll m_1. \quad (88)$$

Example 5 Let

$$m_1(A) = \int_A dm_1 \quad (89)$$

and

$$m_2(A) = \int_A \varrho(\omega) dm_1(\omega) \text{ with } \varrho(\omega) \geq 0. \quad (90)$$

If the first integral vanishes then so does the second: m_2 is absolutely continuous with respect to m_1 . Conversely:

Theorem 7 ("Radon-Nikodym") Let m_2 be absolutely continuous with respect to m_1 . Then there exists

$$0 \leq \varrho \in M(\Omega, \mathfrak{B}) \quad (91)$$

such that

$$\int f dm_2 = \int f \varrho dm_1. \quad (92)$$

Notation:

$$\varrho = \frac{dm_2}{dm_1} \quad (93)$$

"Radon-Nikodym derivative" or "Density" of m_2 with respect to m_1 . The sigma-finiteness is essential.

Counter-Example with a non-sigma-finite measure: Given some Ω and $\mathfrak{B} = \{\emptyset, \Omega\}$. I.e. there is only one non-empty measurable set, so that all measurable functions are constant.

Let

$$m_2(\Omega) = 1 \quad (94)$$

$$m_1(\Omega) = \infty \quad (95)$$

Both have the same zero-set(s) hence are absolutely continuous with respect to each other, and we would have

$$1 = m_2(\Omega) = \int dm_2 = \int \varrho dm_1 = \varrho \int dm_1 = \varrho m_1(\Omega) = \varrho \cdot \infty, \quad (96)$$

which does not make sense.

2.5.3 The Monte-Carlo Method

Let

$$f : [0, 1] \rightarrow [a, b] \quad (97)$$

and I_1, \dots, I_n a decomposition of $[a, b]$ in n subintervals $I_k = [y_k, y_{k+1})$ $y_0 = a$, $y_n = b$.

Let $N_1 = \dots = N_n = 0$

(*) choose $x = RND$

compute $f(x)$

If $f(x) \in I_k$ set $N_k \rightarrow N_k + 1$

GOTO (*)

After N loops compute

$$I = \sum_{k=1}^N y_k \frac{N_k}{N} \quad (98)$$

$$= \sum_{k=1}^N y_k \cdot p(\{x : f(x) \approx y_k\}) \quad (99)$$

$$\approx \int f(x) d\mu(x).. \quad (100)$$

2.6 Some Fundamentals of Probability Theory

Ω : sample space

$A \in \mathcal{B}$: event

$p(A)$: probability of the event A

$f \in M(\Omega, \mathcal{B})$: random variable

$E(f) = \bar{f} = \int_{\Omega} f(\omega) dp(\omega)$: expectation or mean value of the random variable f .

p produces the expectations. Note that conversely

$$E(\chi_A) = p(A) \quad (101)$$

$E((f - \bar{f})^2) = \text{var}(f) = \sigma^2$: Variance, mean square deviation. important to describe fluctuations around the mean value \bar{f} .

Question:

$$p(\{\omega : |f(\omega) - \bar{f}| > \varepsilon\}) = ? \quad (102)$$

To control this, introduce, for $\varepsilon > 0$

$$f_{\varepsilon}(\omega) = \begin{cases} \bar{f} + \varepsilon & \text{if } |f(\omega) - \bar{f}| > \varepsilon \\ \bar{f} & \text{otherwise} \end{cases} \quad (103)$$

i.e.

$$f_{\varepsilon}(\omega) - \bar{f} = \begin{cases} \varepsilon & \text{if } |f(\omega) - \bar{f}| > \varepsilon \\ 0 & \text{otherwise} \end{cases} . \quad (104)$$

Then, for all ω

$$(f_{\varepsilon}(\omega) - \bar{f})^2 \leq (f(\omega) - \bar{f})^2 \quad (105)$$

and further

$$E((f_{\varepsilon} - \bar{f})^2) \leq E((f - \bar{f})^2) = \text{var}(f) \quad (106)$$

i.e.

$$E((f_{\varepsilon} - \bar{f})^2) = \int (f_{\varepsilon}(\omega) - \bar{f})^2 dp(\omega) \quad (107)$$

$$= \varepsilon^2 \cdot p(\{\omega : |f(\omega) - \bar{f}| > \varepsilon\}) \quad (108)$$

$$\leq \text{var}(f) \quad (109)$$

so that

$$p(\{\omega : |f(\omega) - \bar{f}| > \varepsilon\}) \leq \frac{\text{var}(f)}{\varepsilon^2} \quad (110)$$

("Tchebyshev Inequality".)

Notation:

$$E(f^n) = m_n \quad (111)$$

is called the " n^{th} moment of the random variable f "

$$\bar{f} = m_1$$

$$var(f) = m_2 - m_1^2.$$

2.6.1 Covariance and Correlation

Definition 10 For $X, Y \in M(\Omega, \mathfrak{B})$ call

$$cov(X, Y) \equiv E((X - m_X)(Y - m_Y)) \quad (112)$$

their covariance.

$$r_{X,Y} \equiv \frac{cov(X, Y)}{\sigma_X \sigma_Y} \quad (113)$$

is the "correlation coefficient of X and Y ", with

$$-1 \leq r_{X,Y} \leq 1 \quad (114)$$

$$|r_{X,Y}| = 1 \quad (115)$$

iff X and Y are linearly dependent: we have then

$$Y(\omega) = r_{X,Y} \frac{\sigma_Y}{\sigma_X} (X(\omega) - m_X) + m_Y \quad (116)$$

(for almost all ω). Proof?

Important concept: the "distribution of a random variable $y \in M(\Omega, \mathfrak{B})$."

Definition 11 For $A \in \mathfrak{B}_0(R)$ call

$$p(\{\omega : y(\omega) \in A\}) \equiv p_Y(A) \quad (117)$$

the "distribution of the random variable y ".

For given y , p_Y is a probability measure on the Borel space $\{R, \mathfrak{B}_0(R)\}$ and

$$E(g(y)) = \int_{\Omega} g(y(\omega)) dp(\omega) = \int_R g(x) dp_Y(x) \quad (118)$$

Integration over the set R of values instead of over the sample space Ω !

Note:

1.

$$m_n = E(f^n) = \int_R x^n dp_Y(x) \quad (119)$$

2. Generalization: the "joint distribution" of an n -tuple \vec{Y} of random variables: $p_{\vec{Y}}$ a probability measure on $\{R^n, \mathfrak{B}_0(R^n)\}$ with

$$E\left(g(\vec{Y})\right) = \int_{R^n} g(x) dp_{\vec{Y}}(x). \quad (120)$$

3. The moments of a random variable may diverge! Ex. (Cauchy):

$$dp_Y(x) = \rho(x)dx \quad (121)$$

with

$$\rho(y) = \frac{1}{\pi(1+x^2)} \quad (122)$$

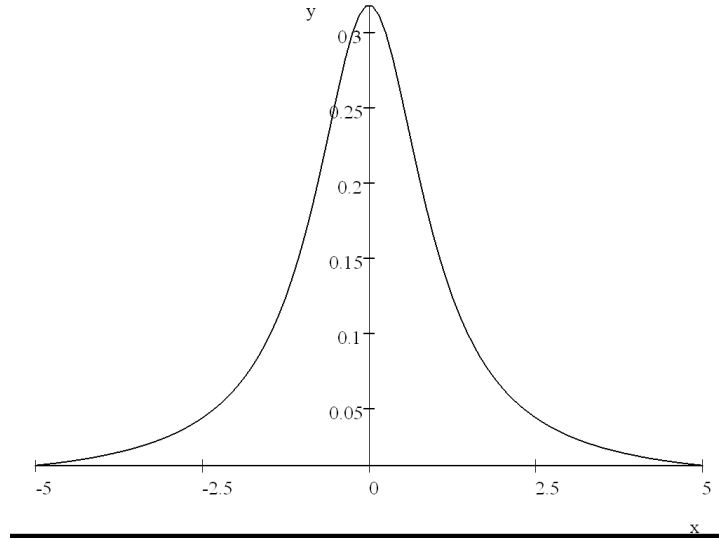


Figure 1: The probability density $\rho(x)$

$$m_n = E(f^n) = \int_R x^n dp_Y(x) \quad (123)$$

$$= \int_R \frac{x^n}{\pi(1+x^2)} dx \quad (124)$$

is divergent for all $n > 0$.

Definition 12 Events A_k $k \in I$, are "independent", if for all subsets J

$$p\left(\bigcap_{k \in J} A_k\right) = \prod_{k \in J} p(A_k). \quad (125)$$

Random variables Y_k are called "independent", if their joint distribution factorizes; the joint distribution factorizes:

$$dp_{\vec{Y}}(\vec{x}) = \prod_{k=1}^n dp_{Y_k}(x_k). \quad (126)$$

The events $A_k : Y_k \in B_k$ are then independent:

$$p\left(\bigcap_{k \in J} A_k\right) = \int dp_{\vec{Y}}(\vec{x}) \prod_{k=1}^n \chi_{B_k}(x_k) \quad (127)$$

$$= \prod_{k=1}^n \int dp_{Y_k}(x_k) \chi_{B_k}(x_k) \quad (128)$$

$$= \prod_{k=1}^n p(A_k). \quad (129)$$

2.7 Characteristic Functions

We have seen that from the distribution of a random variable Y we can determine expectations:

$$E(g(Y)) = \int_R g(x) dp_Y(x) \quad (130)$$

Conversely

$$E(\chi_A(Y)) = p_Y(A). \quad (131)$$

Another class of expectations which completely characterizes a distribution:

Definition 13

$$C_Y(\lambda) \equiv E(e^{i\lambda Y}) = \int_R e^{i\lambda x} dp_Y(x). \quad (132)$$

If.

$$dp_Y(x) = \rho(x)dx, \quad (133)$$

then

$$C_Y(\lambda) = \int_R e^{i\lambda x} \rho(x)dx, \quad (134)$$

is the "Fourier transform" of the density ρ . More generally, for finite measures m we also call its "characteristic function"

$$C(\lambda) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} dm(x) \quad (135)$$

the "Fourier transform of the measure m ".

Example 6 ("Gaussians")

Remember:

$$\int_{\mathbb{R}} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \quad (136)$$

Rescale, with $a > 0$:

$$\int_{\mathbb{R}} e^{-\frac{a}{2}x^2} dx = \sqrt{\frac{2\pi}{a}} \quad (137)$$

Translate $x \rightarrow x - b$:

$$\int_{\mathbb{R}} e^{-\frac{a}{2}(x-b)^2} dx = \sqrt{\frac{2\pi}{a}} \quad (138)$$

i.e.

$$\int_{\mathbb{R}} e^{-\frac{a}{2}x^2 + bx} dx = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}\frac{b^2}{a}} \quad (139)$$

Multiply:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_1^n a_k x_k^2 + b_k x_k} d^n x = \prod_k \sqrt{\frac{2\pi}{a_k}} e^{\frac{1}{2} \sum_1^n \frac{b_k^2}{a_k}} \quad (140)$$

Simplify:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x, Ax) + (b, x)} d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}} e^{\frac{1}{2}(b, A^{-1}b)} \quad (141)$$

Continue analytically and normalize:

$$C(\lambda) \equiv \sqrt{\frac{\det(A)}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x, Ax) + i(\lambda, x)} d^n x \quad (142)$$

$$= e^{-\frac{1}{2}(\lambda, A^{-1}\lambda)} \quad (143)$$

is characteristic function of a probability measure on \mathbb{R}^n , with density

$$\rho(x) = \left(\sqrt{2\pi}\right)^{-n} e^{-\frac{1}{2}(x, Ax)} \quad (144)$$

whenever A is a positive symmetric matrix.

Problem 7 *Compute the characteristic function of the Dirac measure μ_a on \mathbb{R}^1 .*

Problem 8 *Let p be the restriction of the Lebesgue measure to the interval from minus one to one. Compute its characteristic function.*

Example 7 ("Queueing"):

Let $P(n, t)$ be the probability to have n people waiting at the bus stop at time t after departure of the previous one, and $p(n \rightarrow n+1 \text{ in } \Delta t)$ the probability for the arrival of one more person in the time interval Δt .

We suppose that for small Δt

1.

$$p(n \rightarrow n+1 \text{ in } \Delta t) = k \cdot \Delta t \quad (145)$$

2.

$$p(n \rightarrow n+2 \text{ in } \Delta t) = 0 \text{ (etc.)}. \quad (146)$$

3. Also postulate

$$P(0, 0) = 1. \quad (147)$$

Now we can determine $P(n, t)$ as follows:

$$P(n, t + \Delta t) \quad (148)$$

$$= P(n-1, t)p(n-1 \rightarrow n \text{ in } \Delta t) \quad (149)$$

$$+ P(n, t)(1 - p(n-1 \rightarrow n \text{ in } \Delta t)) \quad (150)$$

$$= P(n, t)(1 - k \cdot \Delta t) + P(n-1, t) \cdot k \cdot \Delta t \quad (151)$$

hence have

$$\frac{P(n, t + \Delta t) - P(n, t)}{\Delta t} = k \cdot (P(n-1, t) - P(n, t)) \quad (152)$$

or in the limit of $\Delta t \rightarrow 0$ the differential equation

$$\frac{dP(n, t)}{dt} = k \cdot (P(n-1, t) - P(n, t)). \quad (153)$$

We solve it using the characteristic function

$$C_N(\lambda; t) \equiv E(e^{i\lambda N}) = \sum_{n=0}^{\infty} e^{i\lambda n} P(n, t). \quad (154)$$

It obeys the differential equation

$$\frac{d}{dt} C_N(\lambda; t) = \sum_{n=0}^{\infty} e^{i\lambda n} \frac{d}{dt} P(n, t) \quad (155)$$

$$= k \sum_{n=0}^{\infty} e^{i\lambda n} (P(n-1, t) - P(n, t)) \quad (156)$$

$$= k (e^{i\lambda} - 1) \sum_{n=0}^{\infty} e^{i\lambda n} P(n, t) \quad (157)$$

$$= k (e^{i\lambda} - 1) C_N(\lambda; t). \quad (158)$$

Solution:

$$C_N(\lambda; t) = c \cdot e^{k(e^{i\lambda} - 1)t} \quad (159)$$

and the constant c is

$$C_N(\lambda; 0) = c = \sum_{n=0}^{\infty} e^{i\lambda n} P(n, 0) \quad (160)$$

$$= P(0, 0) = 1 \quad (161)$$

So

$$C_N(\lambda; t) = e^{k(e^{i\lambda} - 1)t} = e^{-kt} \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} e^{i\lambda n}, \quad (162)$$

and comparing coefficients with equation (154), we obtain the "Poisson distribution with intensity k "

$$P(n, t) = e^{-kt} \frac{(kt)^n}{n!}. \quad (163)$$

Now we can e.g. compute the probability for an even number of people waiting:

$$\sum_{n \text{ even}} P(n, t) = e^{-kt} \sum_{n=0}^{\infty} \frac{(kt)^{2n}}{(2n)!} = e^{-kt} \frac{1}{2} (e^{kt} + e^{-kt}) = \frac{1}{2} (1 + e^{-2kt}). \quad (164)$$

Characteristic functions are a useful description of measures. How do I know that some function is the characteristic function of a measure?

Note its properties:

1. Normalization:

$$\underline{C(0)} = \int_{\mathbb{R}^n} dm(x) = m(\Omega) = \underline{M} < \infty \quad (165)$$

2. Continuity near $\lambda = 0$:

$$C(\lambda) - m(\Omega) = \int_{\mathbb{R}^n} (e^{i(\lambda, x)} - 1) dm(x) \quad (166)$$

$$= \int_{\mathbb{R}^n} (\cos((\lambda, x)) - 1) dm(x) \quad (167)$$

$$+ i \int_{\mathbb{R}^n} \sin((\lambda, x)) dm(x) \quad (168)$$

Using dominated convergence we may exchange limits and integral and see that, for $\lambda \rightarrow 0$ the rhs vanishes.

3. "Positive Definiteness": let a_1, \dots, a_n be complex numbers, and $\lambda_1, \dots, \lambda_n$ real vectors, then we always have

$$\sum_{k,l} a_k^* a_l C(\lambda_k - \lambda_l) = \int_{\mathbb{R}^n} \left| \sum_l a_l e^{i(\lambda_l, x)} \right|^2 dm(x) \geq 0 \quad (169)$$

These hold for all characteristic functions. Conversely

Theorem 8 (Bochner). *Any normalized function, continuous at zero and positive definite*

$$C : \mathbb{R}^n \rightarrow \mathbb{C} \quad (170)$$

is the characteristic function of a finite measure on the Borel-Algebra $\mathfrak{B}(\mathbb{R}^n)$.

2.7.1 Properties of characteristic functions

1. Products of characteristic functions are again characteristic functions

$$C(\lambda) \equiv \prod_i C_{Y_i}(\lambda) = \prod_i \int e^{i\lambda y} dp_{Y_i}(y) \quad (171)$$

$$= \int e^{i\lambda \sum_i y_i} \prod_i dp_{Y_i}(y_i) \quad (172)$$

I.e., this is the characteristic function of the random variable

$$Y = \sum_i Y_i \quad (173)$$

with independent Y_i .

2. Positive linear combinations of characteristic functions are characteristic functions of finite measures. For $a_i > 0$ have

$$C_i \leftrightarrow \mu_i \quad (174)$$

$$\sum_i a_i C_i(\lambda) \leftrightarrow \sum_i a_i \mu_i \quad (175)$$

3. For Gauss and Poisson measures have: for all $r > 0$, C^r is again a Gauss resp. a Poisson characteristic function. In such a case we speak of "infinitely divisible" distributions and random variables. Reason:

For $r = 1/n$, $n > 0$ set

$$(C_Y(\lambda))^{1/n} = C_X(\lambda). \quad (176)$$

Then

$$C_Y(\lambda) = (C_X(\lambda))^n, \quad (177)$$

i.e.

$$Y = \sum_i X_i \quad (178)$$

with independent, equally distributed random variables X_i .

2.8 Law of Large Numbers and Central Limit Theorem

Now let us have a look at two items of probabilistic "folk lore":

1. The larger the number of observations, the "better the statistics".
2. For large data sets the fluctuations around the average follows a Gaussian "bell curve".

These notions were subject of early investigations even in the 18th century but are of such central importance in theory and applications that results have been refined and extended ever since:

1. The law of large numbers from Bernoulli 1713 to Kolmogorov 1928.
2. The central limit theorem of de Moivre 1733 via Gauß ("error calculus") all the way to Feller, Lévy, Khinchin et al..

2.8.1 The law of large numbers

In the simplest of setting both theorems consider sequences X_k of independent and equally distributed random variables, think of the repeated throws of dice.

With arithmetic mean

$$A_n \equiv \frac{1}{n} \sum_{k=1}^n X_k. \quad (179)$$

Do not confuse this with the expectation; $A_n = A_n(\omega)$ is a random variable, just as the $X_k = X_k(\omega)$!

Essentially the LLN says that for large n the random variables A_n converge to a constant, more precisely

$$A_n(\omega) \rightarrow E(X_k). \quad (180)$$

Dice: for large n the arithmetic mean of n throws will get closer and closer to the constant average $E(X_k) = 3\frac{1}{2}$, the bigger n , *the less the fluctuations*.

Theorem 9 (*Law of Large Numbers*) Let X_k be independent and equally distributed random variables with finite moments m_1 and m_2 and

$$A_n \equiv \frac{1}{n} \sum_{k=1}^n X_k \quad (181)$$

Then have

$$p(\{\omega : |A_n(\omega) - m_1| > \varepsilon\}) \xrightarrow{n \rightarrow \infty} 0. \quad (182)$$

Note first that

$$E(A_n) = \frac{1}{n} \sum_{k=1}^n E(X_k) = \frac{1}{n} n m_1 = m_1. \quad (183)$$

Hence the probability p above is what is estimated by the Tchebychev inequality:

$$p(\{\omega : |A_n(\omega) - m_1| > \varepsilon\}) < \frac{\sigma^2(A_n)}{\varepsilon^2}. \quad (184)$$

Hence must show

$$\sigma^2(A_n) \xrightarrow{n \rightarrow \infty} 0 \quad (185)$$

$$\begin{aligned} \sigma^2(A_n) &= E\left((A_n(\omega) - m_1)^2\right) = E\left(\left(\frac{1}{n} \sum_{k=1}^n (X_k - m_1)\right)^2\right) \\ &= \frac{1}{n^2} \sum_{k,l=1}^n E((X_k - m_1)(X_l - m_1)). \end{aligned}$$

Independence of the $(X_k - m_1)$ implies

$$E((X_k - m_1)(X_l - m_1)) = E(X_k - m_1) E(X_l - m_1) = 0. \quad (186)$$

So

$$\sigma^2(A_n) = \frac{1}{n^2} \sum_{k=1}^n E((X_k - m_1)(X_k - m_1)) \quad (187)$$

$$= \frac{1}{n^2} n \sigma^2(X_k) = \frac{1}{n} \sigma^2(X_k) \xrightarrow{n \rightarrow \infty} 0. \quad (188)$$

Another approach is via characteristic functions and will lead us to the central limit theorem. As before the X_k are equally distributed, for simplicity we set $m_1 = 0$. Then their characteristic function

$$C_{X_k}(\lambda) = E(e^{iX_k}) = \sum \frac{(i\lambda)^n}{n!} E(X_k^n) = 1 - \sigma^2 \frac{\lambda^2}{2!} + \dots \quad (189)$$

and

$$\begin{aligned} C_{A_n}(\lambda) &= C_{\frac{1}{n} \sum_{k=1}^n X_k}(\lambda) = E\left(e^{i\frac{\lambda}{n} \sum_{k=1}^n X_k}\right) = \prod_k E\left(e^{i\frac{\lambda}{n} X_k}\right) \\ &= \prod_k C_{X_k}\left(\frac{\lambda}{n}\right) \approx \left(1 - \sigma^2 \frac{\lambda^2}{2n^2}\right)^n \approx e^{-\sigma^2 \frac{\lambda^2}{2n}} \xrightarrow{n \rightarrow \infty} 1. \end{aligned}$$

$C_{A_n}(\lambda) \xrightarrow{n \rightarrow \infty} 1$ implies $p_{A_n} \xrightarrow{n \rightarrow \infty} \delta_0$, i.e.

$$A_n \xrightarrow{n \rightarrow \infty} 0 \quad (190)$$

or in general:

$$A_n \xrightarrow{n \rightarrow \infty} m_1. \quad (191)$$

2.8.2 The central limit theorem

We consider once more independent and equally distributed random variables X_k with finite moments m_1 and m_2 ; without loss of generality we fix $E(X_k) = m_1 = 0$. We have seen that then the fluctuations (around zero) of the arithmetic mean

$$A_n \equiv \frac{1}{n} \sum_{k=1}^n X_k \quad (192)$$

go to zero. In fact

$$C_{A_n}(\lambda) \approx e^{-\sigma^2 \frac{\lambda^2}{2n}} \quad (193)$$

corresponds to the probability density

$$\rho(x) = \text{const.} \cdot e^{-\frac{n\lambda^2}{2\sigma^2}}, \quad (194)$$

a more and more narrow bell curve as $n \rightarrow \infty$.

A Gauss distribution is obtained for $n \rightarrow \infty$, if we amplify the fluctuations by a factor \sqrt{n} ; thus we consider

$$Y_n = \sqrt{n} A_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k. \quad (195)$$

If we now investigate its characteristic function as before, we find

$$\begin{aligned} C_{Y_n}(\lambda) &= C_{\frac{1}{\sqrt{n}} \sum_{k=1}^n X_k}(\lambda) = E\left(e^{i \frac{\lambda}{\sqrt{n}} \sum_{k=1}^n X_k}\right) = \prod_k E\left(e^{i \frac{\lambda}{\sqrt{n}} X_k}\right) \\ &= \prod_k C_{X_k}\left(\frac{\lambda}{\sqrt{n}}\right) \approx \left(1 - \sigma^2 \frac{\lambda^2}{2n}\right)^n \approx e^{-\sigma^2 \frac{\lambda^2}{2}} \end{aligned} \quad (196)$$

i.e.

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k \approx N(0, \sigma^2).$$

2.9 Simulation of random variables

2.9.1 Simulation of Gaussian Random Variables

Use the central limit theorem!

Let X_1, \dots, X_{12} be 12 independent random variables, with values uniformly distributed in the interval $[0, 1]$, i.e.. for $0 \leq a_i \leq b_i \leq 1$ have

$$p(\{\omega : X_i \in [a_i, b_i]\}) = \prod_i (b_i - a_i). \quad (197)$$

Set

$$Y(\omega) = \sum_{i=1}^{12} X_i(\omega) - 6. \quad (198)$$

Claim: Y is approximately normally distributed, with mean 0 and variance $\sigma^2 = 1$

$$Y \approx N(0, 1). \quad (199)$$

Proof. Have

$$Y(\omega) = \frac{1}{\sqrt{12}} \sum_1^{12} Z_i(\omega) \quad (200)$$

with independent and equally distributed

$$Z_i = \sqrt{12} \left(X_i - \frac{1}{2} \right). \quad (201)$$

For them we compute

$$E(Z_i) = \sqrt{12} \left(E(X_i) - \frac{1}{2} \right) = 0 \quad (202)$$

and

$$E(Z_i^2) = 12 \left(E(X_i^2) - \frac{1}{4} \right). \quad (203)$$

$$E(X_i^2) = \int_0^1 x^2 dx = \frac{1}{3} \quad (204)$$

so that

$$E(Z_i^2) = 12 \left(\frac{1}{3} - \frac{1}{4} \right) = 1. \quad (205)$$

I.e. by the central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_1^n Z_i(\omega) \approx N(0, 1). \quad (206)$$

■

2.9.2 Simulation of general continuous distributions

Again we use a standard random number generator; X will have the distribution density

$$\rho_X = I_{[0,1]} \quad (207)$$

i.e. the distribution function

$$F_x(s) = \begin{cases} 0 & s < 0 \\ s & s \in [0, 1] \\ 1 & s > 1 \end{cases} \quad (208)$$

Consider now a strictly increasing function G :

$$G : \mathbb{R} \rightarrow [0, 1] \nearrow \nearrow \quad (209)$$

We can invert it on the unit interval:

$$G^{-1} : [0, 1] \rightarrow \mathbb{R}, \quad (210)$$

and we now form the random variable

$$Y(\omega) = G^{-1}(X(\omega)). \quad (211)$$

What is its distribution function?

$$F_Y(x) = p(\{\omega : Y(\omega) < x\}) \quad (212)$$

$$= p(\{\omega : G^{-1}(X(\omega)) < x\}) \quad (213)$$

$$= p(\{\omega : X(\omega) < G(x)\}) \quad (214)$$

$$= F_X(G(x)) = G(x), \quad (215)$$

i.e. the random variable we constructed $Y(\omega) = G^{-1}(X(\omega))$ has the distribution function G .

Problem 9 Consider the routine

10 $x = \text{RND}$

20 $y = \tan \pi(x - 1/2)$

30 $\text{GOTO } 10$

What is the probability density of the values y it produces?

3 Generalized functions (Distributions)

Refs.: [1][4][6][9]

3.1 Introduction

We know functions first of all as mappings such as

$$\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \quad (216)$$

$$\varphi : x \rightarrow \varphi(x) \quad (217)$$

Lebesgue-measurable functions φ also play another role: they produce mappings T_φ from ("test"-) functions f onto \mathbb{R} :

$$T_\varphi : f \rightarrow \int \varphi(x)f(x)d^n x \quad (218)$$

In this sense we define: distributions are continuous linear functionals on certain spaces of differentiable ("test"-)functions.

Example 8

$$T_\varphi : f \rightarrow \int \varphi(x)f(x)dx \quad (219)$$

More general linear mappings such as e.g.

$$T : f \rightarrow f(0) \quad (220)$$

are *not* generated by functions φ , hence the name "generalized functions".

Applications: solving PDEs, particularly in the mathematical physics, electrodynamics, quantum field theory.

3.2 Test function spaces and distributions

In what follows we shall focus on linear subspaces of $C^\infty(\mathbb{R}^n)$, the arbitrarily often continuously differentiable real- or complex valued functions.

3.2.1 The space D .

Definition 14 By $D(\mathbb{R}^n)$ one denotes the space of functions from $C^\infty(\mathbb{R}^n)$ which vanish outside some bounded region. We can define convergence $f_n \rightarrow f$ to be equivalent with

1. all f_n vanish outside some (common!) bounded domain $B \subset \mathbb{R}^n$, and
2. inside B the f_n and all their derivatives converge uniformly.

Linear functionals T are called continuous if

$$f_n \rightarrow f \Rightarrow T f_n \rightarrow T f. \quad (221)$$

Definition 15 The set D' of all continuous functionals on D (the "dual" space) is called the "space of distributions over D ".

Notation:

$$T f = \langle T, f \rangle \quad (222)$$

("dual product").

Exercise 1

$$f(x) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & \text{for } x^2 < a^2 \\ 0 & \text{otherwise} \end{cases} \quad (223)$$

is in D , likewise

$$h(x) \equiv f(x)g(x) \text{ for any } g \in C^\infty(\mathbb{R}^n).$$

Exercise 2 ("Dirac distribution") Show that

$$\langle \delta_a, f \rangle \equiv f(a) \quad (224)$$

defines a distribution $\delta_a \in D'$

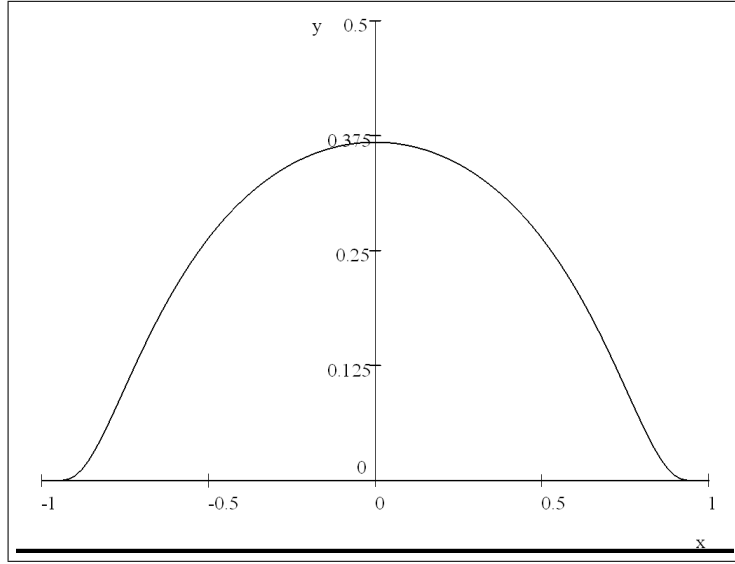


Figure 2: A test function in $D(\mathbb{R})$

δ_a can be represented by a measure:

$$\langle \delta_a, f \rangle = \int f(x) d\mu_a(x) \quad (225)$$

where μ_a is the Dirac measure concentrated on $a \in \mathbb{R}^n$. Notation:

$$\int \delta(x-a) f(x) dx = f(a) \quad (226)$$

Approximation e.g. by Gauss functions

$$\delta_\varepsilon(x-a) = \left(\frac{1}{\sqrt{2\pi\varepsilon}} \right)^n e^{-\frac{(x-a)^2}{2\varepsilon}}. \quad (227)$$

For small ε

$$\int \delta_\varepsilon(x-a) f(x) dx \approx f(a), \quad (228)$$

more precisely

$$\lim_{\varepsilon \rightarrow 0} \int \delta_\varepsilon(x-a) f(x) dx = f(a), \quad (229)$$

if f is a test function. **But there is no (Lebesgue integrable) limit function δ** , such that (226) holds.

More generally a Borel measure μ produces a distribution

$$\langle T, f \rangle = \int f(x) d\mu(x) \quad (230)$$

if $\mu(B)$ is finite for bounded regions B . Such distributions are "positive", in the sense that

$$f \geq 0 \Rightarrow \langle T, f \rangle \geq 0 \quad (231)$$

and one can show conversely that all positive distributions have a representation (230) with a suitable locally finite measure μ .

Exercise 3

$$T_x : f \rightarrow f'(x) \quad (232)$$

is also a continuous linear functional on the space $D(R)$.

"Regular" distributions: If g is Borel-integrable on bounded regions ("locally integrable"), then

$$\langle G, f \rangle \equiv \int g(x)f(x)dx \quad (233)$$

defines a distribution $G \in D$. If g is non-negative, then G is positive.

In particular the "Heaviside function"

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (234)$$

is a distribution $\Theta \in D'(R)$:

$$\langle \Theta, f \rangle = \int \Theta(x)f(x)dx = \int_0^\infty f(x)dx. \quad (235)$$

3.3 The Schwartz Space $S(\mathbb{R}^n)$

Definition 16 Consider functions which, with all their derivatives, decrease faster than polynomially at infinity:

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : |x^k D^p f(x)| < \text{const}_{k,p} \forall k, p\} \quad (236)$$

Convergence $f_n \rightarrow f$ shall mean:

1. all f_n obey estimates

$$|x^k D^p f_n(x)| < \text{const}_{k,p} \quad \forall x \quad (237)$$

(independent of n !)

2. Inside any bounded domain B the f_n and all their derivatives converge uniformly.

For D and S one can show that all limit functions are again in D resp. S : D and S are closed.

Example 9 :

$$f(x) = e^{-ax^2} \quad a > 0$$

1. If f is a test function, then so are all its derivatives

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x).$$

Assume we expand functions in terms of an orthogonal basis of Hermite functions

$$f(x) = \sum \lambda_n e_n(x)$$

with

$$e_n(x) = c_n \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}.$$

We get the differentiability and rapid decrease that are required of test functions if we admit only rapidly decreasing sequences of coefficients (λ_n) . In fact we have

$$\begin{aligned} f &\in S(R) \leftrightarrow \sum n^k \lambda_n^2 < \infty \text{ for all } k, \\ \text{whereas } f &\in L^2(R) \leftrightarrow \sum \lambda_n^2 < \infty. \end{aligned}$$

Exercise 4 Let H be the harmonic oscillator Hamiltonian. Show that

$$\langle \psi | H^k | \psi \rangle < \infty$$

for all $k > 0$ if and only if

$$\psi \in S(R).$$

Definition 17 The dual space S' of continuous linear functionals on S is called the "space of tempered distributions" or Schwartz distributions.

Remark 2 1. D' and S' are vector spaces, if we define $a\Phi_1 + b\Phi_2$ via

$$\langle a\Phi_1 + b\Phi_2, f \rangle = a\langle \Phi_1, f \rangle + b\langle \Phi_2, f \rangle \quad (238)$$

2. Have $D \subset S$. And $f_n \rightarrow 0$ in D , implies $f_n \rightarrow 0$ in S , so that $S' \subset D'$.

Example 10

$$g(x) = e^{x^2} \quad (239)$$

is a regular distribution $G \in D'$, but not in S' .

Definition 18 If for a sequence Φ_n of (tempered) distributions, and for all test functions f

$$\lim_n \langle \Phi_n, f \rangle = \langle \Phi, f \rangle \quad (240)$$

then we say that Φ_n "converges weakly to the (tempered) distribution Φ ".

Weak convergence in S' implies weak convergence in D' . For the spaces S' and D' one can show that convergence of the lhs (for all test functions f) implies the existence of a limit distribution Φ : the spaces S' and D' are "complete".

If for all test functions f which are zero outside an open set U we find

$$\langle T, f \rangle = 0 \quad (241)$$

then we say that " T vanishes on U ".

If a point a has no neighborhood of on which T vanishes, then a is called an "essential point of T ".

The set of all essential points is called the "support of T ".

Example 11 *The support of the Dirac-distribution δ_a is the point $a \in \mathbb{R}^n$.*

Can show: if the support of a distribution T is just one point $a \in \mathbb{R}^n$ then it is a finite linear combination of derivatives of the Dirac distribution.

$$T = \sum^N a_k \delta_a^{(k)} \quad (242)$$

Any distribution with bounded support is tempered.

3.4 Distribution Calculus

Have

1.

$$\langle a\Phi_1 + b\Phi_2, f \rangle = a\langle \Phi_1, f \rangle + b\langle \Phi_2, f \rangle \quad (243)$$

2. $\Phi_n \rightarrow \Phi$ if $\lim_n \langle \Phi_n, f \rangle$ always exists:

$$\lim_n \langle \Phi_n, f \rangle = \langle \Phi, f \rangle \quad (244)$$

3. Translation and scaling of variables *as for regular distributions*:

a) Translation

$$\langle \Phi_{-a}, f \rangle = \int \varphi(x-a)f(x)dx = \int \varphi(x)f(x+a)dx = \langle \Phi, f_{+a} \rangle \quad (245)$$

the latter expression defines a continuous linear functional Φ_{-a}

b) Scaling

$$\langle \Phi_{\cdot a}, f \rangle = \int \varphi(ax)f(x)dx = \frac{1}{|a|^n} \int \varphi(y)f\left(\frac{y}{a}\right)dy = \frac{1}{|a|^n} \langle \Phi, f_{\cdot 1/a} \rangle \quad (246)$$

3.4.1 Differentiation

We define

$$\langle \Phi', f \rangle \equiv -\langle \Phi, f' \rangle \quad (247)$$

Note:

a) for regular differentiable distributions this is the usual derivative:

$$\langle \Phi', f \rangle = - \int \varphi(x) f'(x) dx \quad (248)$$

$$= \int \varphi'(x) f(x) dx \quad (249)$$

b) the rhs defines a continuous linear functional for any given distribution $\Phi \in D'$, i.e., all distributions are arbitrarily often differentiable.

Example 12 *What is the derivative of the step function?*

$$\langle \Theta', f \rangle = - \int \Theta(x) f'(x) dx = - \int_0^\infty f'(x) dx \quad (250)$$

$$= f(0) = \langle \delta_0, f \rangle \quad (251)$$

The derivative of the Heaviside function is the delta function. Now we differentiate the delta function:

$$\langle \delta'_0, f \rangle = -\langle \delta_0, f' \rangle = -f'(0) \quad (252)$$

Example 13 *Let*

$$\langle G, f \rangle = \int \log |x| f(x) dx \quad (253)$$

Then

$$\langle G', f \rangle = - \int \log |x| f'(x) dx \quad (254)$$

$$= -\lim_{\varepsilon \rightarrow +0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \log |x| f'(x) dx \quad (255)$$

$$= \lim_{\varepsilon \rightarrow +0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{1}{x} f(x) dx = P \int \frac{1}{x} f(x) dx \quad (256)$$

i.e. in the sense of distributions

$$(\log |x|)' = P \frac{1}{x} \quad (257)$$

where the distribution on the rhs is Cauchy's "principal value integral".

Likewise one can define all higher inverse powers as (higher derivatives of regular) distributions

$$x^{-n} = \frac{(-1)^{n-1}}{(n-1)!} (\log |x|)^{(n)} \quad (258)$$

the lhs will not be integrable; the derivative on the rhs is well defined in the sense of distributions!

- For partial derivatives of distributions on \mathbb{R}^n have

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_k} = \frac{\partial^2 \Phi}{\partial x_k \partial x_i}. \quad (259)$$

Limit and Differentiation can always be interchanged!

Proof: Let $\Phi = \lim_n \Phi_n$

$$\langle \lim_n (\Phi'_n), f \rangle = \lim_n \langle \Phi'_n, f \rangle = - \lim_n \langle \Phi_n, f' \rangle \quad (260)$$

$$= - \langle \Phi, f' \rangle = \langle \Phi', f \rangle \quad (261)$$

$$= \langle \left(\lim_n \Phi_n \right)', f \rangle \quad (262)$$

Exercise 5 Compute the derivative of the distribution

$$T : f \rightarrow \int |x| f(x) dx.$$

3.4.2 Multipliers

Contrary to usual functions, multiplication of generalized functions - ie. of linear functionals - does not make sense a priori. However one can try to generalize from the case of regular distributions:

$$\langle \Phi, \alpha \cdot f \rangle = \int \varphi(x) \alpha(x) f(x) dx \equiv \langle \alpha \cdot \Phi, f \rangle \quad (263)$$

For which multiplier functions α might this work?

Surely whenever $\alpha \cdot f$ is in D or S respectively and the lhs is a continuous functional on D resp. on S .

Theorem 10 1. Let Φ be a distribution and $\alpha \in C^\infty$. Then the mapping

$$\alpha \cdot \Phi : D(1R^n) \rightarrow K \quad (264)$$

$$\alpha \cdot \Phi : f \rightarrow \langle \Phi, \alpha \cdot f \rangle \quad (265)$$

is a distribution.

2. Let Φ be a tempered distribution and $\alpha \in C^\infty$ such that all its derivatives are polynomially bounded, i.e.

$$|D^k \alpha(x)| < p_k(x) \quad (266)$$

for some polynomials p_k . Then the mapping

$$\alpha \cdot \Phi : S(\mathbb{R}^n) \rightarrow K \quad (267)$$

$$\alpha \cdot \Phi : f \rightarrow \langle \Phi, \alpha \cdot f \rangle \quad (268)$$

is a tempered distribution.

3.4.3 Convolutions

The composition of two functions

$$(f * g)(x) = \int f(x-y)g(y)dy \quad (269)$$

is called "convolution". Have

$$(f * g)(x) = \int f(x-y)g(y)dy \quad (270)$$

$$= \int f(z)g(x-z)dz \quad (271)$$

$$= (g * f)(x). \quad (272)$$

the convolution is commutative.

How can we define the convolution

$$\Phi * f \quad (273)$$

of a distribution and a test function?

Motivation: Consider partial differential equations, e.g. in physics, such as

$$\sum a_k D^k f(x) = \rho(x) \quad (274)$$

to be solved for functions f , for given $a_k, \rho(x)$. Instead of doing this for each given ρ , look for the "fundamental solution" φ of

$$\sum a_k D^k \varphi(x) = \delta(x) \quad (275)$$

and compute for "any" ρ the solution

$$f = \varphi * \rho \quad (276)$$

$$= \int \varphi(x-y)\rho(y)dy \quad (277)$$

because

$$\sum a_k D^k f(x) = \int \sum a_k D^k \varphi(x-y) \rho(y) dy \quad (278)$$

$$= \int \delta(x-y) \rho(y) dy \quad (279)$$

$$= \rho(x). \quad (280)$$

Hence want to study convolutions with generalized functions.

Again we start with a regular distribution for inspiration.

$$(\Phi * f)(x) = \int \varphi(x-y) f(y) dy \quad (281)$$

$$= \int \varphi(-y) f(y+x) dy \quad (282)$$

$$= \int \varphi(y) f(x-y) dy \quad (283)$$

$$= \langle \Phi, (f^{(x)}) \rangle \quad (284)$$

where $f^{(x)}(y) \equiv f(x-y) \in D$, i.e. the rhs exists for **any** distribution Φ and so defines the convolution.

Example 14 *Convolution with the delta function*

$$(\delta_a * f)(x) = \langle \delta_a, f^{(x)} \rangle = f(x-a). \quad (285)$$

Properties:

$\Phi * f \in C^\infty(\mathbb{R}^n)$. For $n=1$:

$$\frac{(\Phi * f)(x+\varepsilon) - (\Phi * f)(x)}{\varepsilon} \quad (286)$$

$$= \frac{1}{\varepsilon} \left(\langle \Phi, f^{(x+\varepsilon)} \rangle - \langle \Phi, f^{(x)} \rangle \right) \rightarrow \langle \Phi, \partial_x f^{(x+\varepsilon)} \rangle \quad (287)$$

Higher derivatives by iteration.

Important approximations for the Delta function δ_0 are

$$\lim_{\varepsilon} \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}, \quad \frac{1}{\pi} \lim_{\varepsilon} \frac{\varepsilon}{x^2 + \varepsilon^2}, \quad \lim_n \frac{\sin nx}{\pi x} \quad (288)$$

For $n=1$ (one independent variable) any distribution can be written as boundary value of analytic functions:

Theorem 11 *Let $\Phi \in D'$. Then there is a function g , analytic outside the support of Φ , such that*

$$\langle \Phi, f \rangle = \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} (g(x+i\varepsilon) - g(x-i\varepsilon)) f(x). \quad (289)$$

If Φ has bounded support then

$$g(z) = \frac{1}{2\pi i} \left\langle \Phi, \frac{1}{\cdot - z} \right\rangle. \quad (290)$$

Example 15

$$\frac{1}{x \pm i0} \equiv \lim_{\varepsilon \rightarrow +0} \frac{1}{x \pm i\varepsilon} = \lim_{\varepsilon \rightarrow +0} \frac{d}{dx} \log(x \pm i\varepsilon) \quad (291)$$

$$= \frac{d}{dx} (\log|x| \pm i\pi\theta(-x)) = \frac{P}{x} \mp i\pi\delta(x) \quad (292)$$

Hence

$$\frac{P}{x} = \frac{1}{2} \lim_{\varepsilon \rightarrow +0} \left(\frac{1}{x + i\varepsilon} + \frac{1}{x - i\varepsilon} \right) \quad (293)$$

and

$$\delta(x) = -\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow +0} \left(\frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon} \right) \quad (294)$$

In the latter case the analytic representation is given by

$$g(z) = -\frac{1}{2\pi iz}, \quad (295)$$

and for P/x by

$$g(z) = \begin{cases} \frac{1}{2z} & \text{for } \text{Im}z > 0 \\ -\frac{1}{2z} & \text{for } \text{Im}z < 0 \end{cases}. \quad (296)$$

3.4.4 Fourier transform

On the space $S(\mathbb{R}^n)$ of Schwartz test functions we consider the linear transformation

$$F : f \rightarrow \tilde{f} \quad (297)$$

with

$$\tilde{f}(p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{ipx} d^n x. \quad (298)$$

Clearly this Fourier transform \tilde{f} is also arbitrarily often differentiable, with

$$D^k \tilde{f}(p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) (ix)^k e^{ipx} d^n x, \quad (299)$$

writing

$$(ix)^k \equiv \prod_{l=1}^n (ix_l)^{k_l} \quad (300)$$

and we have

$$p^k \tilde{f}(p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) (-iD)^k e^{ipx} d^n x = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (iD)^k f(x) e^{ipx} d^n x. \quad (301)$$

Theorem 12

$$F : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n) \quad (302)$$

Proof. (Exercise!) ■

Theorem 13 *The Fourier transform is continuous on $S(\mathbb{R}^n)$:*

$$f_n \rightarrow f \Rightarrow \tilde{f}_n \rightarrow \tilde{f} \quad (303)$$

The inverse Fourier transform is given by

$$F^{-1} : \tilde{f} \rightarrow f \quad (304)$$

and one finds

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \tilde{f}(p) e^{-ipx} d^n p. \quad (305)$$

3.4.5 Fourier transforms of distributions

For regular distributions

$$\langle \tilde{G}, f \rangle = \int dy \tilde{g}(y) f(y) \quad (306)$$

$$= \frac{1}{(2\pi)^{n/2}} \int dy f(y) \int dx e^{ixy} g(x) \quad (307)$$

$$= \int dx g(x) \frac{1}{(2\pi)^{n/2}} \int dy f(y) e^{ixy} \quad (308)$$

$$\int dx g(x) \tilde{f}(x) \quad (309)$$

$$= \langle G, \tilde{f} \rangle. \quad (310)$$

Extend this to general tempered distributions G (\tilde{f} is continuous!).

Remark 3 *The important relation*

$$\int dx g(x) \tilde{f}(x) = \int dy \tilde{g}(y) f(y)$$

is the "Parseval Formula", holding not only for functions from $S(\mathbb{R}^n)$ but whenever

$$f, g \in L^2(\mathbb{R}^n, dx) \quad (311)$$

i.e. when

$$\int dx |f(x)|^2 < \infty \quad (312)$$

and

$$\int dx |g(x)|^2 < \infty. \quad (313)$$

Example 16 1.

$$\langle \tilde{\delta}_0, f \rangle = \langle \delta_0, \tilde{f} \rangle = \tilde{f}(0) = \frac{1}{(2\pi)^{n/2}} \int f(x) dx \quad (314)$$

i.e. $\tilde{\delta}_0$ is a regular distribution:

$$\tilde{\delta}_0(x) = \frac{1}{(2\pi)^{n/2}} \quad (315)$$

Likewise

$$\tilde{1} = (2\pi)^{n/2} \delta_0 \quad (316)$$

2. The Heaviside distribution Θ

$$\langle \tilde{\Theta}, f \rangle = \langle \Theta, \tilde{f} \rangle \quad (317)$$

$$= \int_0^\infty dp \tilde{f}(p) \quad (318)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \int_{-\infty}^\infty dx e^{ipx} f(x) \quad (319)$$

Note: Here you may not switch the order of integrations! In

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dx \left(\int_0^\infty dp e^{ipx} \right) f(x) \quad (320)$$

the integral over p is divergent! In other words: $\tilde{\Theta}$ is not a regular distribution. We use instead the following trick (Regularization):

$$\langle \tilde{\Theta}, f \rangle = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \int_{-\infty}^\infty dx e^{ipx} f(x) \quad (321)$$

$$= \lim_{\varepsilon \rightarrow +0} \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \int_{-\infty}^\infty dx e^{ip(x+i\varepsilon)} f(x) \quad (322)$$

$$= \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^\infty dx \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty dp e^{ip(x+i\varepsilon)} \right) f(x) \quad (323)$$

where

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty dp e^{ip(x+i\varepsilon)} = \frac{1}{i(x+i\varepsilon)} \frac{e^{ip(x+i\varepsilon)}}{\sqrt{2\pi}} \Big|_{p=0}^\infty \quad (324)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{x+i\varepsilon}. \quad (325)$$

so that finally

$$\tilde{\Theta}(x) = \frac{i}{\sqrt{2\pi}} \lim_{\varepsilon \rightarrow +0} \frac{1}{x+i\varepsilon}. \quad (326)$$

Exercise 6 Compute the Fourier transforms of the following distributions on \mathbb{R}^1

1.

$$T = \delta_a$$

2.

$$T : f \rightarrow f'(a)$$

3.

$$T : f \rightarrow \int x f(x) dx$$

3.4.6 Fourier transform of products and convolutions

The Fourier transform for the product of two test functions from $S(\mathbb{R}^n)$

$$h(x) = f(x)g(x) \quad (327)$$

is

$$\begin{aligned} \tilde{h}(p) &= (2\pi)^{-n/2} \int dx e^{ipx} f(x)g(x) = \frac{1}{(2\pi)^{3n/2}} \int dx e^{ipx} \int dq e^{-iqx} \int dk e^{-ikx} \tilde{f}(q)\tilde{g}(k) \\ &= \frac{1}{(2\pi)^{n/2}} \int dq \int dk \tilde{f}(q)\tilde{g}(k) \frac{1}{(2\pi)^n} \int dx e^{i(p-q-k)x} \end{aligned} \quad (328)$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{n/2}} \int dq \int dk \tilde{f}(q)\tilde{g}(k) \delta(p-q-k) = \frac{1}{(2\pi)^{n/2}} \int dk \tilde{f}(p-k)\tilde{g}(k) \\ \widetilde{fg} &= \frac{1}{(2\pi)^{n/2}} \tilde{f} * \tilde{g}. \end{aligned} \quad (329)$$

Likewise one shows

$$\widetilde{f * g} = (2\pi)^{n/2} \tilde{f} \cdot \tilde{g}. \quad (330)$$

The same rules extend to products and convolutions of distribution whenever these are well-defined.

3.4.7 Solving linear PDEs

Example 17

$$(-\Delta + m^2)g(x) = \delta(x) \quad (331)$$

where

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \quad (332)$$

the "Laplace-Operator". Fourier transform produces

$$(k^2 + m^2)\tilde{g}(k) = \tilde{\delta}(k) = (2\pi)^{-n/2} \quad (333)$$

Hence

$$\tilde{g}(k) = (2\pi)^{-n/2} \frac{1}{k^2 + m^2}. \quad (334)$$

By inverse Fourier transform we get

$$g(x) = (2\pi)^{-n} \int \frac{1}{k^2 + m^2} e^{-ikx} d^n k. \quad (335)$$

For $n=3$ explicitly :

$$g(x) = (2\pi)^{-3} \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\Theta \int_0^\infty dk k^2 \frac{1}{k^2 + m^2} e^{-ik|x|\cos\Theta} \quad (336)$$

$$= (2\pi)^{-2} \int_0^\infty dk k^2 \frac{1}{k^2 + m^2} \frac{1}{-ik|x|} \left(e^{-ik|x|} - e^{ik|x|} \right) \quad (337)$$

$$= \frac{1}{2i(2\pi)^2 |x|} \int_{-\infty}^\infty dk k \frac{1}{k^2 + m^2} \left(e^{ik|x|} - e^{-ik|x|} \right) \quad (338)$$

$$= \frac{1}{2i(2\pi)^2 |x|} \int_{-\infty}^\infty dk k \frac{1}{k^2 + m^2} e^{ik|x|} + c.c. \quad (339)$$

This integral we compute by closing the integration path in the upper half plane. There the integrand has a pole of 1. degree at $k = im$. Hence

$$g(x) = \frac{1}{2i(2\pi)^2 |x|} 2\pi i \text{Res} \frac{k}{(k + im)(k - im)} e^{ik|x|} \Big|_{k=im} + c.c. \quad (340)$$

$$= \frac{1}{4\pi |x|} \frac{im}{2im} e^{-m|x|} + c.c. = \frac{1}{4\pi |x|} e^{-m|x|}. \quad (341)$$

Example 18 The inhomogeneous equation

$$(-\Delta + m^2) f(x) = h(x). \quad (342)$$

Fourier transform:

$$(k^2 + m^2) \tilde{f}(k) = \tilde{h}(k) \quad (343)$$

$$\tilde{f}(k) = \frac{1}{k^2 + m^2} \tilde{h}(k) = (2\pi)^{n/2} \tilde{g}(k) \tilde{h}(k). \quad (344)$$

Hence

$$f = g * h \quad (345)$$

$$f(x) = \int \frac{1}{4\pi |x - y|} e^{-m|x-y|} h(y) d^3 y. \quad (346)$$

Example 19 "Heat Equation"

$$\partial_t g(x, t) = \frac{\Delta}{2} g(x, t) \quad (347)$$

with "initial value"

$$g(x, 0) = \delta(x). \quad (348)$$

Fourier transform:

$$\partial_t \tilde{f}(k, t) = -\frac{k^2}{2} \tilde{f}(k, t) \quad (349)$$

$$\tilde{g}(k, 0) = (2\pi)^{-n/2} \quad (350)$$

is solved by

$$\tilde{g}(k, t) = e^{-\frac{k^2}{2}t} \tilde{g}(k, 0) = (2\pi)^{-n/2} e^{-\frac{k^2}{2}t} \quad (351)$$

Hence

$$\begin{aligned} g(x, t) &= (2\pi)^{-n} \int e^{-\frac{k^2}{2}t} e^{-ikx} d^n k = \prod_{r=1}^n (2\pi)^{-1} \int e^{-\frac{k_r^2}{2}t} e^{-ik_r x_r} dk_r \\ &= \prod_{r=1}^n (2\pi)^{-1} \sqrt{\frac{2\pi}{t}} e^{-\frac{k_r^2}{2t}} = (2\pi t)^{-n/2} e^{-\frac{k^2}{2t}}. \end{aligned} \quad (352)$$

The "heat kernel" g , fundamental solution of the heat equation, describes the distribution of heat in a linear rod at time t , which for $t = 0$ was concentrated at $x = 0$ (Dirac distribution), it is a Gauss distribution with width $\sigma = \sqrt{t}$. Heat spreads like the square root of time!

3.4.8 Random walk and Brownian motion

Construction of the Wiener process as limit of random walks.

The drunk on the street randomly makes steps either forward or back. What is the probability $p_t(x)$ to find him at the the point x at time t ?

Or equivalently: in a coin toss I either win or lose one Peso each time. What will be my wealth at time t ?

Evidently

$$p_{t+1}(x) = \frac{1}{2} (p_t(x+1) + p_t(x-1)), \quad (354)$$

hence

$$p_{t+1}(x) - p_t(x) = \frac{1}{2} (p_t(x+1) - p_t(x) - (p_t(x) - p_t(x-1))). \quad (355)$$

Now we rescale the time and space steps

$$t+1 \rightarrow t + \Delta t \quad (356)$$

$$x+1 \rightarrow x + \Delta x \quad (357)$$

and set

$$(\Delta x)^2 = \Delta t. \quad (358)$$

Then our equation becomes

$$\frac{p_{t+\Delta t}(x) - p_t(x)}{\Delta t} = \frac{1}{2} \frac{\frac{p_t(x+\Delta x) - p_t(x)}{\Delta x} - \frac{p_t(x) - p_t(x-\Delta x)}{\Delta x}}{\Delta x}. \quad (359)$$

For $(\Delta x)^2 = \Delta t \rightarrow 0$ this turns into the heat equation (347)

$$\frac{d}{dt}p_t(x) = \frac{1}{2} \frac{d^2}{dx^2}p_t(x) \quad (360)$$

with initial value

$$p_0(x) = \delta(x), \quad (361)$$

all paths start at $x=0$. Solution:

$$p_t(x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}}. \quad (362)$$

Successive coin flip games where one either wins or loses one "Peso" will produce such a random "walk" of ups and downs. The following figures show an such up-and-down random walk as a function of time, scaled successively, from $\Delta t = 1$ to 0.1, 0.01, 0.001.

Note that the last of the rescalings looks pretty much like the previous one: we observe the "self-similarity" which is typical of fractals!

Also a simple calculation would show that the "paths" of successive graphs become longer and longer, in fact infinitely long in the limit of many rescalings. Finally, note that ups and downs alternate more and more closely: The limiting path - while still continuous - would fail to have a tangent or derivative!

We could also have gotten

$$p_t(x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}} \quad (363)$$

from the central limit theorem!. The random walker at time $[t]$ is at

$$X(t) = \sum_{k=1}^{[t]} Y_k, \quad (364)$$

where the Y_k are independent, identically distributed random variables, with

$$\text{prob}(Y_k = \pm 1) = 1/2 \quad (365)$$

and with $\sigma_{Y_k} = 1$.

Scaling produces a process

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} Y_k \approx \frac{\sqrt{t}}{\sqrt{[nt]}} \sum_{k=1}^{[nt]} Y_k \xrightarrow{n \rightarrow \infty} \sqrt{t} N(0, 1) \quad (366)$$

$$= N(0, \sigma^2 = t) \quad (367)$$

with the density

$$\rho(x, t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}. \quad (368)$$

We have calculated the (Gaussian) probability distribution for the position of a (1-dimensional) Brownian path at a fixed time $t > 0$, a snapshot so to speak. To discuss the statistics of paths *as a whole* we would need to start from a sample space

$$\Omega = C([0, T])$$

of continuous functions of "time" t and attribute a probability ("Wiener measure") to suitable sets of such paths $B = \{B(t) : 0 < t < T\}$.

We would be led to its characteristic properties such as

$$E(B(t)) = 0 \quad \forall t > 0$$

and

$$E(B(s)B(t)) = \min(s, t).$$

This is the starting point for a mathematical construction of the Wiener process and for large parts of what is called "Stochastic Analysis". A central topic is Ito's theory of stochastic integrals:

$$\int X(t) dB(t)$$

for which a Riemann sum approximation

$$\int X(t) dB(t) \approx \sum X(\tau_k) (B(t_{k+1}) - B(t_k))$$

is complicated by the fact that it depends on the choice of τ_k :

-

$$\tau_k = t_k$$

produces the *Ito integral*

- a midpoint choice

$$\tau_k = \frac{t_k + \tau_{k+1}}{2}$$

gives a different result, the so-called *Stratonovich integral*.

On these notions there is then built the theory of stochastic differential equations such as

$$dX(t) = \beta dt + \sigma dB(t),$$

a shorthand for an integral equation

$$X(t) - X(0) = \int_0^t \beta(X, s) ds + \int_0^t \sigma(X, s) dB(s),$$

involving those stochastic integrals.

A prototype (Ornstein-Uhlenbeck process) is given by

$$\begin{aligned} \beta &= a(b - X) \\ \sigma &> 0 \text{ (const.)} \end{aligned}$$

i.e.

$$dX(t) = a(b - X(t))dt + \sigma dB(t),$$

widely used in biological and financial modelling. It is "mean reverting", i.e. in the absence of random perturbations σdB the first term on the rhs pushes X towards its "mean value" b (a desirable feature if one wants e.g. to model the long-term behavior of interest rates and their fluctuations), in fact the solution turns out to be

$$X(t) = b + (X(0) - b) e^{-at} + \int_0^t e^{-a(t-s)} dB(s).$$

We shall not struggle with the definition of that stochastic integral but take a different approach to stochastic analysis, one where the sample space is not one of Brownian paths but one of Brownian motion *velocity*, aka. "white noise".

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Infinite Dimensional Analysis

The program:

Calculus, involving generalized functions, for infinitely many variables, and their applications

Central questions:

- What is a good set of independent coordinates?
- Calculus is about differentiation and integration. Differentiation carries over rather straightforwardly as we shall see, but by what should we replace Lebesgue integration in infinite dimension?

Much speaks in favour of an infinite dimensional Gaussian measure, the "Gaussian White Noise". But we shall see that other infinite dimensional measure spaces occur naturally in certain applications. As a consequence we shall proceed as follows:

- Gaussian White Noise Analysis
- Generalization: Non-Gaussian Measures
- A Special Case: Poisson Analysis

4 Gaussian White Noise Analysis

Recall Bochner's theorem for functions

$$c : \mathbb{R}^n \rightarrow \mathbb{C}$$

They are Fourier transforms of measures on the Borel algebra over \mathbb{R}^n if and only if

1.

$$c(0) = 1$$

2. c is continuous in the vectors λ .

3. Finally, for any complex a_1, \dots, a_n , and real vectors $\lambda_1, \dots, \lambda_n$

$$\sum_{k,l} a_k^* a_l c(\lambda_k - \lambda_l) \geq 0. \quad (369)$$

For infinite dimensional vector spaces there are extensions such as

Theorem 14 [6]. *Any normalized continuous positive definite complex function $C(f)$ on test function space $S(\mathbb{R}^n)$ is the Fourier transform of a probability measure μ on distribution space $S^*(\mathbb{R}^n)$*

$$C(f) = \int_{S^*} e^{i\langle \omega, f \rangle} d\mu(\omega).$$

In this generalization of the Bochner theorem, the two spaces $S(\mathbb{R}^n)$ and $S^*(\mathbb{R}^n)$ arise as two different infinite dimensional extensions of the Euclidean space \mathbb{R}^n , and one wonders why we could not just have used Hilbert space functions in both cases.

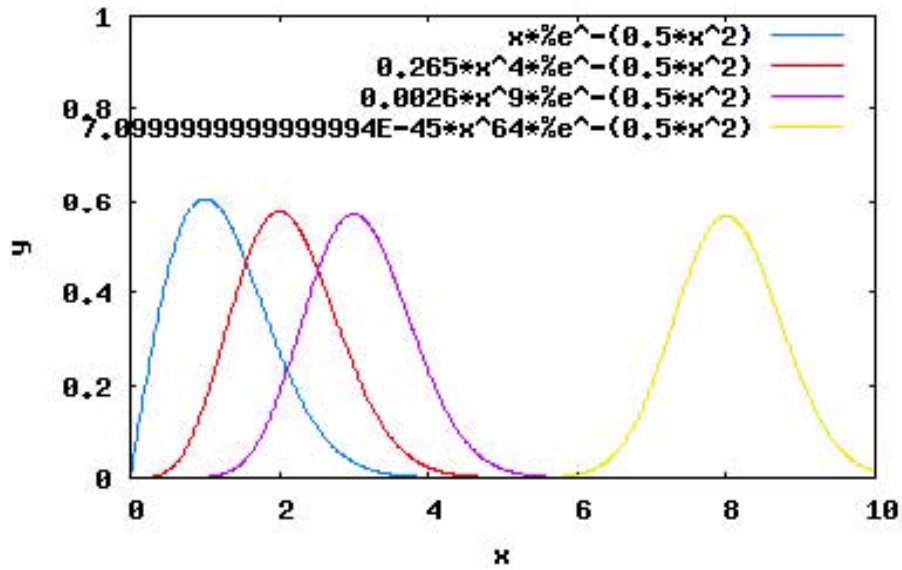
It is rather simple and quite instructive to see why this cannot work. To this end let us look at the finite dimensional Gaussian density

$$\rho(x) d^n x = \sqrt{\frac{1}{(2\pi)^n}} e^{-\frac{1}{2}r^2} r^{n-1} dr dS_n.$$

Disregarding integration over the sphere S_n , we focus on the radial density

$$\rho_n(r) \sim r^{n-1} e^{-\frac{1}{2}r^2}.$$

This is bell-shaped only for $n = 1$; graphically these densities look like this, for $n - 1 = 1, 4, 9, 64$:



As n becomes large, the probability densities $\rho_n(r)$ are essentially concentrated near the surface of a sphere $S_n(R)$ with radius $R = \sqrt{n-1}$.

Hence our limiting measure μ **will be zero for vectors of finite length r , i.e. for all vectors in Hilbert space**. Professor Hida has often underlined this point by saying: " μ is concentrated on $S_\infty(\sqrt{\infty})$, on an infinite dimensional sphere with radius $\sqrt{\infty}$ ".

Technically this means the following. Expanding test functions in terms of a basis, we put

$$f(x) = \sum \lambda_n e_n(x).$$

We get the differentiability and rapid decrease that are required of test functions if we choose Hermite functions as a base and admit only rapidly decreasing sequences of coefficients (λ_n) . In fact we have

$$f \in L^2(\mathbb{R}) \leftrightarrow \sum \lambda_n^2 < \infty$$

whereas

$$f \in S(\mathbb{R}) \leftrightarrow \sum n^k \lambda_n^2 < \infty$$

for all $k > 0$. Now let us expand the ω :

$$\omega(x) = \sum \omega_n e_n(x).$$

Our previous discussion tells us that the coefficients ω_n are not square summable as would be the case for ω in Hilbert space. Equivalently the functions $\omega(x)$ fail to be square integrable: they are "generalized functions", with

$$\langle \omega, f \rangle = \sum \omega_n \lambda_n = \int f(x) \omega(x) dx.$$

The $\omega(x)$ on the right may fail to exist pointwise, but the sum is well defined and finite: the rapid decrease of the λ_n takes care of this even for unbounded ω_n .

Now turn to "Gaussian White Noise" with the probability measure μ on distribution space $S^*(\mathbb{R}^d)$ given by its Fourier transform

$$C(f) = e^{-\frac{\overline{\lambda}^2}{2}} = e^{-\frac{1}{2} \int f^2(x) dx} = \int_{S^*} e^{i\langle \omega, f \rangle} d\mu(\omega).$$

This probability measure, defined on an infinite dimensional linear space, plays an important role in mathematics and physics.

Gaussian white noise models random events occurring independently at different points in time (and/or space). Informally we write

$$\begin{aligned} E(\omega(t)) &\equiv 0 \\ E(\omega(s)\omega(t)) &= \delta(s-t). \end{aligned}$$

Technically, we consider, for test functions $f \in S(\mathbb{R})$, the random variables $\langle \omega, f \rangle$ as Gaussian, with mean zero and covariance $\int f(t)g(t)dt$:

$$\begin{aligned} E(\langle \omega, f \rangle) &\equiv 0 \\ E(\langle \omega, f \rangle \langle \omega, g \rangle) &= \int f(t)g(t)dt. \end{aligned}$$

Exercise 7 Derive the two previous formulas from the knowledge of the WN characteristic function.

Gaussian WN is intimately related to Brownian motion modelled by the Wiener process $B(t)$. Informally, white noise is the velocity of Brownian motion:

$$\omega(t) = \frac{d}{dt} B(t). \quad (370)$$

Exercise 8 *For the random variables*

$$B(t) \equiv \langle \omega, 1_{[0,t]} \rangle$$

verify that they are Gaussian, with mean zero and covariance

$$E(B(s)B(t)) = \min(s, t).$$

The L^2 space of square integrable functions

$$L^2(d\mu) \equiv (L^2)$$

will be fundamental for our construction of (Gaussian) infinite dimensional analysis.

Remark 4 *A more general setting for Gaussian analysis may be based on a "Gelfand triple"*

$$N \subset H \subset N^* \quad (371)$$

of a Hilbert space H with an embedded nuclear space N and its dual N^ , and a measure given via Bochner-Minlos on N^* :*

$$C(f) = e^{-\frac{1}{2}\langle f, f \rangle_H} = \int_{N^*} e^{i\langle \omega, f \rangle} d\mu(\omega).$$

4.1 Chaos Expansion

Ref.[hkps], Ch. 2.

Our goal is to get an explicit description of L^2 -functions of white noise, i. e. of nonlinear functionals

$$\varphi \in (L^2).$$

The strategy will be modelled on one-dimensional Gaussian analysis, where

$$\varphi \in L^2(\mathbb{R}, e^{-\frac{1}{2}\omega^2} d\omega)$$

iff

$$\varphi(\omega) = \sum_k a_k h_n(\omega)$$

with

$$\sum_k k! a_k^2 < \infty.$$

Recall that the Hermite polynomials arise when we construct polynomials in ω which are to be orthogonal with respect to the Gaussian measure $e^{-\frac{1}{2}\omega^2} d\omega$. They can be computed recursively:

$$h_n(\omega) = \omega h_{n-1}(\omega) - (n-1) h_{n-2}(\omega).$$

Polynomial expressions of white noise ω will look like

$$\varphi(\omega) = \sum_{n=0}^N \int d^n t \omega(t_1) \dots \omega(t_n) f_n(t_1, \dots, t_n),$$

more precisely

$$\varphi(\omega) = \sum_{n=0}^N \langle \omega^{\otimes n}, f_n \rangle,$$

and as in the one-dimensional case, terms of different order n will not be orthogonal to one another.

But now we replace the generalized functions $\omega^{\otimes n}$ by $:\omega^{\otimes n}:$, defined such that we have an orthogonality relation

$$\begin{aligned} & E(\langle :\omega^{\otimes m}:, f_m \rangle \langle :\omega^{\otimes n}:, g_n \rangle) \\ &= \int d\mu(\omega) \langle :\omega^{\otimes m}:, f_m \rangle \langle :\omega^{\otimes n}:, g_n \rangle \\ &= \delta_{mn} n! \int d^n t f_n(t) g_n(t) \\ &= \delta_{mn} n! (f_n, g_n)_{Sym L^2(R^n)}. \end{aligned}$$

We could get these $:\omega^{\otimes m}:$ recursively by orthogonalizing monomials of increasing order à la Gram-Schmidt but here is a neat trick using "generating functions". Define

$$e(f, \omega) \equiv e^{\langle \omega, f \rangle - \frac{1}{2}(f;f)}$$

and calculate the (L^2) scalar product

$$\left(\left(\frac{d}{dx} \right)^m e(xf) \Big|_{x=0}, \left(\frac{d}{dy} \right)^m e(yg) \Big|_{y=0} \right)_{(L^2)} = \delta_{mn} n! (f, g)^n$$

(*Exercise!*). Note that by its definition, $\left(\frac{d}{dx} \right)^m e(xf) \Big|_{x=0}$ is of m^{th} order in f and of up to m^{th} order in ω :

$$\begin{aligned} \left(\frac{d}{dx} \right)^m e(xf) \Big|_{x=0} &= \int d^n t : \omega(t_1) \dots \omega(t_n) : f(t_1) \dots f(t_n) \\ &= \langle :\omega^{\otimes n}:, f^{\otimes n} \rangle \end{aligned}$$

The generalized functions

$$:\omega^{\otimes n}: \in Sym S^*(R^n)$$

are called "normal ordered polynomials" and obey a recurrence relation

$$:\omega(t_1) \dots \omega(t_n): =: \omega(t_1) \dots \omega(t_{n-1}) : \omega(t_n)$$

$$- \sum_{i=1}^{n-1} \delta(t_n - t_i) : \prod_{k \neq i}^{n-1} \omega(t_k) : . \quad (372)$$

In terms of distributions, orthogonality reads

$$E(: \omega^{\otimes n}(s) :: \omega^{\otimes m}(t) :) = \delta_{m,n} \sum_{perm} \prod_{\sigma} \delta(s_i - t_{\sigma(i)}).$$

For $F_n \in SymS(R^n)$ we write

$$\langle : \omega^{\otimes n} :, F_n \rangle =: \int d^n t F_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : .$$

(For the quantum field theorist this is exactly the definition of Wick polynomials.)

Exercise 9 Calculate the generalized function $: \omega(s_1) \dots \omega(s_4) : .$

For

$$\varphi(\omega) = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, F_n \rangle ,$$

using orthogonality, the L^2 norm of φ is calculated as

$$\|\varphi\|_{L^2(d\mu)}^2 = E(\varphi^* \varphi) = \sum_{n=0}^{\infty} n! \int d^n t |F_n(t_1, \dots, t_n)|^2 \quad (373)$$

and we may extend this expansion to $F_n \in L^2(R^n)$ by (L^2) continuity.

It is not hard to show that such monomials span (L^2) . Hence for

$$\varphi \in (L^2)$$

we have

$$\begin{aligned} \varphi(\omega) &= \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, F_n \rangle \\ &= \sum_{n=0}^{\infty} \int d^n t F_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : . \end{aligned} \quad (374)$$

with

$$\begin{aligned} \|\varphi\|_{(L^2)}^2 &= \sum_{n=0}^{\infty} n! \int d^n t |F_n(t_1, \dots, t_n)|^2 \\ &= \sum_{n=0}^{\infty} n! \|F_n\|_{SymL^2(R^n, n! d^n t)}^2 . \end{aligned}$$

On the rhs of (373) we have a Fock space norm:

$$\varphi \in (L^2) \sim \{F_n\} \in \mathcal{F} = \bigoplus_n \text{Sym } L^2(R^n, n!d^n t).$$

This is the famous *Gelfand-Ito-Segal isomorphism* between the L^2 -space with Gaussian measure μ and symmetric or "Bose" Fock space:

$$L^2(S^*(R), d\mu) \simeq \bigoplus_{n=0}^{\infty} \text{Sym } L^2(R^n, n!d^n t).$$

4.1.1 Recalling Fock Space:

For $n = 0$ have zero-particle vectors F_0 which are just constant multiples of the vacuum state Ω .

$$F_0 = c\Omega$$

with

$$\|F_0\|^2 = |c|^2.$$

Together, they span the Fock space

$$\mathcal{F} = \{F : F = (F_0, F_1, \dots, F_n, \dots)\}$$

with norm

$$\|F\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} n! (F_n, F_n)_{L^2}.$$

Spanned by certain n-particle vectors, obtained from the vacuum by applying creation operators $a^*(f)$ n times to the vacuum:

$$\Psi_n(f^{\otimes n}) = (a^*(f))^n \Omega.$$

CCR:

$$\begin{aligned} [a(f), a(g)] &= 0 = [a^*(f), a^*(g)] \\ [a(f), a^*(g)] &= (f, g) \end{aligned}$$

Consider vectors

$$e(f) = \sum_{n=0}^{\infty} \frac{1}{n!} (a^*(f))^n \Omega = e^{a^*(f)} \Omega$$

with scalar product

$$(e(f), e(g))_{\mathcal{F}} = e^{\int f(x)g(x)dx}. \quad (375)$$

"Coherent states" (See e.g. [5]) are the corresponding unit vectors

$$\Psi = \exp(-\frac{1}{2}|f|^2) e^{a^*(f)} \Omega.$$

They are eigenvectors of the annihilation operators $a(g)$:

$$a(g)e(f) = (g, f) \cdot e(f) \quad (376)$$

with eigenvalue (g, f) .

Exercise 10 *Prove the formulas (375) and (376).*

4.2 Smooth and Generalized Functionals

Recall the definition of test functions in finite dimensional analysis:

$$f \in S(R) \text{ iff } |f|_p < \infty \text{ for all } p. \quad (377)$$

where the $|f|_p$ $p = 0, 1, 2, \dots$ are suitable increasing "Sobolev" norms; a popular choice is

$$|f|_p^2 = \int \left| \left(-\frac{d^2}{dx^2} + x^2 + 1 \right)^p f(x) \right|^2 dx = |H^p f|_2^2$$

where H is the harmonic oscillator Hamiltonian of quantum mechanics (suitably normalized).

In analogy to this we shall define a doubly infinite sequence of increasing norms for white noise functionals

$$\|\varphi\|_{p,q}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |F_n|_p^2 \quad (378)$$

with

$$|F_n|_p^2 = \int \left| \prod_{k=1}^n \left(-\frac{d^2}{dx_k^2} + x_k^2 + 1 \right)^p F_n(x_1, \dots, x_n) \right|^2 d^n x$$

for $p, q > 0$. For many applications, the most interesting cases are $\beta = 0, 1$.

With $\beta = 0$ define the space (S) of smooth functionals by

$$(S) = \left\{ \varphi : \|\varphi\|_{p,q} < \infty \text{ for all } p, q \right\}$$

and the space $(S)^*$ of generalized functionals (Hida distributions) as its dual: $(S) \subset (L^2) \subset (S)^*$. We note that this does not depend on the specific choice of the norms in (378). The resulting $F_n(t_1, \dots, t_n)$ are not only rapidly decreasing and arbitrarily often differentiable in t , but also rapidly decreasing in n , since the sum (378) must converge for any q .

Similarly, for $\beta = 1$, the "Kondratiev spaces" $(S)^{\pm 1}$, with

$$(S)^1 \subset (S) \subset L^2(d\mu) \subset (S)^* \subset (S)^{-1}.$$

Remark 5 *More generally, can construct analogously*

$$(N)^1 \subset (N) \subset (H) \subset (N)^* \subset (N)^{-1}.$$

4.2.1 Characterization of Generalized Functions

In many cases Φ will be given in terms of what physicists might call a "source functional", such as

$$\begin{aligned} T\Phi(f) &= E\left(\Phi(\omega)e^{i\int \omega(t)f(t)dt}\right) \\ &= \int d\mu(\omega)\Phi(\omega)e^{i\langle\omega,f\rangle}. \end{aligned}$$

It looks much like a Fourier transform, but in view of the Gaussian integration measure μ it does not intertwine differentiation and multiplication operators; we should rather call it a Gauss-Fourier transform.

A related quantity is the so-called S-transform. Consider

$$\begin{aligned} e_f(\omega) &= : e^{\int \omega(t)f(t)dt} : \\ &= \sum_k \frac{1}{k!} \int d^k t f(t_1) \dots f(t_n) : \omega(t_1) \dots \omega(t_n) : \\ &= \frac{e^{\langle\omega,f\rangle}}{E(e^{\langle\omega,f\rangle})} \in (L^2) \text{ whenever } f \in L^2(dt). \end{aligned} \tag{379}$$

Here $\langle\cdot,\cdot\rangle$ denotes the bilinear extension of the scalar product of L^2 .

The last equality is the infinite dimensional analogue of the well-known generating function for Hermite polynomials h_n

$$\sum_k \frac{f^k}{k!} h_n(\omega) = e^{\omega f - \frac{f^2}{2}}.$$

Hence we may calculate, first for $\varphi \in (L^2)$

$$S\varphi(f) = E\left(\varphi(\omega) : e^{\int \omega(t)f(t)dt} : \right) = (\varphi, e_f)_{(L^2)}.$$

One use of this is that it allows us to determine the kernel functions F_n in the chaos expansion

$$\varphi(\omega) = \sum_{n=0}^{\infty} \int d^n t F_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : . \tag{380}$$

of φ . They are just the n^{th} order terms of the S-transform of φ , since from the orthogonality of the Wick products one has immediately that

$$S\varphi(f) = \sum_n \int d^n t F_n(t_1, \dots, t_n) f(t_1) \dots f(t_n).$$

Exercise 11 Verify the previous formula!

Exercise 12 Calculate the kernel functions F_n for

$$\varphi = \int_0^T \delta(B(t) - a) dt$$

Exercise 13 Calculate the S-transform of

$$\varphi_A(\omega) = e^{-\frac{1}{2}(\omega, A\omega)}$$

for suitable non-negative operators A in trace class.

Finally we note

$$S\varphi(f) = E(\varphi(\cdot + f))$$

as a consequence of the Gaussian nature of the white noise measure. For a one-dimensional normal distribution one finds immediately

$$\begin{aligned} E(\varphi \cdot e_f) &= \int_{\mathbb{R}^1} d\mu \cdot \varphi(\omega) \cdot e_f(\omega) \\ &= \int \frac{d\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \cdot \varphi(\omega) \cdot \frac{e^{\omega \cdot f}}{e^{\frac{f^2}{2}}} \\ &= \int \frac{d\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \varphi(\omega + f) \\ &= \int_{\mathbb{R}^1} d\mu(\omega) \varphi(\omega + f) = E(\varphi(\cdot + f)). \end{aligned}$$

For test functions $f \in S(\mathbb{R}^1)$ one verifies that $e_f \in (S)$, hence we can extend the S-transform to $\Phi \in (S)^*$ by setting

$$S\Phi(f) = \langle\langle \Phi, e_f \rangle\rangle.$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the bilinear extension of the scalar product of (L^2) . Note also the relation

$$T\Phi(f) = C(f) \cdot S\Phi(if).$$

It is not hard to verify that for generalized white noise functionals $\Phi \in (S)^*$

1. $S\Phi(zf_1 + f_2)$ is analytic in the whole complex z -plane, and
- 2.

$$|S\Phi(zf)| < ae^{b|zf|_p^2}.$$

Remarkably, these two conditions are not only necessary but in fact sufficient to characterize generalized white noise functions by their S - or T - transforms.

Theorem 15 [6]

(1) A functional $G(f)$, $f \in S(\mathbb{R}^1)$, is the S -transform of a unique generalized white noise functional $\Phi \in (S)^*$ iff for all $f_i \in S(\mathbb{R}^1)$, $G(zf_1 + f_2)$ is analytic in the whole complex z -plane and of second order exponential growth

$$|G(zf)| < ae^{b|zf|_p^2}.$$

(2) A functional $G(f)$, $f \in S(\mathbb{R}^1)$, is the T -transform of a unique generalized white noise functional $\Psi \in (S)^*$ iff for all $f_i \in S(R)$, $G(zf_1 + f_2)$ is analytic in the whole complex z -plane and of second order exponential growth

$$|G(zf)| < ae^{b|zf|_p^2}.$$

Exercise 14 (Gauss kernels). Consider

$$N \exp\left(-\frac{1}{2}(\omega, A\omega)\right) \equiv \frac{\varphi_A(\omega)}{E(\varphi_A)}.$$

To which operators A can you extend this as a Hida distribution?

As a consequence of the characterization we have also an existence criterion for converging sequences and for integrals with respect to an additional parameter.

Proposition 16 Let $(\phi_k)_{k \geq 0} \in (S)^*$, then the following are equivalent:

1. The (ϕ_k) converge in $(S)^*$.
2. For any $f \in S(\mathbb{R}^1)^d$,
 - $(S\phi_k(f))_{k \geq 0}$ is a Cauchy sequence, and
 - There exist $c_1, c_2, p \geq 0$ such that

$$|S\phi_k(f)| \leq c_1 e^{c_2 |H^p f|_0^2}, \forall f \in S(\mathbb{R}^1)^d.$$

Proposition 17 Let be (Ω, \mathcal{B}, m) a measure space, and $\phi_\lambda \in (S)^*$ for $\lambda \in \Omega$. Suppose that

1. the transformation $S\phi_\lambda$ is measurable in λ , for any $f \in S(\mathbb{R}^1)^d$
2. there exists $p > 0$ independent of λ , and functions $c_1 \in L^1(\Omega)$, $c_2 \in L^\infty(\Omega)$ such that

$$|S\phi_\lambda(f)| \leq c_1(\lambda) e^{c_2(\lambda) |H^p f|_0^2} \quad \forall f \in S(\mathbb{R}^1)^d$$

Then ϕ_λ is integrable

$$\int_\Omega \phi_\lambda dm(\lambda) \in (S)^*$$

(in the sense of Bochner) in some Hilbert space $(S)_{-q}$, $(L^2) \subset (S)_{-q} \subset (S)^*$, and

$$S\left\{\int_\Omega \phi_\lambda d\lambda\right\}(f) = \int_\Omega S\phi_\lambda(f) d\lambda.$$

Elements of the space $(S)^{-1}$ are called Kondratiev distributions, the well known Hida distributions form a subspace [KLS96]. The test functions $\exp(i\langle \cdot, f \rangle)$ and $e_f \in (S)$ are not in $(S)^1$, but we can still characterize these distributions by their T - or S -transforms for small enough argument f : Let $\Phi \in (S)^{-1}$ then there exist p, ε such that we can define for every

$$f \in \left\{ f \in S(\mathbb{R}^1) \mid |f|_p^2 < \varepsilon \right\} \quad (381)$$

the T -transform by

$$T\Phi(f) := \langle\langle \Phi, \exp(i\langle \cdot, f \rangle) \rangle\rangle \quad (382)$$

Conversely, a function G for which $G(f_0 + zf)$ is holomorphic and bounded for small enough arguments, is the T -transform of a Kondratiev distribution; the same holds for the S - instead of the T -transform [7].

1. We can define an infinite dimensional Fourier transform by

$$T^{-1}S = F \quad (383)$$

2. Algebraic structure, via

$$S^{-1}(S(\Phi)S(\Psi)) =: \Phi \diamond \Psi. \quad (384)$$

This product is the Wick product:

$$: \omega^{\otimes n}(s) : \diamond : \omega^{\otimes m}(t) :=: \omega^{\otimes n}(s) \omega^{\otimes m}(t) :,$$

hence, for

$$\begin{aligned} \Phi(\omega) &= \sum_{n=0}^{\infty} \int d^n t F_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : \\ \Psi(\omega) &= \sum_{n=0}^{\infty} \int d^n t G_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : \end{aligned}$$

we have

$$(\Phi \diamond \Psi)(\omega) = \sum_{n=0}^{\infty} \int d^n t H_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : \quad (385)$$

with

$$H_n = \sum_{k=1}^n F_k G_{n-k}. \quad (386)$$

3. For the Kondratiev space, analytic functions g of S -transforms are again admissible; this admits an analytic Wick calculus on distribution space

$$g(S(\Phi)) = S(g^\diamond(\Phi))$$

with

$$g^\diamond(\Phi) \equiv \sum a_n \Phi^{\diamond n} \text{ for } g(z) = \sum a_n z^n$$

Wick Calculus in Gaussian Analysis (Y. G. Kondratiev, P. Leukert, L. S. [7])

Triples characterized by holomorphic functions of other than 2nd order growth are studied by

R. Jenane-Gannoun, R. Hachaichi, H. Ouerdiane, A. Rezgui: Un théorème de dualité entre espaces de fonctions holomorphes à croissance eponentielle' JFA 171, 1-14 (2000) [4]

Exercise 15 Consider the spaces

$$(S)^1 \subset (S) \subset (L^2) \subset (S)^* \subset (S)^{-1}.$$

In which of these spaces is

$$\varphi = \delta(B(t) - a)$$

an element?

Hints:

1. use the Fourier representation of the δ -function.
2. get the kernels of the chaos expansion from the S-transform.

4.3 Regular generalized functions

Consider square integrable white noise functionals φ for which the chaos expansion

$$\begin{aligned} \varphi(\vec{\omega}) &= \sum_n \langle : \omega^{\otimes n} :, F_n \rangle \\ &\equiv \sum_n \int_{\mathbb{R}^n} d^n t F_n(t_1, \dots, t_n) : \omega^{\otimes n} : (t_1, \dots, t_n) \end{aligned}$$

converges rapidly, i.e.,

$$\|\varphi\|_q^2 \equiv \sum_{\vec{n}} (n!)^2 2^{q_n} |F_n|_{L^q(\mathbb{R}^n)}^2 < \infty.$$

In our previous notation

$$\|\varphi\|_{p,q}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |F_n|_p^2 \quad (387)$$

this corresponds to the case $\beta = 1$ and $p = 0$, i.e. the F_n are simply L^2 -functions with rapidly decreasing norms. Define the Hilbert space G_q^1 as

$$G_q^1 = \left\{ \varphi \in (L^2) : \|\varphi\|_q^2 < \infty \right\}.$$

The space of test functions \mathcal{G}^1 is their intersection:

$$\mathcal{G}^1 = pr - \lim_{q \rightarrow \infty} G_q^1.$$

and \mathcal{G}^{-1} the dual space of \mathcal{G}^1 with respect to (L^2) .

The corresponding bilinear dual pairing $\ll \cdot, \cdot \gg$ extends the sesquilinear inner product on (L^2)

$$\ll \Phi, \varphi \gg = (\bar{\Phi}, \varphi)_{(L^2)}, \text{ if } \Phi \in (L^2).$$

The constant function 1 is in \mathcal{G}^1 , we may extend the definition of the expectation $E(\cdot)$ to distributions $\Phi \in \mathcal{G}^{-1}$:

$$E(\Phi) = \ll \Phi, 1 \gg.$$

4.4 2nd Quantization, Conditional Expectations

Turning to Fock space and operators A on $L^2(\mathbb{R})$, the linear map which transforms each sequence (F_n) of functions to the sequence $(A^{\otimes n} F_n)$, in particular

$$F_n = f(t_1) \cdot \dots \cdot f(t_n) \rightarrow F_n = Af(t_1) \cdot \dots \cdot Af(t_n)$$

is called the *second quantization of A* , denoted by $\Gamma(A)$:

$$\Gamma(A) = \bigoplus_n A^{\otimes n}.$$

One also often needs Fock space operators which are additive on multiparticle wave functions, mapping e.g. $F_n = f(t_1) \dots f(t_n)$ to

$$(Af(t_1)) \cdot f(t_2) \cdot \dots \cdot f(t_n) + \dots + f(t_1) \cdot \dots \cdot f(t_{n-1}) \cdot Af(t_n)$$

(such as e.g. kinetic energy as sum of one particle energies). This is formalized by

$$d\Gamma(A) = \bigoplus_n (A \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes A)$$

For a detailed discussion see Reed and Simon [9].

Exercise 16 Calculate $d\Gamma(1)$.

Exercise 17 Calculate $\Gamma(A)e(f)$.

Denote by Θ_t the Heaviside function

$$\Theta_t(s) \equiv \begin{cases} 1 & \text{if } s \leq t \\ 0 & \text{if } s > t \end{cases}$$

and also the linear operator

$$\Theta_t : f(\cdot) \rightarrow \Theta_t(\cdot) f(\cdot).$$

The second quantization $\Gamma(\Theta_t)$ of Θ_t , $t \in \mathbb{R}$, is a projection operator in (L^2) , with

$$\begin{aligned} \Gamma(\Theta_t)\varphi(\vec{\omega}) &= \sum_n \langle : \omega^{\otimes n} :, \Theta_t^{\otimes n} F_n \rangle \\ &= \sum_n \int_{-\infty}^t d^n t F_n(t_1, \dots, t_n) : \omega^{\otimes n} : (t_1, \dots, t_n) \end{aligned}$$

Clearly this extends to the space \mathcal{G}^{-1} .

Remark 6 Denote by \mathcal{F}_t the σ -algebra generated by $\{B(s), s \leq t\}$. The conditional expectation with respect to \mathcal{F}_t is the expectation of Φ if you know $B(s)$ up to time t , i.e. it will be a function of white noise up to time t .

Have

$$E(\Phi | \mathcal{F}_t) = \Gamma(\Theta_t)\Phi = \sum_n \int_{-\infty}^t d^n t F_n(t_1, \dots, t_n) : \omega^{\otimes n} : (t_1, \dots, t_n)$$

($\Gamma(\Theta_t)\Phi$ extends the conditional expectation with respect to \mathcal{F}_t to $\Phi \in \mathcal{G}^{-1}$.)

Theorem 18 Brownian martingales $\Phi_t \in (L^2)$ are characterized by

$$\Gamma(\Theta_s)\Phi_t = \Phi_s \text{ if } s < t \quad (388)$$

or, equivalently

$$\Phi_t(\vec{\omega}) = \sum_n \int_{-\infty}^t d^n t F_n(t_1, \dots, t_n) : \omega^{\otimes n} : (t_1, \dots, t_n)$$

where the F_n do not depend on t .

4.5 Calculus

Test functionals $\varphi \in (S)$ admit directional ("partial", "Gateaux") derivatives:

$$D_h \varphi(\omega) = \lim_{\varepsilon \rightarrow 0} \frac{\varphi(\omega + \varepsilon h) - \varphi(\omega)}{\varepsilon}$$

exists for any generalized function $h \in S^*(\mathbb{R}^1)$. Hence the adjoint D_h^* acts continuously on $(S)^{-1}$.

Exercise 18 Verify

$$D_h e(f) = (h, f) e(f)$$

.

Recall that in Fock space we have

$$a(f)e_{Fock}(f) = (h, f)e_{Fock}(f)$$

I.e. under the Wiener-Ito-Segal isomorphism we have

$$\begin{aligned} (L^2) &\leftrightarrow \mathcal{F} \\ D_f &\leftrightarrow a(f) \end{aligned}$$

the derivative D_h corresponds to the annihilation operator, and hence

$$D_h^* \leftrightarrow a^*(f).$$

Exercise 19 *Verify*

$$D_h^* = -D_h - \langle \omega, h \rangle$$

Hint: Study

$$(e(f), D_h^* e(g))_{L^2(d\mu)}.$$

In particular for $h = \delta_t$, we introduce the Hida derivative

$$D_h \equiv \partial_t.$$

Exercise 20 *Show that*

$$S(D_h \varphi)(f) = D_h S(\varphi)(f)$$

$$S(\partial_t^* \varphi)(f) = f(t) S(\varphi)(f)$$

On Wick products the action of ∂_t is

$$\partial_t : \omega(t_1) \dots \omega(t_n) := \sum_k \delta(t - t_k) : \prod_{l \neq k} \omega(t_l) :$$

while

$$\partial_t^* : \omega(t_1) \dots \omega(t_n) := \omega(t) \omega(t_1) \dots \omega(t_n) :$$

Comparing this with the recursion formula (372) one finds

$$\partial_t + \partial_t^* = \omega(t).$$

Exercise: Show for $f \in S(R)$

$$(D_f^*)^n 1 = \int d^n t f(t_1) \dots f(t_n) : \omega(t_1) \dots \omega(t_n) :$$

and

$$\prod_{k=1}^{\infty} \partial_{t_k}^* 1 = : \omega(t_1) \dots \omega(t_n) : .$$

∂_t gives rise to a natural gradient (Frechet derivative):

$$\nabla\varphi = \{\partial_t\varphi : t \in \mathbb{R}^1\}.$$

I.e.

$$\nabla : (L)^2 \rightarrow L^2(\mathbb{R}^1) \otimes (L)^2$$

and

$$\nabla : (S) \rightarrow S(\mathbb{R}^1) \otimes (S).$$

The corresponding carré du champ functional is a test functional for all $\varphi \in (S)$

$$\begin{aligned} |\nabla\varphi|^2 &= (\nabla\varphi, \nabla\varphi)_{L^2(R)} \\ &= \int dt |\partial_t\varphi|^2 \in (S). \end{aligned}$$

4.5.1 Skorohod and Itô integrals, extended

Definition 19 Given Φ an element from $L^2(\mathbb{R}) \otimes G_{-q}^{-1}$, for some $q \in \mathbb{N}_0$, we call generalized Skorohod integral of Φ the distribution on G^{-1} , $I(\Phi)$, defined as the unique regular generalized function from G^{-1} for which the following equality

$$\langle\langle I(\Phi), \psi \rangle\rangle = \langle\langle \Phi, \nabla\psi \rangle\rangle$$

holds for every test function ψ from G^1 .

This definition generalizes the notion of Skorohod integral. Informally we have

$$I(\Phi) = \int \partial_t^* \Phi dt.$$

In fact, if $\Phi \in L^2(\mathbb{R}) \otimes D$,

$$D \equiv \left\{ F \in (L^2) : \sum_n n!n |F_n|_2^2 < \infty \right\},$$

the generalized Skorohod integral $I(\Phi)$ coincides with the Skorohod integral.

Proposition 19 For $t \in \mathbb{R}$, let \mathcal{F}_t denote the σ -algebra generated by $\{B(s), s \leq t\}$. If

(1) $\varphi \in L^2(\mathbb{R}) \otimes (L^2)$ and

(2) φ adapted to $(\mathcal{F}_t)_{t \in \mathbb{R}}$,

then the generalized Skorohod integral $I(\varphi)$ is equal to the Itô integral of φ :

$$I(\varphi)(\omega) = \int \varphi(t, \omega) dB(t, \omega)$$

Remark 7 Without (1) we speak of a generalized Itô integral.

4.6 Applications, Examples

4.6.1 δ -Functions

E.g. in the Edwards polymer model: polymer as Brownian motion path with partition function

$$Z(a) = E \left(e^{-a \int d^2 t \delta(B(t_1) - B(t_2))} \right)$$

Must make sense of self intersection local time

$$L(T) = \int_0^T d^2 t \delta(B(t_1) - B(t_2))$$

Since (Lévy) 1940, more than $.5 \cdot 10^2$ publications.
The contributions of white noise analysis:

- $\delta(B(t_1) - B(t_2))$ and L are well defined generalized functions of white noise
- kernels of chaos expansion may be calculated in closed form and exhibit explicitly the increasing singularity as the dimension d increases.
- The Yor renormalization limit of self-intersection local times for $d = 3$

$$r(\varepsilon)(L_\varepsilon - E(L_\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} \beta \quad (389)$$

can be understood in terms of chaos expansion, and extended to $d > 3$.

Recent progress: Martingale Approximation for Intersection Local Time of Brownian Motion (M. Faria, A. Rezgui, L. S. [2])

- *higher order self intersections: R. Jenane-Gannoun: Ph.D. thesis, Tunis, March 2001.*
- *intersections of different Brownian motions, etc.*
- *For diffusions*

$$dY = \beta(Y, t)dt + \sigma(Y, t)dB(t)$$

have

$$\delta(Y(t) - x) \in (S)^*$$

and with an explicit Isobe-Sato expansion [Isobe] in terms of white noise (M. Gordon, tese, U. Madeira).

4.7 Stochastic Partial Differential Equations

Example 20 (Burgers equation):

$$u_t + u \cdot u_x = \nu u_{xx} + F(x, t, \omega) \quad (390)$$

Problem: Nonlinear expressions for space-time dependent noise produce products of generalized functions. Renormalization procedure required for nonlinearities. Possible ansatz (Øksendal et al. [12]) is

$$u_t + u \diamond u_x = \nu u_{xx} + F(x, t, \omega) \quad (391)$$

In terms of S-transforms (denoted by \tilde{u} etc.)

$$\tilde{u}_t + \tilde{u} \tilde{u}_x = \nu \tilde{u}_{xx} + \tilde{F}(x, t, f) \quad (392)$$

Note: $f \in S(R)$ instead of $\omega \in S^*(R)$, hence solution via Cole-Hopf transform and perturbation theory, with

$$u(x, t, \cdot) \in (S)^{-1} \quad (393)$$

Much more on this in Øksendal [12].

4.8 A generalized Clark-Ocone formula

The Clark-Ocone formula: given φ , represent it as an Ito integral

$$\varphi = E(\varphi) + \int m_t dB_t \quad (394)$$

with m a function of φ . (E.g. in mathematical finance: Given the value φ , determine the corresponding hedging strategy m_t .)

Theorem 20 Let $\Phi \in \mathcal{G}^{-1}$. Then it can be written as a generalized Itô integral

$$\Phi = E(\Phi) + I(m)$$

with

$$m(t) = \Gamma(\Theta_t) \partial_t \Phi$$

In terms of the S-transform:

Theorem 21 Given a regular generalized function Φ from \mathcal{G}^{-1} and $q \in \mathbb{N}_0$ such that $\Phi \in G_{-q}^{-1}$, its S-transform is equal to

$$S\Phi(\eta) = E(\Phi) + \int_{\mathbb{R}} d\tau \eta(\tau) \frac{\delta}{\delta \eta(\tau)} S(\Phi)(\Theta_{\tau} \bar{\eta}).$$

Remark: The Clark-Ocone formula is useful to calculate variances: no need to calculate the expectation, can use Ito-Segal isomorphism on $E(m^2(\tau))$ in

$$E\left((\Phi - E(\Phi))^2\right) = \int_{\mathbb{R}} E(m^2(\tau)) d\tau.$$

*Ref. [1]: A Generalized Clark-Ocone Formula (M. de Faria, M. J. Oliveira, L. S.) Random Oper. Stoch. Eqs., **8**, no. 3 (2000).*

An example: Donsker's delta function

$$\delta(B(t) - a) \in \mathcal{G}^{-1},$$

with S -transform

$$\begin{aligned} & (S\delta(B(t) - a))(f) \\ &= (2\pi t)^{-1/2} \exp \left(-\frac{\left(\int_0^t f(s) ds - a \right)^2}{2t} \right) \end{aligned}$$

From the above Theorems

$$\delta(B(t) - a) = E(\delta(B(t) - a)) + \int dB(\tau) m(\tau)$$

with

$$Sm(\tau)(f) = \frac{\delta}{\delta f(\tau)} S(\Phi)(\Theta_\tau f).$$

The functional derivative is

$$\begin{aligned} & \left(\frac{\delta}{\delta f(\tau)} S(\Phi) \right) (f) = -1_{[0,t]}(\tau) (2\pi t^3)^{-1/2} \\ & \cdot \left(\int_0^t f(s) ds - a \right) \exp \left(-\frac{\left(\int_0^t f(s) ds - a \right)^2}{2t} \right) \end{aligned} \quad (395)$$

Projecting the f with Θ_τ , we obtain

$$\begin{aligned} & (Sm(\tau))(f) \\ &= -1_{[0,t]}(\tau) (2\pi t^3)^{-1/2} \\ & \left(\int_0^\tau f(s) ds - a \right) \exp \left(-\frac{\left(\int_0^\tau f(s) ds - a \right)^2}{2t} \right). \end{aligned}$$

Inverting the S -transform obtain

$$m(\tau) = -1_{[0,t]}(\tau) \frac{B(\tau) - a}{(t - \tau)^{3/2}} \exp -\frac{(B(\tau) - a)^2}{2(t - \tau)} \quad (396)$$

Note: $m(\tau)$ is an adapted random variable in (L^2) as long as $\tau < t$, permits conventional Itô integration. Hence, as a limit in \mathcal{G}^{-1} ,

$$\delta(B(t) - a) = (2\pi t)^{-1/2} e^{-\frac{a^2}{2t}} + \lim_{\varepsilon \rightarrow +0} \int_0^{t-\varepsilon} dB(\tau) m(\tau).$$

5 Appell Systems: A glimpse at Non-Gaussian Analysis

5.1 A General Framework

Generalizing from the Gaussian white noise measure will consider measures μ on $S^*(R)$ which satisfy two additional assumptions. The first one concerns some analyticity of the Laplace transformation

$$\begin{aligned} l_\mu(f) &= \int_{S^*(R)} \exp \langle \omega, f \rangle d\mu(\omega) \\ &= E(\exp \langle \cdot, f \rangle), \quad f \in S_C(R). \end{aligned}$$

Assumption 1 (Analyticity) The measure μ has an analytic Laplace transform in a neighborhood of zero, i.e., $l_\mu \in \text{Hol}_0(S_C(R))$.

This is equivalent to a bound on moments: $\exists p \in \mathbb{N}, \exists C > 0$ such that

$$\left| \int_{S^*(R)} \langle x, f \rangle^n d\mu(x) \right| \leq n! C^n |f|_p^n$$

Assumption 2 (Non-Degeneracy)

Consider "continuous polynomials"

$$\varphi(\omega) = \sum_{n=0}^N \langle \omega^{\otimes n}, F_n \rangle$$

If $\int_A \varphi d\mu = 0$ for all $A \in \mathfrak{B}(S^*(R))$ then $\varphi \equiv 0$.

5.2 The Appell system

Recall the fundamental notion of the normal ordered or Wick polynomials in the Gaussian case. We had

$$\begin{aligned} e_f(\omega) &= \frac{e^{\langle \omega, f \rangle}}{E(e^{\langle \omega, f \rangle})} : \\ &= \sum_k \frac{1}{k!} \langle P_k(\omega), f^{\otimes k} \rangle \end{aligned} \tag{397}$$

Recall also the formula

$$(D_f^*)^k 1 = \langle Q_k(\omega), f^{\otimes k} \rangle$$

where of course

$$P_k(\omega) = Q_k(\omega) =: \omega^{\otimes k} :$$

were the orthogonal Wick polynomials.

Now in the non-Gaussian case we can do the same constructions. But

- the polynomials P_k are no more orthogonal. nevertheless we can, as before use them to construct test functions

•

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, F_n \rangle \in (S)^1$$

using again $F_n(t_1, \dots, t_n)$ which are rapidly decreasing and arbitrarily often differentiable in t , and also rapidly decreasing in n , just as in the Gaussian case.

- the Q_k are not polynomials any more: . On \mathbb{R} consider a measure $d\mu(\omega) = \rho(\omega) d\omega$ where ρ is a positive density function on \mathbb{R} such that assumptions 1 and 2 are fulfilled. In this setting the adjoint of the differentiation operator is given by

$$\left(\frac{d}{d\omega}\right)^* f(\omega) = - \left(\left(\frac{d}{d\omega}\right) + \beta(\omega) \right) f(\omega),$$

where β is the logarithmic derivative of the measure μ and given by

$$\beta = \frac{\rho'}{\rho}.$$

This enables us to calculate the Q^μ -system. One has

$$\begin{aligned} Q_n^\mu(\omega) &= \left(\left(\frac{d}{d\omega}\right)^* \right)^n 1 \\ &= (-1)^n \left(\frac{d}{d\omega} + \beta(\omega) \right)^n 1 \\ &= (-1)^n \frac{\rho^{(n)}(\omega)}{\rho(\omega)}, \end{aligned}$$

where the last equality can be seen by induction.

Note that, for ρ non smooth, this construction produces **generalized** functions Q_n^μ even in this one-dimensional case. If $\rho(\omega) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\omega^2)$ is the Gaussian density, then Q_n^μ is related to the n -th Hermite polynomial:

$$Q_n^\mu(\omega) = 2^{-n/2} H_n \left(\frac{\omega}{\sqrt{2}} \right).$$

- On the other hand, the P, Q form a "biorthogonal system" (Appell system):

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(G_n)$$

and

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, F_n \rangle$$

obey the orthogonality relation

$$(\Phi, \varphi)_\mu = \sum_{n=0}^{\infty} n! (G_n, F_n),$$

and for test functions φ , with smooth kernel functions F_n , we can extend this to generalized functions of non Gaussian noise, with distribution valued kernels G_n .

- As before we can define the S -transform of generalized functions

$$S_\mu \Phi(f) := \langle \Phi, e_f \rangle$$

and the biorthogonality of \mathbf{P} and \mathbf{Q} implies

$$S_\mu \Phi(f) = \sum_{n=0}^{\infty} \langle G_n, f^{\otimes n} \rangle,$$

5.3 A Remark on Generalized Appell Systems

To generalize the Appell system we consider more general generating functions

$$e_\mu^\alpha(f, \omega) = \frac{\exp \langle \omega, \alpha(f) \rangle}{l_\mu(\alpha(f))},$$

obtained from mappings

$$\alpha : S_{\mathbb{C}} \rightarrow S_{\mathbb{C}}$$

which we assume

- holomorphic near zero
- invertible
- and such that $\alpha(0) = 0$.

We write for the inverse function $\alpha^{-1} = \beta$.

Using the same procedure as above we define $P_n^{\mu, \alpha}(\omega) \in S^*$, called **generalized Appell polynomials** such that

$$e_\mu^\alpha(f, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu, \alpha}(\omega), f^{\otimes n} \rangle, \quad f \in \mathcal{U}'_\alpha, \quad \omega \in S^*. \quad (398)$$

Likewise, we define a generalized function $Q_n^{\mu, \alpha}(\Phi^{(n)})$ via the S_μ -transform

$$S_\mu \left(Q_n^{\mu, \alpha} \left(\Phi^{(n)} \right) \right) (f) := \left\langle \Phi^{(n)}, \beta(f)^{\otimes n} \right\rangle. \quad (399)$$

Again one can show

Theorem 22 (Biorthogonality of $Q^{\mu,\alpha}$ and $P^{\mu,\alpha}$ with respect to μ)

$$\langle Q_n^{\mu,\alpha}(G_n), P_m^{\mu,\alpha}(F_m) \rangle_\mu = \delta_{nm} n! \langle G_n, F_m \rangle, \quad (400)$$

In a small number of special cases, one can choose the transformation α in such a way that the P -system is in fact orthogonal. In the one-dimensional case this is known as the "Meixner class" [2]. The Poisson measure is one of them as we shall see in the next chapter.

6 Configuration Space and Poisson Analysis

6.0.1 Processes in Discrete Configuration Spaces

Ising model: spins with values $+1$ or -1 on the sites of a lattice

Also: interpretation as "lattice gas". Particle is present resp. absent at the vertex.

Spin flip $+1 \rightarrow -1$: particle is gone ("death")

Spin flip $-1 \rightarrow +1$: particle appears ("birth")

Vast literature on various possible dynamical processes (see e.g. [3])

Large variety of models.

Example: Independent births and deaths: "*Glauber dynamics*"

Example: Simultaneous death and birth at two neighboring sites: "*Kawasaki dynamics*". Particles hop from one site to a neighboring one. Particle number conserved.

For configurations in the continuum much less is known. Recent results can be found

- for Glauber dynamics e.g. in [1]
- for Kawasaki in [5][6]

6.1 Continuous Configuration Spaces

We want to describe infinite systems of particles: "configurations" of indistinguishable point particles in R^d or in some subset $X \subseteq R^d$.

The *configuration space* $\Gamma := \Gamma_X$ is the set of all **locally finite subsets** of X , i.e.,

$$\Gamma := \{\gamma \subset X : \#(\gamma \cap K) < \infty \text{ for bounded } K \subset X\}.$$

For a given configuration $\gamma = \{x_1, x_2, \dots\}$ we denote

$$\langle \gamma, f \rangle = \sum_{x \in \gamma} f(x) = \sum_{x \in \gamma} \int \delta(x - x') f(x') dx'.$$

This is well defined if f is continuous and zero outside a finite volume: the sum is then finite - no problem of convergence arises.

6.2 Dynamics on Configurations

Example 21 *Kawasaki dynamics:*

$$\partial_t F(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x-y) (F(\gamma \setminus x \cup y) - F(\gamma))$$

Particles are hopping from x to y , with rate $a(x-y)$. How to calculate $F = F(\gamma, t)$ for a given initial state $F(\gamma)$?

Example 22 *Glauber "Birth and Death":*

$$\begin{aligned} \partial_t F(\gamma) &= \int_{\mathbb{R}^d} dy b(y) (F(\gamma \cup y) - F(\gamma)) \\ &\quad + \sum_{x \in \gamma} d(x) (F(\gamma \setminus x) - F(\gamma)) \end{aligned}$$

6.3 Poisson Measures

- Consider n -point configurations $\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\}$.
- We can relate them to non-coinciding n -tuples

We want to attribute probabilities to configurations, and begin by considering configurations in a domain X of space with finite volume:

$$|X| = V < \infty$$

For configurations of only one point $x \in \mathbb{R}^d$ the obvious choice will be a probability proportional to the volume element dv .

For n -point configurations, elements of $\Gamma_X^{(n)}$ we shall use

$$dm_n = \frac{1}{n!} (dv)^n$$

with the combinatorial $1/n!$ factor for the indistinguishability of the n particles.
- But we are interested in configurations of arbitrary many particles, i.e. we want a probability measure on

$$\Gamma_X = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}.$$

We first extend the measures m_n to a measure m on Γ_X , simply by setting

$$m|_{\Gamma_X^{(n)}} = m_n.$$

This is not a probability:

$$\begin{aligned} m(\Gamma_X) &= m\left(\bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}\right) = \sum_n m(\Gamma_X^{(n)}) \\ &= \sum_n \frac{1}{n!} \left(\int_X dv\right)^n = \exp(V). \end{aligned}$$

Must normalize to get a probability measure on Γ

$$\pi \equiv \exp(-V) \cdot m$$

6.3.1 The Characteristic Function

$$\begin{aligned} E(\exp(i \langle \gamma, f \rangle)) &= \int_{\Gamma} \exp(i \langle \gamma, f \rangle) d\pi(\gamma) \\ &= \sum_n \int_{\Gamma^{(n)}} \exp(i \langle \gamma, f \rangle) d\pi(\gamma) \\ &= \exp(-V) \sum_n \frac{1}{n!} \left(\int_{X^n} \exp(i \sum_{k=1}^n f(x_k)) \prod_k (dx_k) \right) \\ &= e^{-V} \sum_n \frac{1}{n!} \left(\int_X \exp(i f(x)) dx \right)^n = e^{-V} \left(\int_X \exp(i f(x)) dx \right) \\ &= \exp \left(\int_X (\exp(i f(x)) - 1) dx \right). \end{aligned}$$

We have (re)discovered the "characteristic function" of the Poisson White Noise probability measure:

$$\begin{aligned} E(\exp(i \langle \gamma, f \rangle)) &= \exp \left(\int_X (\exp(i f(x)) - 1) dx \right) \\ &= C_{\pi}(f) = \int e^{i \langle \omega, f \rangle} d\pi(\omega) \end{aligned}$$

Note: No need to restrict ourselves to a space of finite volume -

$$C_{\pi}(f) = \left(\int_{R^d} (\exp(i f(x)) - 1) dx \right)$$

is well defined even in the limit where $X = R^d$, and we have a limiting measure

$$\pi = \lim_{X \rightarrow R^d} \pi|_{\Gamma_X}$$

Likewise for more general densities, with

$$dv = z(x)dx$$

where z is a non-negative "intensity":

$$C_{\pi_z}(f) = \exp \left(\int_{R^d} (\exp(i f(x)) - 1) z(x) dx \right)$$

6.3.2 Bochner-Minlos

Recall that the Bochner and Minlos theorem guarantees the existence of a probability measure on the space of distributions such that

$$C_{\pi_z}(f) = \int_{D^*} e^{i\langle \omega, f \rangle} d\pi_z(\omega)$$

see e.g. [6]. - In our explicit construction we have used the formula

$$\langle \gamma, f \rangle = \sum_{x \in \gamma} f(x) = \sum_{x \in \gamma} \int \delta(x - x') f(x') dx'.$$

We see from this that the measure is concentrated only on those distributions which are sums of Dirac δ -functions

$$\omega_\gamma = \sum_{x \in \gamma} \delta_x.$$

6.4 The Poisson L^2 - Space

Introduce the L^2 space of functions F on configurations γ with Poisson measure:

$$F = F(\gamma) \in L^2(d\pi_z).$$

6.4.1 Charlier Polynomials

In $L^2(d\pi_z)$ consider

$$e(f, \omega) = \exp(\langle \omega, \ln(1 + f) \rangle - \langle f \rangle), \quad \omega = \omega_\gamma, \quad (401)$$

with

$$\langle f \rangle = \int f(x) z(x) dx.$$

For $\omega = \omega_\gamma = \sum_{x \in \gamma} \delta_x$, find

$$e(f, \omega_\gamma) = \exp(-\langle f \rangle) \prod_{x \in \gamma} (1 + f(x)).$$

Their scalar product is

$$(e(f), e(g))_{L^2(d\pi_z)} = e^{(f, g)_{L^2(dv)}}.$$

Note that this is exactly the scalar product of two coherent states in Fock space!

$e(f, \omega_\gamma)$ is generating function of orthogonal polynomials (“Charlier polynomials”) in ω .

Expanding $e(f, \omega_\gamma)$ in orders of f :

$$e(f, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n(\omega), f^{\otimes n} \rangle,$$

we get the orthogonality relation

$$(\langle C_n(\omega), f^{\otimes n} \rangle, \langle C_m(\omega), g^{\otimes m} \rangle)_{L^2(d\pi)} = \delta_{mn} n! (f^{\otimes n}, g^{\otimes n})_{L^2}$$

6.4.2 Another kind of Fock

Extend from

$$f^{\otimes n} = f(x_1) \dots f(x_n)$$

to symmetric functions

$$f_n = f_n(x_1, \dots, x_n)$$

With these we can express any square integrable F as

$$F(\gamma) = \sum_{n=0}^{\infty} \langle C_n(\omega_\gamma), f_n \rangle$$

and obtain an isomorphism of Hilbert spaces $L^2(\Gamma, d\pi(\gamma)) \simeq \mathcal{F}$:

$$\int F(\gamma) G(\gamma) d\pi(\gamma) = \sum_{n=0}^{\infty} n! \int f_n(x_1, \dots, x_n) g_n(x_1, \dots, x_n) d^n v.$$

6.5 Annihilation and Creation Operators on Poisson Space

6.5.1 Annihilation Operators

Many interesting questions arise. What about the (images of) annihilation and creation operators in Poisson space?

Exercise 21 *Show*

$$\begin{aligned} (a(h)F)(\gamma) &= \int_X (F(\gamma \cup \{x\}) - F(\gamma)) h(x) dx \\ &\stackrel{\text{def.}}{=} \int_X D_x F(\gamma) h(x) dx. \end{aligned}$$

Hint: Recall that coherent states are eigenstates of annihilation operators

$$a(h)e(f) = (h.f)e(f)$$

and verify the above for

$$F(\gamma) = e(f, \gamma) = \exp(\langle \gamma, \ln(1+f) \rangle - \langle f \rangle).$$

6.5.2 Creation Operators

For the adjoint one finds similarly

$$(a^*(g)F)(\gamma) := \sum_{x \in \gamma} F(\gamma \setminus \{x\})g(x) - \langle g \rangle F(\gamma).$$

To determine the action of the adjoint operator $a^*(g)$, we use the "Mecke Identity"

$$\begin{aligned} & \int_{\Gamma} \sum_{x \in \gamma} H(\gamma, x) d\pi(\gamma) \\ &= \int_X \int_{\Gamma} H(\gamma \cup \{x\}, x) d\pi(\gamma) dx \end{aligned}$$

(see, e.g., [Mec67]).

For the adjoint of the operator $a(g)$ have

$$\begin{aligned} (G, a^*(g)F)_{L^2(\pi)} &= (a(g)G, F)_{L^2(\pi_\sigma)} \\ &= \int_X \int_{\Gamma} G(\gamma \cup \{x\})F(\gamma)g(x) d\pi(\gamma) dx \\ &\quad - \left(\int_{\Gamma} G(\gamma)F(\gamma) d\pi(\gamma) \right) \langle g \rangle. \end{aligned}$$

Now use the Mecke identity on the 1st integral and get

$$\begin{aligned} (G, a^*(g)F)_{L^2(\pi)} &= \int_{\Gamma} G(\gamma) \left(\sum_{x \in \gamma} F(\gamma \setminus \{x\})g(x) \right) d\pi(\gamma) \\ &\quad - \left(\int_{\Gamma} G(\gamma)F(\gamma) d\pi(\gamma) \right) \langle g \rangle \\ &= \left(G, \sum_{x \in \cdot} (F(\cdot \setminus \{x\})g(x)) - \langle g \rangle F \right)_{L^2(\pi)}. \end{aligned}$$

I. e. the action of $a^*(g)$ is

$$(a^*(g)F)(\gamma) = \sum_{x \in \gamma} (F(\gamma \setminus \{x\})g(x)) - \langle g \rangle F(\gamma).$$

Remark 8 By the definition of the creation operators $a^*(\varphi)$ ($\varphi \in \mathcal{D}$) on Fock space, one has

$$\sum \frac{(a^*(\varphi))^n}{n!} 1 = e_{Fock}(\varphi).$$

Therefore

$$\sum \frac{(a_{\pi}^*(\varphi))^n}{n!} 1 = e_{\pi}(\varphi)$$

and

$$(a_{\pi}^*(\varphi))^n 1 = \langle C_n^{\sigma}, \varphi^{\otimes n} \rangle$$

for each $n \in \mathbb{N}$.

6.5.3 Poisson Stochastic Integrals

As in the case of White Noise analysis, we can define a Poisson-Skorohod integral (over $X = \mathbb{R} \ni t$) for suitable integrands Φ [IK88]

$$I(\Phi) = \int_X D_t^* \Phi dt.$$

6.5.4 Bogoliubov Exponentials

For later reference we finally introduce

$$e_B(f, \omega_\gamma) = \exp \langle f \rangle e_B(f, \omega_\gamma) = \prod_{x \in \gamma} (1 + f(x)).$$

Their expectations with respect to suitable measures μ on configuration space

$$E(e_B(f)) = \int_\Gamma e_B(f, \omega_\gamma) d\mu(\gamma) = \sum_n \frac{1}{n!} \langle k_n^\mu, f^{\otimes n} \rangle$$

are called Bogoliubov functionals and are the generators of the n^{th} order correlation functions for the distribution μ .

6.6 Lebesgue-Poisson Measure on Finite Configurations - yet another Fock space.

Definition 20 The space of finite configurations Γ_0 is

$$\Gamma_0 := \{\gamma \in \Gamma : |\gamma| < \infty\} \subset \Gamma.$$

$$\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}. \quad (402)$$

Define a measure $\sigma^{(n)}$ on $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ by

$$\sigma^{(n)} : = \sigma^{\otimes n}, \quad (403)$$

$$\sigma^{(0)}(\emptyset) : = 1 \quad (404)$$

Definition 21 The Lebesgue-Poisson measure with intensity measure σ is defined by the sum of measures

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_X^{(n)}.$$

- Functions G on Γ_0 :

$$G|_{\Gamma^{(n)}}(\gamma) = G(\{x_1, \dots, x_n\}) \equiv G^{(n)}(x_1, \dots, x_n)$$

$G^{(n)}(x_1, \dots, x_n)$ is by definition a symmetric function!

- Have

$$\begin{aligned} \int_{\Gamma_0} G(\eta) d\lambda_\sigma(\eta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_X^{(n)}} G(\eta) d\sigma^{(n)}(\eta) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} G^{(n)}(x_1, \dots, x_n) d\sigma^{\otimes n}(x_1, \dots, x_n) \end{aligned}$$

- Define

$$\begin{aligned} n!f^{(n)}(x_1, \dots, x_n) &:= G^{(n)}(x_1, \dots, x_n) \\ f^{(0)} &:= G(\emptyset), \end{aligned}$$

Have

$$\begin{aligned} &\int_{\Gamma_0} |G(\eta)|^2 d\lambda_\sigma(\eta) \\ &= \sum_{n=0}^{\infty} n! \int_{X^n} |f^{(n)}(x_1, \dots, x_n)|^2 d\sigma^{\otimes n}(x_1, \dots, x_n) \end{aligned}$$

i.e.

$$\|G\|_{L^2(\lambda_\sigma)} = \left\| (f^{(n)})_{n=0}^\infty \right\|_{\text{Exp}L^2(\sigma, dx)}.$$

Theorem 23 The Hilbert space $L^2(\Gamma_0, d\lambda_\sigma)$ is isomorphic to the symmetric Fock space $\text{Exp}L^2(X, d\sigma)$ through the mapping I_λ given by

$$\begin{aligned} I_\lambda : L^2(\Gamma_0, d\lambda_\sigma) &\leftrightarrow \text{Exp}L^2(X, d\sigma) \\ G &\leftrightarrow (f^{(n)})_{n=0}^\infty \end{aligned}$$

Remark 9 Definition 22 Recall the total set of coherent states e_f in Gaussian analysis (eq. 379):

$$\begin{aligned} e_f(\omega) &= : e^{\int \omega(t)f(t)dt} : \\ &= \sum_n \frac{1}{n!} \int d^n t f(t_1) \dots f(t_n) : \omega(t_1) \dots \omega(t_n) : \end{aligned}$$

i.e. with Fock space image

$$e(f) = \left(\frac{1}{n!} f^{\otimes n} \right)_{n=0}^{\infty}$$

and note that their image in Lebesgue-Poisson space is then given by

$$e_{\lambda}(f, \{x_1, \dots, x_n\}) = \begin{cases} \prod_{i=1}^n f(x_i) & \text{if } n > 0 \\ 1 & \text{otherwise} \end{cases}.$$

Definition 23

$$\begin{aligned} e_{\lambda}(f, \eta) &\equiv \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\} \\ e_{\lambda}(f, \emptyset) &\equiv 1 \end{aligned}$$

is called the (Lebesgue-Poisson) coherent state corresponding to the one-particle vector f .

The importance of the (Lebesgue-Poisson) coherent states is based on the fact that a family of coherent states $\{e_{\lambda}(f) : f \in \mathcal{L}\}$ is total in $L^2(\lambda_{\sigma})$ whenever \mathcal{L} is a dense subspace in $L^2(\sigma)$. Also, it is not hard to show that they are in $L^p(\lambda_{\sigma})$.

Proposition 24 Let $p \geq 1$. For all $f \in L^p(\sigma)$ we have $e_{\lambda}(f) \in L^p(\lambda_{\sigma})$ and, moreover,

$$\|e_{\lambda}(f)\|_{L^p(\lambda_{\sigma})}^p = \exp \left(\|f\|_{L^p(\sigma)}^p \right).$$

Given a $f \in L^p(\sigma)$, $p \geq 1$, a direct application of the definition of the Lebesgue-Poisson measure λ_{σ} yields

$$\begin{aligned} \int_{\Gamma_0} |e_{\lambda}(f, \eta)|^p d\lambda_{\sigma}(\eta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_X^{(n)}} |e_{\lambda}(f, \eta)|^p d\sigma^{(n)}(\eta) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} |f(x_1) \dots f(x_n)|^p d\sigma^{\otimes n}(x_1, \dots, x_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_X |f(x)|^p d\sigma(x) \right)^n = \exp \left(\int_X |f(x)|^p d\sigma(x) \right). \end{aligned}$$

■

In particular, for a function $f \in L^1(\sigma)$, the expectation of the corresponding coherent state $e_{\lambda}(f)$ is given by

$$\begin{aligned} \int_{\Gamma_0} e_{\lambda}(f, \eta) d\lambda_{\sigma}(\eta) &= \exp \left(\int_X f(x) d\sigma(x) \right) \\ &= \exp \left(\langle f \rangle_{\sigma} \right). \end{aligned} \tag{405}$$

6.7 Three Unitary Isomorphisms

A natural unitary isomorphism exists between the spaces $L^2(\pi_\sigma)$ and $L^2(\lambda_\sigma)$.

Recall that we have defined the unitary isomorphism I_λ between the space $L^2(\lambda_\sigma)$ and the symmetric Fock space $\text{Exp}L^2(\sigma)$:

$$\begin{aligned} I_\lambda : L^2(\lambda_\sigma) &\rightarrow \text{Exp}L^2(\sigma) \\ G &\mapsto (f^{(n)})_{n=0}^\infty. \end{aligned} \quad (406)$$

Combining the previous statements concerning unitary isomorphisms we have

Proposition 25 The linear mapping

$$I_{\lambda\pi} = I_\pi^{-1} \circ I_\lambda : L^2(\lambda_\sigma) \rightarrow L^2(\pi_\sigma)$$

defined by

$$\begin{aligned} G &\rightarrow I_{\lambda\pi}(G) = \sum_{n=0}^\infty \langle C_n^\sigma, f^{(n)} \rangle, \\ \text{with } f^{(n)}(x_1, \dots, x_n) &= \frac{1}{n!} G(\{x_1, \dots, x_n\}), \\ f^{(0)} &:= G(\emptyset) \end{aligned}$$

is a unitary isomorphism with inverse mapping $I_{\pi\lambda} := I_\lambda^{-1} \circ I_\pi$ given by

$$\begin{aligned} I_{\pi\lambda} &: \left(\sum_{n=0}^\infty \langle C_n^\sigma, f^{(n)} \rangle \right) \rightarrow G, \\ \text{with } G(\{x_1, \dots, x_n\}) &= n! f^{(n)}(x_1, \dots, x_n). \end{aligned}$$

Summarizing

$$\begin{array}{ccccc} & & \text{Exp}L^2(\sigma) & & \\ & \nearrow I_\lambda & & \searrow I_\pi^{-1} & \\ L^2(\lambda_\sigma) & & \xrightarrow{I_{\lambda\pi}} & & L^2(\pi_\sigma) \end{array}$$

In particular

$$\begin{array}{ccccc} & & e(f) & & \\ & \nearrow I_\lambda & & \searrow I_\pi^{-1} & \\ e_\lambda(f) & & \xrightarrow{I_{\lambda\pi}} & & e_\pi(f) \end{array}$$

with

$$\begin{aligned} e_\lambda(f, \gamma) &= \prod_{x \in \gamma} f(x) \quad \gamma \in \Gamma_0 \\ e(f) &= \left(\frac{1}{n!} f^{\otimes n} \right)_{n \in \mathbb{N}} \\ e_\pi^\alpha(f, \gamma) &= \exp(-\langle f \rangle_\sigma) \prod_{x \in \gamma} (1 + f(x)) \quad \gamma \in \Gamma \end{aligned}$$

Remark 10 Clearly these isomorphisms extend to test and generalized functions.

Remark 11 Recall the definition of the S_π -transform

$$\begin{aligned} (S_\pi \Psi)(f) &= (\Psi, e_\pi^\alpha(f, \cdot))_{L^2(\Gamma, d\pi_\sigma)} \\ &= \sum_{n=0}^{\infty} \left(f^{\otimes n}, \psi^{(n)} \right)_{L^2(\sigma^{\otimes n})} \end{aligned}$$

if

$$\Psi = \sum_{n=0}^{\infty} \langle C_n^\sigma, \psi^{(n)} \rangle.$$

Setting

$$S_\lambda = S_\pi \circ I_{\lambda\pi}$$

one has

$$(S_\lambda \Psi)(f) = (\Psi, e_\lambda(f, \cdot))_{L^2(\Gamma_0, d\lambda_\sigma)}.$$

6.8 Algebraic Structures

As in Gaussian analysis - see e.g. eq. (384) - we can introduce a Wick product in $L^2(\pi_\sigma)$ by setting

$$F_1 \diamond F_2 = S_\pi^{-1} (S_\pi F_1 \cdot S_\pi F_2).$$

As a consequence - see e.g. eq. (386) - the Wick product $F_1 \diamond F_2$ for

$$F_i = \sum_n \left\langle C_n^\sigma, f_i^{(n)} \right\rangle \in L^2(\pi_\sigma) \quad (407)$$

is defined by

$$F_1 \diamond F_2 := \sum_{n=0}^{\min\{N_1, N_2\}} \left\langle C_n^\sigma, \sum_{k=0}^n f_1^{(k)} \hat{\otimes} f_2^{(n-k)} \right\rangle.$$

What then is the product $*$ induced on $L^2(\lambda_\sigma)$ by

$$F_1 * F_2 = I_{\lambda\pi} (F_1 \diamond F_2)?$$

To this end we use

$$I_{\lambda\pi} = S_\lambda^{-1} S_\pi$$

and consider

$$F_i = e_\pi^\alpha(f_i) \xrightarrow{S_\pi} \left(\frac{1}{n!} f^{\otimes n} \right)_{n \in N}$$

so that

$$\begin{aligned} e_\pi^\alpha(f_1) \diamond e_\pi^\alpha(f_2) &\xrightarrow{S_\pi} \left(\sum_{k=0}^n \frac{f_1^{\otimes k} \hat{\otimes} f_2^{\otimes (n-k)}}{k!(n-k)!} \right)_{n \in N} \\ &= \left(\frac{(f_1 + f_2)^{\otimes n}}{n!} \right)_{n \in N} = e(f_1 + f_2) \end{aligned}$$

and

$$e_\pi(f_1) \diamond e_\pi(f_2) \xrightarrow{S_\lambda^{-1} S_\pi} S_\lambda^{-1} e(f_1 + f_2) = e_\lambda(f_1 + f_2).$$

I.e.

$$\begin{aligned} (e_\lambda(f) * e_\lambda(g))(\eta) &= e_\lambda(f + g, \eta) \\ &= \prod_{x \in \eta} (f(x) + g(x)) \\ &= \sum_{\xi \subset \eta} \left(\prod_{x \in \xi} f(x) \right) \left(\prod_{x \in \eta \setminus \xi} g(x) \right) \\ &= \sum_{\xi \subset \eta} e_\lambda(f)(\xi) \cdot e_\lambda(g)(\eta \setminus \xi). \end{aligned}$$

This extends through bilinearity to

$$\begin{array}{ccc} & \begin{array}{c} \nearrow \\ I_\lambda \end{array} & \left(\sum_{k=0}^n f^{(k)} \hat{\otimes} g^{(n-k)} \right)_{n \in N} \\ & & \searrow \\ & \begin{array}{c} I_{\lambda\pi} \\ \rightarrow \end{array} & I_\pi^{-1} \end{array}$$

$$\sum_{\xi \subset \eta} F(\xi) G(\eta \setminus \xi) \quad \quad \quad F \diamond G$$

Exercise 22 Show that

$$C_n^\sigma(\gamma, \varphi^{\otimes n}) = (\langle \gamma, \varphi \rangle - \langle \varphi \rangle)^{\diamond n}.$$

6.9 Return to Kawasaki

Recall the Kawasaki dynamics:

$$\partial_t F(\gamma) = HF(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy g(x - y) (F(\gamma \setminus x \cup y) - F(\gamma))$$

In terms of creation and annihilation operators one finds

$$H = \int dx z(x) \int dy (g(x-y) - g_0 \delta(x-y)) (a^*(x)a(y) - a(y)).$$

Clearly, in Fock space language this corresponds to a quadratic Hamiltonian and time development can be calculated in closed form.

6.9.1 Time Evolution:

Time evolution of Bogoliubov exponentials takes on a particularly simple form:

$$e^{Ht} e_B(f) = e_B(e^{tA} f)$$

$$Af(x) := \int_{\mathbb{R}^d} dy g(x-y) (f(y) - f(x)).$$

Evolution of the initial (Poisson) distribution

$$\pi_z \rightarrow P_{\pi_z, t}$$

under the adjoint of e^{Ht} is characterized by

$$\int e_B(\varphi, \gamma) P_{\pi_z, t}(d\gamma) = \int e_B(e^{tA} \varphi, \gamma) \pi_z(d\gamma) = \exp \left(\int_{\mathbb{R}^d} e^{tA} \varphi(x) z(x) dx \right).$$

The Process

A Markov process \mathbf{X}_t on Γ , associated with the Kawasaki dynamics

$$\mathbf{E} \left[F(\mathbf{X}_t) - F(\mathbf{X}_0) - \int_0^t ds H F(\mathbf{X}_s) \right] = 0. \quad (408)$$

The transition probability $(\mathbf{P}_t)_{t \geq 0}$ of the process $(\mathbf{X}_t)_{t \geq 0}$ is just the product of the one-particle transition probabilities $e^{tA}(x-y)$, i.e., $\prod_{n=1}^{\infty} e^{tA}(x_n - y_n) dy_n$.

Technical restriction: the process may not start at any arbitrary initial configuration $\gamma \in \Gamma$. Consider the set Θ of all $\gamma \in \Gamma$ such that, for some $m \in \mathbb{N}$ (depending on γ),

$$|\gamma_{B(n)}| \leq m \text{vol}(B(n)), \quad \forall n \in \mathbb{N}.$$

Have $\mu(\Theta) = 1$ for every probability measure μ on Γ whose correlation functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, fulfill the Ruelle bound

$$k_\mu^{(n)} \leq C^n$$

i.e. for Poisson measures with bounded intensity, and for Gibbs measures with suitable potentials, cf. [Ruelle; 1970].

Shall use the so-called empirical field corresponding to a $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$n_t(\varphi, \mathbf{X}) := \langle \varphi, \mathbf{X}_t \rangle = \sum_{x \in \mathbf{X}_t} \varphi(x).$$

Evolution of Distributions The distribution at time t , $P_{\pi_z, t}(d\gamma)$ is again Poissonian, with intensity $z_t \in L^\infty(\mathbb{R}^d, dx)$, given by

$$\int_{\mathbb{R}^d} dx e^{tA} f(x) z(x) = \int_{\mathbb{R}^d} dx f(x) z_t(x), \quad (409)$$

for all $f \in L^1(\mathbb{R}^d, dx)$. Since e^{tA} is positivity preserving in $L^1(\mathbb{R}^d, dx)$, it follows from (409) that $z_t \geq 0$.

Invariant Distributions Proof. Poisson distributions are invariant under free Kawasaki dynamics iff their intensity

$$z(x) = \text{const.}$$

Proof. $H^*1 = 0$ iff the linear annihilation term in H vanishes:

$$\int dy a(y) \left(\int dx z(x) (g(x-y) - g(x)) \right) \stackrel{!}{=} 0$$

■

Using Fourier transforms one sees that this requires $z = \text{const.}$

The Symmetric Case

For g even and constant $z > 0$, H gives rise to a symmetric Dirichlet form on $L^2(\Gamma, \pi_z)$,

$$(F, HF) = -\frac{1}{2} \int_{\Gamma} \pi_z(d\gamma) \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy g(x-y) |F(\gamma)|^2.$$

This allows to derive a Markov process on Γ with cadlag paths and having π_z as an invariant measure [5]. In this setting H is a negative essentially self-adjoint operator on $L^2(\Gamma, \pi_z)$, and the generator of a contraction semi-group on $L^2(\Gamma, \pi_z)$.

6.10 Asymptotics

6.10.1 Large Time Asymptotics

- Any Poisson state of constant intensity is invariant under the evolution (equilibrium).
- “Local Equilibrium”: Poisson states with non-constant intensity $z = z(x)$.

Recall that the state at $t \geq 0$ is again Poissonian, with intensity $z_t \in L^\infty(\mathbb{R}^d, dx)$, given by

$$\int_{\mathbb{R}^d} dx e^{tA} f(x) z(x) = \int_{\mathbb{R}^d} dx f(x) z_t(x).$$

Arithmetic Mean

One says that a function $z \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ has arithmetic mean whenever

$$\lim_{R \rightarrow +\infty} \frac{1}{\text{vol}(B(R))} \int_{B(R)} dx z(x) \equiv \text{mean}(z) \quad (410)$$

exists.

Theorem 26 Let $z \geq 0$ be a bounded measurable function whose Fourier transform \tilde{z} is a signed measure. Then z has arithmetic mean and the one-dimensional distribution $P_{\pi_z, t}$ converges weakly to $\pi_{\text{mean}(z)}$ as t goes to infinity.

Proof (Outline):

In this case $\text{mean}(z) = \tilde{z}(\{0\})$,

$$\int_{\mathbb{R}^d} dx f(x) z_t(x) = \int_{\mathbb{R}^d} dx e^{tA} f(x) z(x) \rightarrow \text{mean}(z) \int_{\mathbb{R}^d} dx f(x),$$

and

$$\pi_{z_t} \rightarrow \pi_{\text{mean}(z)}$$

weakly, because of convergence of characteristic functions.

Remark:

1. the same conclusion holds e.g. for

$$z(x) = \begin{cases} z_1 & \text{if } x_1 \geq 0 \\ z_0 & \text{otherwise} \end{cases}.$$

with $\text{mean}(z) = \frac{z_0 + z_1}{2}$, by explicit calculation although in this case \hat{z} is not a signed measure. It seems natural to expect that the large time asymptotic exists for all bounded intensities which have arithmetic mean.

2. On the other hand, not all measurable bounded non-negative functions z have an arithmetic mean. Counterexamples are slowly oscillating functions such as

$$z(x) = c + \cos(\ln(1 + |x|)), \quad x \in \mathbb{R}^d,$$

where $c \neq 1$. Then for large R

$$\frac{1}{\text{vol}(B(R))} \int_{B(R)} z(x) dx \sim c + \frac{1}{\sqrt{1+d^2}} \sin(\ln(R) + \arctan(d)).$$

3. The non-ergodicity of the infinite particle processes is reflected in the non-ergodicity of the one particle processes in this class of initial intensities.

6.10.2 The hydrodynamic limit

The first correlation function $\rho_t(x)$ is the density of the first moment measure of the empirical field:

$$E(n_t(\varphi, \mathbf{X})) = E(\langle \varphi, \mathbf{X}_t \rangle) = \int \varphi(x) \rho_t(x) dx.$$

Consider space-time scale transformation given by $\langle \varphi, \gamma \rangle \rightarrow \varepsilon^d \langle \varphi(\varepsilon \cdot), \gamma \rangle$, $t \rightarrow \varepsilon^{-\kappa} t$ for suitable $\kappa > 0$, $z \rightarrow z(\varepsilon \cdot)$.

1. If

$$g_i^{(1)} := \int_{\mathbb{R}^d} dx x_i g(x) \neq 0,$$

then for $\kappa = 1$

$$\int_{\mathbb{R}^d} dx \rho_t(x) \varphi(x) = \int_{\mathbb{R}^d} dx z(x + t g^{(1)}) \varphi(x),$$

so that, if the intensity z is smooth enough

$$\frac{\partial}{\partial t} \rho_t(x) = g^{(1)} \cdot \nabla \rho_t(x) = \operatorname{div}(g^{(1)} \rho_t(x))$$

with the initial condition $\rho_0 = z$.

2. If $g^{(1)} = 0$, and

$$g_{ij}^{(2)} := \int_{\mathbb{R}^d} dx x_i x_j g(x)$$

then for $\kappa = 2$

$$\int_{\mathbb{R}^d} dx \rho_t(x) \varphi(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx z(x) \int_{\mathbb{R}^d} dk e^{ik \cdot x} e^{-\frac{t}{2} \langle g^{(2)} k, k \rangle} \hat{\varphi}(k),$$

solution of the partial differential equation

$$\frac{\partial}{\partial t} \rho_t(x) = \frac{1}{2} \sum_{i,j=1}^d g_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j} \rho_t(x).$$

3. Consider weak asymmetries, decomposing g into a sum of an even function p and an odd function q , and use the scaling

$$g_\varepsilon := p + \varepsilon q$$

and $\kappa = 2$.

The limiting density ρ_t is solution of the partial differential equation

$$\frac{\partial}{\partial t} \rho_t(x) = \operatorname{div}(g^{(1)} \rho_t(x)) + \frac{1}{2} \sum_{i,j=1}^d g_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j} \rho_t(x).$$

6.10.3 Far from Equilibrium

More General Initial States

The construction of the free Kawasaki process and its scaling limits are not restricted to Poissonian initial distributions. Sufficient conditions for admissible measures can be stated in terms of their correlation functions and are in particular fulfilled for Gibbs measures at high temperatures.

Gibbs Measures A probability measure μ on Γ is called a Gibbs measure for V , intensity function $z \geq 0$, and inverse temperature β if it fulfills the Georgii-Nguyen-Zessin equation [NZ79]

$$\int_{\Gamma} \mu(d\gamma) \sum_{x \in \gamma} H(x, \gamma) = \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} dx z(x) H(x, \gamma \cup \{x\}) e^{-\beta E(x, \gamma)} \quad (411)$$

with

$$E(x, \gamma) := \begin{cases} \sum_{y \in \gamma} V(x - y), & \text{if } \sum_{y \in \gamma} |V(x - y)| < \infty \\ +\infty, & \text{otherwise} \end{cases}.$$

(Equivalent to DLR-equation, see [Georgii, Nguyen-Zessin].)

The correlation functions corresponding to such measures fulfill a Ruelle bound, and thus, the measures are supported on Θ , but are neither reversible nor invariant initial distributions for the free Kawasaki dynamics.

Theorem: Consider a Gibbs measure with translation invariant potential V , temperature and activity z which is in the high temperature low activity regime, and let the Fourier transform of z be a bounded signed measure. Then

1. the first correlation function has arithmetic mean, and the one-dimensional distribution $P_{\pi_z, t}$ converges weakly to $\pi_{\text{mean}(z)}$ when t goes to infinity.
2. Hydrodynamic scaling PDEs hold as before, where now the initial value $\rho_0(x)$ is a scaling limit of the first correlation function.

Specifically, because of translation invariance the 1st correlation function for a constant activity c is a constant

$$\rho^{(1)} = \rho^{(1)}(c).$$

For $z = z(x)$ have

$$\rho_0(x) = \rho^{(1)}(z(x)).$$

Proof and more details:

Yu. Kondratiev, T. Kuna, M.J. Oliveira, J.L. Silva, L. Streit, Hydrodynamic limits for free Kawasaki dynamics of continuous particle systems. To appear 2008.

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