An Introduction to Infinite Dimensional Analysis for Physicists.

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1 Introduction

These lectures are addressed to a physics student audience. We would expect them to be acquainted with the basics of Fock space, and will go from there to have a look at the tool box of modern infinite dimensional analysis, with statistical and quantum physics applications in mind. For the benefit of students with a limited access to literature we emphasize reference to internet resources.

Since the advent of particle physics, but also e.g. of hydrodynamics, physicists are concerned with systems that have infinitely many degrees of freedom; technically speaking they would need mathematics with infinitely many variables, in particular, infinite dimensional analysis.

While (some) mathematicians slowly and carefully began to develop these tools, physicists could not wait for all the i's to be dotted and all t's crossed. They discovered Fock space. Later on the mathematicians disccovered it too. Which is good. Since the days of the ancient Greeks we know that

$$o\psi\varepsilon\ \delta\varepsilon\omega\nu\ \alpha\lambda\varepsilon o\upsilon\sigma\iota\ \mu\upsilon\lambda o\iota,\ \alpha\lambda\varepsilon o\upsilon\sigma\iota\ \delta\varepsilon\ \lambda\varepsilon\pi\tau\alpha$$

i.e. the mills of the Gods grind late but very, very fine.

So do the mills of the mathematicians, and their output is far more reliable than that of the physicists who, more often than not, get themselves into a mess for lack of dotted i's and crossed t's.

Which also is good, for undaunted they will look for a quick fix, often to get the desired results, and often, after a decade or so, the physicists' quick fix becomes a solid piece of new mathematics in the hands of the slow but careful millers.

2 Recall Fock Space

Bosonic n-particle wavefunctions g, symmetric in the variables $x_1, \ldots, x_n \in \mathbb{R}^d$ and square integrable, give rise to Fock space vectors $\Psi_n(g)$, normed by

$$\|\Psi_n\|^2 = n!\,(g_n, g_n)_{L^2}$$

with

$$(g_n, g_n)_{L^2} = \left| \int \widetilde{g}_n(k_1, \ldots, k_n) \right|^2 \prod \frac{dk_i}{\omega(k_i)}$$

For relativistic particles, to implement Poicaré invariance, one uses $\omega(k) = \sqrt{k^2 + m^2}$.

For $n = 0$ we introduce the zero-particle vectors which are just constant multiples of the vacuum state Ω.

$$\Psi_0 = c\Omega$$

with

$$\|\Psi_0\|^2 = |c|^2$$

Together, they span the Fock space

$$\mathcal{H} = \{\Psi : \Psi = (\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots)\}$$

with norm

$$\|\Psi\|^2_{\mathcal{H}} = \sum_{n=0}^{\infty} n!\,(g_n, g_n)_{L^2}\,.$$

2.1 Creation and Annihilation Operators, Canonical Fields, Normal Ordering

Annihilation operators remove one of the particles, through the operation

$$\begin{aligned} (a(f)g_n)\,(x_1, \ldots, x_{n-1}) &= \int dx_n f(x_n) g_n(x_1, \ldots, x_n) \\ a(f)\Psi_0 &= 0 \end{aligned}$$

Their adjoints are the creation operators which add an extra particle with wave function f

$$\begin{aligned} a^*(f)\Psi_n \quad & \to Sym\ \ (f \cdot g_n)\,(x_1, \ldots, x_{n+1}) \\ &= \frac{1}{(n+1)!} \sum_{\sigma} f(x_{\sigma(1)}) g_n(x_{\sigma(2)}, \ldots, x_{\sigma(n+1)}) \end{aligned}$$

Note that we obtain certain n-particle vectors from the vacuum by applying $a^*(f)$ n times to the vacuum:

$$(a^*(f))^n\,\Omega = \Psi\left(f^{\otimes n}\right)$$

with

$$f^{\otimes n}(x_1, \ldots, x_n) = f(x_1) \cdot \ldots \cdot f(x_n)$$

and scalar product

$$\left(\Psi\left(f^{\otimes m}\right), \Psi\left(g^{\otimes n}\right)\right)_{\mathcal{H}} = \delta_{mn} n!(f,g)^n$$

It is not hard to verify that these span the whole Fock space.

One finds further the commutation relations

$$\begin{aligned} [a(f), a(g)] &= 0 = [a^*(f), a^*(g)] \\ [a(f), a^*(g)] &= (f, g) \end{aligned}$$

We can introduce a selfadjoint φ by defining

$$\varphi(f) = a(f) + a^*(f)$$

if f is real. Note that, contrary to the $(a^*(f))^n\,\Omega$, the $(\varphi(f))^n\,\Omega$ are **not** orthogonal for different n.

Exercise 1 *Check wether the vectors* $(\varphi(f))^n \Omega$ *are orthogonal to*

- *the vacuum state*
- *to one-particle states*

The procedure of orthogonalizing them is the via the well known Wick or normal ordering. The usual iterative orthogonalization of the vectors Ω, $\varphi(f)\Omega$, $(\varphi(f))^2 \Omega$, $(\varphi(f))^3 \Omega, \ldots$ produces vectors denoted by Ω, $:\varphi(f):\Omega$, $:(\varphi(f))^2:\Omega$, $:(\varphi(f))^3:\Omega, \ldots$ which are given by the recursion formula

$$:(\varphi(f))^{n+1}: = \varphi(f)\cdot \ :(\varphi(f))^n: - n\cdot(f,f)\cdot \ :(\varphi(f))^{n-1}:$$

Wick's rule states that we obtain: $(\varphi(f))^n$: by writing $(\varphi(f))^n$ in terms of creation and annihilation operators

$$(\varphi(f))^n = (a(f) + a^*(f))^n$$

but with all the creation operators to the left of all the annihilation operators

$$:(\varphi(f))^n: = \sum_k \binom{n}{k} (a^*(f))^k (a(f))^{n-k}.$$

Exercise 2 *Verify that the Wick ordered expression*

$$:(\varphi(f))^n: = \sum_k \binom{n}{k} (a^*(f))^k (a(f))^{n-k}$$

obeys the recursion relation

$$:(\varphi(f))^{n+1}: = \varphi(f)\cdot \ :(\varphi(f))^n: - n\cdot(f,f)\cdot \ :(\varphi(f))^{n-1}:$$

Clearly, all the terms except k=n give zero when aplied to the vacuum, hence

$$:(\varphi(f))^n:\Omega = (a^*(f))^n \Omega = \Psi\left(f^{\otimes n}\right)$$

are orthogonal for different n.

2.2 Coherent States

For these vectors we have a generating function

$$\begin{aligned} : \quad e^{(\varphi(f))}:\Omega &= \sum_{n=0}^{\infty} \frac{1}{n!} :(\varphi(f))^n:\Omega \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (a^*(f))^n \Omega = e^{a^*(f)}\Omega \end{aligned}$$

in the sense that

$$: (\varphi(f))^n : \Omega = \left(\frac{d}{d\lambda}\right)^n : e^{(\varphi\lambda(f))} : \Omega\Big|_{\lambda=0}.$$

The norm of $\Psi(f^{\otimes n})$ is

$$\begin{aligned} \left\|\Psi(f^{\otimes n})\right\|^2 &= \left\|\sum_{n=0}^{\infty} \frac{1}{n!} (a^*(f))^n \Omega\right\|^2 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(f,f)^n = e^{(f,f)}, \end{aligned}$$

i.e.

$$\left\|\Psi(f^{\otimes n})\right\| = \exp(\frac{1}{2}\left|f\right|^2),$$

and we obtain unit vectors

$$e(f) = \exp(-\frac{1}{2}\left|f\right|^2) e^{a^*(f)}\Omega.$$

if we divide by this norm. These are called "coherent states" and play a central role particularly in quantum optics [12].

Exercise 3 *Verify that the coherent states are eigenstates of the annihilation operators:*

$$a(g)e(f) = (g,f) \cdot e(f)$$

Exercise 4 *Show that, for operators A, B with $[A,B] = c \cdot 1$*

1.

$$[e^A, B] = c \cdot e^A$$

2.

$$e^{-\lambda A} e^{\lambda(A+B)} e^{-\lambda B} = e^{-\frac{1}{2}\lambda^2 c}$$

(Hint: Differentiate w.r.t. to λ.)

3. Prove

$$\begin{aligned} e^{\varphi(f)} &= e^{\frac{1}{2}(f,f)} e^{a^*(f)} e^{a(f)} = e^{\frac{1}{2}(f,f)} : e^{\varphi(f)} : \\ &= \left(\Omega, e^{\varphi(f)}\Omega\right) : e^{\varphi(f)} : \end{aligned}$$

Furthermore, by analytical continuation, one finds

$$\left(\Omega, e^{i\varphi(f)}\Omega\right) = e^{-\frac{1}{2}(f,f)}.$$

2.3 Local Fields

Of course one would like to localize these operators, such as in

$$\varphi(x) = a^*(x) + a(x).$$

The corresponding commutation relations would then have to be

$$[a(x), a^*(y)] = \triangle_+(x-y),$$

where

$$\triangle_+(x-y) = (\Omega, \varphi(x)\varphi(y)\Omega)$$

is the Fourier transform of $\omega(k)^{-1}$, in particular for $\omega(k) \equiv 1$ it is just the Dirac δ-function.

Clearly this has to be understood in the sense of distributions. The commutation relation, without smearing out, would imply

$$\|a^*(x)\Omega\|^2 = [a(x), a^*(x)] = \infty.$$

Check how this problem goes away if we consider instead the smeared out operators such as

$$a^*(f) = \int a^*(x) f(x) dx.$$

With these local operators we have

$$:(\varphi(f))^n := \int d^n x \; f(x_1)\dots f(x_n) \;\; :\varphi(x_1)\dots\varphi(x_n):.$$

The local Wick products obey the recursion relation

$$:\varphi(x_1)\dots\varphi(x_{n+1}): =: \varphi(x_1)\dots\varphi(x_n): \cdot\varphi(x_{n+1}) - \sum_{k=1}^{n} \triangle_+(x_{n+1}-x_k) : \prod_{j\neq k} \varphi(x_j):.$$

Recall the orthogonal vectors

$$:(\varphi(f))^n:\Omega = \int d^n x \; f(x_1)\dots f(x_n) \;\; :\varphi(x_1)\dots\varphi(x_n):\Omega$$

which describe n-particle states with wave function $f^{\otimes n}$. More generally, n-particle vectors are of the form

$$\Psi(g) = \int d^n x \; g(x_1,\dots,x_n) \;\; :\varphi(x_1)\dots\varphi(x_n):\Omega,$$

where g is the n-particle wave function, symmetric in $x_1, \dots, x_n$, with scalar product

$$(\Psi(g_1), \Psi(g_2)) = n! \; (g_1, \; g_2).$$

2.4 Summary

Fock space

$$\{\mathcal{H}, \Omega, \varphi\}$$

is a Hilbert space $\mathcal{H}$ with field operators φ and a vector Ω ("vacuum", "ground state") such that

$$C(f) \equiv \left(\Omega, e^{i\varphi(f)}\Omega\right) = e^{-\frac{1}{2}(f,f)}.$$

Clearly, the field φ may be characterized by its "n-point functions":

$$(\Omega, \varphi^n(f)\Omega) = \left(-i\frac{d}{d\lambda}\right)^n C(\lambda f)|_{\lambda=0}$$

There are (generalized) vectors

$$:\varphi(x_1)\dots\varphi(x_n):\Omega$$

in $\mathcal{H}$ obeying the orthogonality relation

$$\begin{aligned} & (:\varphi(x_1)\dots\varphi(x_m):\Omega, :\varphi(y_1)\dots\varphi(y_n):\Omega)_H \\ = \ & \delta_{mn}\sum_\sigma \prod_{k=1}^n \triangle_+\left(x_k - y_{\sigma(k)}\right). \end{aligned}$$

An arbitrary vector $\Psi \in \mathcal{H}$ always has has an expansion

$$\Psi = \sum_n \int d^n x \; g_n(x_1, \dots, x_n) \ :\varphi(x_1)\dots\varphi(x_n):\Omega,$$

in terms of n-particle wave functions g_n.

Given $\Psi \in \mathcal{H}$ we have a useful tool to find the kernel functions by calculating the scalar product

$$S(\Psi)(f) \equiv \left(\Psi, :e^{\varphi(f)}:\Omega\right) = \frac{\left(\Psi, e^{\varphi(f)}\Omega\right)}{\left(\Omega, e^{\varphi(f)}\Omega\right)},$$

since

$$\begin{aligned} \left(\Psi, :e^{\varphi(f)}:\Omega\right) &= \sum_n \int d^n x \; g_n(x_1, \dots, x_n) \sum_m \frac{1}{m!}\int d^m y \; f(y_1)\dots f(y_n) \\ & (:\varphi(x_1)\dots\varphi(x_m):\Omega, :\varphi(y_1)\dots\varphi(y_n):\Omega)_H . \end{aligned}$$

Orthogonality then gives

$$\left(\Psi, :e^{\varphi(f)}:\Omega\right) = \sum_n \left(g_n, \ f^{\otimes n}\right)_{L^2}$$

i.e. the wave function g_n is simply the kernel of the nth order term of $S(\Psi)(f)$. We call

$$S(\Psi)(f) \equiv \frac{\left(\Psi, e^{\varphi(f)}\Omega\right)}{\left(\Omega, e^{\varphi(f)}\Omega\right)}$$

the *S-transfom of the Fock space vector* Ψ.

3 Remember One Integral

Note that the Fock space is an "abstract" Hilbert space, its vectors Ψ are not square integrable functions in some L^2-space as would be the case e.g. in Schroedinger theory; we cannot lay our hands on e.g. "the wave function of the vacuum", desirable as such a notion might be. The situation is rather akin to treatments of the harmonic oscillator in terms of

- energy eigenstates ("kets") $|n\rangle$
- creation operator $a^* : |n\rangle \to \sqrt{n+1}\,|n+1\rangle$
- annihilation operator $a : |n\rangle \to \sqrt{n}\,|n-1\rangle$
- position operator $Q = a + a^*$.
- momentum operator $P = \frac{i}{2}\left(a - a\right)^*$

This framework is equivalent to the one in terms of Schroedinger wave function, in particular one can e.g. calculate from it the ground state distribution $\varrho(x)$ in position space from its Fourier transform:

$$C(\lambda) \equiv \langle 0|\, e^{i\lambda Q}\, |0\rangle = e^{-\frac{1}{2}\lambda^2\cdot} = \int \rho(x) e^{i\lambda x} dx.$$

Inverting the Fourier transform one finds a Gaussian ground state density

$$\varrho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{1}$$

This brings me to the one most memorable integral, which I claim to be

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

(If you hate memorizing, there is an easy way to calculate it: take the square of the integral and introduce polar coodinates, the rest is staightforward.)

Why is this so good?

Here we go.

- First we rescale

$$\int_{\mathbb{R}} e^{-\frac{a}{2}x^2} dx = \sqrt{\frac{2\pi}{a}}$$

- Then we translate $x \to x - b/a$:

$$\int_{\mathbb{R}} e^{-\frac{a}{2}(x-b)^2} dx = \sqrt{\frac{2\pi}{a}} \tag{2}$$

i.e.

$$\int_{\mathbb{R}} e^{-\frac{a}{2}x^2 + bx} dx = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}\frac{b^2}{a}}$$

- Then we multiply:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_1^n a_k x_k^2 + b_k x_k} d^n x = \prod_k \sqrt{\frac{2\pi}{a_k}} e^{\frac{1}{2}\sum_1^n \frac{b_k^2}{a_k}}$$

- and simplify:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+(b,x)} d^n x = \sqrt{\frac{(2\pi)^n}{det\,(A)}} e^{\frac{1}{2}(b,A^{-1}b)}$$

  Note that for this to be true, the matrix A need not be diagonal. Whenever A is positive and self-adjoint, a rotation will bring it into diagonal form while leaving the integration volume $d^n x$ invariant.

- Continue analytically and normalize

$$C(\lambda) \equiv \sqrt{\frac{det\,(A)}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+i(\lambda,x)} d^n x \quad (3)$$

$$= e^{-\frac{1}{2}(\lambda,A^{-1}\lambda)} \quad (4)$$

  to obtain the Fourier transform of the probability density

$$\rho(x) = \sqrt{\frac{det\,(A)}{(2\pi)^n}} e^{-\frac{1}{2}(x,Ax)} \quad (5)$$

This is in fact what allowed us to conclude in equation (1) that our abstractly defined harmonic oscillator had ground state density

$$\varrho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

or, equivalently, ground state wave function

$$\varphi_0(x) = \rho^{1/2}(x).$$

Recall that the higher eigenstates are given by suitably normalized Hermite polynomials:

$$\varphi_n(x) = H_n(x)\rho^{1/2}(x).$$

3.1 The Ground State Representation [1]

Before we go on further, let me introduce, first for the harmonic oscillator, yet another Hilbert space description, again equivalent to the ones on abstract Hilbert space and the Schroedinger type description by the $\varphi_n \in L^2(\mathbb{R}, dx)$.

We are thinking of the so-called "ground state representation", given by the following isomorphism:

$$L^2(\mathbb{R}, dx) \leftrightarrow L^2\left(\mathbb{R}, \varphi_0^2(x)dx\right)$$

where we set $\varphi_0^2(x)dx = d\mu = \rho dx$.

Practically, this amounts to dividing each wave function φ by the ground state wave function φ_0, as an example, the nth eigenstate is now represented by the Hermite function $H_{n.}$:

$$\varphi_n \in L^2(dx) \leftrightarrow \frac{\varphi(x)}{\varphi_0(x)} = H_n(x) \in L^2(d\mu).$$

Clearly, this mapping

$$L^2(\mathbb{R}, dx) \leftrightarrow L^2(\mathbb{R}, \rho(x)\,dx)$$

is not restricted to the harmonic oscillator. Essentially all that matters is to avoid trouble with the division of wave functions which might arise if the density ρ has zeros.

Why should one want to study this? Part of the answer is contained in the following

Exercise 5 *Consider a Schroedinger Hamiltonian in* $L^2(\mathbb{R}^n, d^n x)$ *of the form*

$$H = -\Delta_x + V(x) \geq 0$$

with a zero-energy solution

$$(-\Delta_x + V(x))\,\varphi_0 = 0.$$

With this φ_0 *we construct the ground state representation as above*

$$\begin{aligned} L^2(\mathbb{R}^n, d^n x) &\leftrightarrow L^2\left(\mathbb{R}^n, \varphi_0^2(x)\,d^n x\right) \\ \varphi_n &\leftrightarrow \psi(x) = \frac{\varphi(x)}{\varphi_0(x)} \\ H &\leftrightarrow H' \end{aligned}$$

Show that the Hamiltonian in the ground state representation

$$(\varphi, (H_0 + V)\,\varphi)_{L^2(dx)} \stackrel{!}{=} (\psi, H'\psi)_{L^2(d\mu)}$$

is given by

$$\begin{aligned} (\psi, H'\psi)_{L^2(d\mu)} &= \int (\nabla\psi)^2\,\varphi_0^2(x)\,d^n x \\ &= \|\nabla\psi\|^2_{L^2(d\mu)} \end{aligned}$$

This is a very remarkable result: we are describing a quantum mechanical system with interaction V. Yet, in the (equivalent) ground state representation, the interaction seems to have disappeared from the definition of the Hamiltonian. In fact it is present in the ground state density $\rho = \varphi_0^2$:

While H' has the universal form

$$H' = \nabla^* \cdot \nabla$$

the interaction is completely encoded in the representation, i.e. in the ground state measure $d\mu(x) = \varphi_0^2(x)\, d^n x$: "*the vacuum contains everything!*".

This is particularly important in the realm of interactions which are too strong to admit a perturbative treatment, most prominently in quantum field theory [10] [23] where Haag's theorem mandates a representation change whenever one considers interacting particles.

As an aside only, we want to point out that there are diffusion processes naturally associated with the operator $H' = \nabla^* \cdot \nabla$ in $L^2\left(\varphi_0^2(x)\, d^n x\right)$ as follows. We have

$$e^{-H't}\psi(x) = E\left(\psi(X_t)\right)$$

where the expectation is w.r. to a diffusion process starting at x and solving the stochastic differential equation

$$dX_t = \beta(X_t)dt + dB(t).$$

Here B is the standard n-dimensional Brownian motion and the drift is again given by the ground state density

$$\beta(x) = \frac{\nabla\varphi_0^2(x)}{\varphi_0^2(x)}.$$

3.2 1, 2, 3, ∞.

Let us now return to our "characteristic function"

$$C(\lambda) \equiv \sqrt{\frac{det\,(A)}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+i(\lambda,x)} d^n x \quad (6)$$

$$= e^{-\frac{1}{2}(\lambda, A^{-1}\lambda)} \quad (7)$$

It has this name because it is the Fourier transform of a probability measure

$$d\mu(x) = \varrho(x)dx,$$

in fact we determined the probability density

$$\varrho(x) = \sqrt{\frac{det\,(A)}{(2\pi)^n}} e^{-\frac{1}{2}(x,Ax)}$$

by performing an inverse Fourier transform on C.

More often than not, such an explicit calculation is not possible. How do we then know wether a given function C is the Fourier transform of a probability measure?

Note that our function

$$C(\lambda) \equiv \int_{\mathbb{R}^n} \varrho(x) e^{i(\lambda,x)} d^n x$$

has three fundamental properties which arise not from the specific form of the probability density ϱ, but just from its general properties as a probability density.

1. C is *normalized*:
$$C(0) = \int_{\mathbb{R}^n} \varrho(x) d^n x = 1$$

2. C is *continuous* at zero, since
$$\begin{aligned} C(\lambda) - 1 &= \int_{\mathbb{R}^n} \left(e^{i(\lambda,x)} - 1\right) \varrho(x) d^n x && (8)\\ &= \int_{\mathbb{R}^n} (\cos((\lambda,x)) - 1)\, \varrho(x) d^n x && (9)\\ &\quad + i \int_{\mathbb{R}^n} \sin((\lambda,x))\, \varrho(x) d^n x && (10) \end{aligned}$$
and the principle of "dominated convergence" permits us to do the limits $\lambda \to 0$ inside the integrals.

3. Finally, for any complex $a_1, \ldots, a_n$, and real $\lambda_1, \ldots, \lambda_n$
$$\int_{\mathbb{R}^n} \left| \sum_l a_l e^{i(\lambda_l,x)} \right|^2 \varrho(x) d^n x = \underline{\sum_{k,l} a_k^* a_l C(\lambda_k - \lambda_l) \geq 0} \qquad (11)$$
as a consequence of the positivity of ϱ ("*Positive definiteness*" of C).

This is all it takes:

Theorem 6 *(Bochner): Any normalized continuous positive definite complex function on* $\mathbb{R}^n$ *is the Fourier transform of a probability measure* μ *on* $\mathbb{R}^n$.

Clearly,

$$C(\lambda) \equiv e^{-\frac{1}{2}\lambda^2} \qquad (12)$$

has all of these properties. Hence we have a probability measure, in fact, as we know, even a probability density ϱ such that

$$C(\lambda) \equiv \int_{\mathbb{R}} \varrho(x) e^{i\lambda x} dx.$$

Specifically we know that

$$\rho(x) = \sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2}x^2}.$$

We can then consider the Hilbert space $\mathcal{H} = L^2\left(\varrho(x)dx\right)$.

A natural base in $\mathcal{H}$ can be obtained from the monomials $f_n(x) = x^n$, n=0,1,...

If we orthogonalize them in $\mathcal{H}$ it is just the Hermite polynomials which we obtain.

Recall their generating function

$$e^{\lambda x - \frac{\lambda^2}{2}} = \sum \frac{\lambda^n}{n!} H_n(x)$$

In harmonic oscillator notation we can rewrite this as

$$\frac{e^{\lambda Q}}{\langle 0|\, e^{\lambda Q}\, |0\rangle} = \sum \frac{\lambda^n}{n!} H_n(Q).$$

(Analogous expressions will hold for an n-dimensional harmonic oscillator.)

Comparing this to our Fock space result

$$\begin{aligned} \frac{e^{\varphi(f)}}{\left(\Omega, e^{\varphi(f)}\Omega\right)} &= \; :e^{\varphi(f)}: = \sum \frac{1}{n!} :\varphi(f)^n: \\ &= \sum \frac{1}{n!} \int d^n x \; f(x_1)\ldots f(x_n) \;\; :\varphi(x_1)\ldots\varphi(x_n): \end{aligned}$$

we see that, as we move from finitely position operators Q to fields φ, it is the Wick polynomials which generalize the well-known Hermite polynomials.

We can develop this analogy further by noting that the field $\varphi(f)$ can be seen as coordinates of an infinite dimensional oscillator; expanding the test function f in terms of an orthonormal basis e_n we have

$$f(x) = \sum \lambda_n e_n(x)$$

and consequently

$$\left(\Omega, e^{i\varphi(f)}\Omega\right) = e^{-\frac{1}{2}(f,f)}$$

is equal to

$$\left(\Omega, e^{i\sum \lambda_n \varphi(e_n)}\Omega\right) \equiv \left(\Omega, e^{i(\lambda,Q)}\Omega\right) = e^{-\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^2}$$

i.e. the field modes $\varphi(e_n) = Q_n$ are distributed like those of an infinite dimensional harmonic oscillator!

Can we now write

$$\left(\Omega, e^{i\sum \lambda_n \varphi(e_n)}\Omega\right) = N \int_{\mathbb{R}^\infty} e^{i(\lambda,x)} e^{-\frac{1}{2}(x,x)^2} d^\infty x \;?$$

This would then allow a "ground state representation" for fields as in Schroedinger theory, with

$$\begin{aligned} H_{Fock} &\leftrightarrow L^2\left(\rho(x)d^\infty x\right) \\ \Omega &\leftrightarrow 1 \\ \varphi(f) &\leftrightarrow (\lambda, x) \\ H &\leftrightarrow H' = \nabla^* \cdot \nabla, \end{aligned}$$

with an infinite dimensional gradient, of course.

Well we are not quite there but almost. The problems come of course from trying to extend

$$\rho\left(x\right)d^{n}x=\sqrt{\frac{1}{\left(2\pi\right)^{n}}}e^{-\frac{1}{2}(x,x)}d^{n}x$$

to $n=\infty$. Mathematicians assure us that $d^{\infty}x$ will not make sense, $(x,x)=\sum_{1}^{\infty}x_{n}^{2}$ would require this infinite sum to be convergent, and finally the factor up front will simply go to zero as n$\rightarrow\infty$. On the other hand

$$\begin{aligned}\lim_{n\rightarrow\infty}\left(\Omega,e^{i(\lambda,Q)}\Omega\right) &= \left(\Omega,e^{i\varphi(f)}\Omega\right)=\left(\Omega,e^{i\sum\lambda_{n}\varphi(e_{n})}\Omega\right)\\ &= e^{-\frac{1}{2}\sum_{k=1}^{\infty}\lambda_{k}^{2}}=e^{-\frac{1}{2}\int f^{2}(x)dx}\end{aligned}$$

looks perfectly sound. In particular we note that for

$$C(f)=\left(\Omega,e^{i\varphi(f)}\Omega\right)=e^{-\frac{1}{2}\int f^{2}(x)dx}$$

1. $$C\left(0\right)=1$$

2. C is continuous in the test functions f.

3. Finally, for any complex $a_1,\ldots,a_n$, and real test functions $f_1,\ldots,f_n$

$$\sum_{k,l}a_{k}^{*}a_{l}C(f_{k}-f_{l})\geq 0. \tag{13}$$

This will be all we need to complete our ground state representation of fields in an L^2- space! All we need is to invoke the following generalization of Bochner's theorem [5, 9].

Theorem 7 *(Bochner-Minlos) Any normalized continuous positive definite complex function on test function space $S(1\!R^n)$ is the Fourier transform of a probability measure μ on distribution space $S^*(1\!R^n)$.*

Heureka, at long last. For now we may write

$$C(f)=\left(\Omega,e^{i\varphi(f)}\Omega\right)=\int_{S^{*}}e^{i\langle\omega,f\rangle}d\mu(\omega)$$

where $\langle\omega,f\rangle$ is of course the application of the generalized function $\omega\in S^{*}$ to the test function f. So here now is the proper wave function representation of fields and Fock states:

$$\begin{aligned}\Psi\in H_{Fock} &\leftrightarrow \Psi(\omega)\in L^{2}\left(d\mu(\omega)\right)\\ \Omega &\leftrightarrow 1\\ \varphi(f) &\leftrightarrow \langle\omega,f\rangle\,.\end{aligned}$$

For better understanding we pause here for a moment to ask why this trouble with test functions $f \in S$ and generalized functions $\omega \in S^*$.

As we generalized the Bochner theorem these two spaces arose as two different infinite dimensional extensions of the Euclidean space $\mathbb{R}^n$, and one wonders why we could not just have used Hilbert space functions in both cases.

It is rather simple and quite instructive to see why this cannot work. To this end let us look once more at the finite dimensional Gaussian density

$$\rho(x)\, d^n x = \sqrt{\frac{1}{(2\pi)^n}} e^{-\frac{1}{2}r^2} r^{n-1} dr dS_n.$$

Disregarding integration over the sphere S_n, we focus on the radial density

$$\rho_n(r) \sim r^{n-1} e^{-\frac{1}{2}r^2}.$$

This is bell-shaped only for n=1; graphically these densities look like this: As n becomes large, the probability densities $\rho_n(r)$ are essentially concentrated near the surface of a sphere $S_n(R)$ with radius $R = \sqrt{n}$. Hence our limiting measure μ **will be zero for vectors of finite length r, i.e. for all vectors in Hilbert space**. Professor Hida has often underlined this point by saying: "μ is concentrated on $S_\infty(\sqrt{\infty})$, on an infinite dimensional sphere with radius $\sqrt{\infty}$".

Technically this means the following. Recall our expansions of test functions in terms of a basis, where we had put

$$f(x) = \sum \lambda_n e_n(x).$$

We get the differentiability and rapid decrease that are required of test functions if we choose Hermite functions as a base and admit only rapidly decreasing sequences of coefficients (λ_n). In fact we have

$$f \in L^2(R) \leftrightarrow \sum \lambda_n^2 < \infty$$

whereas

$$f \in S(R) \leftrightarrow \sum n^k \lambda_n^2 < \infty \text{ for all k}$$

Now let us expand the ω:

$$\omega(x) = \sum \omega_n e_n(x).$$

Our previous discussion tells us that the coefficients ω_n are not square summable. Equivalently the functions $\omega(x)$ fail to be square summable: they are "generalized functions", with

$$\langle \omega, f \rangle = \sum \omega_n \lambda_n \text{"} = \text{"} \int f(x)\omega(x)dx.$$

The $\omega(x)$ on the right may fail to exist pointwise, but the sum is well defined and finite: the rapid decrease of the λ_n take care of this even for unbounded ω_n .

3.3 Summary

In this chapter we have obtained an L^2 realization of Fock space

$$\begin{array}{rcl} \Psi \in H_{Fock} & \leftrightarrow & \Psi(\omega) \in L^2\left(d\mu(\omega)\right) \\ \Omega & \leftrightarrow & 1 \\ \varphi(f) & \leftrightarrow & \langle \omega, f \rangle . \end{array}$$

where the probability measure μ on distribution space $S^*(R^d)$ given by its Fourier transform

$$C(f) = \int_{S^*} e^{i\langle \omega, f\rangle} d\mu(\omega) = \left(\Omega, e^{i\varphi(f)}\Omega\right) = e^{-\frac{1}{2}\int f^2(x)dx}.$$

This probability measure, defined on an infinite dimensional linear space, plays an important role in mathematics and physics, way beyond our quantum field theory starting point. The following section will shed some light on this.

4 White Noise and Brownian Motion.

4.1 What is Brownian Motion?

Exercise 8 *In this exercise we consider the one dimensional case $d = 1$, i.e.*

$$C(f) = \int_{S^*} e^{i} d\mu(\omega) = E\left(e^{i\langle \omega, f\rangle}\right) = e^{-\frac{1}{2}\int_{-\infty}^{\infty} f^2(s)ds}.$$

Note that we have also changed notation: instead of $x \in 1R^n$ we now use $s \in 1R$.

Let us now define

$$B(t) \equiv \langle \omega, f_t \rangle$$

where

$$f_t(s) = \begin{cases} 1 & \text{if } 0 < s < t \\ 0 & \text{otherwise} \end{cases}$$

Show that

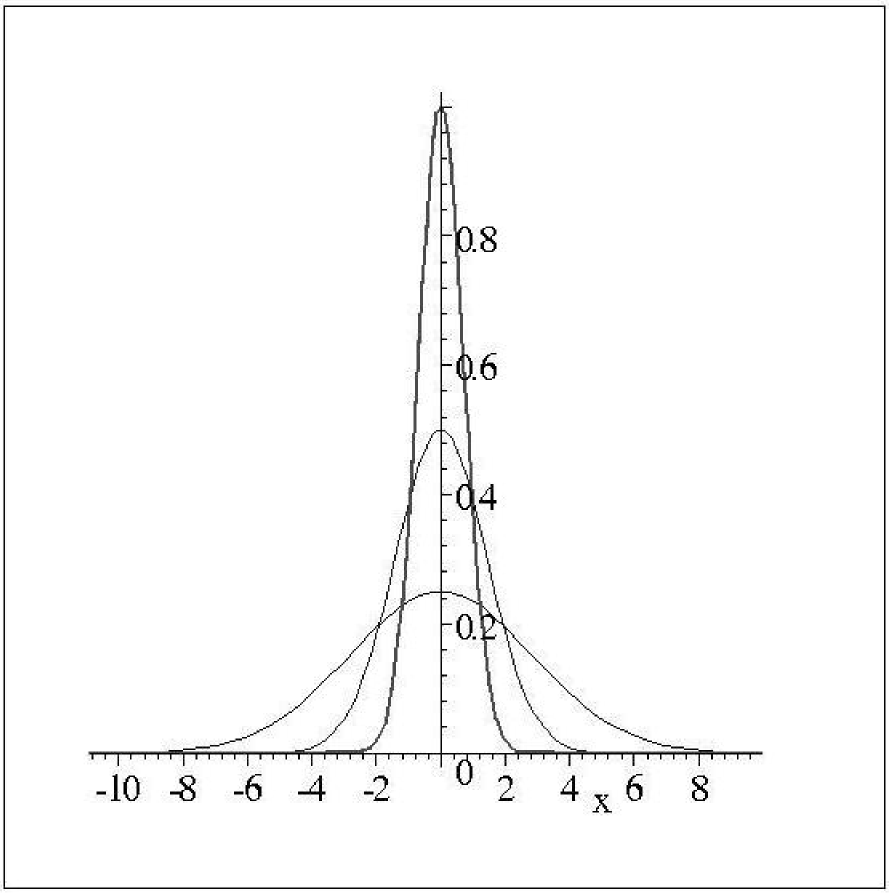
1. *$B(t) = N(0, t)$ i.e. a Gaussian random variable with mean zero and variance $\sigma^2 = t$.*
2. *Let $t_1 < t_2 < t_3 < t_4$.. Then*

$$E\left(e^{i\langle B(t_4)-B(t_3)\rangle} e^{i\langle B(t_2)-B(t_1)\rangle}\right) = E\left(e^{i\langle B(t_4)-B(t_3)\rangle}) E(e^{i\langle B(t_2)-B(t_1)\rangle}\right).$$

This Section will tell us about the physical meaning of these random variables B(t): we have constructed a mathematical model of Brownian motion!

The history of Brownian motion goes back (at least) to the year of 1827 when the biologist Robert Brown observed under the microscope that fine grains of dust suspended in liquid exhibit a random lifelike motion. What is it? Where does it come from, he wonders. His own account transmits a vivid picture of how his hypotheses evolved through refined observation, from animated motion of organic matter to a general behavior of any microscopic body. It can be enjoyed through a mouse click [?].

Then, in the famous year of 1905, came Einstein's contribution [3]. Not being completely sure that he was discussing the well-known Brownian motion, he built a model for the random displacement of microscopic bodies coming from the molecular shocks of the fluid in which they are imbedded.

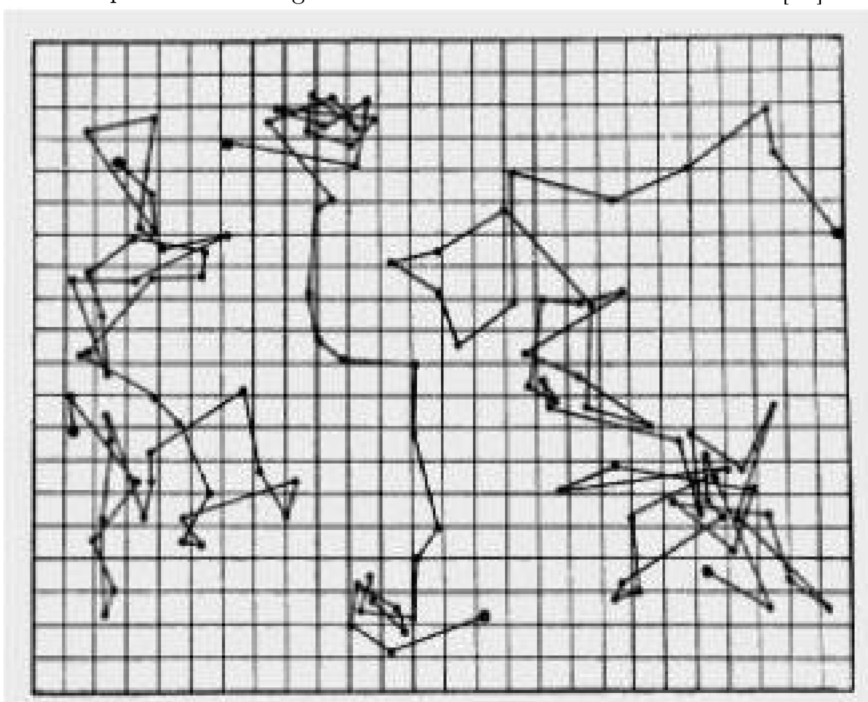


The above graph shows the Gaussian probability densities for the random displacement at consecutive times t=1,4,16 (in suitable units): note that *the displacement grows like the square root of time!*

$$x^2 \sim t. \tag{14}$$

The immediate importance of Einstein's formula was that observation of this displacement would allow for a determination of Avogadro's number, i.e.

to “count” molecules, at a time when their existence was still doubted by some. Four years later, J. B. Perrin published his experimental findings [21]. In 1926 he was awarded the Nobel prize "for his work on the discontinuous structure of matter...". But he did more than just measure the average displacement after an interval of time: he tried to trace the actual paths of the grains he saw under the microscope: the following rendition is from his book “Les atomes” [22].

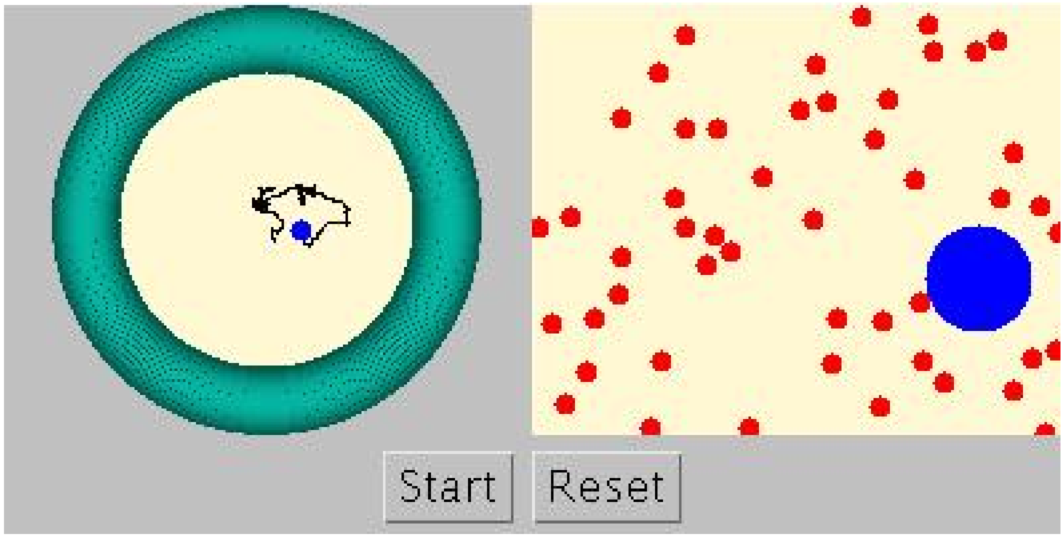


Of these observations he writes in the book’s preface: ” ... putting the eye to the microscope we observe the Brownian motion which agitates any small particle in liquid suspension. To attach a tangent to its trajectory we should find at least an approximate limit for the direction of the straight line joining the positions of the particle at two very close instants of time. Now as far as experiment can go, this direction varies like crazy if one goes to shorter and shorter time intervals. So what this suggests to the unbiased observer, is a non-differentiable curve and not at all one which would admit a tangent.” He had seen a “fractal” under the microscope much before that term was coined in by Mandelbrot (see e.g. [19]) in 1975, - and he knew it, giving various other examples where, in his words, natural phenomena appear “infinitely differentiated”.

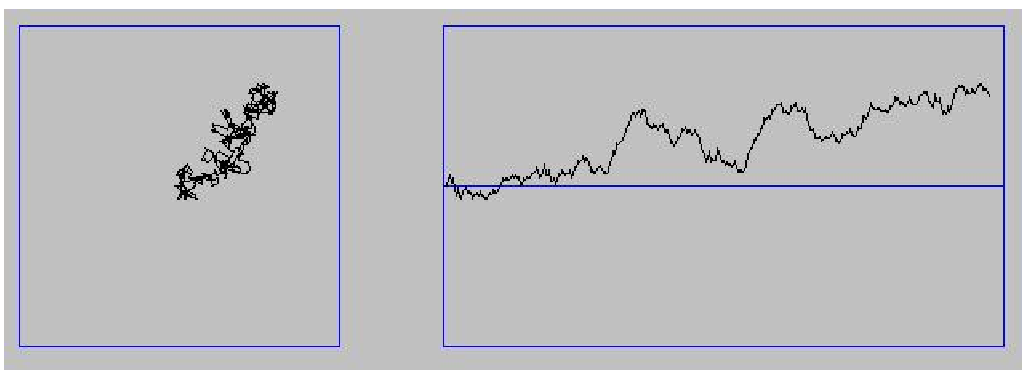
The Brownian motion story does not end here. It plays an important role in the contemporary mathematical research on stochastic processes, its impact in applied fields is important in such different fields as the analysis of stock exchange prices, or nanotechnology. for a recent review see e.g.

If you can go on-line, here are some pretty computer simulations of Brownian motion:

- The first one shows the erratic movement and its causes [25]:



- The second shows the two-dimensional motion in the plane and, on the right hand side, the one-dimensional up-and-down motion as a function of time [26]:

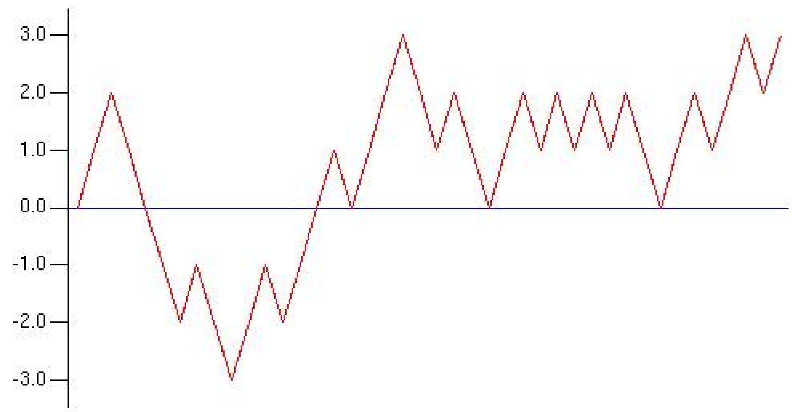


- Finally let me point you to a simulation [27] into which you can zoom in, to discover the "infinite differentiation" which Perrin talks about.

It is instructive to explore a do-it-yourself version of a Brownian motion simulation. All you need for this is pencil and paper, and a coin (even if for efficiency, you will surely want to replace these tools by a few lines of programming on your PC!).

It goes like this:

1. Plot a graph for your wealth in consecutive coin toss games: it will rise by one unit whenever you win and decrease whenever you lose. For forty games it would look something like this:



2. Next, compress your graph by a factor of 10 to accomodate ten times as many coin tosses (now at the latest you may want to use the help of your PC ...) .

3. Iterate this a couple of times.

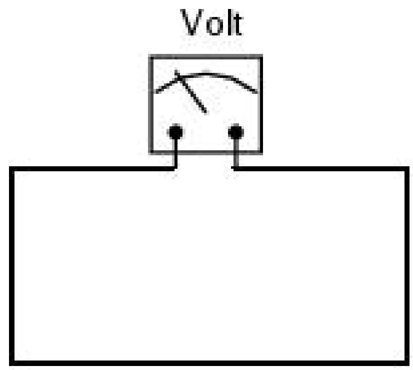
Of course wins and losses will reach bigger values when you play more games. But to keep the graph from exploding it is sufficient to compress the wins and losses by a factor of $\sqrt{10}$ whenever you compress the time axis by a factor of 10. This clearly reflects the behavior in formula (14), people in the know will recognize the workings of the *Central Limit Theorem* of probability theory.

Note that after a few iterations the apparent jaggedness of the graph will stay the same. Conversely, zooming into the first tenth of it - by going back one iteration - produces a similar graph, just as in the simulation [27] above, we have obtained a "self-similar" curve.

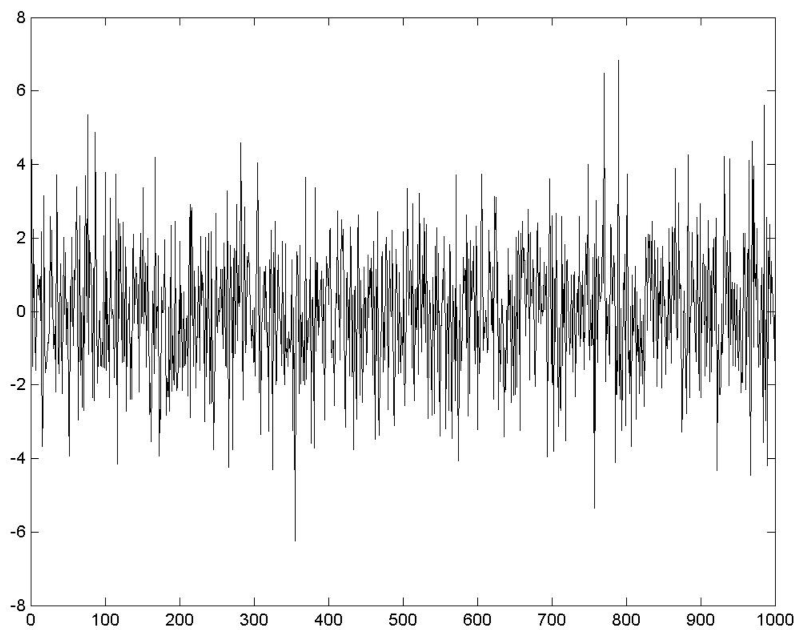
The limit curve is continuous, but too rough to admit a derivative.

4.2 What is White Noise?

In the 1920's telephones were beginning to be omnipresent, and unwanted noise in telephone communicationbecame an important issue. It was addressed by two Bell Labs scientists, Johnson and Nyquist, in a pair of experimental and theoretical studies [11]. A spectral analysis revealed a (more or less) equal presence of all frequencies in that noise - hence the name "white". To understand its origin, let us have a look at the simplest possible electric circuit:



This is so dull you would expect the voltmeter to show Zero. But it doesn't! If suitably sensitive it will exhibit rapidly varying voltage fluctuations. On an oscilloscope they would look something like this graph:



And what you see there is the effect of the irregular thermal movement of the electrons in the wire, you see (approximately) the velocity of Brownian motion.

If you prefer to hear rather than see it, tune your radio to a frequency where there is no station and turn up the volume: the muted hiss you hear is an audio version, again approximate, of white noise.

5 Back to Fock - and Beyond

Recall that we introduced Fock space as an "abstract" Hilbert space. As in the case of the harmonic oscillator, we now want to realize it as a space of square integrable functions.

In Fock space the n-particle states

$$\Psi(g) = \int d^n x \; g(x_1, \ldots, x_n) \;\; :\varphi(x_1) \ldots \varphi(x_n) : \Omega$$

play a central role. Let us explore their images in (L^2). This is not hard. Recall

$$\begin{array}{rcl} \Psi \in H_{Fock} & \leftrightarrow & \Psi(\omega) \in \left(L^2\right) \\ \Omega & \leftrightarrow & 1 \\ \varphi(f) & \leftrightarrow & \langle \omega, f \rangle \end{array}$$

$$: \; e^{\varphi(f)} := \frac{e^{\varphi(f)}}{\left(\Omega, e^{\varphi(f)}\Omega\right)} \leftrightarrow \frac{e^{\langle \omega, f \rangle}}{E\left(e^{\langle \omega, f \rangle}\right)} \equiv e_f(\omega)$$

$$\int d^n x \; f(x_1) \ldots f(x_n) \;\; : \;\; \varphi(x_1) \ldots \varphi(x_n) : \leftrightarrow \int d^n s \; f(s_1) \ldots f(s_n) : \omega(s_1) \ldots \omega(s_n) :$$

Evidently the Wick powers $: \omega(s_1) \ldots \omega(s_n) :$ are defined recursively just like those of fields:

$$: \omega(s_1) \ldots \omega(s_{n+1}) : =: \omega(s_1) \ldots \omega(s_n) : \omega(s_{n+1}) \\ - \sum_{k=1}^{n} \delta(s_{n+1} - s_k) : \prod_{j \neq k}^{n} \omega(s_j) : .$$

Exercise 9 *Caculate the generalized function* $: \omega(s_1) \ldots \omega(s_4) :$.

Recall that any Fock space vector admits an expansion in terms of Wick polynomials. Likewise we have for any square integrable function of white noise

$$\Psi\left(\omega\right) = \sum_n \int d^n s \; g_n(s_1, \ldots, s_n) \;\; : \omega(s_1) \ldots \omega(s_n) :$$

and

$$\|\Psi\|^2_{(L^2)} = \sum_n n! \int d^n s \;\; |g_n(s_1, \ldots, s_n)|^2$$

this is the infinite dimensional analogue of an expansion in terms of Hermite functions which we would have for functions of one Gaussian variable!

Technically we observe here the so-called "Ito-Wiener-Segal isomorphism"

$$L^2\left(d\mu(\omega)\right) \simeq \bigoplus_{n=0}^{\infty} L^2_{symm}(R^n, n! d^n s)$$

between square integrable functions of white noise on the one hand, and sequences of square integrable symmetric functions on the other.

5.1 Infinite Dimensional Calculus

White noise forms the coordinate system for our space (L^2). We are interested in an - infinite dimensional - calculus based on these coordinates. To get there we take a look at the following question:

How are the creation and annihilation operators represented in (L^2)? This is in fact easy. Recall that expopnential vectors are eigenstates of the annihilation operators

$$a(g) : e^{\varphi(f)} : \Omega = (f, g) : e^{\varphi(f)} : \Omega$$

These vectors have images in (L^2) as follows

$$: e^{\varphi(f)} : \Omega = \frac{e^{\varphi(f)}}{\left(\Omega, e^{\varphi(f)}\Omega\right)}\Omega \leftrightarrow \frac{e^{\langle\omega, f\rangle}}{E\left(e^{\langle\omega, f\rangle}\right)} \equiv e_f(\omega).$$

Exercise 10 *Verify that*

$$\frac{d}{d\lambda} e_f(\omega + \lambda g))|_{\lambda=0} = (f, g) e_f(\omega)$$

We conclude that in (L^2) the annihilation operators are partial ("directional") derivatives. For suitable $F \in (L^2)$

$$\begin{aligned} a(g)F(\omega) &= \frac{d}{d\lambda} F(\omega + \lambda g))|_{\lambda=0}, \\ a(x)F(\omega) &= \frac{d}{d\lambda} F(\omega + \lambda\delta_x))|_{\lambda=0} \end{aligned}$$

Monographs which elaborate the material of this Section are [17, 20, 8].

Summary:

6 Waiting for the Bus

When you look at the people waiting at the bus stop you will surely be tormented by the following question:

Which is more probable - an even or an odd number of persons in the queue?

Let $P(n, t)$ denote the probability to find n people at the bus stop when a time t has elapsed since the passage of the previous bus, and let

$$p(n \to n + 1 \text{ in } \Delta t)$$

denote the probability that one more person arrives within the time interval Δt .

We assume for small Δt

1. The probability of one more person arriving at the bus stop is proportional to Δt:
$$p(n \to n + 1 \text{ in } \Delta t) = k \cdot \Delta t \qquad (15)$$

2. The probability of more than one person arriving at the bus stop is zero

$$p(n \to n+2 \text{ in } \Delta t) = 0 \ (etc.).$$

3. At time zero, nobody was waiting yet:

$$P(0,0) = 1. \tag{16}$$

This allows us to determine $P(n,t)$ as follows

$$\begin{aligned} & P(n, t+\Delta t) && (17)\\ = \ & P(n-1,t)p(n-1 \to n \text{ in } \Delta t) && (18)\\ & +P(n,t)\left(1 - p(n-1 \to n \text{ in } \Delta t)\right) && (19)\\ = \ & P(n,t)\left(1 - k \cdot \Delta t\right) + P(n-1,t) \cdot k \cdot \Delta t && (20) \end{aligned}$$

hence

$$\frac{P(n,t+\Delta t) - P(n,t)}{\Delta t} = k \cdot \left(P(n-1,t) - P(n,t)\right) \tag{21}$$

and for $\Delta t \to 0$

$$\frac{dP(n,t)}{dt} = k \cdot \left(P(n-1,t) - P(n,t)\right). \tag{22}$$

Introducing the generating function

$$C(\lambda;t) \equiv \sum_{n=0}^{\infty} e^{i\lambda n} P(n,t) = E\left(e^{i\lambda N}\right)$$

we find the differential equation

$$\begin{aligned} \frac{d}{dt} C(\lambda;t) &= \sum_{n=0}^{\infty} e^{i\lambda n} \frac{d}{dt} P(n,t) && (23)\\ &= k \sum_{n=0}^{\infty} e^{i\lambda n} \left(P(n-1,t) - P(n,t)\right) && (24)\\ &= k\left(e^{i\lambda} - 1\right) \sum_{n=0}^{\infty} e^{i\lambda n} P(n,t) && (25)\\ &= k\left(e^{i\lambda} - 1\right) C(\lambda;t), && (26) \end{aligned}$$

solved by

$$C(\lambda) = e^{kt\left(e^{i\lambda}-1\right)} \tag{27}$$

By comparison of coefficients we finally obtain for the random variable N the "Poisson distribution with intensity $\sigma = kt$":

$$P(n,t) = e^{-kt} \frac{(kt)^n}{n!}. \tag{28}$$

Exercise 11 *Show that, at any time t the probability for an even number of people at the bus stop is bigger than 1/2:*

$$P(even,t)=\frac{1}{2}\left(1+e^{-2kt}\right).$$

Actually queues, important as their study is e.g. for the adequate design of supermarkets, are not our main focus. What we want to retain is that our example has furnished us with another class of characteristic functions

$$C_\sigma(\lambda)=e^{\sigma\left(e^{i\lambda}-1\right)} \tag{29}$$

This one is of course not related to a Gaussian: the underlying probability distribution is not continuous but concentrated on the integers $n=0,1,2,\ldots$

As in the Gaussian case however, we can get a multi-dimensional generalization by vectorizing since products of characteristic functions are again characteristic functions (of product measures) :

$$C_\sigma(\lambda)=\prod_k e^{\sigma_k\left(e^{i\lambda_k}-1\right)}=e^{\sum\sigma_k\left(e^{i\lambda_k}-1\right)}$$

As in the Gaussian case, we would like to extend to infinite dimensional probability distributions. Again, the vectors become functions, sums become integrals and we are led to

$$C(f)=e^{\int\left(e^{if(x)}-1\right)z(x)dx},$$

(we use test functions f$\in D$, i.e. of bounded support, to ensure convergence of the integral), and again the Bochner-Minlos theorem allows us to conclude that there is a probability measure on distribution space of which C is the Fourier transform. We say that

$$C(f)=e^{\int\left(e^{if(x)}-1\right)dx}$$

is the "characteristic function of Poisson White Noise".

7 Infinitely Many Particles - Another Kind of Fock

In this final chapter we will be interested to describe infinite systems of particles: "configurations" of indistinguishable point particles in R^d or in some subset $X\subseteq R^d$.

The *configuration space* $\Gamma:=\Gamma_X$ over X is the set of all **locally finite subsets** of X, *i.e.*,

$$\Gamma:=\{\gamma\subset X:\#(\gamma\cap K)<\infty\text{ for bounded }K\subset X\}.$$

For a given configuration $\gamma=\{x_1,x_2,\ldots\}$ we denote

$$\langle\gamma,f\rangle=\sum_{x\in\gamma}f(x)=\sum_{x\in\gamma}\int\delta(x-x')f(x')dx'.$$

This is well defined if f is continuous and zero outside a finite volume: the sum is then finite - no problem of convergence arises.

Our first task will be to attribute probabilities to ensembles of configurations. We begin by considering configurations in a finite volume:

$$|X| = V < \infty$$

For configurations of only one point $x \in R^d$ the obvious choice will be a probability proportional to the volume element dv.

For n-point configurations we shall use

$$dm_n = \frac{1}{n!} (dv)^n$$

the combinatorial 1/n! factor takes into account the indistinguishability of the n particles.

But we are interested in configurations of arbitrary many particles, i.e. we want a probability measure on

$$\Gamma_X = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}.$$

We first extend the measures m_n to a measure m on Γ_X, simply by setting

$$m|_{\Gamma_X^{(n)}} = m_n.$$

This is not a probability:

$$\begin{aligned} m(\Gamma_X) &= m\left(\bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}\right) \\ &= \sum_n m\left(\Gamma_X^{(n)}\right) \\ &= \sum_n m_n\left(\Gamma_X^{(n)}\right) \\ &= \sum_n \frac{1}{n!}\left(\int_X dv\right)^n \\ &= \exp(|X|). \end{aligned}$$

Now it is clear how we should define a probability measure on all configurations in X:

$$\pi \equiv \exp(-|X|) \cdot m$$

To get acquainted with these constructs we shall calculate an expectation.

$$\begin{aligned}
E\left(\exp(i\left\langle\gamma,f\right\rangle)\right) &= \int_\Gamma \exp(i\left\langle\gamma,f\right\rangle)d\pi(\gamma) \\
&= \sum_n \int_{\Gamma^{(n)}} \exp(i\left\langle\gamma,f\right\rangle)d\pi(\gamma) \\
&= \exp\left(-\left|X\right|\right)\sum_n \frac{1}{n!}\left(\int_{X^n} \exp(i\sum_{k=1}^n f(x_k))\prod{}_k\,(dv_k)\right) \\
&= \exp\left(-\left|X\right|\right)\sum_n \frac{1}{n!}\left(\int_X \exp(if(x))dv\right)^n \\
&= \exp\left(-\left|X\right|\right)\exp\left(\int_X \exp(if(x))dv\right) \\
&= \exp\left(\int_X \left(\exp(if(x)-1\right)dv\right).
\end{aligned}$$

Heureka, once more!

We have (re)discovered the probability measure of Poisson White Noise. Furthermore we note that

- no need to restrict ourselves to a space of finite volume

  $$C_\pi(f) = \exp\left(\int_{R^d} \left(\exp(if(x)-1\right)dv\right)$$

  is well defined even in the limit where $X = R^d$, and we have a limiting measure

  $$\pi = \lim_{X\to R^d} \pi|_{\Gamma_X}$$

- Recall that Bochner and Minlos guarantee the existence of a probability measure on the space of distributions such that

  $$\begin{aligned}
  C_\pi(f) &= \exp\left(\int_{R^d} \left(\exp(if(x)-1\right)dv\right) \\
  &= \int_{D^*} e^{i\langle\omega,f\rangle} d\pi(\omega).
  \end{aligned}$$

  In our explicit construction we have used the formula

  $$\langle\gamma,f\rangle = \sum_{x\in\gamma} f(x) = \sum_{x\in\gamma}\int \delta(x-x')f(x')dx'.$$

  We see from this that the measure is concentrated only on those distributions which are sums of Dirac δ-functions

  $$\omega_\gamma = \sum_{x\in\gamma}\delta_x.$$

7.1 Charlier Polynomials - Another Fock Space

As in the Gaussian case, we can get orthogonal polynomials from a generating function. Consider

$$e(f,\omega) = \exp\left(\langle\omega, \ln(1+f)\rangle - \langle f\rangle\right), \quad \omega = \omega_\gamma, \tag{30}$$

with

$$\langle f\rangle = \int f(x)dx.$$

Exercise 12 *For* $\omega = \omega_\gamma = \sum_{x\in\gamma}\delta_x$, *show*

$$e(f,\omega_\gamma) = \exp\left(-\langle f\rangle\right)\prod_{x\in\gamma}(1+f(x)).$$

Exercise 13 *Show*

$$\left(e(f), e(g)\right)_{L^2(d\pi)} = e^{\int f(x)g(x)dx}$$

From this we easily conclude that $e(f,\omega_\gamma)$ is generating function of orthogonal polynomials in ω.

If we expand $e(f,\omega_\gamma)$ in orders of f :

$$e(f,\omega) = \sum_{n=0}^{\infty}\frac{1}{n!}\langle C_n(\omega), f^{\otimes n}\rangle,$$

and insert this series into the result of the last Exercise we get the orthogonality relation

$$\left(\langle C_n(\omega), f^{\otimes n}\rangle, \langle C_m(\omega), g^{\otimes m}\rangle\right)_{L^2(d\pi)} = \delta_{mn} n! \left(f^{\otimes n}, g^{\otimes n}\right)_{L^2}$$

The obvious extension is from

$$f^{\otimes n} = f(x_1)\dots f(x_n)$$

to symmetric functions

$$f_n = f_n(x_1,\dots,x_n)$$

With these we can express any square integrable

$$F(\gamma) = \sum_{n=0}^{\infty}\langle C_n(\omega_\gamma), f_n\rangle$$

and as in the Gaussian case we have an isomorphism of Hilbert spaces:

$$\begin{aligned}\int F(\gamma)\,G(\gamma)\,d\pi(\gamma) &= \sum_{n=0}^{\infty} n!\int f_n(x_1,\dots,x_n)g_n(x_1,\dots,x_n)d^n v \\ L^2(\Gamma, d\pi(\gamma)) &\simeq H_{Fock}\end{aligned}$$

and as in the Gaussian case we have an isomorphism of Hilbert spaces:

$$\int F(\gamma) G(\gamma) d\pi(\gamma) = \sum_{n=0}^{\infty} n! \int f_n(x_1, ..., x_n) g_n(x_1, ..., x_n) d^n x$$

$$L^2(\Gamma, d\pi(\gamma)) \simeq H_{Fock}.$$

this is far from obvious: Recall that any state vector in Poisson space describes *infinitely* many particles.

Many interesting questions arise. What about the (images of) annihilation and creation operators in Poisson space?

Exercise: Show

$$\begin{aligned} (a(h) F)(\gamma) &= \int_X (F(\gamma \cup \{x\}) - F(\gamma)) h(x) dx \\ &\stackrel{def.}{=} \int_X D_x F(\gamma) h(x) dx \end{aligned}$$

Hint: Use $F(\gamma) = e(f, \gamma) = \exp(\langle \gamma, \ln(1+f) \rangle - \langle f \rangle)$.

For the adjoint one finds similarly

$$(a^*(f) F)(\gamma) := \sum_{x \in \gamma} (F(\gamma \diagdown \{x\}) f(x)) - \langle f \rangle F(\gamma),$$

and the adjoint of the "gradient" D_x can serve as starting point for stochastic integration in Poisson space (see e.g. Y. Ito and I. Kubo, 1988).

Remark *Another very active field of research is that of Gibbs states on configurations*

$$d\mu = \lim_{\Lambda \nearrow R^d} \frac{1}{Z_\Lambda} \exp\left(-\beta \sum_{\{x,y\} \subset \gamma \cap \Lambda} \phi(x-y)\right) d\pi_z$$

where $\phi(x-y)$ are pair interactions between particles at points x, y. Furthermore, dynamics of infinite particle systems, such as birth and death, particle exchange, etc., find a proper description in Poisson space; the Poisson-Fock duality then offers a helpful tool for their analysis.

Remark 2 *There are various extensions of our formalism: for one point configurations $x \in R^d$ a more general choice would have been a probability with non-uniform distribution $\rho(x) dv(x)$.*

For n-point configurations we would then have

$$dm_n = \frac{1}{n!} (\rho(x) dv)^n$$

etc., with, finally $C_{\pi,\rho}(f) = \exp\left(\int_{R^d} (\exp(if(x) - 1)\rho(x)\,dv)\right)$. *It is not hard to generalize the whole formalism to this more general setting.*

Remark *Another generalization: configurations of different type of particles. Etc...etc.*

Many recent developments can be studied in papers such as [6, 24, 14, 15, 13, 16] (and their references) available at the site
http://www.uma.pt/Investigacao/Ccm/publica.html, more still at
http://www.physik.uni-bielefeld.de/bibos/preblank.html.
Enjoy!

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