Stochastic convolution-type heat equations with nonlinear drift

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Abstract

In this paper we study the solution of a class of stochastic convolution-type heat equations with nonlinear drift. For general initial condition and coefficients we prove existence and uniqueness using the characterization theorem and Banach's fixed point theorem. We also give an implicit solution which is a well defined generalized stochastic process in a suitable distribution space. Finally we investigate the continuous dependence of the solution on the initial data as well as the dependence on the coefficient.

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1 Introduction

This paper is inspired by the articles of Ouerdiane et al. [1], [2] and more recently [3] in which certain (linear) stochastic convolution equations are treated within the framework of convolution calculus. The present work extends the work in [1] and [3], namely we allow non-linearities in the equation. Our method is an extension of the ideas in [1] where the combination of the Laplace transform with the classical fixed point theorems and the so-called characterization theorems (see [4], [5] and references quoted there) which serve to reverse the Laplace transform.

We consider the following class of Cauchy problems

$$\begin{cases} \frac{\partial}{\partial t} X_t(\omega, x) = a\Delta X_t(\omega, x) + V_t(\omega, x) * X_t(\omega, x) + H(X_t(\omega, x)) \\ \\ X_0(\omega, x) = f(\omega, x), \end{cases} \tag{1}$$

where $a \in \mathbb{R}_+$, $t \in [0, \infty)$ is the time parameter, $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ is the spatial variable, $r \in \mathbb{N}$, and $\Delta = \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the generalized sense on $\mathbb{R}^r$, $\omega = (\omega_1, \ldots, \omega_d)$ is the stochastic vector variable in the tempered Schwartz distribution space $S'_d := S'(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$, and $*$ is the convolution product between generalized functions on $\mathcal{F}'_\theta(S'_d \times \mathbb{R}^r)$. The drift $H : \mathcal{F}'_{\theta_1}(S'_d \times \mathbb{R}^r) \to \mathcal{F}'_{\theta_2}(S'_d \times \mathbb{R}^r)$ is (possibly non-linear) mapping of the solution X_t. For a more precise formulation of the problem we refer to Section 4. We prove existence and uniqueness results for these Cauchy problems under various conditions of Lipschitz type on the non-linearity H. Hence, after applying the Laplace transform to the Cauchy problem, we use the contraction method to apply the Banach's fixed point theorem. This is accomplished on various spaces of holomorphic functions which are images of generalized random variables under the Laplace transform. As in the earlier works by Ouerdiane et al. [1] we allow general potentials V_t as well as initial conditions f. We would like to mention also the works of Benth et al. [6], Deck [7], [8] and Potthoff et al. [9] for related works in the framework of white noise analysis and references therein.

The paper is organized as follows. In Section 2 we provide the mathematical background needed to solve the Cauchy problem stated above, namely spaces of test and generalized functions and the characterization theorem of generalized functions. In Section 3 we introduce the definition of convolution product and some of its properties. In Section 4 we combine the convolution calculus and the characterization theorem in order to find an implicit solution of the problem. To this end we need to introduce an appropriate Banach space of entire functions and apply Banach's fixed point theorem. All conditions on the coefficients and the drift term are stated in this section. Finally, as a by-product of our method in Section 5 we study the continuous dependence of the solution on the initial data f as well as on the drift term H.

2 Preliminaries

In this section we introduce the framework need later on. We start with a real Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^d) \oplus \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. More precisely, if $(f, x) = ((f_1, \ldots, f_d), (x_1, \ldots, x_r)) \in \mathcal{H}$, then the Hilbertian norm of (f, x) is given by

$$|(f,x)|^2 := \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(u) du + \sum_{i=1}^{r} x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R}, \mathbb{R}^d) \oplus \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R}, \mathbb{R}^d) \oplus \mathbb{R}^r = \mathcal{M}. \tag{2}$$

The pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle (\omega, x), (\xi, y) \rangle := (\omega, \xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x, y)_{\mathbb{R}^r}$, $(\omega, x) \in \mathcal{M}'$ and $(\xi, y) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_n(\mathbb{R}, \mathbb{R}^d) \oplus \mathbb{R}^r = \bigcap_{n=0}^{\infty} \mathcal{M}_n,$$

where $S_n(\mathbb{R}, \mathbb{R}^d) \oplus \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2_{\mathbb{R}^r}$, see [10] and references therein. We will consider the complexification of the triple (2) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{3}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos' theorem via its characteristic functional for every $(\xi, p) \in \mathcal{M}$ by

$$C_\mu(\xi, p) = \int_{\mathcal{M}'} \exp(i\langle (\omega, x), (\xi, p) \rangle) d\mu((\omega, x)) = \exp(-\frac{1}{2}(|\xi|^2 + |p|^2)).$$

In order to solve the Cauchy problem (1) we need to introduce an appropriate space of generalized functions for which we follow closely the construction in [11]. Let $\theta = (\theta_1, \theta_2) : \mathbb{R}_+^2 \to \mathbb{R}_+$, $(t_1, t_2) \mapsto \theta_1(t_1) + \theta_2(t_2)$ where θ_1, θ_2 are two Young functions, i.e., θ_i is a continuous, convex, increasing, $\theta_i(0) = 0$ and $\lim_{t\to\infty} \frac{\theta_i(t)}{t} = \infty$, $i = 1, 2$. For every pair $m = (m_1, m_2)$ where m_1, m_2 are strictly positive real numbers, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{N}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f : \mathcal{N}_{-n} \to \mathbb{C},\ \textit{entire},\ |f|_{\theta,m,n} = \sup_{z\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta(m|z|_{-n})) < \infty\},$$

where for each $z = (\omega, x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Here $|\omega|_{-n}$ is the norm in the dual space $S'_n(\mathbb{R}, \mathbb{R}^d) =: S_{-n}(\mathbb{R}, \mathbb{R}^d)$. Now we consider as test function space as the space of entire functions on $\mathcal{N}'$ of (θ_1, θ_2)-exponential growth and minimal type given by

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to construct the triplet of the complex Hilbert space $L^2(\mathcal{M}', \mu)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need to add a condition on the pair of Young functions (θ_1, θ_2). Namely, $\lim_{t\to\infty} \frac{\theta_i(t)}{t^2} < \infty$, $i = 1, 2$. This is enough to obtain the following Gelfand triplet

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \mu) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{4}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \mu)$ endowed with the inductive limit topology which coincides with the strong topology since $\mathcal{F}_\theta(\mathcal{N}')$ is a nuclear space, see [12] for more details on this subject. We denote the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ which is the extension of the inner product in $L^2(\mathcal{M}', \gamma)$.

Remark 2.1 *For every entire function $f : \mathcal{N}' \to \mathbb{C}$ we have the Taylor expansion*

$$f(z) = \sum_{k\in\mathbb{N}_0^2} \langle z^{\otimes k}, f_k \rangle,$$

where $z^{\hat{\otimes} k} \in \mathcal{N}'^{\hat{\otimes} k}$. This allowed us to identify each entire function f with the corresponding Taylor coefficients $\vec{f} = (f_k)_{k=0}^\infty$. The mapping $f \mapsto T(f) = \vec{f}$ is called Taylor series map.

Using the mapping T we can construct a topological isomorphism between the test function space $\mathcal{F}_\theta(\mathcal{N}')$ and the formal power series space $F_\theta(\mathcal{N})$ defined by

$$F_\theta(\mathcal{N}) = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} F_{\theta,m}(\mathcal{N}_n), \tag{5}$$

where

$$F_{\theta,m}(\mathcal{N}_n) := \left\{ \vec{f} = (f_k)_{k\in\mathbb{N}_0^2},\ f_k \in \mathcal{N}_n^{\hat{\otimes}k} \,|\, |\vec{f}|^2 := \sum_{k\in\mathbb{N}_0^2} \theta_k^{-2} m^{-k} |f_k|_n^2 < \infty \right\},$$

here $\theta_k^{-2} = \theta_{1,k_1}^{-2}\theta_{2,k_2}^{-2}$, with $\theta_{i,k_i} := \inf_{u>0} \frac{\exp(\theta_i(u))}{u^{k_i}}$, $i = 1, 2$. In the case where $\theta(x) = x^2$, then $F_{\theta,1}(\mathcal{N}_n)$ is nothing than the usual Bosonic Fock space associated to $\mathcal{N}_n$, see [10] for more details.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. This will be done in Theorem 2.2 with the help of the Laplace transform. Therefore, let us first define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ we define the exponential function $\exp((\xi, p))$ by

$$\mathcal{N}' \ni (\omega, x) \mapsto \exp(\langle\omega, \xi\rangle + (p, x)). \tag{6}$$

It is not hard to verify that for every element $(\xi, p) \in \mathcal{N}$ $\exp((\xi, p)) \in \mathcal{F}_\theta(\mathcal{N}')$. With the help of this function we can define the Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\langle \Phi, \exp((\xi, p))\rangle\rangle. \tag{7}$$

The Laplace transform is well defined because $\exp((\xi, p))$ is a test function. In order to obtain the characterization theorem we need to introduce another space of entire functions on $\mathcal{N}$ with θ^*-exponential growth and arbitrary type, where θ^* is another Young function (called polar functions associated to θ) defined by

$$\theta^*(x) := \sup_{t>0}(tx - \theta(t)).$$

The next characterization theorem is essentially based on the topological dual of the formal power series space $F_\theta(\mathcal{N})$ defined in (5) and the inverse Taylor series map T^{-1}, see [4] or [11] for details. In the white noise analysis framework this theorem is known as Potthoff-Streit characterization theorem, see [13] [14] for details and historical remarks.

Theorem 2.2 *The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the Banach space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g,\ |g(\xi,p)| \le k\exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)),\ (\xi,p)\in\mathcal{N}_n$$

and norm defined by

$$|g|_{\theta^*,m,n} := \sup_{z\in\mathcal{N}_n} |g(z)|\exp(-\theta^*(m|z|_n)).$$

3 The Convolution Product $*$

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator, see [15]. This notion generalizes the differential equations with constant coefficients in finite dimensional case. If we consider the space of test functions $\mathcal{F} = \mathcal{F}_\theta(\mathcal{N}')$, then we can show that each convolution operator is associated with a generalized function from $\mathcal{F}'_\theta(\mathcal{N}')$ and vice-versa, see [16].

Let us define the convolution between a generalized and a test function on $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$, respectively. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi\rangle\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega+\eta, x+y).$$

It is not hard to see that $\Phi * \varphi$ is an element of $\mathcal{F}_\theta(\mathcal{N}')$. Note that the dual pairing between $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is given in terms of the convolution product of Φ and φ applied at $(0,0)$, i.e., $(\Phi * \varphi)(0,0) = \langle\langle \Phi, \varphi\rangle\rangle$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given. Then $\Phi * \Psi$ is defined as

$$\langle\langle \Phi * \Psi, \varphi\rangle\rangle := \langle\langle \Phi, \Psi * \varphi\rangle\rangle,\ \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{8}$$

This definition of convolution product for generalized functions will be used on Section 4 in order to solve the heat stochastic equation with non-linear drift stated in (1). We have the following connection between the Laplace transform and the convolution product. The simple proof can be seen in [1].

Proposition 3.1 *Let* $(\xi, p) \in \mathcal{N}$ *be given and consider the exponential function* $\exp((\xi, p))$ *defined on (6). Then for every* $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ *we have*

$$\Phi * \exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p) \exp((\xi, p)).$$

As a consequence of the above proposition and the definition in (8) we obtain the following corollary which says that the Laplace transform maps the convolution product in $\mathcal{F}'_\theta(\mathcal{N}')$ into the usual pointwise product in the function space $\mathcal{G}_{\theta^*}(\mathcal{N})$.

Corollary 3.2 *For every generalized functions* $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi, \tag{9}$$

and equality (9) may be taken as an alternative definition of the convolution product between two generalized functions.

In order to solve the Cauchy problem (1) we need to handle non-linear functionals $K : \mathcal{F}'_\theta(\mathcal{N}') \to \mathcal{F}'_\lambda(\mathcal{N}')$ for certain Young functions θ, λ given.

Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function verifying the following growth condition: $|g(z)| \leq C \exp(\gamma(m|z|))$, where $C, m > 0$ and γ is a Young function which not necessary satisfies the condition $\lim_{x\to\infty} \frac{\gamma(x)}{x} = \infty$. Then for each $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ the convolution functional $g^*(\Phi)$ defined by:

$$\mathcal{L}(g^*(\Phi)) = g(\mathcal{L}\Phi)$$

belongs to the space $\mathcal{F}'_\lambda(\mathcal{N}')$, where $\lambda = (\gamma \circ e^{\theta^*})^*$, see [5] for the proof. A typical example of a non-linear functional on $\mathcal{F}'_\theta(\mathcal{N}')$ is $K(\Phi) = g^*(\Phi)$, cf. Example 4.7.

In particular if $g(z) = \exp(z)$ and $\gamma(x) = x$, then the convolution exponential

$$\exp^*(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!}(\Phi^*)^n$$

is a well defined element in $\mathcal{F}'_\lambda(\mathcal{N}')$, where $\lambda = (e^{\theta^*})^*$. The convolution exponential just defined will be the main object in solving the stochastic differential equation in (1), cf. (22).

If $g(z) = \sum_{k=0}^n g_k z^k$ is a polynomial of order $n \in \mathbb{N}$, then the corresponding convolution functional $g^*(\Phi) = \sum_{k=0}^n g_k (\Phi^*)^k$ is clearly an element in $\mathcal{F}'_\theta(\mathcal{N}')$, whenever $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$. This follows from the fact that $\mathcal{F}'_\theta(\mathcal{N}')$ is topological isomorphic to $\mathcal{G}_{\theta^*}(\mathcal{N})$ via Laplace transform (cf. Theorem 2.2) and because $\mathcal{G}_{\theta^*}(\mathcal{N})$ is an algebra. Notice that the corresponding functional $K(\Phi) = g^*(\Phi)$ is a mapping from $\mathcal{F}'_\theta(\mathcal{N}')$ into itself.

4 Stochastic heat equation with non-linear drift

4.1 Generalized $\mathcal{F}'_\theta(\mathcal{N}')$-valued stochastic processes

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi_t,\ t \in I\} \subset \mathcal{F}'_\theta(\mathcal{N}')$, where I is an interval from $\mathbb{R}$, without loss generality we may assume that $0 \in I$. The process Φ_t is said to be continuous if the map $t \longmapsto \Phi_t$ is continuous. In order to introduce generalized stochastic integrals, we need the following result proved in [17].

Proposition 4.1 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of generalized functions on $\mathcal{F}'_\theta(\mathcal{N}')$. Then the following two conditions are equivalent:*

1. *The sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}'_\theta(\mathcal{N}')$ strongly.*
2. *The sequence $(\hat{\Phi}_n = \mathcal{L}(\Phi_n))_{n\in\mathbb{N}}$ of Laplace transform of $(\Phi_n)_{n\in\mathbb{N}}$ satisfies the following two conditions:*
   - *(a) There exists $p \in \mathbb{N}$ and $m \in (\mathbb{R}_+^*)^2$ such that the sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{G}_{\theta^*,m}(\mathcal{N}_p)$ and is bounded in this Banach space.*
   - *(b) For every point $z \in \mathcal{N}$, the sequence of complex numbers $(\hat{\Phi}_n(z))_{n=0}^\infty$ converges.*

Let $\{\Phi_t\}_{t\in I}$ be a continuous $\mathcal{F}'_\theta(\mathcal{N}')$-process and put

$$\Phi_n = \frac{t}{n}\sum_{k=0}^{n-1} \Phi_{\frac{tk}{n}}, \quad n \in \mathbb{N}^* := \mathbb{N}\backslash\{0\},\ t \in I.$$

It is easy to prove that the sequence $(\hat{\Phi}_n)$ is bounded in $\mathcal{G}_{\theta^*}(\mathcal{N})$ and for every $\xi \in \mathcal{N}$, $p \in \mathbb{C}^r$ $(\hat{\Phi}_n(\xi,p))_n$ converges to $\int_0^t \hat{\Phi}_s(\xi,p)ds$. Thus we conclude by Proposition 4.1 that (Φ_n) converges in $\mathcal{F}'_\theta(\mathcal{N}')$. We denote its limit by

$$\int_0^t \Phi_s ds := \lim_{n\to\infty} \Phi_n \ \ in \ \mathcal{F}'_\theta(\mathcal{N}').$$

The following result is widely used in this remaining of this paper, the proof is given in [1].

For a given continuous generalized stochastic process X_t we define the generalized function

$$Y_t(x,\omega) = \int_0^t X_s(x,\omega)ds \in \mathcal{F}'_\theta(\mathcal{N}') \tag{10}$$

by

$$\mathcal{L}\left(\int_0^t X_s(x,\omega)ds\right)(\xi,p) := \int_0^t \mathcal{L}X_s(p,\xi)ds. \tag{11}$$

Moreover, the generalized stochastic process $Y_t(x,\omega)$ is differentiable in $\mathcal{F}'_\theta(\mathcal{N}')$ and we have $\frac{\partial}{\partial t}Y_t(x,\omega) = X_t(x,\omega)$.

4.2 Existence and uniqueness of solution

We are now ready to solve the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}X_t(\omega,x) = a\Delta X_t(\omega,x) + V_t(\omega,x) * X_t(\omega,x) + H(X_t(x,\omega)) \\ \\ X_0(\omega,x) = f(\omega,x). \end{cases} \tag{12}$$

The different terms in (12) are as follows: a is a constant, Δ is the Laplacian in the generalized sense with respect to the spatial variable $x \in \mathbb{R}^r$, H is a non-linear mapping $H : \mathcal{F}'_\beta(\mathcal{N}') \to \mathcal{F}'_\lambda(\mathcal{N}')$, where the solution X_t of (12) belongs to $\mathcal{F}'_\beta(\mathcal{N}')$. The initial conditions f and the generalized stochastic process V_t verify the following growth condition: there exists $m > 0$, $n \in \mathbb{N}$ and $K_f, K_V > 0$ such that

$$|\hat{V}_s(\xi,p)| \ \leq \ K_V\beta^*(m|(\xi,p)|_n), \tag{13}$$

$$|\hat{f}(\xi,p)| \ \leq \ K_f\beta^*(m|(\xi,p)|_n). \tag{14}$$

It is clear that by Theorem 2.2 the above conditions implies that f and V_t belongs to an appropriate space of generalized functions, namely $\mathcal{F}'_\theta(\mathcal{N}')$, where $\theta = (\log(1+\beta^*))^*$.

We prove the existence and uniqueness result for the Cauchy problem under certain assumptions on the non-linear term H, see (H1) and (H2) below. That is, we consider mappings $t \mapsto X_t$ from $[0,T]$ into $\mathcal{F}'_\beta(\mathcal{N}')$. We have to discuss the continuity and differentiability properties of these mappings. Assume that for any $\varphi \in \mathcal{F}_\beta(\mathcal{N}')$ the mapping $t \mapsto \langle X_t, \varphi\rangle$ has derivative at t and this expression is linear and continuous in $\varphi \in \mathcal{F}_\beta(\mathcal{N}')$. Then the corresponding element in $\mathcal{F}'_\beta(\mathcal{N}')$ is denoted by $\frac{\partial}{\partial t}X_t$.

We apply the Laplace transform to (12) and obtain

$$\begin{cases} \frac{\partial}{\partial t}\hat{X}_t(\xi,p) = ap^2\hat{X}_t(\xi,p) + \hat{V}_t(\xi,p)\hat{X}_t(\xi,p) + \widehat{H(X_t)}(\xi,p) \\ \\ \hat{X}_0(\xi,p) = \hat{f}(\xi,p). \end{cases} \tag{15}$$

Consider the subspace $U_{\beta^*,m,n}$ of continuous $\mathcal{G}_{\beta^*}(\mathcal{N})$-valued functions on $[0,T]$ for which the following norm

$$\|u\|_{\beta^*,m,n} := \sup_{t\in[0,T], z\in\mathcal{N}_n} |u(t,z)|\tau(t,z)$$

is finite. The weight function τ is defined for certain $C > 0$ by

$$\tau(t,z) := \frac{\exp(-tC(\beta^*(m|z|_n))}{1+\beta^*(m|z|_n)}.$$

We may adapt the proof of Proposition 3 in [6] to see that $(U_{\beta^*,m,n}, \|\cdot\|_{\beta^*,m,n})$ is a Banach space.

Remark 4.2 *Each $u \in U_{\beta^*,m,n}$ satisfies the following bound*

$$\begin{aligned} |u(t,z)| &\leq \|u\|_{\beta^*,m,n} \exp(tC(\beta^*(m|z|_n))(1+\beta^*(m|z|_n)) \\ &\leq \|u\|_{\beta^*,m,n} \exp((1+Ct)\beta^*(m|z|_n)). \end{aligned}$$

Using the properties of Young functions if $m' = (1+CT)m$ then we have

$$(1+CT)\beta^*(m|z|_n) \leq \beta^*(m'|z|_n).$$

Therefore, $u \in C([0,T], \mathcal{G}_{\beta^*}(\mathcal{N}))$ and hence there exists $\Phi_u \in C([0,T], \mathcal{F}'_\beta(\mathcal{N}'))$, such that $(\mathcal{L}\Phi_u(t))(z) = u(t,z)$.

In order to obtain the solution of (15) we need the following assumptions: let h be defined by

$$(hu)(t,\xi,p) := \mathcal{L}H(\mathcal{L}^{-1}u(t))(\xi,p), \quad u \in C([0,T];\mathcal{G}_{\beta^*}(\mathcal{N}))$$

which satisfies the following two conditions: for any $u, v \in C([0,T];\mathcal{G}_{\beta^*}(\mathcal{N}))$

(H1) there exists $m \in (\mathbb{R}_+^*)^2$, $n \in \mathbb{N}_0$ and $K_h > 0$ such that

$$|(hu)(t,\xi,p) - (hv)(t,\xi,p)| \le K_h(\beta^*(m|(\xi,p)|_n))|u(t,\xi,p) - v(t,\xi,p)|.$$

(H2) $|(hu)(t,\xi,p)| \le K_h(\beta^*(m|(\xi,p)|_n))(1 + |u(t,\xi,p)|)$.

With this notation the system (15) becomes

$$\begin{cases} \frac{\partial}{\partial t}\hat{X}_t(\xi,p) = ap^2\hat{X}_t(\xi,p) + \hat{V}_t(\xi,p)\hat{X}_t(\xi,p) + h(\hat{X})(t,\xi,p) \\ \hat{X}_0(\xi,p) = \hat{f}(\xi,p). \end{cases} \quad . \tag{16}$$

To solve the equation in (16) we proceed as follows. We define the following operator Γ on $C([0,T];\mathcal{G}_{\beta^*}(\mathcal{N}))$, as follows

$$\begin{aligned}(\Gamma u)(t,\xi,p) &= \hat{f}(\xi,p) + \int_0^t ap^2 u(s,\xi,p)ds + \int_0^t \hat{V}_s(\xi,p)u(s,\xi,p)ds \\ &\quad + \int_0^t (hu)(s,\xi,p)ds. \end{aligned} \tag{17}$$

Proposition 4.3 *Under the assumptions **(H1)** and **(H2)** the operator Γ is a strict contraction on $U_{\beta^*,m,n}$.*

Proof. First of all let us show that Γ maps $U_{\beta^*,m,n}$ into itself. Let $u \in U_{\beta^*,m,n}$ be given, then it is easy to see that $(\Gamma u)(t,\cdot,\cdot)$ is an entire function on $\mathcal{N}_n$. We can estimate (17) as follows:

$$\begin{aligned}|(\Gamma u)(t,\xi,p)| &\le |\hat{f}(\xi,p)| + \int_0^t |a|p^2|u(s)(\xi,p)|ds + \int_0^t |\hat{V}_s(\xi,p)||u(s)(\xi,p)|ds \\ &\quad + \int_0^t |(hu)(s,\xi,p)|ds.\end{aligned}$$

We estimate each term of the right hand side of the above equality using the hypothesis on f, V_t in (13), (14) and (H2) as

$$\begin{aligned}
|\hat{f}(\xi,p)| &= K_f\beta^*(m|(\xi,p)|_n), \\
\int_0^t |a|p^2|u(s,\xi,p)|ds &\leq K\beta^*(m|(\xi,p)|_n)\int_0^t |u(s,\xi,p)|ds \\
\int_0^t |\hat{V}_s(\xi,p)u(s,\xi,p)|ds &\leq K_V\beta^*(m|(\xi,p)|_n)\int_0^t |u(s,\xi,p)|ds \\
\int_0^t |(hu)(s,\xi,p)|ds &\leq K_h\beta^*(m|(\xi,p)|_n)\int_0^t (1+|u(s,\xi,p)|)ds.
\end{aligned}$$

In the second estimate we have used the fact that $\lim_{x\to\infty}\frac{\beta(x)}{x^2}<\infty$ which implies that $\beta(x)\leq cx^2$, $c>0$. Then it follows that $\beta^*(x)\geq c'x^2$, $c'>0$. Putting all this together we obtain the following estimate for $|(\Gamma u)(t,\xi,p)|$:

$$\begin{aligned}
|(\Gamma u)(t,\xi,p)| &\leq (K_f+TK_h)\beta^*(m|(\xi,p)|_n) \\
&\quad +\tilde{K}\beta^*(m|(\xi,p)|_n)\int_0^t |u(s,\xi,p)|ds, \qquad (18)
\end{aligned}$$

where $\tilde{K}=K+K_V+K_h$. Taking into account that $u\in U_{\beta^*,m,n}$ and the fact that

$$\int_0^t \exp(sC(\beta^*(m|(\xi,p)|_n))ds \leq \frac{\exp(tC(\beta^*(m|(\xi,p)|_n))}{C(\beta^*(m|(\xi,p)|_n)}$$

we arrive at

$$|(\Gamma u)(t,\xi,p)|\leq (K_f+TK_h)\tau^{-1}(t,\xi,p)+\frac{\tilde{K}}{C}\|u\|_{\beta^*,m,n}\,\tau^{-1}(t,\xi,p).$$

Hence if we take $K':=\max\left\{K_f+TK_h,\frac{\tilde{K}}{C}\right\}$ we have $\|\Gamma u\|_{\beta^*,m,n}\leq K'(1+\|u\|_{\beta^*,m,n})$. This shows that Γ maps $U_{\beta^*,m,n}$ on itself. Now we proceed in order to prove that Γ is a contraction on $U_{\beta^*,m,n}$.

$$\begin{aligned}
|\Gamma u(t,\xi,p)-\Gamma v(t,\xi,p)| &\leq \int_0^t |a|p^2|u(s,\xi,p)-v(s,\xi,p)|ds \\
&\quad +\int_0^t |\hat{V}_s(\xi,p)||u(s,\xi,p)-v(s,\xi,p)|ds \\
&\quad +\int_0^t |(hu)(s,\xi,p)-(hv)(s,\xi,p)|ds.
\end{aligned}$$

Using the same estimates as before we obtain

$$|\Gamma u(t,\xi,p)-\Gamma v(t,\xi,p)|\leq \frac{\tilde{K}}{C}\left\|u-v\right\|_{\beta^*,m,n}\tau^{-1}(t,\xi,p).$$

This implies that

$$\left\|\Gamma u-\Gamma v\right\|_{\beta^*,m,n}\leq \frac{\tilde{K}}{C}\left\|u-v\right\|_{\beta^*,m,n}.$$

It is obvious that we can choose C (e.g., $C=2\tilde{K}$) such that Γ becomes a contraction on $U_{\beta^*,m,n}$. ■

It follows from Banach's fixed point theorem that Γ has a unique fixed point $Y_{\cdot}\in U_{\beta^*,m,n}$ which is the solution of (16) given by

$$Y_t(\xi,p)=\hat{f}(\xi,p)+\int_0^t ap^2Y_s(\xi,p)ds+\int_0^t \hat{V}_s(\xi,p)Y_s(\xi,p)ds+\int_0^t (hY)(s,\xi,p)ds. \tag{19}$$

By Remark 4.2 there is a unique $X\in C([0,T],\mathcal{F}'_\beta(\mathcal{N}'))$ such that $(LX_t)(\xi,p)=Y_t(\xi,p)$. But we also know that the Laplace transform commutes with the integral (cf. 10 and 11). Therefore, by Theorem 2.2 equation (19) is equivalent to

$$X_t=f+\int_0^t a\Delta X_sds+\int_0^t V_s*X_sds+\int_0^t H(X_s)ds. \tag{20}$$

On the other hand, we can show that X_t is weakly continuously differentiable in t, the proof is an easy adaption of the proof given in [18] with $\theta^*(x)=x^2$. Hence taking derivative of equality (20) with respect to t we obtain that X_t is the solution of the Cauchy problem (12). In order to write X_t implicitly we may solve (19) with respect to $Y_t(\cdot,\cdot)$ by the classical methods of differential equations and obtain

$$\begin{aligned}Y_t(\xi,p) &= \hat{f}(\xi,p)\exp\left(ap^2t+\int_0^t\hat{V}_s(\xi,p)ds\right)\\ &\quad+\int_0^t\exp\left(\int_s^t(ap^2+\hat{V}_r(\xi,p))dr\right)\hat{H}(Y_s)(\xi,p)ds. \end{aligned}\tag{21}$$

We summarize in the following theorem.

Theorem 4.4 *Under assumptions **(H1)**, **(H2)** the Cauchy problem (12) has a unique continuous solution X_t which is a generalized $\mathcal{F}'_\beta(\mathcal{N}')$-valued stochastic process. The implicit solution X_t is obtained by applying $\mathcal{L}^{-1}$ to equation (21) and reads as*

$$\begin{aligned} X_t(\omega,x) &= f(\omega,x) * \exp^*\left(\int_0^t V_s(\omega,x)ds\right) * \gamma_{2at} \\ &\quad + \int_0^t \exp^*\left(\int_s^t V_r(\omega,x))dr\right) * \gamma_{2a(t-s)} * H(X_s)(\omega,x)ds, \end{aligned} \tag{22}$$

where γ_σ is the Gaussian measure on $\mathbb{R}^r$ with variance σ.

Remark 4.5 1. *The existence of the solution in (22) of course supposes that the potential V is such that*

$$|\hat{V}_s(z)| \leq l(s)\beta^*(m|z|_n),$$

where $l \in L^1([0,T])$. In this way we do not need the continuity of the stochastic process $[0,T] \ni t \mapsto V_t \in \mathcal{F}'_\beta(\mathcal{N}')$.

2. *The above scheme also applies with minor changes for time dependent coefficient $a(t)$ such that $a \in L^1([0,T])$, $a > 0$.*

3. *We would like to emphasize that the method of the paper applies also to the more general problem*

$$\begin{cases} \frac{\partial}{\partial t} X_t(\omega,x) = LX_t(\omega,x) + H(X_t(x,\omega)) \\ \\ X_0(\omega,x) = f(\omega,x), \end{cases} , \tag{23}$$

where the operator L has the form

$$L = \sum_{i,j=1}^r a_{ij}(t,\omega,x) * \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^r b_i(t,\omega,x) * \frac{\partial}{\partial x_i} + c(t,\omega,x) * \cdot,$$

where the coefficients $a_{i,j}(t), b_i(t), c(t)$ are $\mathcal{F}'_\theta(\mathcal{N}')$-valued generalized stochastic processes. Under the same conditions (H1) and (H2) we have to assume that the coefficients a_{ij}, b_i, c have to fulfill the following assumption: for every $i,j \in \{1,\ldots,r\}$

$$|\hat{a}_{ij}(t,\xi,p)| + |\hat{b}_i(t,\xi,p)| + |\hat{c}(t,\xi,p)| \leq K\beta^*(m|(\xi,p)|_n).$$

In this case the Laplace transform of the solution of the Cauchy problem (23) $\hat{X}_\cdot \in U_{\beta^*,2m,n}$.

We now give two examples to illustrate our method.

Example 4.6 *1. First consider the linear case: $H(X_t) = bX_t$, $b \in \mathbb{R}$. The Cauchy problem (12) with initial condition $f \in \mathcal{F}'_\theta(\mathcal{N}')$ and V_t $\mathcal{F}'_\theta(\mathcal{N}')$-valued stochastic process has an explicit solution, see e.g., Proposition 12 in [1] with V_t replaced by $V_t + b\delta$, where δ is the Dirac measure at 0. In this case the solution X_t belongs to the space $\mathcal{F}'_\beta(\mathcal{N}')$, where $\beta = (e^{\theta^*} - 1)^*$.*

2. Consider now the convolution polynomial case: $H(X_t) = \sum_{k=0}^{n} a_k (X_t^*)^k$, $a_k \in \mathbb{R}$, $k = 1, \ldots, n$. Notice that $\widehat{H(X_t)} = H(\hat{X}_t)$ from which follows that if $X_t \in \mathcal{F}'_\beta(\mathcal{N}')$, then $H(X_t) \in \mathcal{F}'_\beta(\mathcal{N}')$. In other words, H maps $\mathcal{F}'_\beta(\mathcal{N}')$ into itself. We can give an explicit relation between β and θ, namely $\beta = (e^{\theta^*})^*$, see Example 4.7 for more details.

Example 4.7 *A more general class of non-linearities is obtained as follows: suppose that $f, V_t \in \mathcal{F}'_\theta(\mathcal{N}')$ for a certain fixed Young function θ. Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be an entire function on $\mathbb{C}$ verifying the following growth condition: $|g(z)| \leq C \exp(\gamma(m|z|))$, where $C, m > 0$ and γ is another Young function which not necessary satisfies $\lim_{x\to\infty} \frac{\gamma(x)}{x} = \infty$. In addition assume that the non-linear drift term H is given by*

$$H(X_t) = g^*(X_t) = \sum_{n=0}^{\infty} g_n (X_t^*)^n.$$

In this case we can find the distribution space where the solution X_t exists. In fact, if we suppose that $X_t \in \mathcal{F}'_\beta(\mathcal{N}')$ for a certain Young function β, then we can express β in terms of γ and θ. To this end, taking into account equality (21) we have the following growth condition for Y_t: with $|z| = |(\xi, p)|_n$

$$\begin{aligned} |Y_t(z)| \leq\ & K_1 \exp(\theta^*(m_1|z|) + ap^2 T + TK_2 \exp(\theta^*(m_2|z|))) \\ & + T \exp(\exp(ap^2 T + K_2 \exp(\theta^*(m_2|z|))))K_3 \exp(\gamma \circ e^{\beta^*(m_3|z|)}). \end{aligned} \tag{24}$$

If we take $\beta = (\gamma \circ e^{\theta^*})^*$, then there exists $K, m_4 > 0$ such that,

$$|Y_t(z)| \leq K \exp(\gamma \circ e^{\theta^*}(m_4|z|)).$$

This proves that the solution X_t of the Cauchy problem (12) is localized in the distribution space $\mathcal{F}'_\beta(\mathcal{N}')$. Notice that the conditions (13) and (14) are automatically satisfied and conditions (H1) and (H2) may be weakness replacing β^* by e^{θ^*} since $\beta^* > e^{\theta^*}$.

5 Properties of the solution

In this section we will investigate the continuous dependence of the solution (22) on the initial data as well as the dependence on the coefficient H.

Proposition 5.1 *Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}'_\beta(\mathcal{N}')$ and $f \in \mathcal{F}'_\beta(\mathcal{N}')$. Denote by X_t^j, X_t the corresponding solutions of (12) with initial data f_j, f, respectively. If f_j converges strongly to f in $\mathcal{F}'_\beta(\mathcal{N}')$, then X_t^j converges strongly to X_t in $\mathcal{F}'_\beta(\mathcal{N}')$.*

Proof. We need to show assertion 1. of that Proposition 4.1. To this end we first show that there exists m'' such that for all $t \in [0,T]$ $(\widehat{X_t^j})_{j\in\mathbb{N}} \subset \mathcal{G}_{\beta^*,m''}(\mathcal{N}_n)$, and is bounded in that space. Indeed using (18) with $\widehat{X_t^j}$ replacing $u(t)$ and Proposition 4.3 we have

$$\begin{aligned}|\widehat{X_t^j}(\xi,p)| &\leq (K_f + TK_h)\beta^*(m|(\xi,p)|_n) \\ &\quad +\tilde{K}\beta^*(m|(\xi,p)|_n)\int_0^t |\widehat{X_s^j}(\xi,p)|ds.\end{aligned}$$

From Gronwall's lemma and the properties of Young's functions we can find m'' such that

$$\begin{aligned}\sup_{j\in\mathbb{N}}\sup_{t\in[0,T]}|\widehat{X_t^j}(\xi,p)| &\leq (K_f + TK_h)\beta^*(m|(\xi,p)|_n) \\ &\quad \times\exp(T\tilde{K}\beta^*(m|(\xi,p)|_n)) \\ &\leq (K_f + TK_h)\exp(\beta^*(m''|(\xi,p)|_n)).\end{aligned}$$

This shows 1-a of Proposition 4.1. Next we prove the convergence of $\widehat{X_t^j}(\xi,p)$ for each (ξ,p):

$$\begin{aligned}|\widehat{X_t^j}(\xi,p) - \hat{X}_t(\xi,p)| &\leq |(\widehat{f_j} - \hat{f})(\xi,p)| + \int_0^t |a|p^2|\widehat{X_s^j}(\xi,p) - \hat{X}_s(\xi,p)|ds \\ &\quad + \int_0^t |\hat{V}_s(\xi,p)||\widehat{X_s^j}(\xi,p) - \hat{X}_s(\xi,p)|ds \\ &\quad + \int_0^t |(\hat{H}(X_s^j) - \hat{H}(X_s))(\xi,p)|ds.\end{aligned}$$

Using the hypothesis **(H1)** we obtain

$$\begin{aligned}|\widehat{X_t^j}(\xi,p)-\hat{X}_t(\xi,p)| &\leq |(\widehat{f_j}-\hat{f})(\xi,p)|+\tilde{K}\beta^*(m|(\xi,p)|_n)\\ &\quad\times\int_0^t|\widehat{X_s^j}(\xi,p)-\hat{X}_s(\xi,p)|ds.\end{aligned}$$

Applying again Gronwall's inequality it yields

$$|\widehat{X_t^j}(\xi,p)-\hat{X}_t(\xi,p)|\leq|(\widehat{f_j}-\hat{f})(\xi,p)|\exp(TK\beta^*(m|(\xi,p)|_n)).$$

Using Proposition 4.1 we conclude the result. ∎

Remark 5.2 *It follows from the above proof that if $\hat{f},\hat{g}\in U_{\beta^*,m,n}$, then the corresponding solutions satisfies*

$$\left\|\hat{X}(\cdot,\cdot)(\hat{f})-\hat{X}(\cdot,\cdot)(\hat{g})\right\|_{\beta^*,m,n}\leq C(\beta^*,m,n,T)\left\|\hat{f}-\hat{g}\right\|_{\beta^*,m,n},$$

for a certain constant $C(\beta^*,m,n,T)>0$ and this says that $\hat{f}\mapsto\hat{X}(\cdot,\cdot)(\hat{f})$ is Lipschitz. Now using the inverse Laplace transform we have the same type of result for the solution X_t.

The next proposition states that the solution of (22) depends continuously on the coefficient H. To do this we need the notion of the convergence on $\mathcal{F}'_\beta(\mathcal{N}')$, see [19] for more details.

Definition 5.3 *Let H_j,H, $j\in\mathbb{N}$ be mappings from $\mathcal{F}'_\beta(\mathcal{N}')$ onto $\mathcal{F}'_\lambda(\mathcal{N}')$. Then we say $(H_j)_{j\in\mathbb{N}}$ converges to H if and only if for each $\Phi\in\mathcal{F}'_\beta(\mathcal{N}')$, $H_j(\Phi)$ converges strongly to $H(\Phi)$ in $\mathcal{F}'_\lambda(\mathcal{N}')$ in the sense of Definition 4.1.*

Proposition 5.4 *Let H_j,H, $j\in\mathbb{N}$ as in Definition 5.3 be given which satisfies **(H1)** and **(H2)** with the same constant K_h. Denote by X_t^j,X_t the corresponding solutions of (12) with initial data f, respectively. If H_j converges to H (in the sense of Definition 5.3), then X_t^j converges strongly to X_t in $\mathcal{F}'_\beta(\mathcal{N}')$.*

Proof. As in Proposition 5.1 we need to show assertion 2. of Proposition 4.1. We also have that there exists m'' such that for all $t\in[0,T]$ $(\widehat{X_t^j})_{j\in\mathbb{N}}\subset\mathcal{G}_{\beta^*,m''}(\mathcal{N}_n)$. Hence we estimate the following difference:

the

$$
\begin{aligned}
|\widehat{X_t^j}(\xi,p)-\hat{X}_t(\xi,p)| &\leq \int_0^t |a|p^2|\widehat{X_s^j}(\xi,p)-\hat{X}_s(\xi,p)|ds \\
&+\int_0^t |\hat{V}_s(\xi,p)||\widehat{X_s^j}(\xi,p)-\hat{X}_s(\xi,p)|ds \\
&+\int_0^t |(\widehat{H_j(X_s^j)}-\widehat{H_j(X_s)})(\xi,p)|ds \\
&+\int_0^t |(\widehat{H_j(X_s)}-\widehat{H(X_s)})(\xi,p)|ds.
\end{aligned}
$$

The same procedure as in the proof of the last proposition yields

$$
\begin{aligned}
|\widehat{X_t^j}(\xi,p)-\hat{X}_t(\xi,p)| &\leq \tilde{K}\beta^*(m|(\xi,p)|_n)\int_0^t |\widehat{X_s^j}(\xi,p)-\hat{X}_s(\xi,p)|ds \\
&+\int_0^t |(\widehat{H_j(X_s)}-\widehat{H(X_s)})(\xi,p)|ds.
\end{aligned}
$$

Using **(H2)** and the fact that

$$\sup_{t\in[0,T]}|\widehat{X_t}(\xi,p)|\leq (K_f+TK_h)\exp(\beta^*(m''|(\xi,p)|_n))$$

it follows from the Lebesgue dominated convergence theorem that

$$\int_0^T |(\widehat{H_j(X_s)}-\widehat{H(X_s)})(\xi,p)|ds=\varepsilon_j\to 0,\ j\to\infty.$$

The result of the proposition follows from the Gronwall inequality. ■

Combining Proposition 5.1 and Proposition 5.4 we obtain the we obtain the same convergence result for the solution of the Cauchy problem (12).

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