On regular generalized functions in white noise analysis and their applications

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We dedicate this work to the memory of a great mathematician - Yuri Daletskii (December 16, 1926 – December 12, 1997), Member of the National Academy of Sciences of Ukraine.

Abstract

The concepts of regular generalized functions in Gaussian analysis are presented. Spaces of regular generalized functions are characterized and equipped with an infinite dimensional calculus. Finally, these concepts are applied in the theory of stochastic (partial) differential equations of Wick type and a generalized Clark-Ocone formula is given.

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1 Introduction

In this note we give a review on the concepts and applications of regular generalized functions in white noise analysis. White noise analysis is an infinite dimensional calculus which includes various generalizations of concepts known from finite dimensional analysis. Among them are Fourier transform, differential operators, and generalized functions. For a detailed exposition of white noise analysis we refer to the monographs [BK95], [Hid80], [HKPS93], [HØUZ96],[Kuo96] and [Oba94].

The concepts of regular generalized functions have been introduced in [GKS99], see also [Gro98]. Their development was motivated by the results and insights in the theory of stochastic (partial) differential equations of Wick type, see [HØUZ96] and the reference therein. Solutions of such equations turned out to be generalized stochastic processes, i.e., processes which are point-wise defined in space and/or time and generalized functions in the variable giving the probability. Many concrete examples show that processes solving nonlinear Wick type differential equations are processes in the Kondratiev space $(S)^{-1}$ and not in the space of Hida distributions $(S)' \subset (S)^{-1}$.

It is of interest to study probabilistic properties of such solutions. This a priori causes difficulties, because they are generalized functions in the variable giving the probability. A first step in tackling this problem has been done in [BP96]. There the authors have generalized basic concepts from the theory of stochastic processes such as measurability and martingale property to mapping from an interval of the real line to $\mathcal{G}'$ (the space of regular Hida distributions $\mathcal{G}'$ has been introduced in [PT95]). It turns out that $\mathcal{G}'$ is also a realization of a space of regular generalized functions introduced in [GKS99].

After recalling some preliminaries about white noise analysis we present the construction of spaces of regular generalized and show how to equip them with the tools from the theory of stochastic processes.

In Section 3 we give the characterization theorems of the spaces $\mathcal{G}'$ and $\mathcal{G}^{-1}$ of regular generalized functions. These theorems have been proved in [GKS97] and [GKS99], respectively. The characterization is done under use of the S-transform which maps spaces of generalized functions to holomorphic functions on an infinite dimensional space. Elements from $\mathcal{G}'$ and $\mathcal{G}^{-1}$ can equivalently be described as functions from the Bargmann-Segal space and Hardy space, respectively. Both are spaces of square-integrable functions or can equivalently described by such spaces.

In Section 4 we present the infinite dimensional calculus in $\mathcal{G}^{-1}$. We start with the Wick calculus. Originally, this calculus has been developed in the space of Kondratiev distributions $(S)^{-1}$, see [KLS96], where $\mathcal{G}^{-1} \subset (S)^{-1}$. Then in [GKU99] it has been proved that the Wick calculus leaves $\mathcal{G}^{-1}$ invariant, i.e., Wick analytic functions are mappings from $\mathcal{G}^{-1}$ into itself. This calculus turned out to be useful in applications to stochastic (partial) differential equations of Wick type. Furthermore, Wick analytic functions preserve measurability and martingale property, see [GKU99]. Hence, this calculus can be utilized to check these properties. Then we show how to generalize the Skorohod and Itô integrals to regular generalized processes and the Hida gradient to regular generalized functions, following the ideas of [dFOS00].

Finally, we present applications of these concepts. In Section 5.1 we give the generalization of the Clark-Ocone formula to functions from $\mathcal{G}^{-1}$ which has been derived in [dFOS00]. The Clark-Ocone formula is useful in the determination of hedging portfolios, see e.g. [AØU98] and [Øks96].

In Section 5.2 we look at regularity properties for solutions of stochastic (partial) differential equations of Wick type. In [GKU99] it has been proved that the processes solving the stochastic Verhulst equation, the heat equation with a stochastic potential, and the viscous Burgers equation with a stochastic source are regular generalized processes in $\mathcal{G}^{-1}$. Furthermore, it has been shown there that the solution of the stochastic Verhulst equation is a martingale w.r.t. the filtration generated by Brownian motion. And, it has been proved that both, the solution of the heat equation with a stochastic potential and the viscous Burgers equation with a stochastic source, are adapted to this filtration.

In Section 5.3 we present the investigation of scaling limits for the solution of the Wick type Burgers equation performed in [GKS98]. These scaling limits give information about the behavior of the solution for large times and are of interest e.g. in the discussion of the large scale structure of the universe, see [AMS94]. As result of their investigations in [GKS98] the authors found that the limiting distribution for the solution of Wick type Burgers equation with Gaussian initial data is the same as that for the solution of Burgers equation with ordinary product calculated in [LPW96].

2 Regular generalized functions in white noise analysis

2.1 Preliminaries

We work in the framework of white noise analysis where $S'(\mathbb{R},\mathbb{R})$ is the space of real valued tempered distributions equipped with the weak topology and choose the basic measurable space $(S'(\mathbb{R},\mathbb{R}),\mathcal{F})$, where $\mathcal{F}$ is the Borel σ-algebra. On this measurable space we define the Gaussian probability measure μ with characteristic function $\exp(-\frac{1}{2}|\cdot|^2)$, where $|\cdot|$ is the L^2-norm of $L^2(\mathbb{R},\mathbb{R})$, the space of real valued square-integrable functions on $\mathbb{R}$ w.r.t. the Lebesgue measure. In this framework a Brownian motion B_t is defined by

$$B_t(\omega) = \operatorname{sign}(t)\langle \omega, \mathbf{1}_{[t\wedge 0, 0\vee t]}\rangle, \quad \omega \in S'(\mathbb{R},\mathbb{R}),\ t\in\mathbb{R},$$

where $\mathbf{1}_I$ is the indicator function of the interval I, $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $S'(\mathbb{R})$ and $S(\mathbb{R})$ and $\operatorname{sign}(t)$ gives the sign of $t\in\mathbb{R}$. The Brownian motion has to be understood as the $L^2(\mu)$-limit of a sequence of measurable linear functions $(\langle\cdot, f_n\rangle)_{n\in\mathbb{N}}$, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of Schwartz test functions converging to $\mathbf{1}_{[t\wedge 0,0\vee t]}$ in $L^2(\mathbb{R},\mathbb{R})$.

Each square-integrable function $F\in L^2(\mu)$ has the chaos decomposition

$$F = \sum_{n=0}^{\infty} I_n(F^{(n)}), \quad F^{(n)} \in L^2\widehat{(\mathbb{R}^n,\mathbb{C})}, \tag{1}$$

where I_n maps a symmetric complex valued square-integrable function $F^{(n)}\in L^2\widehat{(\mathbb{R}^n,\mathbb{C})}$ to its n-fold iterated Wiener integral. The function $F^{(n)}$ is called the n-th kernel corresponding to F. The space of smooth Wick polynomials $\mathcal{P}$ is defined as the space of functions φ as in (1) having smooth kernels, i.e., $\varphi^{(n)}\in \widehat{S(\mathbb{R}^n,\mathbb{C})}$, and only a finite number of them are different from zero.

2.2 Regular generalized functions

Next we recall the construction of spaces of regular generalized functions, see [GKS97]. For any given $q\in\mathbb{Z}$ and $\beta\in[0,1]$ we define the following Hilbert norm for smooth Wick polynomials $\varphi=\sum_{n=0}^{N} I_n(\varphi^{(n)})\rangle\in\mathcal{P}$:

$$\|\varphi\|^2_{q,\pm\beta} := \sum_{n=0}^{\infty}(n!)^{1\pm\beta}\,2^{nq}\left|\varphi^{(n)}\right|^2.$$

Then, for $q \in \mathbb{N}_0$ and $\beta \in [0,1]$ we define the Hilbert spaces G_q^β as the completion of $\mathcal{P}$ w.r.t. $\| \cdot \|_{q,\beta}$. Or, equivalently,

$$G_q^\beta \;=\; \Big\{ F \in L^2(\mu) \Big| F = \sum_{n=0}^{\infty} I_n(f^{(n)}),\ \| F \|_{q,\beta}^2 < \infty \Big\}.$$

The space of test functions $\mathcal{G}^\beta$ are defined as the projective limit of the spaces G_q^β:

$$\mathcal{G}^\beta = \bigcap_{q \geq 0} G_q^\beta.$$

Let $G_{-q}^{-\beta}$ be the dual of G_q^β and $\mathcal{G}^{-\beta}$ the dual of $\mathcal{G}^\beta$, both w.r.t. $L^2(\mu)$. We know from general duality theory that

$$\mathcal{G}^{-\beta} = \bigcup_{q \geq 0} G_{-q}^{-\beta}.$$

Finally, we construct the following chain of spaces

$$\mathcal{G}^1 \subset \mathcal{G}^\beta \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' \subset \mathcal{G}^{-\beta} \subset \mathcal{G}^{-1}.$$

The spaces $\mathcal{G}^0$ and $\mathcal{G}^{-0}$ have been introduced in [PT95] and are denoted by $\mathcal{G}$ and $\mathcal{G}'$, respectively.

The bilinear dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $\mathcal{G}^1$ and $\mathcal{G}^{-1}$ is connected to the sesquilinear inner product on $L^2(\mu)$ by

$$\langle\!\langle F, \varphi \rangle\!\rangle = (\overline{F}, \varphi)_{L^2(\mu)}, \quad F \in L^2(\mu),\, \varphi \in \mathcal{G}^1.$$

Since the constant function 1 is in $\mathcal{G}^1$ we may extend the concept of expectation from integrable functions to distributions $\Phi \in \mathcal{G}^{-1}$:

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle.$$

It is not hard to see that $G_{\pm q}^{\pm\beta}$ is a Hilbert space which can be described as follows

$$G_{\pm q}^{\pm\beta} = \Big\{ \Phi = \sum_{n=0}^{\infty} I_n(\Phi^{(n)}) \Big| \Phi^{(n)} \in L^2 \widehat{(\mathbb{R}^n, \mathbb{C})}, \| \Phi \|_{\pm q, \pm\beta}^2 < \infty \Big\}.$$

This description of $G^{\pm\beta}_{\pm q}$ (and therefore also of $\mathcal{G}^{\pm\beta}$) shows that its elements have the property that the Wick monomials in their (generalized) chaos decomposition are square-integrable functions. This is the characteristic feature of so called regular generalized function which we use in the next section to endow spaces of them with a probabilistic structure.

Additionally, one can define the second quantized operator $\Gamma(A)$ of a bounded operator $A \in L^2(\mathbb{R}, \mathbb{C})$ as an operator on the space $\mathcal{G}^{\pm\beta}$, $\beta \in [0, 1]$. It is defined as the mapping which transforms the sequence of kernels $(\Phi^{(n)})_{n\in\mathbb{N}_0}$ corresponding to $\Phi \in \mathcal{G}^{\beta}$ to the sequence $(A^{\otimes n}\Phi^{(n)})_{n\in\mathbb{N}_0}$.

2.3 Probabilistic properties

Here we recall how to extend tools from probability theory such as conditional expectation and Martingale property to generalized stochastic functions and generalized stochastic processes, respectively. Although one can consider more general σ-algebras, for simplicity we concentrate on the σ-algebras σ_J generated by Brownian motion $(B_t)_{t\in J}$ where J is an interval of the real line which may be unbounded. The essential feature of these σ-algebras is that measurability and conditional expectation of a square-integrable function F w.r.t. these σ-algebras equivalently can be described via the kernels of F. Let $P_J^{\otimes n}$ denote the projection in $L^2\widehat{(\mathbb{R}^n, \mathbb{C})}$ given by

$$P_J^{\otimes n} g = \mathbf{1}_{J^n} g, \quad g \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}.$$

The following proposition has been proved in [Hid80] Propositions 4.5 and 4.7:

Proposition 2.1 *The function $F = \sum_{n=0}^{\infty} I_n(F^{(n)}) \in L^2(\mu)$ is measurable w.r.t. σ_J if and only if for all $n \in \mathbb{N}_0$*

$$P_J^{\otimes n} F^{(n)} = F^{(n)} \quad \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}$$

almost everywhere w.r.t. the Lebesgue measure. Furthermore, the conditional expectation $\mathbb{E}_\mu(F|\sigma_J)$ is given by

$$\mathbb{E}_\mu(F|\sigma_J) = \sum_{n=0}^{\infty} I_n(P_J^{\otimes n} F^{(n)}).$$

Hence, taking conditional expectation w.r.t. σ_J coincides with applying the second quantized operator $\Gamma(P_J)$ corresponding to the projection P_J, see Section 2.2.

By a generalized stochastic process we mean a mapping from an interval I of the real line to $\mathcal{G}^{-\beta}, \beta \in [0,1]$. Furthermore, let $(\mathcal{F}_t)_{t\in I}$ be the filtration generated by Brownian motion $(B_t)_{t\in I}$, i.e., $\mathcal{F}_t = \sigma_{(\inf I, t)}$, and denote $P_{(\inf I,t)}$ by P_t. Proposition 2.1 suggests the following natural definition, see [BP96] and [GKS97].

Definition 2.2 *(i) A regular generalized stochastic process $(\Phi_t)_{t\in I}$ is called adapted to the filtration $(\mathcal{F}_t)_{t\in I}$ generated by Brownian motion if for all $t \in I$ the regular generalized function $\Phi_t = \sum_{n=0}^{\infty} I_n(\Phi_t^{(n)})$ is measurable w.r.t. $\mathcal{F}_t$. That is for all $n \in \mathbb{N}_0$ and all $t \in I$*

$$P_t^{\otimes n}\Phi_t^{(n)} = \Phi_t^{(n)} \quad \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}$$

almost everywhere w.r.t. the Lebesgue measure.
(ii) A regular generalized stochastic process $(\Phi_t)_{t\in I}$ is called a martingale w.r.t. $(\mathcal{F}_{t_1,t})_{t\in I}$ if for all $n \in \mathbb{N}_0$ and $s \le t$, $s, t \in I$, the following equality holds:

$$P_s^{\otimes n}\Phi_t^{(n)} = \Phi_s^{(n)} \quad \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}$$

almost everywhere w.r.t. the Lebesgue measure.

3 Characterization theorems

In order to characterize $\mathcal{G}'$ and $\mathcal{G}^{-1}$ we need to introduce the S-transform. The S-transform of generalized functions Φ is defined as the dual paring with the Wick exponential, i.e.,

$$S\Phi(g) := \langle\!\langle \Phi, :\exp(\langle \cdot, g\rangle): \rangle\!\rangle, \quad g \in S(\mathbb{R}, \mathbb{C}),$$

where the Wick exponential is defined as

$$:\exp(\langle \omega, g\rangle): := \exp(\langle \omega, g\rangle - \tfrac{1}{2}\langle g, g\rangle), \quad \omega \in S'(\mathbb{R}). \tag{2}$$

Since the Wick exponential is in $\mathcal{G}$ the S-transform is well defined for all $\Phi \in \mathcal{G}'$. For $\Phi \in \mathcal{G}^{-1}$ the S-transform a priori is not well defined because the Wick exponential (2) for $g \neq 0$ is not in $\mathcal{G}^1$. But, since every distribution is of finite order, the S-transform of $\Phi \in \mathcal{G}^{-1}$ can be defined locally, i.e., for each $\Phi \in \mathcal{G}^{-1}$ there exists $\epsilon > 0$ such that its S-transform is defined for all $g \in S(\mathbb{R}, \mathbb{C})$ with $|g| < \epsilon$.

Regular generalized functions can be characterized via holomorphy and an integrability condition of their S-transform. Hence, next we have to recall what we understand by a measurable extension. Let $\mathbb{P}$ denote the set of all finite dimensional projections $P : S'(\mathbb{R}, \mathbb{C}) \to S(\mathbb{R}, \mathbb{C})$. A function H on $S(\mathbb{R}, \mathbb{C})$ is said to have a measurable extension if for each ordered sequence of projections $(P_n)_{n\in\mathbb{N}}$ in $\mathbb{P}$, (i.e., $P_n P_m = P_m$ for $n \geq m$)

$$H(z) := \lim_{n\to\infty} H(P_n z), \quad z \in S'(\mathbb{R}, \mathbb{C}),$$

exists as a measurable function on $(S'(\mathbb{R}), \mathcal{F}_{\mathbb{C}})$, $\mathcal{F}_{\mathbb{C}}$ is the complexification of $\mathcal{F}$.

3.1 Characterization of the space $\mathcal{G}'$ via the Bargmann-Segal space

The Bargmann-Segal space $E^2(\nu)$ on $S'(\mathbb{R}, \mathbb{C})$ is defined as the L^2-closure of the vector space generated by the measurable polynomials

$$\langle z^{\otimes n}, H^{(n)} \rangle, \quad H^{(n)} \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}, \quad z \in S'(\mathbb{R}, \mathbb{C}),$$

w.r.t. the product measure $\nu(z) := \mu_{1/2}(x) \times \mu_{1/2}(y), z = x + iy, x, y \in S'(\mathbb{R}, \mathbb{R})$, where $\mu_{1/2}$ is the centered Gaussian measure with covariance operator $1/2\, Id$. Thus, each $H \in E^2(\nu)$ can be written as

$$H(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, H^{(n)} \rangle, \quad H^{(n)} \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}, \quad z \in S'(\mathbb{R}, \mathbb{C}). \tag{3}$$

The following characterization theorem has been proved in [GKS97].

Theorem 3.1 *(i) For each generalized function Φ from $\mathcal{G}'$ there exists $\epsilon > 0$ such that the measurable extension of $S\Phi(\epsilon\cdot)$ exists and is an element of $E^2(\nu)$.*
(ii) Let H be a function on $S(\mathbb{R}, \mathbb{C})$ having a measurable extension to $S'(\mathbb{R}, \mathbb{C})$ such that there exists $\epsilon > 0$ for which $H(\epsilon\cdot) \in E^2(\nu)$. Then there exists a unique $\Phi \in \mathcal{G}'$ such that $H = S\Phi$.

Using the series expansion of $H \in E^2(\mu)$ as in (3) together with the summability property of its kernel given by the square-integrability of H, one can restrict H to holomorphic function on $L^2(\mathbb{R}, \mathbb{C})$. This mapping can be used to construct, see [GKS97], a natural isomorphism between $E^2(\mu)$ and the in literature well known Bargmann-Segal space on a Hilbert space, see [Seg60], [Bar61], [Bar62], [Seg62], and [Seg78]. In applications of the characterization theorem this isomorphism turned out to be useful, see [GKS97].

3.2 Characterization of the space $\mathcal{G}^{-1}$ via the Hardy space

The infinite dimensional Hardy space $\mathcal{H}^2(U)$, $U := \{z \in L^2(\mathbb{R}, \mathbb{C}) | \, |z| < 1\}$, is defined as the space of holomorphic functions $H : U \to \mathbb{C}$, which can written as

$$H(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, H^{(n)} \rangle, \quad H^{(n)} \in L^2\widehat{(\mathbb{R}^n, \mathbb{C})}, \quad z \in L^2(\mathbb{R}, \mathbb{C}),$$

where $\sum_{n=0}^{\infty} |H^{(n)}|^2 < \infty$. In the one dimensional case, i.e., U is the unit disc in the complex plane, this definition coincides with the original definition of the Hardy space, see [Koo80], [Rud69], and [Sha93].

The following isomorphism has been proven in [GKS97].

Theorem 3.2 *(i) For each generalized function Φ from $\mathcal{G}^{-1}$ there exists $\epsilon > 0$ such that the continuous extension of $S\Phi(\epsilon \cdot)$ to U exists and is an element of $\mathcal{H}^2(U)$.*
(ii) Let H be a function defined on a neighborhood of $0 \in S(\mathbb{R}, \mathbb{C})$ open w.r.t. the norm of $L^2(\mathbb{R}, \mathbb{C})$. Furthermore, assume that for some $\epsilon > 0$ H has a continuous extension to U such that $H(\epsilon \cdot) \in \mathcal{H}^2(U)$. Then there exists a unique $\Phi \in \mathcal{G}^{-1}$ such that $H = S\Phi$.

In the one dimensional case there had been worked out an equivalent representation of the Hardy spaces as a space of square-integrable functions on the unit circle S^1 in the complex plane, see [Koo80], [Rud69], and [Sha93]. Generalizing their ideas, in [GKS97] the authors have developed an equivalent description of the infinite dimensional Hardy space in terms of the space $\mathcal{H}^2(\mathbb{T})$ of "holomorphic" square-integrable functions w.r.t. the flat measure on the infinite dimensional torus

$$\mathbb{T} := \times_{n=1}^{\infty} T_n, \quad T_n = S^1.$$

Hence, both spaces, $\mathcal{G}'$ and $\mathcal{G}^{-1}$, are characterized by holomorphy and an integrability (implying measurability) condition.

4 Wick calculus, Itô integral, and Hida gradient

4.1 Wick calculus

Here we summarize some facts on Wick calculus for regular generalized functions, see [GKU99]. Let $\Phi, \Psi \in \mathcal{G}^{-1}$. Then the Wick product is defined by

$$\Phi \diamond \Psi := S^{-1}(S\Phi \cdot S\Psi) \in \mathcal{G}^{-1}.$$

This multiplication is clearly associative and for deterministic functions it coincides with the point-wise product. In terms of the chaos decomposition the Wick product may be described as follows: Let $\Phi, \Psi \in \mathcal{G}^{-1}$ correspond to the sequences of kernels $(\Phi^{(n)})_{n\in\mathbb{N}_0}$ and $(\Psi^{(n)})_{n\in\mathbb{N}_0}$, respectively. Then the Wick product $\Phi \diamond \Psi$ corresponds to $(\Xi^{(n)})_{n\in\mathbb{N}_0}$ where

$$\Xi^{(n)} = \sum_{k+l=n} \Phi^{(k)} \hat{\otimes} \Psi^{(l)}.$$

By induction, we can define Wick powers

$$\Phi^{\diamond n} = S^{-1}((S\Phi)^n)$$

in $\mathcal{G}^{-1}$ and, by taking finite linear combinations of them, also Wick polynomials of finite order $\sum_{n=1}^{N} a_n \Phi^{\diamond n}$ can be defined in $\mathcal{G}^{-1}$.

Not only Wick polynomials can be defined in $\mathcal{G}^{-1}$, it is even possible to define Wick analytic functions in $\mathcal{G}^{-1}$ under very general assumptions, see [GKU99] for a proof.

Theorem 4.1 *Let F be analytic in a neighborhood of the point $z_0 = \mathbb{E}_\mu(\Phi)$ in $\mathbb{C}$, $\Phi \in \mathcal{G}^{-1}$. Then $F^\diamond(\Phi)$ defined as $F^\diamond(\Phi) := S^{-1}(F(S\Phi))$ exists in $\mathcal{G}^{-1}$.*

If F is analytic at $z_0 = \mathbb{E}_\mu(\Phi)$, $\Phi \in \mathcal{G}^{-1}$, then

$$F^\diamond(\Phi) = \sum_{n=1}^{\infty} a_n (\Phi - z_0)^{\diamond n},$$

where the series converges in $\mathcal{G}^{-1}$.

In applications to stochastic partial differential equations of Wick type, see Section 5.2, the Wick calculus turned out to be useful in order to identify solutions of such equations as regular generalized stochastic processes. Furthermore, Wick analytic functions preserve probabilistic properties such as measurability or martingale property, see [GKU99]. Hence, the Wick calculus can be utilized in order to check such properties.

4.2 Extension of the Skorohod and Itô integrals

The Skorohod integral in white noise analysis and its extensions to certain spaces of generalized functions has been discussed in [Ben93], [HKPS93], and [LØU92]. Here we recall the definition given for regular generalized functions developed in [dFOS00].

Considering an element $\Phi.$ from $L^2(\mathbb{R})\otimes\mathcal{G}^{-1}$ characterized by the sequence $\Phi^{(n)}_{\cdot}(\cdot) \in L^2(\mathbb{R},\mathbb{C}) \otimes L^2(\mathbb{R}^n,\mathbb{C})$, $n \in \mathbb{N}_0$. Let us consider the generalized function characterized by the sequence

$$\begin{aligned}\Psi^{(0)} &= 0\\ \Psi^{(n)}(\cdot,\cdot) &= \widehat{\Phi}^{(n-1)}_{\cdot}(\cdot) \in L^2(\mathbb{R},\mathbb{C}) \otimes L^2(\mathbb{R}^{n-1},\mathbb{C}) \simeq L^2(\mathbb{R}^n,\mathbb{C}),\end{aligned}$$

where $\widehat{\Phi}^{(n-1)}$ denotes the symmetrization of $\Phi^{(n-1)}$ in the variables $t, s_1, \ldots, s_{n-1}$. Ψ is an element from $\mathcal{G}^{-1}$ and is denoted by $I(\Phi)$.

This mapping generalizes the Skorohod integral. In fact, in the particular situation $\Phi \in L^2(\mathbb{R},\mathbb{C}) \otimes D$, $D = \{F \in L^2(\mu) | \sum_{n=0}^{\infty} n\, n! |F^{(n)}|^2 < \infty\}$, the mapping $I(\Phi)$ coincides with the Skorohod integral. If, additionally, the process $\Phi.$ is adapted to the filtration generated by Brownian motion, $I(\Phi)$ coincides with the Itô integral.

Definition 4.2 *For a given $\Phi.$ from $L^2(\mathbb{R},\mathbb{C}) \otimes \mathcal{G}^{-1}$ we call $I(\Phi)$ the generalized Skorohod integral of Φ and if, additionally, the generalized process is adapted to the filtration generated by Brownian motion, we call $I(\Phi)$ generalized Itô integral.*

4.3 Generalization of the Hida gradient operator

We begin with the observation that the Hida derivative ∂_t fails to be pointwise defined on the spaces $\mathcal{G}^{\pm1}$. However, we still may consider the gradient

$\partial.$ as an operator

$$\partial. : \mathcal{G}^{\pm 1} \to L^2(\mathbb{R}, \mathbb{C}) \otimes \mathcal{G}^{\pm 1}.$$

Given a (generalized) function Ψ from $\mathcal{G}^{\pm 1}$ with kernel functions $(\Psi^{(n)})_{n \in \mathbb{N}_0}$, we define the generalized gradient $\partial.\Psi \in L^2(\mathbb{R}, \mathbb{C}) \otimes \mathcal{G}^{\pm 1}$ as the operator characterized by the sequence

$$(\partial.\Psi)^{(n)}(\cdot) = \Psi^{(n+1)}(\cdot, \cdot) \in L^2(\mathbb{R}^{n+1}, \mathbb{C}) \simeq L^2(\mathbb{R}, \mathbb{C}) \otimes L^2(\mathbb{R}^n, \mathbb{C}).$$

Furthermore, the generalization of the Skorohod integral is the adjoint operator to the generalization of the gradient, i.e.,

$$\langle\!\langle I(\Phi), \psi \rangle\!\rangle = \int_{\mathbb{R}} \langle\!\langle \Phi_t, \partial_t \psi \rangle\!\rangle \, dt, \quad \Phi. \in L^2(\mathbb{R}, \mathbb{C}) \otimes \mathcal{G}^{-1}, \psi \in \mathcal{G}^1,$$

see [dFOS00] for a detailed discussion.

5 Applications

5.1 Generalized Clark-Ocone formula

We denote by Θ_t the Heaviside function

$$\Theta_t(s) = \begin{cases} 1 & \text{for} \quad s < t \\ 0 & \text{for} \quad s \geq t \end{cases}$$

as well as the projection operator in $L^2(\mathbb{R}, \mathbb{C})$ given by

$$\Theta_t : g \mapsto \Theta_t g.$$

Taking conditional expectation w.r.t. $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t \in \mathbb{R}}$ is the filtration generated by Brownian motion $(B_t)_{t \in \mathbb{R}}$, then coincides with applying the second quantized operator $\Gamma(\Theta_t)$, see Section 2.2. The following generalization of the Clark-Ocone formula has been given in [dFOS00].

Theorem 5.1 *Let Φ be a regular generalized function, $\Phi \in \mathcal{G}^{-1}$. Then it can be written as a generalized Itô integral*

$$\Phi = E(\Phi) + I(m)$$

with

$$m(\cdot) = \Gamma(\Theta.)\partial.\Phi.$$

5.2 Stochastic (partial) differential equations of Wick type

Here we present the applications of the above discussed concepts to Wick type stochastic (partial) differential equations. For a detailed introduction to this theory we refer to the monograph [HØUZ96] and the references therein. These differential equations are in general posed in the space of generalized functions $(S)^{-1}$ which is larger than the spaces of regular generalized functions discussed in the previous sections. The essential difference of this so called Kondratiev space w.r.t. spaces of regular generalized functions is that the kernels of an element $\Phi \in (S)^{-1}$ may be elements of $\widehat{S'(\mathbb{R}^n, \mathbb{C})}$. Hence, the Gaussian white noise process $\omega_t := \langle \omega, \delta_t \rangle$, where $\delta_t \in S'(\mathbb{R}, \mathbb{C})$ is the Dirac delta at $t \in \mathbb{R}$, is a process in $(S)^{-1}$. Using the Wick product in $(S)^{-1}$, this fact enables us to give an even more general extension of the Skorohod integral than the one of Section 4.2. For $\Phi. \in L^2(\mathbb{R}, \mathbb{C}) \otimes \mathcal{G}^{-1}$ we have

$$I(\Phi) = \int_{\mathbb{R}} \Phi_t \diamond \omega_t \, dt, \tag{4}$$

where the integral on the right hand side of (4) is a Pettis integral in $(S)^{-1}$ which can be extended to certain processes in $(S)^{-1}$. For a more detailed discussion on the relation of the Pettis integral in (4) to the Skorohod and Itô integral we refer to [Ben93], [HKPS93], and [LØU92] .

5.2.1 Stochastic Verhulst equation

In this section we discuss some regularity properties of the solution of the Wick type stochastic Verhulst equation

$$X_t = X_0 + r \int_0^t X_s \diamond (1 - X_s) \, ds + \alpha \int_0^t X_s \diamond (1 - X_s) \diamond \omega_s \, ds, \quad t \geq 0, \tag{5}$$

where $r > 0$ and $\alpha \in \mathbb{R}$ are fixed. In [Ben96], Theorem 3.1, it has been proved that, for $X_0 \in (S)^{-1}$ with $\mathbb{E}_\mu(X_0) > 0$, the generalized stochastic process

$$X_t(\omega) = (1 + Y_0 \diamond \exp^\diamond(-rt - \alpha B_t(\omega)))^{\diamond(-1)}, \quad \omega \in S'(\mathbb{R}), \tag{6}$$

with

$$Y_0 = (X_0)^{\diamond(-1)} - 1$$

is the unique weakly continuously differentiable $(S)^{-1}$-process solving the stochastic Verhulst equation (5).

Utilizing the Wick calculus in $\mathcal{G}^{-1}$ in [GKU99], Theorems 5.4 and 5.5, the following regularity property for the solution of the Verhulst equation has been proved, see also [GKS99], Theorem 8.3 and Theorem 8.5, and [Us97], Theorem 6.1.

Theorem 5.2 *(i) Suppose $X_0 \in \mathcal{G}^{-1}$ with $\mathbb{E}_\mu(X_0) > 0$. Then the generalized stochastic process (6) solving the stochastic Verhulst equation (5) is regular, i.e., a process in $\mathcal{G}^{-1}$.*
(ii) For deterministic initial values $X_0 > 0$ the solution of the stochastic Verhulst equation is a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by Brownian motion.

5.2.2 Heat equation with a stochastic potential

Stochastic partial differential equations are often also formulated in the space of Kondratiev distributions. The partial derivatives have to be understood w.r.t. the weak topology of $(S)^{-1}$.

Let us consider the following stochastic heat equation with a stochastic potential:

$$\begin{aligned} \frac{\partial Y_t}{\partial t}(x) &= \frac{1}{2}\nu \Delta Y_t(x) + a(x) Y_t(x) \diamond \omega_t \quad t \geq 0,\ x \in \mathbb{R}^d, \\ Y_0(x) &= M(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{7}$$

Here $\nu > 0$ and $d \in \mathbb{N}$. The equality is formulated in $(S)^{-1}$ and the noise $(\omega_t)_{t\geq 0}$ is the Gaussian white noise process.

Theorem 5.3 *(i) Assume that there exist $q \in \mathbb{N}$ such that $\{M(x) | x \in \mathbb{R}^d\} \subset G_{-q}^{-1}$ is bounded. In addition, let M be weakly continuous w.r.t. $x \in \mathbb{R}^d$ and $a \in C(\mathbb{R}^d, \mathbb{R})$ be bounded. Then there is a unique $\mathcal{G}^{-1}$-valued $C^{1,2}$ solution Y of the stochastic heat equation (7), namely*

$$\begin{aligned} Y_t(x) &= \int_{S'(\mathbb{R},\mathbb{R}^d)} M(\sqrt{2\nu}(B_t(\tilde{\omega}) + x)) \\ &\diamond \ \exp^{\diamond}\Big(\int_0^t a(\sqrt{2\nu}(B_s(\tilde{\omega}) + x)) \langle \cdot, \delta_{t-s} \rangle \, ds \Big) \, d\mu_d(\tilde{\omega}). \end{aligned} \tag{8}$$

(ii) Additionally, let M be a deterministic function (the kernels of order higher than 0 vanish). Then the regular generalized stochastic process (8) solving the stochastic heat equation (7) is square-integrable and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by Brownian motion.

Remark 5.4 *(i) Formula (8) is a consequence of the Feynman-Kac formula. The integrand*

$$\begin{aligned} J(\tilde{\omega}) &= M(\sqrt{2\nu}(B_t(\tilde{\omega}) + x)) \\ &\diamond \ \exp^{\diamond}\Big(\int_0^t a(\sqrt{2\nu}(B_s(\tilde{\omega}) + x))\langle\cdot, \delta_{t-s}\rangle\, ds\Big), \quad \tilde{\omega} \in S'(\mathbb{R}, \mathbb{R}^d), \end{aligned}$$

is a function from $S'(\mathbb{R}, \mathbb{R}^d)$ to G^{-1}_{-q} for some $q \in \mathbb{N}$ and the integral exists as a Bochner integral w.r.t. the standard Gaussian measure μ_d on $S'(\mathbb{R}, \mathbb{R}^d)$. The process $B_t(\tilde{\omega})$, $t \geq 0$, $\tilde{\omega} \in S'(\mathbb{R}, \mathbb{R}^d)$, is a vector valued Brownian motion realized in the framework of vector valued white noise, see [SW93].
(ii) The proof of Theorem 5.3 is given in [GKU99], Theorems 5.11 and 5.13.
(iii) Under weaker assumptions on the initial condition M and the noise in [HLØ$^+$95], Theorem 3.1, an analogue to Theorem 5.3 has been proven in the space of Kondratiev distributions.

5.2.3 Viscous Burgers equation with a stochastic source

In [HLØ$^+$95] the authors have introduced the Wick type Cole-Hopf transformation in order to solve the Wick type stochastic Burgers equation with a stochastic source. There the source is given by a function of time and space with values in $(S)^{-1}$. In Theorem 5.1 of that article the authors proved existence and uniqueness of the solution as a process in $(S)^{-1}$. The following regularity has been proved in [GKU99], Theorems 5.17 and 5.19.

Theorem 5.5 *Assume that there exist $q \in \mathbb{N}$ such that*

$$\{M(x)|x \in \mathbb{R}^d\}, \Big\{\frac{\partial M}{\partial x^k}(x)\Big| x \in \mathbb{R}^d\Big\} \subset G^{-1}_{-q}, \quad 1 \leq k \leq d,$$

are bounded subsets and let M be weakly continuously differentiable w.r.t. $x \in \mathbb{R}^d$. Moreover, assume that

$$\mathbb{E}_\mu(M(x)) > 0, \quad x \in \mathbb{R}^d,$$

and let $a \in C^1(\mathbb{R}^d, \mathbb{R})$ be bounded with bounded derivative. Then the generalized stochastic processes

$$U_t^k(x) = -\frac{2\nu}{\lambda} \frac{\partial \log^\diamond(Y_t)}{\partial x^k}(x), \quad 1 \le k \le d, \tag{9}$$

where Y is the solution of the stochastic heat equation (7), belong to $\mathcal{G}^{-1}$ for all $t \ge 0, x \in \mathbb{R}^d$, and $U_t(x) = (U_t^1(x), \ldots, U_t^d(x))$ solves the stochastic Burgers equation

$$\frac{\partial U_t^k}{\partial t}(x) + \lambda \sum_{j=1}^d U_t^j(x) \diamond \frac{\partial U_t^k}{\partial x^j}(x) = \nu \Delta U_t^k(x) - \frac{2\nu}{\lambda} \frac{\partial a}{\partial x^k}(x)\omega_t, \quad t \ge 0,$$
$$U_0^k(x) = G^k(x), \quad x \in \mathbb{R}^d, 1 \le k \le d, \tag{10}$$

where

$$G^k(x) = -\frac{2\nu}{\lambda} \frac{\partial \log^\diamond(M)}{\partial x^k}(x).$$

Here $\lambda, \nu > 0$ and $d \in \mathbb{N}$. The noise $(\omega_t)_{t \ge 0}$ again is the white noise process.
(ii) Additionally, let M be a deterministic function. Then the regular generalized process (9) solving the stochastic Burgers equation (10) is adapted to the filtration $(\mathcal{F}_t)_{t \ge 0}$ generated by Brownian motion.

Remark 5.6 *Theorem 5.5 holds also for $a(x) = \sum_{k=1}^d x_k$, $x \in \mathbb{R}^d$. For this choice of the function a the noise in the Burgers equation (10) is Gaussian white noise.*

5.3 Scaling limits for the solution of Wick type Burgers equation

Let us consider the Burgers equation with random initial data:

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^d U_j(t,x) \diamond \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x), \quad t > 0,$$
$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d, 1 \le k \le d. \tag{11}$$

In order to formulate the conditions on the initial velocity potential G we need the following definition.

Definition 5.7 *A function L is said to be slowly varying at infinity if it is positive and measurable on $[a, \infty)$, for some $a > 0$, and if for each $p > 0$*

$$\lim_{s\to\infty} \frac{L(ps)}{L(s)} = 1.$$

Properties of slow varying functions are discussed in e.g. [Sen76].

Condition (A) The initial velocity potential G is a zero-mean, measurable, mean-square differentiable, homogeneous and isotropic complex Gaussian random field on $\mathbb{R}^d \times S'(\mathbb{R}^d)$. In addition, its variance $\mathbb{E}_\mu(\overline{G(x)}G(x)) = 1$, and its covariance is of the form

$$C(|x|) = \mathbb{E}_\mu(\overline{G(0)}G(x)) = \frac{L(|x|)}{|x|^\alpha}, \quad x \in \mathbb{R}^d,\ 0 < \alpha < d,$$

where the function L is slowly varying at infinity, bounded, and there exist $b, c > 0$ such that $L(t) \geq c$ for all $t \geq b$.

Condition (B) The initial velocity potential G has a spectral density $f(p) = \tilde{f}(|p|) \geq 0$, $p \in \mathbb{R}^d$, i.e.,

$$\begin{aligned} G(x,\omega) &= \langle \omega, \exp(i\cdot x) f^{1/2}(\cdot)\rangle, \quad x \in \mathbb{R}^d,\ \omega \in S'(\mathbb{R}^d, \mathbb{R}) \\ &= \int_{\mathbb{R}^d} \exp(ipx) f^{1/2}(p)\, dB(p,\omega), \end{aligned}$$

where B is a d-parameter Brownian sheet realized in the framework of white noise analysis on $S'(\mathbb{R}^d, \mathbb{R})$. The expression px, $p, x \in \mathbb{R}^d$, symbolizes the scalar product in $\mathbb{R}^d$. As a consequence of condition (A) together with a Tauberian theorem, see [LO91], the spectral density f has a singularity at zero. We assume that this singularity is integrable, the value of the integral of f is one, f is a bounded continuous function on $\mathbb{R}^d \backslash \{p \in \mathbb{R}^d \,|\, |p| < \epsilon\}$ for all $\epsilon > 0$, and there exist $p_0 > 0$ such that f is a decreasing function for $|p| \geq p_0$.

Before stating the main theorem we have to introduce the vector valued Gaussian random field

$$X(t,a,\omega) = i\left(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\right)^{1/2} \langle \omega, t^{d/4} \exp(i\sqrt{t}\cdot a) g(\sqrt{t}\cdot)\rangle, \quad a \in \mathbb{R}^d, \quad (12)$$

$t > 0$, $\omega \in S'(\mathbb{R}^d, \mathbb{R})$, where

$$\begin{aligned} g(p) &= \frac{\exp(-\nu|p|^2)}{|p|^{(d-\alpha)/2}} \frac{p}{\lambda}, \quad p \in \mathbb{R}^d, \qquad (13) \\ c_1(d,\alpha) &= 2^\alpha \Gamma\Big(1+\frac{\alpha}{2}\Big)\Gamma\Big(\frac{d}{2}\Big)\Big/\Big(\frac{d-\alpha}{2}\Big), \\ c_2(d) &= 2\pi^{d/2}/\Gamma(n/2), \end{aligned}$$

the latter constant is the area of the unit sphere in $\mathbb{R}^d$ (Γ is the well-known gamma function).

Remark 5.8 *Calculating the covariance of the Gaussian random vectors $X(t,a_1)$ and $X(t,a_2)$, $a_1, a_2 \in \mathbb{R}^d$, one easily finds that it is independent of t ($t > 0$). Hence, finite dimensional distributions of the random field $X(t,a)$, $a \in \mathbb{R}^d$, coincide with finite dimensional distributions of the random field $X(a) := X(1,a)$, $a \in \mathbb{R}^d$, for all $t > 0$. In particular, the random field*

$$\begin{aligned} X(a,\omega) &= i\Big(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\Big)^{1/2} \langle \omega, \exp(i\cdot a) g(\cdot)\rangle, \quad a \in \mathbb{R}^d, \\ &= i\Big(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\Big)^{1/2} \int_{\mathbb{R}^d} \exp(ipa) g(p)\, dB(p,\omega), \end{aligned}$$

$\omega \in S'(\mathbb{R}^d, \mathbb{R})$, is a homogeneous Gaussian random field with mean zero.

The following theorem has been proved in [GKS98].

Theorem 5.9 *Let $U(t,x)$, $t > 0$, $x \in \mathbb{R}^d$, $d \geq 1$, be the solution of the initial value problem (11) with random initial data satisfying condition (A) and (B). Then for α as in condition (A) we have:*
(i) For any $n \neq 1$ each component of the scaled chaos of order n

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U^{(n)}(t, \sqrt{t}a)$$

converges in mean square to zero, uniformly in $a \in \mathbb{R}^d$, as t tends to infinity.
(ii) For $n = 1$ finite dimensional distributions of the vector valued Gaussian random field

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U^{(1)}(t, \sqrt{t}a), \quad a \in \mathbb{R}^d,$$

converge weakly to finite dimensional distribution of the random field $X(a)$, $a \in \mathbb{R}^d$, as t tends to infinity.
(iii) Each component of the renormalized random process

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} R(N)U(t, \sqrt{t}a) - X(t, a), \quad t \geq t_0,$$

is a process in $L^2(\mu)$. Here

$$R(n) = (n!)^{-1/2}\, 2^{(1-n)q/2},\, n \in \mathbb{N}_0,$$

N is the number operator, i.e., $NU^{(n)} = nU^{(n)}$, and $q \in \mathbb{N}$ is chosen large enough. Moreover, each component of this process converges in mean square to zero, uniformly in $a \in \mathbb{R}^d$, as t tends to infinity.

Remark 5.10 *(i) There are several papers in which scaling limits for the solution of Burgers equation with ordinary product have been calculated, see the introduction of [GKS98] for references. Under the use of condition (A) only in e.g. [LOR94] weak convergence of finite dimensional distributions of the field*

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U(t, \sqrt{t}a), \quad a \in \mathbb{R}^d, \tag{14}$$

as $t \to \infty$ to a centered Gaussian random field has been proved (here the solution of Burgers equation with ordinary product is also denoted by U).
(ii) Note that the solutions of the Wick type Burgers equation and the Burgers equation with ordinary product for finite times $t > 0$ are completely different random processes. Since the solutions of the Wick type Burgers equation are generalized functions in the variable giving the probability, in [GKS98] the authors had to work out another kind of convergence.
(iii) Theorem 5.9 shows that the scaling limit for the solution of Wick type Burgers equation is almost the same, including the type of convergence, as that of the Burgers equation with ordinary product. Under the same assumptions on the initial velocity potential G in [LPW96] it has been shown that finite dimensional distributions of the scaled solution of Burgers equation with ordinary product weakly converge to the random field $X(a)$, $a \in \mathbb{R}^d$, when t tends to infinity. Theorem 5.9 (ii) states that finite dimensional distributions of the first chaos of the scaled solution of the Wick type Burgers

equation weakly converge to the same random field. Since the solution of the Wick type Burgers equation is a generalized random field and therefore in general not measurable in the usual sense one can not prove convergence in distribution for the entire solution. Nevertheless, the limiting behavior of the scaled solution of Wick type Burgers equation is very close to that of the scaled solution of Burgers equation with ordinary product, because all other chaos, see Theorem 5.9 (i), converge to zero in mean square.
(iv) Consider the initial value problem (11). The parameter λ is giving the nonlinearity of the Burgers equation. Notice that the limiting Gaussian random field X still depends on the parameter λ, see (12) and (13).
(v) The proof of Theorem 5.9 can easily be generalized to the case of Poisson analysis. More precisely, if we use a Poisson measure, see e.g. [IK88], instead of a Gaussian measure, substitute the orthogonal system given by Hermite polynomial by Charlier polynomials, and use the Wick calculus in Poisson analysis, see [KSS97], then one can prove an analogous result. The initial velocity potential then is a compensated Poisson field, just as the limiting random field.

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