Generalized fractional evolution equation

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Abstract

In this paper we study the generalized Riemann-Liouville (resp. Caputo) time fractional evolution equation in infinite dimensions. We show that the explicit solution is given as the convolution between the initial condition and a generalized function related to the Mittag-Leffler function. The fundamental solution corresponding to the Riemann-Liouville time fractional evolution equation does not admit a probabilistic representation while for the Caputo time fractional evolution equation it is related to the inverse stable subordinators.

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1 Introduction

The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 1)$, namely

$$D_t^\alpha u(x,t) = \Delta u(x,t), \; t > 0, \quad x \in \mathbb{R}, \tag{1}$$

where D_t^α is the the Riemann-Liouville or Caputo derivative of order α. The main physical purpose for investigating these type of equations is to describe phenomena of anomalous diffusion appearing in transport processes and disordered systems. For a recent and interesting reviews see [20], [23] and references therein. Almost all studies done on this subject treated the finite dimensional case and sufficiently smooth initial condition, cf. [16], [18], [21], [15]. The methods used to find the explicit solution consists in applying in succession the transforms of Fourier in space and Laplace in time. The infinite dimensional case was studied by E. Bazhlekova for a fixed Banach space X using semigroup technics. More precisely, she studied the strong solution of the equation (1) with Δreplaced by a densely defined operator A on X and the time derivative is related to the Riemann-Liouville derivative. We refer the interested reader to [2], [3] and [4] for more details.

The aim of this paper is to consider the above scheme in infinite dimensions. First of all, we notice that the Gross Laplacian Δ_G is the natural generalization of the usual Laplacian to represent the diffusion in infinite dimensions. For $\alpha = 1$ we have constructed a solution of equation (1) and gave a probabilistic representation of it, cf. [8]. Second, the Fourier transform is replaced by the so-called Laplace transform for generalized functions. Thus the same method the solution of the problem

$$D_t^\alpha U(t) = \Delta_G U(t), \; t > 0 \tag{2}$$

is given in terms of the convolution product between the fundamental solution and the initial condition which is a generalized function. The fundamental solution is related to the Mittag-Leffler function through the Laplace transform. For the Riemann-Liouville time derivative problem, we show that the corresponding fundamental solution does not correspond to a density of a measure, cf. Remark 3.11. Hence (2) does not admit a probabilistic interpretation. On the other hand, the same problem with the Caputo derivative may be interpreted as a probability density of a stochastic process. The details of the construction of this process and measure will be the subject of a forthcoming paper.

The paper is organized as follows. In Section 2 we provide the mathematical background needed to solve the problem (2). Namely, we construct the appropriate test functions space $\mathcal{F}_\theta(N')$, $\mathcal{G}_{\theta^*}(N)$ and the associated generalized functions $\mathcal{F}'_\theta(N')$. The elements in $\mathcal{F}_\theta(N')$ (resp. $\mathcal{G}_{\theta^*}(N)$) are entire functions on the co-nuclear space N' (resp. in N) with exponential growth of order θ (a Young function) and of minimal type (resp. maximal type). $\mathcal{F}'_\theta(N')$ is the topological dual of $\mathcal{F}_\theta(N')$. An example of an entire function is

$$N \ni \xi \mapsto E_{\alpha,\beta}(\langle \xi, \xi \rangle) \in \mathbb{C},$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function, see Example 2.2 and Remark 2.5. In Section 3 we solve the problem (2) using the introduced tools and study the stability of the solution. We end the section proving that the solution is continuous with respect to the initial data.

2 Preliminaries

In this section we will introduce the framework which is necessary later on. Let X be a real nuclear Fréchet space with topology given by an increasing family $\{|\cdot|_k;\, k \in \mathbb{N}_0\}$ of Hilbertian norms, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Then X is represented as

$$X = \bigcap_{k \in \mathbb{N}_0} X_k,$$

where X_k is the completion of X with respect to the norm $|\cdot|_k$. We use X_{-k} to denote the dual space of X_k. Then the dual space X' of X can be represented as

$$X' = \bigcup_{k \in \mathbb{N}_0} X_{-k}$$

which is equipped with the inductive limit topology.

Let $N = X + iX$ and $N_k = X_k + iX_k,\ k \in \mathbb{Z}$, be the complexifications of X and X_k, respectively. For $n \in \mathbb{N}_0$, we denote by $N^{\hat{\otimes} n}$ the n-fold symmetric tensor product of N equipped with the π-topology and by $N_k^{\hat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product of N_k. We will preserve the notation $|\cdot|_k$ and $|\cdot|_{-k}$ for the norms on $N_k^{\hat{\otimes} n}$ and $N_{-k}^{\hat{\otimes} n}$, respectively.

2.1 Functional spaces

Let $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty \qquad \text{and} \quad \theta(0) = 0.$$

Such a function is called a Young function. For a Young function θ we define

$$\theta^*(x) := \sup_{t\geq 0}\{tx - \theta(t)\}, \qquad x \geq 0.$$

This is called the polar function associated to θ. It is known that θ^* is again a Young function and $(\theta^*)^* = \theta$, see [17] for more details and general results.

Given a Young function θ, we denote by $\mathcal{F}_\theta(N')$ the space of holomorphic functions on N' with exponential growth of order θ and of minimal type. Similarly, let $\mathcal{G}_\theta(N)$ denote the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type. More precisely, for each $k \in \mathbb{Z}$ and $m > 0$, define $\mathcal{F}_{\theta,m}(N_k)$ to be the Banach space of entire functions f on N_k satisfying the condition

$$|f|_{\theta,k,m} := \sup_{x\in N_k} |f(x)|e^{-\theta(m|x|_k)} < \infty. \tag{3}$$

Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ may be represented as

$$\mathcal{F}_\theta(N') = \bigcap_{k\in\mathbb{N}_0, m>0} \mathcal{F}_{\theta,m}(N_{-k}),$$
$$\mathcal{G}_\theta(N) = \bigcup_{k\in\mathbb{N}_0, m>0} \mathcal{F}_{\theta,m}(N_k)$$

which are equipped with the projective limit topology and the inductive limit topology, respectively. The space $\mathcal{F}_\theta(N')$ is called the space of test functions on N'. For a test function $\varphi \in \mathcal{F}_\theta(N')$ there exists coefficients $\varphi_n \in N^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$ such that φ admits the decomposition

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi_n\rangle.$$

Its dual space $\mathcal{F}'_\theta(N')$, equipped with the strong topology, is called the space of generalized functions. The dual pairing between $\mathcal{F}'_\theta(N')$ and $\mathcal{F}_\theta(N')$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

Later on we need the comparison between test function spaces $\mathcal{F}_\theta(N')$ for different Young functions θ.

Lemma 2.1 *Let θ, γ be two given Young functions and denote*

$$\begin{aligned}\liminf_{x\to\infty} \frac{\theta(x)}{\gamma(x)} &=: c_1 \\ \limsup_{x\to\infty} \frac{\theta(x)}{\gamma(x)} &=: c_2.\end{aligned}$$

1. *If $0 < c_1 \le c_2 < \infty$, then we have $\mathcal{F}_\theta(N') = \mathcal{F}_\gamma(N') \Longrightarrow \mathcal{F}'_\theta(N') = \mathcal{F}'_\gamma(N')$.*
2. *If $c_2 = 0$, then $\mathcal{F}_\theta(N') \subset \mathcal{F}_\gamma(N') \Longrightarrow \mathcal{F}'_\gamma(N') \subset \mathcal{F}'_\theta(N')$.*
3. *If $c_1 = \infty$, then $\mathcal{F}_\gamma(N') \subset \mathcal{F}_\theta(N') \Longrightarrow \mathcal{F}'_\theta(N') \subset \mathcal{F}'_\gamma(N')$.*

Proof. First we notice that the case $c_1 = c_2 = 1$ was studied in [10, Lemme 3]. The idea of the proof follows from the fact that there exists $x_0 > 0$ such that

$$\frac{\gamma(x)}{\theta(x)} \ge \varepsilon, \qquad 0 < \varepsilon < 1, \quad x \ge x_0,$$

then $\mathcal{F}_\theta(N') \subset \mathcal{F}_\gamma(N')$. ■

For $k \in \mathbb{N}_0$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_k) = \left\{ \vec{\varphi} = (\varphi_n)_{n=0}^{\infty};\ \varphi_n \in N_k^{\hat{\otimes} n},\ \sum_{n=0}^{\infty} \theta_n^{-2} m^{-n} |\varphi_n|_k^2 < \infty \right\}, \tag{4}$$

$$G_{\theta,m}(N_{-k}) = \left\{ \vec{\Phi} = (\Phi_n)_{n=0}^{\infty}\ ;\ \Phi_n \in N_k^{\hat{\otimes} n},\ \sum_{n=0}^{\infty} (n!\theta_n)^2 m^n |\Phi_n|_{-k}^2 < \infty \right\}, \tag{5}$$

where

$$\theta_n = \inf_{x>0} \frac{e^{\theta(x)}}{x^n}, \quad n \in \mathbb{N}_0. \tag{6}$$

We define

$$\begin{aligned} F_\theta(N) &:= \bigcap_{k\in\mathbb{N}_0, m>0} F_{\theta,m}(N_k) \\ G_\theta(N') &:= \bigcup_{k\in\mathbb{N}_0, m>0} G_{\theta,m}(N_{-k}). \end{aligned}$$

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Fréchet space, see [10, Proposition 2]. The space $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the bilinear pairing given by

$$\langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \Phi_n, \varphi_n \rangle, \tag{7}$$

where $\vec{\Phi} = (\Phi_n)_{n=0}^{\infty} \in G_\theta(N')$ and $\vec{\varphi} = (\varphi_n)_{n=0}^{\infty} \in F_\theta(N)$.

The Taylor map defined by

$$\mathfrak{T} : \mathcal{F}_\theta(N') \longrightarrow F_\theta(N),\ \varphi \mapsto \left(\frac{1}{n!} \varphi^{(n)}(0) \right)_{n=0}^{\infty}$$

is a topological isomorphism. The same is true between $\mathcal{G}_{\theta^*}(N)$ and $G_\theta(N')$. The action of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ on a test function $\varphi \in \mathcal{F}_\theta(N')$ can be expressed in terms of the Taylor map as follows:

$$\langle\langle \Phi, \varphi \rangle\rangle = \langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle, \tag{8}$$

where $\vec{\Phi} = (\mathfrak{T}^*)^{-1}\Phi$ and $\vec{\varphi} = \mathfrak{T}\varphi$.

Example 2.2 (Mittag-Leffler function) *As an example of an element in $\mathcal{G}_{\gamma^*}(N)$, with the Young function $\gamma^*(x) = x^{2/\alpha}$, $0 < \alpha < 1$ (polar function of $\gamma(x) = x^{2/(2-\alpha)}$) we consider*

$$N \ni \xi \longmapsto E_{\alpha,\beta}(\langle \xi, \xi \rangle) \in \mathbb{C},$$

where $E_{\alpha,\beta}(z)$, $z \in \mathbb{C}$ is the (entire) Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \quad \alpha, \beta > 0. \tag{9}$$

Moreover the corresponding Taylor series is given by

$$E_{\alpha,\beta}(\langle \xi, \xi \rangle) = \sum_{n=0}^{\infty} \langle E_{\alpha,\beta}^n, \xi^{\otimes n} \rangle,$$

where the kernels $E_{\alpha,\beta}^n$ are given in (12) below.

Proof. It is sufficient to prove that there exist $m > 0$ and $k \in \mathbb{N}$ such that $|E_{\alpha,\beta}|_{\gamma^*,k,m} < \infty$. Let us denote the Taylor coefficients of $E_{\alpha,\beta}$ by $(E^n_{\alpha,\beta})_{n=0}^\infty \subset N'$. Then according to the isomorphism between $\mathcal{G}_{\gamma^*}(N)$ and $G_\gamma(N')$, $\gamma^*(x) = x^{2/(2-\alpha)}$ the following series has to be finite

$$\sum_{n=0}^\infty (n!\gamma_n)^2 m^n |E^n_{\alpha,\beta}|^2_{-k}. \tag{10}$$

In order to find the Taylor coefficients $(E^n_{\alpha,\beta})_{n=0}^\infty$, first we introduce the trace operator τ defined as

$$\tau : N^{\otimes 2} \longrightarrow \mathbb{C},\ \xi \otimes \eta \longmapsto \langle \tau, \xi \otimes \eta \rangle := \langle \xi, \eta \rangle. \tag{11}$$

It is clear that there exists $k \in \mathbb{N}$ such that $\tau \in N^{\hat{\otimes} 2}_{-k}$. Then the function $E_{\alpha,\beta}$ can be written as

$$E_{\alpha,\beta}(\langle \xi, \xi \rangle) = \sum_{n=0}^\infty \frac{\langle \xi, \xi \rangle^n}{\Gamma(\alpha n + \beta)} = \sum_{n=0}^\infty \langle E^n_{\alpha,\beta}, \xi^{\otimes n} \rangle,$$

where,

$$\begin{cases} E^{2n}_{\alpha,\beta} & = \frac{\tau^{\otimes n}}{\Gamma(\alpha n + \beta)} \\ E^{2n+1}_{\alpha,\beta} & = 0. \end{cases} \tag{12}$$

Therefore, (10) becomes

$$\sum_{n=0}^\infty (2n)!^2 \gamma_{2n}^2 m^{2n} \frac{1}{\Gamma^2(\alpha n + \beta)} |\tau|^{2n}_{-k} \tag{13}$$

which should be finite. By definition θ_n is given by (6) and it is not hard to prove that θ_n is given by

$$\gamma_n = \left[\frac{e^n}{n^n} \left(\frac{2}{2-\alpha} \right)^n \right]^{(2-\alpha)/2}.$$

Using the inequalities

$$\begin{aligned} |\Gamma(x)| &\geq e^{-x}\sqrt{2\pi} x^{x-1/2}, \quad x > 0 \\ n! &\leq \sqrt{2\pi} e^{-n} n^{n+1/2}, \quad n \in \mathbb{N} \end{aligned}$$

we can prove that there exists a constant $m(\alpha, |\tau|_{-k})$ such that the series (13) converges. ■

With the same calculations of the previous example we have:

Example 2.3 *For any $p \in \mathbb{N}\backslash\{0\}$ we define an element in $\mathcal{G}_{\gamma_p^*}(N)$ with the Young function $\gamma_p^*(x) := x^{2p/\alpha}$ (polar function of $\gamma_p(x) = x^{2p/(2p-\alpha)}$) by*

$$N \ni \xi \longmapsto E_{\alpha,\beta}(\langle \xi, \xi\rangle^p).$$

Moreover its Taylor series is given by

$$E_{\alpha,\beta}(\langle \xi, \xi\rangle^p) = \sum_{m=0}^{\infty} \langle E_{\alpha,\beta}^m, \xi^{\otimes m}\rangle,$$

where

$$\begin{cases} E_{\alpha,\beta}^m & = \frac{\tau^{\otimes m/2}}{\Gamma(\beta+\alpha m/(2p))},\ m \in 2p\mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.4 *The Mittag-Leffler function $E_{\alpha,\beta}$ defined in (9) is an element of $\mathcal{F}_{\theta_r}(\mathbb{C})$ for any Young function $\theta_r(x) = x^r$ where $r > \frac{1}{\alpha}$. For $r = \frac{1}{\alpha}$ we have $E_{\alpha,\beta} \in \mathcal{G}_{\theta_r}(\mathbb{C})$. This can be seen using the same arguments from the previous example.*

2.2 Laplace transform

We write, a for a generic function $f(t)$ the usual Laplace transform as follows

$$Lf(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

For each $\xi \in N$, the exponential function

$$e_\xi(z) = e^{\langle z, \xi\rangle}, \quad z \in N',$$

is a test function in the space $\mathcal{F}_\theta(N')$ for any Young function θ, cf. [10, Lemme 2]. Thus we can define the Laplace transform of a generalized function $\Phi \in \mathcal{F}_\theta'(N')$ by

$$\hat{\Phi}(\xi) := (\mathcal{L}\Phi)(\xi) := \langle\!\langle \Phi, e_\xi \rangle\!\rangle, \quad \xi \in N. \tag{14}$$

The Laplace transform is a topological isomorphism

$$\mathcal{L} : \mathcal{F}_\theta'(N') \longrightarrow \mathcal{G}_{\theta^*}(N), \tag{15}$$

cf. [10, Théorème 1].

Remark 2.5 *1. As in the Example 2.2 we may show that for any $t > 0$*

$$N \ni \xi \longmapsto E_{\alpha,\beta}(\langle \xi, \xi \rangle t^{\alpha}) \in \mathcal{G}_{\gamma^*}(N), \quad \gamma^*(x) = x^{2/\alpha}.$$

Therefore, there exists a unique element $\Psi_{\alpha,\beta,t} \in \mathcal{F}'_{\gamma}(N')$ such that

$$\mathcal{L}\Psi_{\alpha,\beta,t}(\xi) = E_{\alpha,\beta}(\langle \xi, \xi \rangle t^{\alpha}) \tag{16}$$

and $\gamma(x) = x^{2/(2-\alpha)}$.

2. *For $p \in \mathbb{N}\backslash\{0\}$ we can see that there exists an unique element $\Psi_{\alpha,\beta,p,t} \in \mathcal{F}'_{\gamma_p}(N')$ such that*

$$\mathcal{L}\Psi_{\alpha,\beta,p,t}(\xi) = E_{\alpha,\beta}(\langle \xi, \xi \rangle^p t^{\alpha}) \tag{17}$$

and $\gamma_p(x) = x^{2p/(2p-\alpha)}$.

Let T be a fixed positive real number. Consider a family $\{\Phi(t);\ t \in [0,T]\}$ of generalized functions in $\mathcal{F}'_\theta(N')$. We assume that the function $t \mapsto \Phi(t)$ is continuous from $[0,T]$ into $\mathcal{F}'_\theta(N')$. Then the function $t \mapsto \mathcal{L}\Phi(t)$ is continuous from $[0,T]$ into $\mathcal{G}_{\theta^*}(N)$. Thus for each $t \in [0,T]$, the set $\{\mathcal{L}\Phi(s);\ s \in [0,t]\}$ is a compact subset of $\mathcal{G}_{\theta^*}(N)$. In particular, it is bounded in $\mathcal{G}_{\theta^*}(N)$. Hence there exist constants $k \in \mathbb{N}_0$, $m > 0$ and $C(t) > 0$ such that

$$|\mathcal{L}\Phi(s)(\xi)| \le C(t) e^{\theta^*(m|\xi|_k)}, \quad \forall s \in [0,t], \quad \xi \in N_k. \tag{18}$$

This inequality shows that the function $\xi \mapsto \int_0^t \mathcal{L}\Phi(s)(\xi)\, ds$ belongs to the space $\mathcal{G}_{\theta^*}(N)$. Hence there exists a unique generalized function, denoted by $\int_0^t \Phi(s)\, ds$, in $\mathcal{F}'_\theta(N')$ satisfying

$$\mathcal{L}\left(\int_0^t \Phi(s) ds\right)(\xi) = \int_0^t \mathcal{L}\Phi(s)(\xi)\, ds, \quad \xi \in N. \tag{19}$$

Moreover, the generalized function $X(t) := \int_0^t \Phi(s) ds$, $t \in [0,T]$ is differentiable in $\mathcal{F}'_\theta(N')$ and satisfies the equation

$$\frac{\partial}{\partial t} X(t) = \Phi(t). \tag{20}$$

In the following we need to define integrals of $\mathcal{F}'_\theta(N')$-valued generalized functions $(\Phi(t))_{t\ge 0}$ as

$$\int_0^\infty \Phi(s)\, ds.$$

To this end, it is sufficient that, for each $\xi \in N$, the integral

$$\int_0^\infty \mathcal{L}\Phi(s)(\xi)\, ds$$

exists in $\mathcal{G}_{\theta^*}(N)$. In particular, the function $[0,\infty) \ni t \mapsto C(t) \in [0,\infty)$ in (18) is in $L^1([0,\infty), dt)$. In addition, we have

$$\mathcal{L}\left(\int_0^\infty \Phi(s)\, ds\right)(\xi) = \int_0^\infty \mathcal{L}\Phi(s)(\xi)\, ds. \tag{21}$$

As a consequence, the Laplace transform (in t) of $\mathcal{F}'_\theta(N')$-valued generalized functions $(\Phi(t))_{t\geq 0}$ is given by

$$L\Phi(\cdot)(s) = \int_0^\infty e^{-st}\Phi(t)\, dt$$

provided the right hand side exists. In terms of the estimate in (18) this means that the Laplace transform of the map $[0,\infty) \ni t \mapsto C(t) \in [0,\infty)$ exists.

Proposition 2.6 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of generalized functions in $\mathcal{F}'_\theta(N')$. Then the following two conditions are equivalent:*

1. *The sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}'_\theta(N')$ strongly.*
2. *The sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ of Laplace transform of $(\Phi_n)_{n\in\mathbb{N}}$ satisfies the following two conditions:*
   - *(a) There exists $k \in \mathbb{N}$ and $m \in]0,\infty[$ such that the sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{F}_{\theta^*,m}(N_k)$ and is bounded in this Banach space.*
   - *(b) For every point $z \in N$, the sequence of complex numbers $(\hat{\Phi}_n(z))_{n=0}^\infty$ converges.*

2.3 Convolution operators

For $\varphi \in \mathcal{F}_\theta(N')$, the translation $t_x\varphi$ of φ by $x \in N'$ is defined by

$$(t_x\varphi)(y) = \varphi(y-x), \quad y \in N'.$$

It is easy to check that t_x is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself for any $x \in N'$, cf. [9, Proposition 2.1]. By duality we may define the translation

t_x on $\mathcal{F}'_\theta(N')$ as follows: For any $\Phi \in \mathcal{F}'_\theta(N')$ the generalized function $t_x\Phi$ is defined by

$$\langle\langle t_x\Phi, \varphi\rangle\rangle := \langle\langle \Phi, t_{-x}\varphi\rangle\rangle, \quad \mathcal{F}_\theta(N').$$

The convolution $\Phi * \varphi$ of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ and a test function $\varphi \in \mathcal{F}_\theta(N')$ is the function given by

$$(\Phi * \varphi)(x) = \langle\langle \Phi, t_{-x}\varphi\rangle\rangle, \quad x \in N'.$$

Notice that the dual pairing is given in terms of the convolution, namely

$$(\Phi * \varphi)(0) = \langle\langle \Phi, \varphi\rangle\rangle.$$

By a convolution operator on the test space $\mathcal{F}_\theta(N')$ we mean a continuous linear operator C from $\mathcal{F}_\theta(N')$ into itself which commutes with all translation operators t_x, $x \in N'$, see [8, Definition A.1]. Moreover, the mapping C_Φ defined by

$$C_\Phi : \mathcal{F}_\theta(N') \longrightarrow \mathcal{F}_\theta(N'),\ \varphi \mapsto \Phi * \varphi,$$

is a convolution operator, see [8, Proposition A.2]. On the other hand, all convolution operators on $\mathcal{F}_\theta(N')$ occur this way, i.e., if C is a convolution operator on $\mathcal{F}_\theta(N')$, then there exists a unique $\Phi \in \mathcal{F}'_\theta(N')$ such that $C = C_\Phi$, or equivalently,

$$C(\varphi) = C_\Phi(\varphi) = \Phi * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'), \tag{22}$$

see [8, Corollary A.3].

Let $\Phi, \Psi \in \mathcal{F}'_\theta(N')$ be given and C_Φ and C_Ψ be the convolution operators given by Φ and Ψ, respectively, as in equation (22). It is clear that the composition $C_\Phi \circ C_\Psi$ is also a convolution operator on $\mathcal{F}_\theta(N')$. Hence there exists a unique distribution, denoted by $\Phi * \Psi$, in $\mathcal{F}'_\theta(N')$ such that

$$C_\Phi \circ C_\Psi = C_{\Phi * \Psi}. \tag{23}$$

The distribution $\Phi * \Psi$ in equation (23) is called the convolution of Φ and Ψ. In particular for any $p \in \mathbb{N}$ the p-composition of C_Φ is given by

$$C_\Phi \circ \ldots \circ C_\Phi = C_{\Phi^{*p}}. \tag{24}$$

Remark 2.7 *For any $\varphi \in \mathcal{F}_\theta(N')$ and $\Phi, \Psi \in \mathcal{F}'_\theta(N')$ we have*

1. $\langle\langle \Phi * \Psi, \varphi\rangle\rangle = \langle\langle \Phi, \Psi * \varphi\rangle\rangle,$

2. *For any $\xi \in N$ we have*

$$C_\Phi(e_\xi) = \Phi * e_\xi = (\mathcal{L}\Phi)(\xi)e_\xi$$

and this implies the following property

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi. \tag{25}$$

3. *The convolution exponential functional of Φ, denoted by $\exp^* \Phi$, is defined as*

$$\mathcal{L}(\exp^* \Phi) = \exp(\mathcal{L}\Phi)$$

which is an element in $\mathcal{F}'_{(e^{\theta^*})^*}(N')$.

4. *The convolution product of generalized functions is associative due to the associativity of the composition of operators.*

In Section 3 we also need the convolution of two distributions not necessarily on the same space. This is given in the following lemma which is a direct consequence of Lemma 2.1.

Lemma 2.8 *Let θ, γ be two Young functions and c_1, c_2 as in Lemma 2.1. Let $\Phi \in \mathcal{F}'_\theta(N')$ and $\Psi \in \mathcal{F}'_\gamma(N')$ be given. Then we have*

1. *If $0 < c_1 \leq c_2 < \infty$, then $\Phi * \Psi \in \mathcal{F}'_\theta(N') = \mathcal{F}'_\gamma(N')$.*
2. *If $c_2 = 0$, then $\Phi * \Psi \in \mathcal{F}'_\theta(N')$.*
3. *If $c_1 = \infty$, then $\Phi * \Psi \in \mathcal{F}'_\gamma(N')$.*

As an example of convolution operator we give the Gross Laplacian. For $\varphi \in \mathcal{F}_\theta(N')$ of the form

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi^{(n)} \rangle, \tag{26}$$

we define the Gross Laplacian Δ_G of φ at $x \in N'$ by

$$(\Delta_G \varphi)(x) := \sum_{n=0}^{\infty} (n+2)(n+1) \langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle,$$

where the contraction $\langle \tau, \varphi^{(n+2)} \rangle$ is defined by

$$\langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle := \langle x^{\otimes n} \hat{\otimes} \tau, \varphi^{(n+2)} \rangle$$

and τ is the trace operator given by

$$\langle \tau, \xi \otimes \eta \rangle := \langle \xi, \eta \rangle, \quad \xi, \eta \in N. \tag{27}$$

For more information on the Gross Laplacian, see [12], [13]. The Gross Laplacian Δ_G is a convolution operator, namely

$$\Delta_G(\Psi) = C_{\mathcal{T}}(\Psi) = \mathcal{T} * \Psi, \quad \Psi \in \mathcal{F}'_\theta(N'), \tag{28}$$

where $\mathcal{T}$ is the generalized function in $\mathcal{F}'_\theta(N')$ associated with $\vec{\mathcal{T}} = (0, 0, \tau, 0, \cdots) \in G_\theta(N')$.

Since the powers of the Gross Laplacian are convolution operators, then (28) allows us to derive their associated distributions. That is the content of the next lemma.

Lemma 2.9 *For every positive integer $p \in \mathbb{N}$ we have*

$$\Delta_G^p(\Phi) = \mathcal{T}^{*p} * \Phi, \quad \Phi \in \mathcal{F}'_\theta(N'). \tag{29}$$

The generalized function $\mathcal{T}^{*p}$ associated with Δ_G^p is given by

$$\overrightarrow{\mathcal{T}^{*p}} = (0, \ldots, 0, \tau^{\otimes p}, 0, \ldots), \tag{30}$$

where $\tau^{\otimes p}$ is at the $2p$ position.

Proof. Using the definition of the convolution operator (23) and (28) we obtain equation (29). To obtain the associated power series (30), first we notice that

$$\mathcal{L}\mathcal{T}(\xi) = \langle \tau, \xi^{\otimes 2} \rangle = \langle \xi, \xi \rangle.$$

Hence $\mathcal{L}\mathcal{T}^{*p}(\xi) = \langle \xi, \xi \rangle^p$. On the other hand, if $(\mathcal{T}_n^{*p})_{n=0}^\infty$denotes the formal power series of $\mathcal{T}^{*p}$, then

$$\mathcal{L}\mathcal{T}^{*p}(\xi) = \sum_{n=0}^{\infty} \langle \mathcal{T}_n^{*p}, \xi^{\otimes n} \rangle = \langle \xi, \xi \rangle^p$$

which implies the result by identification. ∎

Lemma 2.10 *Let $\Phi, U(t) \in \mathcal{F}'_\theta(N')$ be given such that the Laplace transform (in s) of the function*
$$[0,\infty) \ni t \longrightarrow \mathcal{L}U(t) \in \mathcal{G}_{\theta^*}(N)$$
exists. Then we have
$$[L(\Phi * U(\cdot))](s) = \Phi * [LU(\cdot)](s).$$

Proof. Since the Laplace transform of the function $[0,\infty) \ni t \mapsto \mathcal{L}\Phi\mathcal{L}U(t)$ exists, using (21) we get
$$\begin{aligned}
\mathcal{L}[L(\Phi * U(\cdot))](s)(\xi) &= \int_0^\infty e^{-st}\mathcal{L}(\Phi * U(t))(\xi)\,dt \\
&= \mathcal{L}\Phi(\xi)\int_0^\infty e^{-st}\mathcal{L}U(t)(\xi)\,dt \\
&= \mathcal{L}\Phi(\xi)\mathcal{L}\left(\int_0^\infty e^{-st}U(t)\,dt\right)(\xi) \\
&= \mathcal{L}\Phi * [LU(\cdot)](s)(\xi).
\end{aligned}$$
The result follows by the topological isomorphism (15) induced by $\mathcal{L}$. ■

3 Fractional diffusion equation

In this section we study the generalized fractional diffusion equation associated to Riemann-Liouville and Caputo fractional time derivative. The solution is given in terms of the convolution product between the fundamental solution and the initial data. The Laplace transform of the fundamental solution is expressed in terms of the Mittag-Leffler function which enables us to represent it as a power series with explicit kernels, cf. Remarks 3.5 and 3.10.

3.1 Riemann-Liouville time fractional diffusion equation

In this subsection we are interested in the following Riemann-Liouville time fractional diffusion equation
$$\begin{cases} {}^{RL}D_t^\alpha U(t) = \Delta_G U(t), \quad t > 0 \\ D_t^{\alpha-1}U(t)|_{t=0} = \Phi, \end{cases} \tag{31}$$

where $\Phi \in \mathcal{F}'_\theta(N')$ and ${}^{RL}D_t^\alpha$ is the Riemann-Liouville operator defined by

$$ {}^{RL}D_t^\alpha U(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{U(\tau)}{(t-\tau)^\alpha}\, d\tau, \qquad 0 < \alpha < 1, $$

and $D_t^{\alpha-1} U(t)|_{t=0}$ is given by

$$ D_t^{\alpha-1} U(t)|_{t=0} := \lim_{t \downarrow 0} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{U(\tau)}{(t-\tau)^\alpha}\, d\tau. $$

Lemma 3.1 *For any generalized function $\Psi(t) \in \mathcal{F}'_\theta(N')$ we have*

$$ L({}^{RL}D_\cdot^\alpha \Psi(\cdot))(s) = s^\alpha L(\Psi(\cdot))(s) - D_t^{\alpha-1}\Psi(t)|_{t=0}. \tag{32} $$

Proof. Applying the Laplace transform $\mathcal{L}$ to the left side of (32) gives

$$ \mathcal{L}[L({}^{RL}D_\cdot^\alpha \Psi(\cdot))](\xi) = \langle\langle L({}^{RL}D_\cdot^\alpha \Psi(\cdot)), e_\xi \rangle\rangle = L\langle\langle ({}^{RL}D_\cdot^\alpha \Psi(\cdot)), e_\xi \rangle\rangle. $$

The second equality is obtained using (19). Using (20) and again (19) we obtain

$$ \langle\langle ({}^{RL}D_\cdot^\alpha \Psi(\cdot)), e_\xi \rangle\rangle = {}^{RL}D_\cdot^\alpha \langle\langle \Psi(\cdot), e_\xi \rangle\rangle = {}^{RL}D_\cdot^\alpha \mathcal{L}\Psi(\cdot)(\xi). $$

The Laplace of the function $[0, \infty) \ni t \mapsto {}^{RL}D_t^\alpha \mathcal{L}\Psi(t)(\xi) \in \mathcal{G}_{\theta^*}(N)$ is given by

$$ \int_0^\infty e^{-st}\, {}^{RL}D_t^\alpha \mathcal{L}\Psi(t)(\xi)\, dt = s^\alpha L(\mathcal{L}\Psi(\cdot)(\xi))(s) - D_t^{\alpha-1}\mathcal{L}\Psi(t)(\xi)|_{t=0} $$

which follows from an integration by parts formula, see for example [22]. Finally a similar argument shows that

$$ \mathcal{L}[L({}^{RL}D_\cdot^\alpha \Psi(\cdot))](\xi) = \mathcal{L}[s^\alpha L\Psi(\cdot)(s) - D_t^{\alpha-1}\Psi(t)|_{t=0}](\xi). $$

The result now is a consequence of the fact that the Laplace transform $\mathcal{L}$ is a topological isomorphism between $\mathcal{F}'_\theta(N')$ and $\mathcal{G}_{\theta^*}(N)$. ■

The following theorem gives the existence for the solution of the Riemann-Liouville time fractional diffusion equation.

Theorem 3.2 *Let $\Phi \in \mathcal{F}'_\theta(N')$, $\gamma(x) = x^{2/(2-\alpha)}$ and c_1, c_2 be as in Lemma 2.8. The solution of the fractional diffusion equation (31) is given by*

$$ U(t) = t^{\alpha-1}\Psi_{\alpha,\alpha,t} * \Phi, \tag{33} $$

where $\Psi_{\alpha,t}$ is defined in (16). Moreover the solution $U(t)$ belongs to the space:

1. $\mathcal{F}'_\gamma(N')$ *if* $0 < c_1 \le c_2 < \infty$ *or* $c_1 = \infty$*;*

2. $\mathcal{F}'_\theta(N')$ *if* $c_2 = 0$.

Proof. Having in mind the representation (28) of the Gross Laplacian and applying the Laplace transform (in s) in both sides of (31) we get

$$L({}^{RL}D^\alpha_\cdot U(\cdot))(s) = s^\alpha L(U(\cdot))(s) - \Phi = \mathcal{T} * LU(\cdot)(s).$$

Now applying the Laplace transform (in ξ) yields

$$s^\alpha \mathcal{L}[LU(\cdot)(s)](\xi) - \mathcal{L}\Phi(\xi) = \langle\xi,\xi\rangle \mathcal{L}[LU(\cdot)(s)](\xi)$$

which gives

$$\mathcal{L}[LU(\cdot)(s)](\xi) = \frac{\mathcal{L}\Phi(\xi)}{s^\alpha - \langle\xi,\xi\rangle}.$$

For fixed $\xi \in N$ we notice that, see for example [7]

$$\begin{aligned} L(t^{\alpha-1}E_{\alpha,\alpha}(\langle\xi,\xi\rangle t^\alpha))(s) &= \int_0^\infty e^{-st}t^{\alpha-1}E_{\alpha,\alpha}(\langle\xi,\xi\rangle t^\alpha)\,dt \\ &= \frac{1}{s^\alpha - \langle\xi,\xi\rangle}, \quad \Re(s) > |\langle\xi,\xi\rangle|^{1/\alpha} \end{aligned}$$

and therefore

$$\mathcal{L}U(t)(\xi) = t^{\alpha-1}E_{\alpha,\alpha}(\langle\xi,\xi\rangle t^\alpha)\mathcal{L}\Phi(\xi).$$

Finally, using Remark 2.5 (16), the solution $U(t)$ is given by

$$U(t) = \Phi * (t^{\alpha-1}\Psi_{\alpha,\alpha,t}).$$

To conclude the proof it is sufficient to use Lemma 2.8. ■

Remark 3.3 *We would like to emphasis the stability of the solution given in the theorem above. For any Young function θ such that $\lim_{x\to\infty}\theta(x)/x^{2/(2-\alpha)}$ is constant (resp. infinity) the solution $U(t)$ does not leave the space $\mathcal{F}'_\theta(N')$ (resp. leave the space $\mathcal{F}'_\theta(N')$ to the bigger $\mathcal{F}'_\gamma(N')$).*

Remark 3.4 *1. The Laplace transform (in ξ) of the fundamental solution $U(t)$ given by (notice that δ_0 is the neutral element for the convolution)*

$$\mathcal{L}U(t)(\xi) = t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\langle\xi,\xi\rangle). \tag{34}$$

In a neighborhood of zero the function $[0,\infty) \ni t \mapsto E_{\alpha,\alpha}(t^\alpha\langle\xi,\xi\rangle)$ is equivalent to $\frac{1}{\Gamma(\alpha)}$. Therefore $\mathcal{L}(U(t))(\xi)$ diverges as $t \to 0$. The finite dimensional analogous case was studied by many authors, see for example [6], [14], [16], [21] and references therein.

2. *Setting in (34) $\xi = 0$, then $\mathcal{L}U(t)(0) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and therefore $\mathcal{L}U(t)$ cannot be the Laplace transform of a probability density because its normalization would depend on t. In particular, equation (31) does not admit a probabilistic representation.*

Remark 3.5 *The fundamental solution of (31) is given by $U(t) = t^{\alpha-1}\Psi_{\alpha,\alpha,t}$ and its formal power series $(U_n(t))_{n=0}^{\infty}$ associated to the fundamental solution is*

$$U_n(t) = \begin{cases} U_{2n}(t) = \frac{t^{\alpha(n+1)-1}\tau^{\otimes n}}{\Gamma(\alpha(n+1))} \\ U_{2n+1}(t) = 0. \end{cases}$$

This is a direct consequence of Lemma 2.2.

Now we show that the solution (33) is continuous with respect to the initial data Φ.

Proposition 3.6 *Let $(\Phi_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}'_\theta(N')$. Denote by $U_j(t)$, $t \in [0,T]$ the corresponding solutions of (31) with initial data Φ_j. If $(\Phi_j)_{j\in\mathbb{N}}$ converges in $\mathcal{F}'_\theta(N')$ strongly to $\Phi \in \mathcal{F}'_\theta(N')$, then for each $t \in [0,T]$, $U_j(t)$ converges strongly to $U(t)$ in $\mathcal{F}'_\theta(N')$, where $\{U(t),\ t \in [0,T]\}$ is the solution of (31) with initial condition Φ . Moreover for each $\varepsilon > 0$ and $\xi \in N$ we have*

$$\sup_{t\in[\varepsilon,T]} |(\mathcal{L}U_j(t) - \mathcal{L}U(t))(\xi)| \longrightarrow 0,\ j \to \infty. \tag{35}$$

Proof. First we notice that $U_j(t)$ is given by

$$U_j(t) = t^{\alpha-1}\Psi_{\alpha,\alpha,t} * \Phi_j$$

and its Laplace transform is

$$\mathcal{L}U_j(t)(\xi) = t^{\alpha-1}E_{\alpha,\alpha}(\langle\xi,\xi\rangle t^\alpha)\mathcal{L}\Phi_j(\xi).$$

Using Proposition 2.6 and the hypothesis we derive that for each $t \in (0,T]$, $U_j(t)$ converges strongly to $U(t)$ in $\mathcal{F}'_\theta(N')$. Finally it is clear that

$$\sup_{t\in[\varepsilon,T]} t^{\alpha-1}|E_{\alpha,\alpha}(\langle\xi,\xi\rangle t^\alpha)| \le \varepsilon^{\alpha-1}E_{\alpha,\alpha}(|\langle\xi,\xi\rangle|T^\alpha) < \infty$$

and (41) follows. ■

For any $p \in \mathbb{N}\backslash\{0\}$ we consider the Riemann-Liouville time fractional evolution equation

$$\begin{cases} {}^{RL}D_t^\alpha U(t) = \Delta_G^p U(t), \quad t > 0 \\ D_t^{\alpha-1} U(t)|_{t=0} = \Phi. \end{cases} \tag{36}$$

where $\Phi \in \mathcal{F}'_\theta(N')$. For $p = 1$ (36) is exactly the same as the equation (31).

Proposition 3.7 *Let $\Phi \in \mathcal{F}'_\theta(N')$, $\gamma_p(x) = x^{2p/(2p-\alpha)}$ and c_1, c_2 as in Lemma 2.8.*

1. *The solution of the time fractional diffusion equation (36) is given by*

$$U(t) = t^{\alpha-1}\Psi_{\alpha,\alpha,p,t} * \Phi,$$

   where $\Psi_{\alpha,\alpha,p,t}$ is defined in (17).

2. *Moreover the solution $U(t)$ belongs to the space:*

   (a) $\mathcal{F}'_{\gamma_p}(N')$ if $0 < c_1 \le c_2 < \infty$ or $c_1 = \infty$;

   (b) $\mathcal{F}'_\theta(N')$ if $c_2 = 0$.

3. *The fundamental solution of (36) is given by $U(t) = t^{\alpha-1}\Psi_{\alpha,\alpha,p,t}$. Moreover, its formal power series $(U_n(t))_{n=0}^\infty$ associated to the fundamental solution is*

$$\begin{cases} U_m(t) & = \frac{t^{\alpha(1+m/(2p))-1}\tau^{\otimes m/2}}{\Gamma(\alpha(1+m/(2p))}, \ m \in 2p\mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

3.2 Caputo time fractional diffusion equation

In this subsection we are interested in the following Caputo time fractional diffusion equation

$$\begin{cases} {}^{C}D_t^\alpha U(t) = \Delta_G U(t), \quad t > 0 \\ U(0) = \Phi, \end{cases} \tag{37}$$

where $\Phi \in \mathcal{F}'_\theta(N')$ and ${}^{C}D_t^\alpha$ is the Caputo time fractional derivative defined by

$${}^{C}D_t^\alpha U(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{U(\tau) - U(0)}{(t-\tau)^\alpha}\, d\tau, \qquad 0 < \alpha < 1.$$

The Caputo derivative is a sort of regularization in the time origin of the Riemann-Liouville derivative by incorporating the relevant initial conditions, see for example [11] for detailed considerations and its major applications.

The following lemma is the analogous of Lemma 3.1 for the Caputo derivative.

Lemma 3.8 *For any generalized function $\Psi(t) \in \mathcal{F}'_\theta(N')$ we have*

$$L({}^C D^\alpha_\cdot \Psi(\cdot))(s) = s^\alpha L(\Psi(\cdot))(s) - s^{\alpha-1}\Phi. \tag{38}$$

The existence result for the solution of the Caputo time fractional diffusion equation (37) is given as follows.

Theorem 3.9 *Let $\Phi \in \mathcal{F}'_\theta(N')$, $\gamma(x) = x^{2/(2-\alpha)}$ and c_1, c_2 be as in Lemma 2.8. The solution of the fractional diffusion equation (37) is given by*

$$U(t) = \Psi_{\alpha,1,t} * \Phi, \tag{39}$$

where $\Psi_{\alpha,1,t}$ is defined in (16). Moreover the solution $U(t)$ belongs to the space:

1. *$\mathcal{F}'_\gamma(N')$ if $0 < c_1 \leq c_2 < \infty$ or $c_1 = \infty$;*
2. *$\mathcal{F}'_\theta(N')$ if $c_2 = 0$.*

Proof. The proof follow the same idea as in Theorem 3.2. Applying the Laplace in s and in ξ we obtain

$$s^\alpha \mathcal{L}[LU(\cdot)(s)](\xi) - s^{\alpha-1}\mathcal{L}\Phi(\xi) = \langle \xi, \xi \rangle \mathcal{L}[LU(\cdot)(s)](\xi)$$

which gives

$$\mathcal{L}[LU(\cdot)(s)](\xi) = \frac{s^{\alpha-1}\mathcal{L}\Phi(\xi)}{s^\alpha - \langle \xi, \xi \rangle}.$$

For fixed $\xi \in N$ we notice that, see for example [7]

$$L(E_{\alpha,1}(\langle \xi, \xi \rangle t^\alpha))(s) = \frac{s^{\alpha-1}}{s^\alpha - \langle \xi, \xi \rangle}, \quad \Re(s) > |\langle \xi, \xi \rangle|^{1/\alpha}$$

and therefore

$$\mathcal{L}U(t)(\xi) = E_{\alpha,1}(\langle \xi, \xi \rangle t^\alpha)\mathcal{L}\Phi(\xi).$$

Finally, using Remark 2.5 (16), the solution $U(t)$ is given by

$$U(t) = \Psi_{\alpha,1,t} * \Phi.$$

To conclude the proof it is sufficient to use Lemma 2.8. ■

Remark 3.10 *If the initial condition Φ is equal to the Dirac distribution δ_0, then the fundamental solution of (37) is given by $U(t) = \Psi_{\alpha,1,t}$. Moreover, its formal power series $(U_n(t))_{n=0}^{\infty}$ associated to the fundamental solution is*

$$U_n(t) = \begin{cases} U_{2n}(t) = \frac{t^{\alpha n}\tau^{\otimes n}}{\Gamma(\alpha n+1)} \\ U_{2n+1}(t) = 0. \end{cases}$$

Remark 3.11 *The fundamental solution $\Psi_{\alpha,1,t}$ admits a probabilistic representation. More precisely, we first notice that*

$$\begin{aligned} L(E_{\alpha,1}(\langle\xi,\xi\rangle t^{\alpha}))(s) &= \frac{s^{\alpha-1}}{s^{\alpha} - \langle\xi,\xi\rangle} \\ &= s^{\alpha-1}\int_0^{\infty} e^{-(s^{\alpha}-|\xi|^2)r}\,dr \\ &= \int_0^{\infty} s^{\alpha-1}e^{-s^{\alpha}r}e^{|\xi|^2 r}\,dr. \end{aligned} \tag{40}$$

Let $g_{\alpha}(t)$ be de density of a α-stable random variable, cf. [5], i.e.,

$$(Lg_{\alpha})(s) := e^{-s^{\alpha}}, \quad s \in [0,\infty).$$

It is easy to see that

$$e^{-s^{\alpha}r} = \int_0^{\infty} e^{-st} g_{\alpha}(r^{-1/\alpha}t)r^{-1/\alpha}\,dt.$$

Moreover

$$\begin{aligned} s^{\alpha-1}e^{-s^{\alpha}r} &= -\frac{1}{\alpha r}\frac{d}{ds}\left(e^{-s^{\alpha}r}\right) \\ &= -\frac{1}{\alpha r}\frac{d}{ds}\left(\int_0^{\infty} e^{-st}g_{\alpha}(r^{-1/\alpha}t)r^{-1/\alpha}\,dt\right) \\ &= \frac{1}{\alpha r^{1+1/\alpha}}\int_0^{\infty} te^{-st}g_{\alpha}(r^{-1/\alpha}t)\,dt. \end{aligned}$$

Hence (40) may be written as (using Fubini's theorem)

$$\begin{aligned} L(E_{\alpha,1}(\langle\xi,\xi\rangle t^{\alpha}))(s) &= \int_0^{\infty}\left(\frac{1}{\alpha r^{1+1/\alpha}}\int_0^{\infty} te^{-st}g_{\alpha}(r^{-1/\alpha}t)\,dt\right)e^{|\xi|^2 r}\,dr \\ &= \frac{1}{\alpha}\int_0^{\infty} e^{-st}\int_0^{\infty}\frac{t}{r^{1+1/\alpha}}g_{\alpha}(r^{-1/\alpha}t)e^{|\xi|^2 r}\,dr\,dt \\ & \frac{1}{\alpha}L\left(\int_0^{\infty}\frac{t}{r^{1+1/\alpha}}g_{\alpha}(r^{-1/\alpha}t)e^{|\xi|^2 r}\,dr\right)(s). \end{aligned}$$

Therefore we have

$$\begin{aligned}
E_{\alpha,1}(\langle\xi,\xi\rangle t^{\alpha}) &= \frac{1}{\alpha}\int_0^{\infty}\frac{t}{r^{1+1/\alpha}}g_{\alpha}(r^{-1/\alpha}t)e^{|\xi|^2 r}\,dr \\
&= \int_0^{\infty} g_{\alpha}(u)e^{|\xi|^2(u/t)^{\alpha}}\,du. \\
&= \int_0^{\infty} g_{\alpha}(u)(\mathcal{L}e^{*(u/t)^{\alpha}\mathcal{T}})(\xi)\,du \\
&= \mathcal{L}\left(\int_0^{\infty} g_{\alpha}(u)e^{*(u/t)^{\alpha}\mathcal{T}}\,du\right)(\xi).
\end{aligned}$$

Finally, the fundamental solution is given as

$$\Psi_{\alpha,1,t} = \int_0^{\infty} g_{\alpha}(u)e^{*(u/t)^{\alpha}\mathcal{T}}\,du$$

which is related to the inverse stable subordinators. The finite dimensional case is well studied, see for example [1], [19], and references therein. A rigorous construction of this probabilistic representation in our framework will be the subject of a forecoming paper.

Finally we show the continuity of the solution with respect to the initial data.

Proposition 3.12 *Let $(\Phi_j)_{j\in\mathbb{N}}$, Φ be given in $\mathcal{F}'_{\theta}(N')$. Denote by $U_j(t)$ (resp. $U(t)$) ,$t\in[0,T]$ the corresponding solutions of (37) with initial data Φ_j (resp. Φ). If $(\Phi_j)_{j\in\mathbb{N}}$ converges in $\mathcal{F}'_{\theta}(N')$ strongly to $\Phi\in\mathcal{F}'_{\theta}(N')$, then for each $t\in[0,T]$, $U_j(t)$ converges strongly to $U(t)$ in $\mathcal{F}'_{\theta}(N')$. Moreover for each $\xi\in N$ we have*

$$\sup_{t\in[0,T]}|(\mathcal{L}U_j(t)-\mathcal{L}U(t))(\xi)| \longrightarrow 0,\ j\to\infty. \tag{41}$$

Proof. First we notice that $U_j(t)$ is given by

$$U_j(t) = \Psi_{\alpha,1,t} * \Phi_j$$

and its Laplace transform is

$$\mathcal{L}U_j(t)(\xi) = E_{\alpha,1}(\langle\xi,\xi\rangle t^{\alpha})\mathcal{L}\Phi_j(\xi).$$

Using Proposition 2.6 and the hypothesis we derive that for each $t\in[0,T]$, $U_j(t)$ converges strongly to $U(t)$ in $\mathcal{F}'_{\theta}(N')$. Finally it is clear that

$$\sup_{t\in[\varepsilon,T]}|E_{\alpha,1}(\langle\xi,\xi\rangle t^{\alpha})| \leq E_{\alpha,1}(|\langle\xi,\xi\rangle|T^{\alpha}) < \infty$$

and (41) follows. ■

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