The Renormalization of Self Intersection Local Times of Brownian Motion

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1 Introduction and statement of the results

Self intersection local time of Brownian motion have been the subject of numerous studies of half a century. This was, in part, motivated by the important role played by self intersection local time in the construction of certain Euclidean quantum fields, see for instance [8][22][23][27]. The expression of self intersection local time appears also in the study of polymers, see Edwards [10]. Using a chaos expansion in terms of Malliavin's calculus or, even more appropriately white noise calculus, many results were proven see for instance [4][11][14]. More precisely it was shown in [24] that the self-intersection local time of d-dimensional Brownian motion, renormalized by subtracting some terms in its chaos expansion, was a Hida distribution, we also point out [13][19].

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An informal definition of the self intersection local time of the Brownian motion is

$$L = \int d^2t\, \delta(B(t_2) - B(t_1))\,,$$

where B is a d-dimensional Brownian motion. This could be understood as the amount of time, spent by the Brownian path intersecting itself during the time t. To make sense of this integral one can invoke an approximation

$$L^\epsilon(t) = \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta_\epsilon(B(t_2) - B(t_1))\,,$$

with, for x in $\mathbb{R}^d$

$$\delta_\epsilon(x) = \frac{1}{(2\pi\epsilon)^{d/2}} e^{\frac{-x^2}{2\epsilon}}\,.$$

The central problem is then the control of what happens when $\epsilon \longrightarrow 0$.
As is known, for $d \geq 2$, we need to center L_ϵ by

$$L^\epsilon - \mathbb{E}(L^\epsilon)$$

see [17][20][23].
A Tanaka formula method was initiated in [21], and used in [28] for d=3 with a regularization of type "$\delta_y(x) = \delta(x+y)$" to show that

$$r_3(|y|)(L_y - \mathbb{E}(L_y)) \xrightarrow[y\to 0]{\mathcal{L}} c\beta$$

where β is a one dimensional Brownian motion, independent of the initial one and $r_3(|y|) = |\log|y||^{-1/2}$.
Using an explicit computation of the chaos of L^ϵ_T given in [11], it has been shown in [4] and for $d > 2$, that each chaos of the renormalized self-intersection local time, converges in law to a one dimensional Brownian motion, independent from the initial one.

Our aim in this work is to prove a similar result for the whole renormalized self-intersection local time. Again, using the explicit computation of the chaos of L^ϵ_T, we decompose the self-intersection local time into a martingale part M^ϵ and a negligible one N^ϵ, following in this what has been done in [4].

We prove that for $d > 2$ the renormalized martingale part $r_d(\epsilon)M^\epsilon$ converges in law to a one dimensional Brownian motion, independent of the initial one and that $r_d(\epsilon)N^\epsilon$ converges to zero in mean square.

This entails that the renormalized self-intersection local time $r_d(\epsilon)(L^\epsilon - \mathbb{E}(L^\epsilon))$ converges in finite dimensional distribution to a Brownian motion independent of the initial one.

Before announcing our results, let us give some notations and recall some results.

In [11], the kernels of the chaos expansion of the renormalized self-intersection local time were explicitly computed.
Let $\epsilon > 0$, $\vec{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d - \{0\},\ n = \sum_i n_i$.
The kernels of $L_\epsilon - \mathbb{E}(L_\epsilon)$ in Fock space are, for any $d > 2$:

$$F_{\epsilon,\vec{n}}(s_1, ..., s_n) = (-1)^{n/2} \left(\chi(\chi+1)(2\pi)^{d/2} 2^{n/2} \frac{\vec{n}}{2}! \right)^{-1}$$
$$\theta(u)\theta(t-v) \left((v-u+\epsilon)^{-\chi} + (t+\epsilon)^{-\chi} - (v+\epsilon)^{-\chi} - (t-u+\epsilon)^{-\chi} \right)$$

if all n_i are even (and are zero otherwise) with $v = \max_i s_i,\ u = \min_i s_i$ and $\chi = \frac{n+d}{2} - 2$; θ is the Heaviside function.
Let us set up some notations, for $\vec{n} \in \mathbb{N}^d - \{0\}$ and $\epsilon > 0$

$$M_{\vec{n}}^\epsilon = \int_{[0,t]^n} d^n s (v-u+\epsilon)^{-\chi} : w^{\otimes \vec{n}}(s) : \tag{1}$$

$$N_{\vec{n}}^\epsilon = \int_{[0,t]^n} d^n s \left((t+\epsilon)^{-\chi} - (v+\epsilon)^{-\chi} - (t-u+\epsilon)^{-\chi} \right) : w^{\otimes \vec{n}}(s) : . \tag{2}$$

Note that for each $\vec{n}$ and ϵ the above processes are continuous in time.
For the $\vec{n}^{\text{th}}$ order chaos of the renormalized self-intersection local time we set

$$K_{\vec{n}}^\epsilon = \alpha_{\vec{n}} \left(M_{\vec{n}}^\epsilon + N_{\vec{n}}^\epsilon \right) .$$

It was shown in [4], that for $d \geq 3$ and $\vec{n} \in \mathbb{N}^d - \{0\}$,

$$r_d(\epsilon) K_{\vec{n}}^\epsilon \xrightarrow[\epsilon \to 0^+]{\mathcal{L}} c_{\vec{n}} \beta_{\vec{n}} \tag{3}$$

where for $\vec{n} \neq 0$, $\beta_{\vec{n}}$ is a one dimensional Brownian motion independent of the initial one and among each other, with

$$c_{\vec{n}}^2 = k_n^2 \alpha_{\vec{n}}^2 ,$$

$$k_n^2 = \begin{cases} n(n-1) & d=3 \\ \frac{n!(d-4)!}{(n+d-5)!} & d>3\,, \end{cases}$$

$$\alpha_{\vec{n}} = (-1)^{n/2}\left(\chi(\chi+1)(2\pi)^{d/2}2^{n/2}\frac{\vec{n}}{2}!\right)^{-1}\,,$$

$$r_d(\epsilon) = \begin{cases} |\log\epsilon|^{-1/2} & d=3 \\ \epsilon^{\frac{d-3}{2}} & d>3\,. \end{cases}$$

Let us denote the dominant (martingale) part of the renormalized self-intersection local time by

$$M^\epsilon = \sum_{\vec{n}\neq 0} \alpha_{\vec{n}} M^\epsilon_{\vec{n}}.$$

Our main result is as follows

Theorem 1 *For $d \geq 3$,*

$$r_d(\epsilon) M^\epsilon \xrightarrow[\epsilon\to 0^+]{\mathcal{L}} c\beta\,, \tag{4}$$

where β is a one dimensional standard Brownian motion independent of the initial one and $c = \sqrt{\sum_{\vec{n}\neq 0} c^2_{\vec{n}}}$.

Remark
It was shown in [4], by explicit computation, that for all $\vec{n} \neq 0$ and uniformly in finite t-intervals that

$$r_d(\epsilon) N^\epsilon_{t,\vec{n}} \xrightarrow{(L^2)} 0.$$

Considering the summed up process

$$N^\epsilon_t = \sum_{\vec{n},\vec{n}\neq 0} \alpha_{\vec{n}} N^\epsilon_{t,\vec{n}},$$

we prove

Proposition 1 *For every $t \geq 0$ and $d > 2$*

$$r_d(\epsilon) N^\epsilon_t \xrightarrow{(L^2)} 0,$$

and the convergence is uniform in any finite t-interval.

We set

$$\begin{aligned} K^\epsilon &= L^\epsilon - \mathbb{E}(L^\epsilon) \\ &= M^\epsilon + N^\epsilon. \end{aligned}$$

Then using the theorem 1 and the proposition 1 we obtain

Theorem 2 *For $d \geq 3$,*

$$r_d(\epsilon) K^\epsilon \xrightarrow{\mathcal{L}_f} c\beta,$$

where $\mathcal{L}_f$ means convergence of finite dimensional distributions.

2 Tools from white noise analysis

We quote some white noise analysis concepts as introduced in [4], referring to [12] for a systematic presentation.

Consider a white noise space $(S'(\mathbb{R})^d, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the weak Borel σ-algebra of $S'(\mathbb{R})^d$, and μ is the centered Gaussian measure with covariance given by the inner product of $L^2(\mathbb{R})^d$.
Then a realization of a vector of independent Brownian motions B_i, $i = 1, \cdots, d$, is obtained by

$$B_i(t) = \langle \omega_i, \mathbf{1}_{[0,t]} \rangle = \int_0^t \omega_i(s) ds.$$

Hence we consider independent d-tuples of Gaussian white noise $\omega = (w_1, \cdots, \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, \cdots, f_d) \in S(\mathbb{R}, \mathbb{R}^d)$, and use the following multi-index notation:

$$\vec{n}! = \prod_1^d n_i! \quad n = \sum_{i=1}^d n_i$$

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt f_i^2(t)$$

$$\langle F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} \rangle = \int d^n t F_{\vec{n}}(t_1, \cdots, t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \cdots, t_n)$$

and similarly for $\langle : \omega^{\otimes\vec{n}} :, F_{\vec{n}} \rangle$ where for d-tuples of white noise the usual Wick product $: \cdots :$ (see [12]) generalises to

$$: \omega^{\otimes\vec{n}} := \bigotimes_{i=1}^{d} : \omega_i^{\otimes n_i} : .$$

The vector valued white noise has the characteristic function

$$C(\mathbf{f}) = I\!E(e^{i\langle\omega,\mathbf{f}\rangle}) = \int_{S^*(I\!R,I\!R^d)} d\mu[\omega] e^{i\langle\omega,\mathbf{f}\rangle} = e^{-\frac{1}{2}\langle\mathbf{f},\mathbf{f}\rangle},$$

where $\langle\omega,\mathbf{f}\rangle = \sum_{i=1}^{d}\langle\omega_i, f_i\rangle$ and $f_i \in S(I\!R, I\!R)$.

The Hilbert space

$$(L^2) = L^2(d\mu)$$

is canonically isomorphic to the d-fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq \Big(\bigoplus_{k=0}^{\infty} SymL^2(I\!R^k, k!d^k t)\Big)^{\otimes d} := \mathcal{F}$$

for a general element of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} \langle : \omega^{\otimes\vec{n}} :, F_{\vec{n}}\rangle,$$

the norm of φ is given by

$$\|\varphi\|_{(L^2)}^2 = \sum_{\vec{n}}^{\infty} \vec{n}! |F_{\vec{n}}|_{2,n}^2$$

with kernel functions F in $\mathcal{F}$.

3 Proof of the results

Our strategy to prove the theorem 1 is a classical one. In fact we first prove the tightness of the family of processes $\{r_d(\epsilon)M^\epsilon\}_{\epsilon>0}$, and then we prove the appropriate convergence of the associated sequence of finite dimensional distributions.

3.1 Tightness of $\{r_d(\epsilon)M^\epsilon\}_{\epsilon>0}$

We recall that to prove the tightness of $\{M^\epsilon_{\vec{n}};\ \epsilon>0\}$ for each $\vec{n}\neq 0$, the authors in [4] have used the hypercontractivity of the Ornstein-Uhlenbeck semigroup.
They obtained for $0\le t,s\le T$ and $\alpha>2$ the following inequality

$$\mathbb{E}\left[r_d(\epsilon)|M^\epsilon_{\vec{n}}(t)-M^\epsilon_{\vec{n}}(s)|\right]^\alpha \le C_{T,\vec{n}}|t-s|^{\alpha/2}$$

which essentially implies tightness [15].
Such hypercontractivity argument cannot readily be extended to the process $\{r_d(\epsilon)M^\epsilon_T;\ \epsilon>0\}$. To circumvent this difficulty, we prove a technical lemma.

Lemma 1 *Let $(X^n)_{n\in\mathbb{N}}$, $(X^n_k)_{n,k\in\mathbb{N}}$ be two families of processes with continuous paths and starting from the origin. Suppose that :*
i) For each $k\in\mathbb{N}$, $(X^n_k)_{n\in\mathbb{N}}$ is tight.
ii) For each $T>0$ and $\epsilon>0$,

$$sup_{n\in\mathbb{N}}\mathbb{P}\left(\max_{0\le t\le T}|X^n_k(t)-X^n(t)|>\epsilon\right)\xrightarrow[k\to+\infty]{}0, \tag{5}$$

then the family $(X^n)_{n\in\mathbb{N}}$ is tight.

Proof of lemma 1
Following [2][15], we have to prove for each $\epsilon>0$ and $T>0$, that
$\sup_{n\in\mathbb{N}}\mathbb{P}\left(\max_{|t-s|\le\delta,\,0\le t,s\le T}|X^n(t)-X^n(s)|>\epsilon\right)\xrightarrow[\delta\downarrow 0]{}0.$
Let $\epsilon>0$, $T>0$ and $n\in\mathbb{N}$

$$\begin{aligned}|X^n(t)-X^n(s)|\le\ & |X^n(t)-X^n_k(t)|+|X^n_k(t)-X^n_k(s)|+\\ & |X^n_k(s)-X^n(s)|\end{aligned}$$

so

$$\begin{aligned}\mathbb{P}\left(\max_{|t-s|\le\delta,\,0\le t,s\le T}|X^n(t)-X^n(s)|>\epsilon\right) &\le \mathbb{P}\left(2I_{1,k}+I_{2,k}>\epsilon\right)\\ &\le \mathbb{P}\left(I_{1,k}>\frac{\epsilon}{4}\right)+\mathbb{P}\left(I_{2,k}>\frac{\epsilon}{2}\right)\end{aligned}$$

where

$$I_{1,k}=\max_{0\le t\le T}|X^n_k(t)-X^n(t)|$$

$$I_{2,k} = \max_{|t-s|\leq\delta,\, 0\leq t,s\leq T} |X_k^n(t) - X_k^n(s)|$$

Let $\alpha > 0$. By the second hypothesis, there exists $k_0 \in I\!N$ such that

$$\sup_{n\in I\!N} I\!P\left(I_{1,k_0} > \frac{\epsilon}{4}\right) \leq \frac{\alpha}{2}$$

Choose $k = k_0$; because of the tightness of $\left(X_{k_0}^n\right)_{n\in I\!N}$ there exists $\delta_0 > 0$ such that for every $\delta > \delta_0$

$$\sup_{n\in I\!N} I\!P\left(I_{2,k_0} > \frac{\epsilon}{2}\right) < \frac{\alpha}{2},$$

so the lemma is proved.

Let us now prove the tightness of $(r_d(\epsilon)M^\epsilon)_{\epsilon>0}$.
For each $k \in I\!N^*$, denote by

$$M_k^\epsilon = \sum_{\vec{n}\neq 0, n\leq k} \alpha_{\vec{n}} M_{\vec{n}}^\epsilon$$

so that we have, for $t > 0$

$$\|M^\epsilon(t) - M_k^\epsilon(t)\|_2^2 = \sum_{\vec{n}\neq 0, n>k} \alpha_{\vec{n}}^2 \|M_{\vec{n}}^\epsilon(t)\|_2^2.$$

It was shown in [4] (proof of lemma 3.2), that

$$\|M_{\vec{n}}^\epsilon(t)\|_2^2 = \vec{n}!n(n-1)\epsilon^{3-d} \int_0^t dv \int_0^{v/\epsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \cdot \tag{6}$$

For d=3

$$r_3(\epsilon)^2 \|M_{\vec{n}}^\epsilon(t)\|_2^2 \leq \vec{n}!n(n-1)\left\{\frac{t|\log\epsilon| + (t+\epsilon)\log(t+\epsilon) - t}{|\log\epsilon|}\right\}$$

the function of t and ϵ in the right hand side of the last inequality is uniformly bounded on bounded sets of $(t,\epsilon) \in I\!R_+^2$.
If $d > 3$

$$\int_0^{v/\epsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \leq \int_0^{+\infty} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \leq \int_0^{+\infty} dx \frac{1}{(x+1)^{d-2}} = \text{const.}$$

so in view of (6), if $d > 3$, we obtain

$$r_d(\epsilon)^2 \|M^\epsilon_{\vec{n}}(t)\|_2^2 \leq \text{const}\vec{n}!n(n-1)t.$$

It was proved in [5] that

$$\sum_{\vec{n}\neq 0} \alpha_n^2 \vec{n}!n(n-1) < +\infty \, .$$

Then we have proved that for every $d \geq 3$,

$$r_d^2(\epsilon)\|M^\epsilon(t) - M^\epsilon_k(t)\|_2^2 \xrightarrow[k\to+\infty]{} 0 \tag{7}$$

uniformly in bounded sets of $(t,\epsilon) \in I\!\!R^2_+$.
On the other hand, using the same technique as in [4] it has been proved that $\{r_d(\epsilon)M^\epsilon_{\vec{n}};\ n \leq k\}$ is also tight when $\epsilon \to 0$.
Hence, also for each $k \geq 1$ $r_d(\epsilon)M^\epsilon_k$ is tight when $\epsilon \to 0$.
Let $\eta > 0$ and $T > 0$ fixed,

$$I\!\!P\left(\max_{0\leq t\leq T}\left(r_d(\epsilon)|M^\epsilon(t) - M^\epsilon_k(t)|\right) > \eta\right) \leq \frac{I\!\!E\left(r_d(\epsilon)|M^\epsilon(T) - M^\epsilon_k(T)|\right)}{\eta}$$

because $M^\epsilon(.) - M^\epsilon_k(.)$ is a martingale, see e.g. [15].
Using (7) we find (5) and we prove that $\{r_d(\epsilon)M^\epsilon;\ \epsilon > 0\}$ is tight.

3.2 Convergence of the finite dimensional distributions

We intend to prove that, for every $0 \leq t_1 < ... < t_m$, $m \in I\!\!N - \{0\}$,

$$r_d(\epsilon)\left(M^\epsilon(t_1);...;M^\epsilon(t_m)\right) \xrightarrow[\epsilon\to 0]{\mathcal{L}} c\left(\beta(t_1);...;\beta(t_m)\right),$$

where β is a standard one dimensional Brownian motion, and $c^2 := \lim_{k\to\infty} c_k^2 = \sum_{\vec{n}\neq 0} c^2_{\vec{n}}$.
Using the same technique as in [4] one finds that for every $k \in I\!\!N - \{0\}$

$$r_d(\epsilon)M^\epsilon_k \xrightarrow[\epsilon\to 0]{\mathcal{L}} c_k\beta \tag{8}$$

where $c_k^2 = \sum_{\vec{n}\neq 0; n\leq k} c^2_{\vec{n}}$.
Then for every $k, m \in I\!\!N^*$ and $0 \leq t_1 < ... < t_m$

$$r_d(\epsilon)\left(M^\epsilon_k(t_1);...;M^\epsilon_k(t_m)\right) \xrightarrow[\epsilon\to 0]{\mathcal{L}} c_k\left(\beta(t_1);...;\beta(t_m)\right),$$

Now we vectorize

$$\vec{M}^\epsilon_k = r_d(\epsilon)\left(M^\epsilon_k(t_1); ...; M^\epsilon_k(t_m)\right),$$

$$\vec{M}^\epsilon = r_d(\epsilon)\left(M^\epsilon_{t_1}; ...; M^\epsilon_{t_m}\right),$$

$$\vec{\beta} = \left(\beta(t_1); ...; \beta(t_m)\right).$$

Let X, Y be two random vectors in $I\!R^m$ (not necessarily defined on the same probabilty space), we define the following distance

$$d(X,Y) = \sup_{\|f\|_1 \leq 1} |I\!E(f(X)) - I\!E(f(Y))|$$

with $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ for f in $C^1_b(I\!R^m)$. With this distance the topology of the convergence in law of random vectors on Euclidean space becomes metrisable, see [1]. Then we have

$$d\left(\vec{M}^\epsilon, c\vec{\beta}\right) \leq d\left(\vec{M}^\epsilon, \vec{M}^\epsilon_k\right) + d\left(\vec{M}^\epsilon_k, c_k\vec{\beta}\right) + d\left(c_k\vec{\beta}, c\vec{\beta}\right).$$

In view of (7) the first term of the rhs can be made arbitrarily small for k large enough, uniformly in $\epsilon \geq 0$. The second distance is small when $\epsilon \to 0$ because of (8), and finally the third becomes small when k is large by the definition of c.

Note that this argument does not require us to distinguish the β arising in (8) for different k since they all have the same law.

Proof of proposition 1

In view of (2) we have

$$\|N^\epsilon_{t,\vec{n}}\|^2_{(L^2)} \leq 3\vec{n}!\left(\|(t+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} + \|(v+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} + \|(t-u+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)}\right),$$

with elementary computations we find

$$r_d(\epsilon)^2\|(v+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} \leq \begin{cases} \frac{t}{|\log\epsilon|}\,n & \text{for } d=3 \\ \epsilon\log\frac{t+\epsilon}{\epsilon}\,n & \text{for } d=4 \\ \frac{\epsilon}{d-4}\,n & \text{for } d>4 \end{cases}$$

also

$$\|(t+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} \leq \|(t-u+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} = \|(v+\epsilon)^{-\chi}\|^2_{L^2([0,t]^n)} = o(1)n$$

so we have the same estimation for all three terms. Thus for

$$\|r_d(\epsilon)N_t^\epsilon\|^2_{(L^2)} = r_d(\epsilon)^2 \sum_{\vec{n}\neq 0} \alpha^2_{\vec{n}} \|N^\epsilon_{t,\vec{n}}\|^2_{(L^2)},$$

in view of the last estimation and for t in a finite interval, we have for every $d \geq 3$

$$\|r_d(\epsilon)N_t^\epsilon\|^2_{(L^2)} \leq o(1) \sum_{\vec{n}\neq 0} \alpha^2_{\vec{n}} \vec{n}! n,$$

the last series converges and so the proposition is proved.

Proof of theorem 2

Recall that $K^\epsilon = M^\epsilon + N^\epsilon$, so in view of theorem 4.1 in [2] we only need to show that for every $m \in \mathbb{N}^*$ and every $0 \leq t_1 < ... < t_m$

$$r_d(\epsilon)\left(N^\epsilon(t_1); ...; N^\epsilon(t_m)\right) \xrightarrow[\epsilon \to 0]{} 0$$

in probability. This follows immediately from proposition 1.

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