1 Introduction and statement of the results

Self intersection local time of Brownian motion have been the subject of numerous studies of half a century. This was, in part, motivated by the important role played by self intersection local time in the construction of certain Euclidean quantum fields, see for instance [8][22][23][27]. The expression of self intersection local time appears also in the study of polymers, see Edwards [10]. Using a chaos expansion in terms of Malliavin’s calculus or, even more appropriately white noise calculus, many results were proven see for instance [4][11][14]. More precisely it was shown in [24] that the self-intersection local time of d-dimensional Brownian motion, renormalized by subtracting some terms in its chaos expansion, was a Hida distribution, we also point out [13][19].

*Supported by FCT, Portugal.
An informal definition of the self intersection local time of the Brownian motion is
\[ L = \int d^2t \delta(B(t_2) - B(t_1)), \]
where \( B \) is a d-dimensional Brownian motion. This could be understood as the amount of time, spent by the Brownian path intersecting itself during the time \( t \). To make sense of this integral one can invoke an approximation
\[ L^\epsilon(t) = \int_0^t dt_2 \int_0^{t_2} dt_1 \delta^\epsilon(B(t_2) - B(t_1)), \]
with, for \( x \) in \( \mathbb{R}^d \)
\[ \delta^\epsilon(x) = \frac{1}{(2\pi\epsilon)^{d/2}} e^{-\frac{x^2}{2\epsilon}}. \]
The central problem is then the control of what happens when \( \epsilon \to 0 \).
As is known, for \( d \geq 2 \), we need to center \( L^\epsilon \) by
\[ L^\epsilon = \mathbb{E}(L^\epsilon) \]
see [17][20][23].
A Tanaka formula method was initiated in [21], and used in [28] for \( d=3 \) with a regularization of type ” \( \delta^\epsilon_y(x) = \delta(x + y) \)” to show that
\[ r_3(|y|)(L_y - \mathbb{E}(L_y)) \xrightarrow{y \to 0} c_3 \beta \]
where \( \beta \) is a one dimensional Brownian motion, independent of the initial one and \( r_3(|y|) = |\log|y||^{-1/2} \).
Using an explicit computation of the chaos of \( L^\epsilon_T \) given in [11], it has been shown in [4] and for \( d > 2 \), that each chaos of the renormalized self-intersection local time, converges in law to a one dimensional Brownian motion, independent from the initial one.

Our aim in this work is to prove a similar result for the whole renormalized self-intersection local time. Again, using the explicit computation of the chaos of \( L^\epsilon_T \), we decompose the self-intersection local time into a martingale part \( M^\epsilon \) and a negligible one \( N^\epsilon \), following in this what has been done in [4].

We prove that for \( d > 2 \) the renormalized martingale part \( r_d(\epsilon)M^\epsilon \) converges in law to a one dimensional Brownian motion, independent of the initial one and that \( r_d(\epsilon)N^\epsilon \) converges to zero in mean square.
This entails that the renormalized self-intersection local time \( r_d(\epsilon)(L^\epsilon - E(L^\epsilon)) \) converges in finite dimensional distribution to a Brownian motion independent of the initial one.

Before announcing our results, let us give some notations and recall some results.

In [11], the kernels of the chaos expansion of the renormalized self-intersection local time were explicitly computed.

Let \( \epsilon > 0, \, n = (n_1, \cdots, n_d) \in \mathbb{N}^d - \{0\}, \, n = \sum_i n_i 

The kernels of \( L_\epsilon - E(L_\epsilon) \) in Fock space are, for any \( d > 2 \):

\[
F_{\epsilon, \bar{n}}(s_1, \ldots, s_n) = (-1)^{n/2} \left( \chi(\chi + 1)(2\pi)^{d/2} n^{d/2} \bar{n}^1 \right)^{-1} 
\theta(u)\theta(t - v) \left( (v - u + \epsilon)^{-\chi} + (t + \epsilon)^{-\chi} - (v + \epsilon)^{-\chi} - (t - u + \epsilon)^{-\chi} \right)
\]

if all \( n_i \) are even (and are zero otherwise) with \( v = \max_i s_i, \, u = \min_i s_i \) and \( \chi = \frac{n+d}{2} - 2; \theta \) is the Heaviside function.

Let us set up some notations, for \( \bar{n} \in \mathbb{N}^d - \{0\} \) and \( \epsilon > 0 \)

\[
M^\epsilon_{\bar{n}} = \int_{[0,\epsilon]^n} d^n s (v - u + \epsilon)^{-\chi} : w^{\otimes \bar{n}}(s) : \quad (1)
\]

\[
N^\epsilon_{\bar{n}} = \int_{[0,\epsilon]^n} d^n s \left( (t + \epsilon)^{-\chi} - (v + \epsilon)^{-\chi} - (t - u + \epsilon)^{-\chi} \right) : w^{\otimes \bar{n}}(s) : \quad (2)
\]

Note that for each \( \bar{n} \) and \( \epsilon \) the above processes are continuous in time.

For the \( \bar{n} \)th order chaos of the renormalized self-intersection local time we set

\[
K^\epsilon_{\bar{n}} = \alpha_{\bar{n}} (M^\epsilon_{\bar{n}} + N^\epsilon_{\bar{n}})
\]

It was shown in [4], that for \( d \geq 3 \) and \( \bar{n} \in \mathbb{N}^d - \{0\}, \)

\[
r_d(\epsilon)K^\epsilon_{\bar{n}} \xrightarrow{\epsilon \to 0^+} c_{\bar{n}} \beta_{\bar{n}} \quad (3)
\]

where for \( \bar{n} \neq 0, \beta_{\bar{n}} \) is a one dimensional Brownian motion independent of the initial one and among each other, with

\[
c_{\bar{n}}^2 = k_{\bar{n}}^2 \alpha_{\bar{n}}^2,
\]

3
\[ k_n^2 = \begin{cases} n(n - 1) & d = 3 \\ \frac{n!(d-4)!}{(n+d-5)!} & d > 3 \end{cases}, \]

\[ \alpha_{\vec{n}} = (-1)^{n/2} \left( \chi(\chi + 1)(2\pi)^{d/2}n^{n/2} \right)^{-1}, \]

\[ r_d(\epsilon) = \begin{cases} |\log \epsilon|^{-1/2} & d = 3 \\ \epsilon^{d-3/2} & d > 3 \end{cases}. \]

Let us denote the dominant (martingale) part of the renormalized self-intersection local time by

\[ M^\epsilon = \sum_{\vec{n} \neq 0} \alpha_{\vec{n}} M^\epsilon_{\vec{n}}. \]

Our main result is as follows

**Theorem 1** For \( d \geq 3 \),

\[ r_d(\epsilon) M^\epsilon \xrightarrow{\epsilon \to 0^+} c \beta, \quad (4) \]

where \( \beta \) is a one dimensional standard Brownian motion independent of the initial one and \( c = \sqrt{\sum_{\vec{n} \neq 0} \beta^2}. \)

**Remark**

It was shown in [4], by explicit computation, that for all \( \vec{n} \neq 0 \) and uniformly in finite \( t \)-intervals that

\[ r_d(\epsilon) N^\epsilon_{t,\vec{n}} \xrightarrow{(L^2)} 0. \]

Considering the summed up process

\[ N^\epsilon_t = \sum_{\vec{n}, \vec{\vec{n}} \neq 0} \alpha_{\vec{n}} N^\epsilon_{t,\vec{n}}, \]

we prove

**Proposition 1** For every \( t \geq 0 \) and \( d > 2 \)

\[ r_d(\epsilon) N^\epsilon_t \xrightarrow{(L^2)} 0, \]

and the convergence is uniform in any finite \( t \)-interval.
We set
\[ K^\epsilon = L^\epsilon - \mathbb{E}(L^\epsilon) \]
\[ = M^\epsilon + N^\epsilon. \]

Then using the theorem 1 and the proposition 1 we obtain

**Theorem 2** For \( d \geq 3 \),
\[ r_d(\epsilon)K^\epsilon \xrightarrow{\mathcal{L}_f} c\beta, \]
where \( \mathcal{L}_f \) means convergence of finite dimensional distributions.

## 2 Tools from white noise analysis

We quote some white noise analysis concepts as introduced in [4], referring to [12] for a systematic presentation.

Consider a white noise space \( (S'(\mathbb{R})^d, \mathcal{B}, \mu) \), where \( \mathcal{B} \) is the weak Borel \( \sigma \)-algebra of \( S'(\mathbb{R})^d \), and \( \mu \) is the centered Gaussian measure with covariance given by the inner product of \( L^2(\mathbb{R})^d \).

Then a realization of a vector of independent Brownian motions \( B_i, i = 1, \cdots, d \), is obtained by
\[ B_i(t) = \langle \omega_i, 1_{[0,t]} \rangle = \int_0^t \omega_i(s)ds. \]

Hence we consider independent \( d \)-tuples of Gaussian white noise \( \omega = (\omega_1, \cdots, \omega_d) \) and correspondingly, \( d \)-tuples of test functions \( \mathbf{f} = (f_1, \cdots, f_d) \in S(\mathbb{R}, \mathbb{R}^d) \), and use the following multi-index notation:
\[ \vec{n}! = \prod_{i=1}^d n_i! \quad n = \sum_{i=1}^d n_i \]
\[ \langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt f_i^2(t) \]
\[ \langle F_{\vec{n}}, G_{\vec{m}} \rangle = \int d^n t F_{\vec{n}}(t_1, \cdots, t_n) \bigotimes_{i=1}^d f_i^{n_i}(t_1, \cdots, t_n) \]
and similarly for ⟨: ω⊗n :, F_n⟩ where for d-tuples of white noise the usual
Wick product : : : (see [12]) generalises to

\[ : \omega^\otimes n := \bigotimes_{i=1}^d : \omega_i^\otimes n_i : . \]

The vector valued white noise has the characteristic function

\[ C(\mathbf{f}) = \mathbb{E}(e^{i\langle \omega, \mathbf{f} \rangle}) = \int_{S^*(\mathbb{R}, \mathbb{R}^d)} d\mu[\omega] e^{i\langle \omega, \mathbf{f} \rangle} = e^{-\frac{1}{2}\langle \mathbf{f}, \mathbf{f} \rangle}, \]

where \( \langle \omega, \mathbf{f} \rangle = \sum_{i=1}^d \langle \omega_i, f_i \rangle \) and \( f_i \in S(\mathbb{R}, \mathbb{R}) \).

The Hilbert space \((L^2) = L^2(d\mu)\)

is canonically isomorphic to the d-fold tensor product of Fock spaces of symmetric square integrable functions:

\[ (L^2) \simeq \left( \bigoplus_{k=0}^\infty \text{Sym} L^2(\mathbb{R}^k, k!d^k) \right)^\otimes d := \mathcal{F} \]

for a general element of \((L^2)\) this implies the chaos expansion

\[ \varphi(\omega) = \sum_{n=0}^\infty \langle : \omega^\otimes n :, F_n \rangle, \]

the norm of \( \varphi \) is given by

\[ \| \varphi \|_{L^2}^2 = \sum_{n=0}^\infty \langle \omega^\otimes n \rangle_F^2 = \sum_{n=0}^\infty n!|F_n|_2^2, \]

with kernel functions \( F \) in \( \mathcal{F} \).

3 Proof of the results

Our strategy to prove the theorem 1 is a classical one. In fact we first prove the tightness of the family of processes \( \{r_d(\epsilon)M^\epsilon \}_{\epsilon > 0} \), and then we prove the appropriate convergence of the associated sequence of finite dimensional distributions.
3.1 Tightness of \( \{r_d(\epsilon)M^{\epsilon}\}_{\epsilon>0} \)

We recall that to prove the tightness of \( \{M^{\epsilon}_{\bar{n}}; \epsilon > 0\} \) for each \( \bar{n} \neq 0 \), the authors in [4] have used the hypercontractivity of the Ornstein-Uhlenbeck semigroup.

They obtained for \( 0 \leq t, s \leq T \) and \( \alpha > 2 \) the following inequality

\[
\mathbb{E} [r_d(\epsilon)|M^{\epsilon}_{\bar{n}}(t) - M^{\epsilon}_{\bar{n}}(s)|^\alpha] \leq C_{T,\bar{n}}|t - s|^{\alpha/2}
\]

which essentially implies tightness [15].

Such hypercontractivity argument cannot readily be extended to the process \( \{r_d(\epsilon)M^{\epsilon}_{\bar{n}}; \epsilon > 0\} \). To circumvent this difficulty, we prove a technical lemma.

**Lemma 1** Let \( (X^n)_{n \in \mathbb{N}}, (X^n_k)_{n, k \in \mathbb{N}} \) be two families of processes with continuous paths and starting from the origin. Suppose that:

i) For each \( k \in \mathbb{N}, (X^n_k)_{n \in \mathbb{N}} \) is tight.

ii) For each \( T > 0 \) and \( \epsilon > 0 \),

\[
\sup_{n \in \mathbb{N}} \mathbb{P} \left( \max_{0 \leq t \leq T} |X^n_k(t) - X^n(t)| > \epsilon \right) \xrightarrow{\delta_{\epsilon \to \infty}} 0,
\]

then the family \( (X^n)_{n \in \mathbb{N}} \) is tight.

**Proof of lemma 1**

Following [2][15], we have to prove for each \( \epsilon > 0 \) and \( T > 0 \), that

\[
\sup_{n \in \mathbb{N}} \mathbb{P} \left( \max_{|t-s| \leq \delta, 0 \leq t, s \leq T} |X^n(t) - X^n(s)| > \epsilon \right) \xrightarrow{\delta \downarrow 0} 0.
\]

Let \( \epsilon > 0 \), \( T > 0 \) and \( n \in \mathbb{N} \)

\[
|X^n(t) - X^n(s)| \leq |X^n(t) - X^n_k(t)| + |X^n_k(t) - X^n_k(s)| + |X^n_k(s) - X^n(s)|
\]

so

\[
\mathbb{P} \left( \max_{|t-s| \leq \delta, 0 \leq t, s \leq T} |X^n(t) - X^n(s)| > \epsilon \right) \leq \mathbb{P} (2I_{1,k} + I_{2,k} > \epsilon)
\]

\[
\leq \mathbb{P} \left( I_{1,k} > \frac{\epsilon}{4} \right) + \mathbb{P} \left( I_{2,k} > \frac{\epsilon}{2} \right)
\]

where

\[
I_{1,k} = \max_{0 \leq t \leq T} |X^n_k(t) - X^n(t)|
\]
$I_{2,k} = \max_{|t-s| \leq \delta, 0 \leq t,s \leq T} |X_k^n(t) - X_k^n(s)|$

Let $\alpha > 0$. By the second hypothesis, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}(I_{1,k_0} > \frac{\epsilon}{4}) \leq \frac{\alpha}{2}$$

Choose $k = k_0$; because of the tightness of $(X_{k_0}^n)_{n \in \mathbb{N}}$, there exists $\delta_0 > 0$ such that for every $\delta > \delta_0$

$$\sup_{n \in \mathbb{N}} \mathbb{P}(I_{2,k_0} > \frac{\epsilon}{2}) < \frac{\alpha}{2},$$

so the lemma is proved.

Let us now prove the tightness of $(r_d(\epsilon)M^r)_{\epsilon > 0}$.

For each $k \in \mathbb{N}^+$, denote by

$$M_k^r = \sum_{\vec{n} \neq 0, n \leq k} \alpha_{\vec{n}} M_{\vec{n}}^r$$

so that we have, for $t > 0$

$$\|M^r(t) - M_k^r(t)\|_2^2 = \sum_{\vec{n} \neq 0, n > k} \alpha_{\vec{n}}^2 \|M_{\vec{n}}^r(t)\|_2^2.$$ 

It was shown in [4] (proof of lemma 3.2), that

$$\|M_{\vec{n}}^r(t)\|_2^2 = \vec{n}! n(n - 1) \epsilon^{3-d} \int_0^t dv \int_0^{v/\epsilon} dx \frac{x^{n-2}}{(x + 1)^{n+d-4}}. \quad (6)$$

For $d=3$

$$r_3(\epsilon)^2 \|M_k^r(t)\|_2^2 \leq \vec{n}! n(n - 1) \left\{ \frac{t |\log \epsilon| + (t + \epsilon) \log(t + \epsilon) - t}{|\log \epsilon|} \right\}$$

the function of $t$ and $\epsilon$ in the right hand side of the last inequality is uniformly bounded on bounded sets of $(t, \epsilon) \in \mathbb{R}^2_+$. 

If $d > 3$

$$\int_0^{v/\epsilon} dx \frac{x^{n-2}}{(x + 1)^{n+d-4}} \leq \int_0^{+\infty} dx \frac{x^{n-2}}{(x + 1)^{n+d-4}} \leq \int_0^{+\infty} dx \frac{1}{(x + 1)^{d-2}} = \text{const.}$$
so in view of (6), if $d > 3$, we obtain

$$r_d(\epsilon)^2 \| M_\epsilon^n(t) \|_2^2 \leq \text{const} n(n - 1)t.$$ 

It was proved in [5] that

$$\sum_{n \neq 0} a_n^2 i^n(n - 1) < +\infty.$$ 

Then we have proved that for every $d \geq 3$,

$$r_d^2(\epsilon)^{\| M^\epsilon(t) - M_k^\epsilon(t) \|_2^2 \rightarrow 0 \text{ as } \epsilon \rightarrow 0}$$ 

uniformly in bounded sets of $(t, \epsilon) \in \mathbb{R}_+^2$.

On the other hand, using the same technique as in [4] it has been proved that $\{ r_d(\epsilon)M_\epsilon^n; n \leq k \}$ is also tight when $\epsilon \rightarrow 0$.

Hence, also for each $k \geq 1$ $r_d(\epsilon)M_k^\epsilon$ is tight when $\epsilon \rightarrow 0$.

Let $\eta > 0$ and $T > 0$ fixed,

$$\mathbb{P} \left( \max_{0 \leq t \leq T} \left( r_d(\epsilon) | M^\epsilon(t) - M_k^\epsilon(t) | \right) > \eta \right) \leq \frac{\mathbb{E} ( r_d(\epsilon) | M^\epsilon(T) - M_k^\epsilon(T) | )}{\eta}$$

because $M^\epsilon(\cdot) - M_k^\epsilon(\cdot)$ is a martingale, see e.g. [15].

Using (7) we find (5) and we prove that $\{ r_d(\epsilon)M^\epsilon; \epsilon > 0 \}$ is tight.

### 3.2 Convergence of the finite dimensional distributions

We intend to prove that, for every $0 \leq t_1 < \cdots < t_m$, $m \in \mathbb{N} - \{ 0 \}$,

$$r_d(\epsilon) \left( M^\epsilon(t_1); \ldots; M^\epsilon(t_m) \right) \xrightarrow[\epsilon \to 0]{} c(\beta(t_1); \ldots; \beta(t_m)),$$

where $\beta$ is a standard one dimensional Brownian motion, and $c^2 := \lim_{k \to \infty} c_k^2 = \sum_{n \neq 0} c_n^2$.

Using the same technique as in [4] one finds that for every $k \in \mathbb{N} - \{ 0 \}$

$$r_d(\epsilon)M_k^\epsilon \xrightarrow[\epsilon \to 0]{} c_k \beta$$

where $c_k^2 = \sum_{n \neq 0; n \leq k} c_n^2$.

Then for every $k, m \in \mathbb{N}^*$ and $0 \leq t_1 < \cdots < t_m$

$$r_d(\epsilon) \left( M_k^\epsilon(t_1); \ldots; M_k^\epsilon(t_m) \right) \xrightarrow[\epsilon \to 0]{} c_k (\beta(t_1); \ldots; \beta(t_m)),$$
Now we vectorize
\[ \tilde{M}_k = r_d(\epsilon) \left( M_k(t_1); \ldots; M_k(t_m) \right), \]
\[ \tilde{M}^* = r_d(\epsilon) \left( M_{i_1}^*; \ldots; M_{i_m}^* \right), \]
\[ \vec{\beta} = (\beta(t_1); \ldots; \beta(t_m)). \]

Let \( X, Y \) be two random vectors in \( \mathbb{R}^m \) (not necessarily defined on the same probability space), we define the following distance
\[ d(X, Y) = \sup_{\|f\|_1 \leq 1} |\mathbb{E}(f(X)) - \mathbb{E}(f(Y))| \]
with \( \|f\|_1 = \|f\|_\infty + \|f'\|_\infty \) for \( f \) in \( C_b^1(\mathbb{R}^m) \). With this distance the topology of the convergence in law of random vectors on Euclidean space becomes metrisable, see [1]. Then we have
\[ d\left( \tilde{M}^*, \vec{\beta} \right) \leq d\left( \tilde{M}^*, \tilde{M}^*_k \right) + d\left( \tilde{M}^*_k, c_k \vec{\beta} \right) + d\left( c_k \vec{\beta}, c \vec{\beta} \right). \]

In view of (7) the first term of the rhs can be made arbitrarily small for \( k \) large enough, uniformly in \( \epsilon \geq 0 \). The second distance is small when \( \epsilon \to 0 \) because of (8), and finally the third becomes small when \( k \) is large by the definition of \( c \).

Note that this argument does not require us to distinguish the \( \beta \) arising in (8) for different \( k \) since they all have the same law.

**Proof of proposition 1**

In view of (2) we have
\[ \|N_{r, \epsilon}^\kappa\|_{L^2} \leq 3\pi! \left( \|((v+\epsilon)^{-\kappa} \|_{L^2((0,1)^n)} + \|(v+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} + \|(t-u+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} \right), \]
with elementary computations we find
\[ r_d(\epsilon)^2 \|((v+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} \leq \begin{cases} \frac{t}{|\log \epsilon|} n & \text{for } d = 3 \\ \frac{\epsilon \log \frac{t+\epsilon}{\epsilon}}{\epsilon} n & \text{for } d = 4 \\ \frac{\epsilon}{\epsilon-1} n & \text{for } d > 4 \end{cases} \]
also
\[ \|(t+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} \leq \|(t-u+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} = \|(v+\epsilon)^{-\kappa}\|_{L^2((0,1)^n)} = o(1)n \]
so we have the same estimation for all three terms. Thus for
\[ \| r_d(\epsilon) N_t^\epsilon \|_{(L^2)}^2 = r_d(\epsilon)^2 \sum_{\vec{n} \neq 0} \alpha^2_{\vec{n}} \| N_{t, \vec{n}}^\epsilon \|_{(L^2)}^2, \]
in view of the last estimation and for t in a finite interval, we have for every
\( d \geq 3 \)
\[ \| r_d(\epsilon) N_t^\epsilon \|_{(L^2)}^2 \leq o(1) \sum_{\vec{n} \neq 0} \alpha^2_{\vec{n}} \bar{n} \ln n, \]
the last series converges and so the proposition is proved.

**Proof of theorem 2**
Recall that \( K^\epsilon = M^\epsilon + N^\epsilon \), so in view of theorem 4.1 in [2] we only need to show that for every \( m \in \mathbb{N}^* \) and every \( 0 \leq t_1 < \ldots < t_m \)
\[ r_d(\epsilon) (N^\epsilon(t_1); \ldots; N^\epsilon(t_m)) \xrightarrow{\epsilon \to 0} 0 \]
in probability. This follows immediately from proposition 1.

**References**


