ELECTROMAGNETIC SIGNAL PROCESSING AND NONCOMMUTATIVE TOMOGRAPHY

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Abstract

Generic electromagnetic signal described by Maxwell equations both in vacuum and media is considered in the tomographic representation. The Ville–Wigner phase-space representation of the electromagnetic field is also discussed. Relations between different representations of the electromagnetic signal are elucidated. A connection of the Fourier analysis of the electromagnetic signal and other mathematical approaches like Radon transform of the analytic signal is presented. A distinguished property of the tomogram to coincide with the probability density of a random variable considered in a reference frame in the signal's phase space is pointed out. Entropy of the signal related to the probability density is studied.

keywords: Maxwell equations, analytic signal, noncommutative tomography, Ville–Wigner phase-space representation.

1 Introduction

Electromagnetic signals in vacuum are described by electric **E** and **H** strength vectors obeying to Maxwell equations [1, 2]. The standard analysis of electromagnetic signals is based on Fourier transform [3, 4] of the electric and magnetic vectors. This transform provides information on wavelengths and frequencies associated to the concrete electromagnetic signal under study. The electromagnetic signal processing based on the Fourier representation of electromagnetic fields and electric currents is also connected with language using the frequency content of the objects under study and operations with frequencies. Mathematically the Fourier analysis of electromagnetic fields gives the possibility, in some cases, to reduce differential Maxwell equations to the algebraic form.

On the other hand, there exist other transforms like wavelet transform (see, for example, [5, 6]) and Ville–Wigner transform [7, 8], which provide information on other aspects of electromagnetic signals. Recently Radon transform [9, 6] was discussed to construct the optical tomography [10] of the signal. The optical tomography uses the invertible map of the Wigner function [8] onto a probability density of the random variable, which is rotated homodyne quadrature of the photon [11, 12]. In quantum mechanics, the optical tomography scheme was generalized to the symplectic tomography map of the Wigner function onto the symplectic tomogram of the quantum state [13, 14]. In signal analysis, the corresponding noncommutative tomography of the analytic signal was suggested in [15]. Various aspects of noncommutative tomography of the analytic signals were studied in [16]. A unified view to construct different types of tomograms and other transforms of the analytic signal like Fourier transform, wavelet transform, Ville–Wigner transform was formulated recently in [17].

The noncommutative tomography procedure associates to a signal the signal's tomogram, which is the tomographic probability distribution function. In view of the analogy of the analytic signal to the quantum wave function of the quantum state in quantum mechanics, a notion related to quantum states, i.e., entanglement, was discussed for classical signals [18, 19]. In fact, classical processes are known to demonstrate some properties, which can be considered as quantumlike properties [20]. In the electromagnetic-field context, quantumlike properties were found by Fock and Leontovich [21] for paraxial beams of the electromagnetic radiation. Such properties permit one to employ the methods of the tomography of the quantum state's wave function for studying the electron-optics problems as well [22]. Also the method of time-dependent quantum invariants [23, 24] can be applied and developed to find universal invariants for paraxial electromagnetic beams propagating in media, i.e., in optical fibers [25, 26].

The aim of this work is to review some aspects of the noncommutative tomography approach [15–19] and present the tomographic-map analysis of the generic electromagnetic field described by Maxwell equations both in vacuum and media. We introduce tomograms of the electromagnetic-field potentials and derive equations for the tomograms.

2 Maxwell Equations

In this section, we introduce some notation and review properties of Maxwell equations for the electromagnetic field in vacuum [2]. In this case, the electromagnetic field has the electric-field-strength component $\mathbf{E}(\mathbf{r},t)$ and the magnetic-field-strength component $\mathbf{H}(\mathbf{r},t)$ depending on the space coordinate $\mathbf{r}=(\mathbf{x},\mathbf{y},\mathbf{z})$ and time coordinate t. These fields components satisfy the Maxwell equations with the sources $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$:

$$\begin{aligned}
&\operatorname{div}\mathbf{E} = 4\pi\rho,\\
&\operatorname{div}\mathbf{H} = 0,\\
&\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},\\
&\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j},
\end{aligned} \tag{1}$$

where $\rho(\mathbf{r},t)$ is the charge density, $\mathbf{j}(\mathbf{r},t)$ is the current density, and c is the constant light velocity in vacuum.

Using the scalar potential $\varphi(\mathbf{r},t)$ and the vector potential $\mathbf{A}(\mathbf{r},t)$ such that the electric-field-strength and magnetic-field-strength vectors are given by

$$\begin{aligned}
&\mathbf{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},\\
&\mathbf{H} = \operatorname{rot}\mathbf{A},
\end{aligned} \tag{2}$$

one has Maxwell equations in the form:

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 4\pi\rho,\\
&\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\,\mathbf{j},
\end{aligned} \tag{3}$$

with the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.$$

These equations follow from Maxwell equations (1) and relations (2) provided the Lorentz gauge condition is used

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0. \tag{4}$$

In the case of free fields, i.e., $\rho = 0$, $\mathbf{j} = 0$, Maxwell equations (1) take the form

$$\begin{aligned} &\operatorname{div}\mathbf{E} = 0,\\ &\operatorname{div}\mathbf{H} = 0,\\ &\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\,,\\ &\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\,. \end{aligned} \tag{5}$$

These equations can be reduced to the wave equations for the field strengths:

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta\mathbf{E} = 0,\\ &\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} - \Delta\mathbf{H} = 0. \end{aligned} \tag{6}$$

For the case without sources, one has the wave equation for the scalar potential $\varphi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ as well

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi = 0,\\ &\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = 0. \end{aligned} \tag{7}$$

Maxwell equations (1) can be rewritten in the integral form

$$\begin{aligned} &\oint \mathbf{H}\, d\mathbf{l} = \frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{E}\, d\mathbf{s} + \frac{4\pi}{c}\int \mathbf{j}\, d\mathbf{s},\\ &\int \mathbf{E}\, d\mathbf{s} = 4\pi \int \rho\, dV,\\ &\int \mathbf{H}\, d\mathbf{s} = 0,\\ &\oint \mathbf{E}\, d\mathbf{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{H}\, d\mathbf{s}. \end{aligned} \tag{8}$$

Also Maxwell equations can be written in the relativistic-covariant form. Introducing the four-vector A^k with the components

$$A^0 = c\varphi, \qquad A^1 = A_x, \qquad A^2 = A_y, \qquad A^3 = A_z$$

and the four-vector j^k with the components

$$j^0 = c\rho, \qquad j^1 = j_x, \qquad j^2 = j_y, \qquad j^3 = j_z,$$

along with the relativistic notation for the time and space coordinates

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z,$$

$$x_0 = x^0, \qquad x_1 = -x^1, \qquad x_2 = -x^2, \qquad x_3 = -x^3,$$

one arrives at Eqs. (3) and (4) in the form

$$\begin{aligned} &\sum_{i=0}^{3} \frac{\partial}{\partial x_i} \frac{\partial A^k}{\partial x^i} = \frac{4\pi}{c} j^k, \qquad k = 0, 1, 2, 3, \\ &\sum_{k=0}^{3} \frac{\partial A^k}{\partial x^k} = 0. \end{aligned} \tag{9}$$

The antisymmetric electromagnetic-field tensor is defined as follows

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}, \qquad i, k = 0, 1, 2, 3. \tag{10}$$

In terms of the field tensor components, Maxwell equations (1) take the form

$$\sum_{k=0}^{3} \frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i, \tag{11}$$

$$\sum_{k=0}^{3} \left(\frac{\partial F^{ik}}{\partial x_l} + \frac{\partial F^{kl}}{\partial x_i} + \frac{\partial F^{li}}{\partial x_k} \right) = 0. \tag{12}$$

The current four vector j^k satisfies the continuity condition

$$\frac{1}{c} \frac{\partial \rho}{\partial t} + \operatorname{div} j = 0, \tag{13}$$

or in the relativistic form

$$\sum_{k=0}^{3} \left(\frac{\partial j^k}{\partial x^k} \right) = 0. \tag{14}$$

3 Noncommutative Tomography

In this section, we review properties of the noncommutative tomography scheme for the analytic signal following [15, 16].

Given a function $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_N)$ of N variables $\mathbf{t} = (t_1, t_2, \ldots, t_N)$. Let us define the symplectic tomogram associated to the function $f(\mathbf{t})$

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{k=1}^{N} \frac{1}{2\pi|\nu_k|} \left| \int dt_1\, dt_2 \cdots dt_N f(\mathbf{t}) \exp\left\{ \sum_{k=1}^{N} \left(\frac{i\mu_k}{2\nu_k}\, t_k^2 - \frac{iX_k}{\nu_k}\, t_k \right) \right\} \right|^2, \tag{15}$$

where

$$\mathbf{X} = (X_1, X_2, \ldots, X_N),$$
$$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N),$$
$$\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_N).$$

One has the equality

$$\int w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})\, d\mathbf{X} = \int |f(\mathbf{t})|^2 d\mathbf{t}. \tag{16}$$

The tomogram is the probability distribution function of the random variable $\mathbf{X}$. This probability distribution depends on $2N$ extra real parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.

The map of the signal $f(\mathbf{t})$ onto the tomogram $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is invertible. The signal function $f(\mathbf{t})$ can be associated to the density matrix (see [27, 28])

$$\rho_f(\mathbf{t}, \mathbf{t}') = f(\mathbf{t}) f^*(\mathbf{t}'). \tag{17}$$

The density matrix can be mapped onto the Ville–Wigner function

$$W(\mathbf{q}, \mathbf{p}) = \int \rho_f \left(\mathbf{q} + \frac{\mathbf{u}}{2}, \mathbf{q} - \frac{\mathbf{u}}{2} \right) e^{-i\mathbf{p}\mathbf{u}}\, d\mathbf{u}. \tag{18}$$

This map is invertible and one has

$$\rho(\mathbf{t}, \mathbf{t}') = \frac{1}{(2\pi)^N} \int W \left(\frac{\mathbf{t} + \mathbf{t}'}{2}, \mathbf{p} \right) e^{i\mathbf{p}(\mathbf{t}-\mathbf{t}')}\, d\mathbf{p}. \tag{19}$$

The tomogram (15) is related to the Ville–Wigner function by the equation

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int W(\mathbf{q}, \mathbf{p}) \prod_{k=1}^{N} \delta\Big(X_k - \mu_k q_k - \nu_k q_k \Big) \frac{dq_k\, dp_k}{2\pi}\, . \tag{20}$$

The delta-function in (20) is not equal to zero on the straight lines described by the relations

$$X_k = \mu_k q_k + \nu_k p_k. \tag{21}$$

The Ville–Wigner function can be reconstructed if one knows the tomogram

$$W(\mathbf{q},\mathbf{p}) = \frac{1}{(2\pi)^N} \int w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) \prod_{k=1}^{N} e^{i(X_k-\mu_k q_k-\nu_k p_k)}\, dX_k\, d\mu_k\, d\nu_k. \tag{22}$$

In view of relation (22), one can express the density matrix in terms of the tomogram

$$f(\mathbf{t})f^*(\mathbf{t}') = \frac{1}{(2\pi)^N} \int w(\mathbf{X},\boldsymbol{\mu},\mathbf{t}-\mathbf{t}') \prod_{k=1}^{N} \exp\left\{i\left(X_k-\mu_k\frac{t_k+t_k'}{2}\right)\right\} dX_k\, d\mu_k. \tag{23}$$

Relation (23) determines the analytic signal up to a constant phase factor

$$f(\mathbf{t}) = [f^*(0)]^{-1}\,\frac{1}{(2\pi)^N} \int w(\mathbf{X},\boldsymbol{\mu},\mathbf{t}) \prod_{k=1}^{N} \exp\left\{i\left(X_k-\frac{\mu_k t_k}{2}\right)\right\} dX_k\, d\mu_k. \tag{24}$$

If one uses the quantumlike Dirac notation for the density matrix $\rho_f(\mathbf{t},\mathbf{t}')$ in terms of the density operator $\widehat{\rho}_f$

$$\rho_f(\mathbf{t},\mathbf{t}') = \langle \mathbf{t} \mid \widehat{\rho}_f \mid \mathbf{t}'\rangle, \tag{25}$$

the tomogram can be rewritten in a compact form [17]:

$$w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \Big\langle \prod_{k=1}^{N} \delta\Big(X_k-\mu_k\widehat{q}_k-\nu_k\widehat{p}_k\Big)\Big\rangle, \tag{26}$$

where the mean value of the operator reads

$$\langle\widehat{A}\rangle = \mathrm{Tr}\,\Big(\widehat{\rho}_f\widehat{A}\Big). \tag{27}$$

The operator delta-function of the operator argument $\widehat{s}$ is determined by the Fourier decomposition

$$\delta(\widehat{s}) = \frac{1}{2\pi}\int e^{ik\widehat{s}}\, dk. \tag{28}$$

The position-like operator $\widehat{q}_k$ and the momentum-like operator $\widehat{p}_k$ are determined in the coordinate representation by the formulas

$$\begin{aligned} \widehat{q}_k f(\mathbf{t}) &= t_k f(\mathbf{t}), \\ \widehat{p}_k f(\mathbf{t}) &= -i\,\frac{\partial f(\mathbf{t})}{\partial t_k}. \end{aligned} \tag{29}$$

One can introduce the entropy associated to the signal using the standard definition of entropy related to a probability distribution function

$$S(\boldsymbol{\mu},\boldsymbol{\nu}) = -\int w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu})\,\ln w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu})\,d\mathbf{X}. \tag{30}$$

The tomogram has the property

$$\begin{aligned} &w\Big(\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_N X_N, \lambda_1\mu_1, \lambda_2\mu_2, \ldots, \lambda_N\mu_N, \lambda_1\nu_1, \lambda_2\nu_2, \ldots, \lambda_N\nu_N\Big) \\ &= \frac{1}{|\lambda_1\lambda_2\cdots\lambda_N|}\,w\Big(X_1, X_2, \ldots, X_N, \mu_1, \mu_2, \ldots, \mu_N, \nu_1, \nu_2, \ldots, \nu_N\Big). \end{aligned} \tag{31}$$

This means that the tomographic probability distribution is similar to the probability-distribution density determined by Dirac delta-function. This property is obvious if one uses the tomogram definition given by (26).

Relation (21) shows that we use the lines which are rotated and scaled in the signal's phase space. If one introduces parameters

$$\mu_k = e^{\lambda_k}\cos\theta_k, \qquad \nu_k = e^{-\lambda_k}\sin\theta_k, \tag{32}$$

transformation (21) demonstrates that the physical (geometrical) meaning of the variable X_k is the following one. It is the position in a rotated and scaled reference frame in the phase space, with the angle θ_k being the rotation parameter and the parameter λ_k determines the scaling of the reference frame.

Thus the tomogram is not a single distribution function but it is a $2N$-parametric family of the distribution functions. In fact, due to relation (31) one needs only rotation parameters. This means that taking $\lambda_k = 0$ in (32) one obtains the tomogram

$$w(\mathbf{X},\boldsymbol{\theta}) = w\Big(X_1, X_2, \ldots, X_N, \cos\theta_1, \cos\theta_2, \ldots \cos\theta_N, \sin\theta_1, \sin\theta_2, \ldots, \sin\theta_N\Big). \tag{33}$$

For the one-dimensional case, this tomogram determines the Ville–Wigner function by means of inverse Radon transform in the optical-tomography scheme [10–12]. But due to singularity of Radon transform, the employment of the symplectic tomogram containing extra scaling parameters has an advantage because inverse relation (22) has no singularity.

4 Change of Variables for Tomographic Map

In this section, we discuss how the differential operators defined in $\mathbf{t}$-coordinates are transformed if the tomographic map is used. In view of the integral relation of the density matrix to the tomogram, one can get the correspondence rule:

$$\begin{aligned} t_k\rho_f(\mathbf{t},\mathbf{t}') &\longleftrightarrow \left[-\left(\frac{\partial}{\partial X_k}\right)^{-1}\frac{\partial}{\partial\mu_k}+\frac{i\nu_k}{2}\frac{\partial}{\partial X_k}\right]w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),\\ t'_k\rho_f(\mathbf{t},\mathbf{t}') &\longleftrightarrow \left[-\left(\frac{\partial}{\partial X_k}\right)^{-1}\frac{\partial}{\partial\mu_k}-\frac{i\nu_k}{2}\frac{\partial}{\partial X_k}\right]w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),\\ -i\,\frac{\partial}{\partial t_k}\rho_f(\mathbf{t},\mathbf{t}') &\longleftrightarrow \left[-\frac{i\mu_k}{2}\frac{\partial}{\partial X_k}-\frac{\partial}{\partial\nu_k}\left(\frac{\partial}{\partial X_k}\right)^{-1}\right]w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),\\ -i\,\frac{\partial}{\partial t'_k}\rho_f(\mathbf{t},\mathbf{t}') &\longleftrightarrow \left[-\frac{i\mu_k}{2}\frac{\partial}{\partial X_k}+\frac{\partial}{\partial\nu_k}\left(\frac{\partial}{\partial X_k}\right)^{-1}\right]w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}). \end{aligned} \tag{34}$$

This correspondence rule gives a possibility for known linear equations for the density matrix to write their tomographic counterparts. On the right-hand side of (34), the operator $(\partial/\partial X_k)^{-1}$ is defined by the action on the Fourier component, i.e.,

$$\left(\frac{\partial}{\partial X_k}\right)^{-1}\int\varphi(s)e^{isX_k}\,ds=\int\frac{\varphi(s)}{is}e^{isX_k}\,ds. \tag{35}$$

Let us suppose that the signal function $f(\mathbf{t})$ satisfies the equation

$$\mathcal{L}\left(\mathbf{t},-i\frac{\partial}{\partial\mathbf{t}}\right)f(\mathbf{t})=0, \tag{36}$$

where $\mathcal{L}(\mathbf{a}, \mathbf{b})$ is a real function of two real vectors. The components of the two vectors do not commute. Using Eq. (36) one obtains

$$\mathcal{L}\left(\mathbf{t}', i\frac{\partial}{\partial \mathbf{t}'}\right) f^*(\mathbf{t}') = 0. \tag{37}$$

Equations (36) and (37) can be used to derive the equation for the density matrix $\rho_f(\mathbf{t}, \mathbf{t}')$.

Thus one has two equations

$$\left[\mathcal{L}\left(\mathbf{t}, -i\frac{\partial}{\partial \mathbf{t}}\right) \pm \mathcal{L}\left(\mathbf{t}', i\frac{\partial}{\partial \mathbf{t}'}\right)\right] \rho_f(\mathbf{t}, \mathbf{t}') = 0. \tag{38}$$

In view of the correspondence rule (34), Eqs. (38) can be presented in the tomographic form by the corresponding replacements of $\rho_f(\mathbf{t}, \mathbf{t}')$ and vectors $\mathbf{t}$ and $\mathbf{t}'$, $\partial/\partial \mathbf{t}$ and $\partial/\partial \mathbf{t}'$.

5 Examples of Electrostatics

For electrostatics, Maxwell equations in vacuum have the form

$$\Delta \varphi(x, y, z) = 0. \tag{39}$$

Let us denote

$$x = t_1, \qquad y = t_2, \qquad z = t_3.$$

In this case, one has for the scalar potential

$$\varphi(x, y, z) = \varphi(t_1, t_2, t_3)$$

and the basic equation of electrostatics takes the form

$$\frac{\partial^2 \varphi(\mathbf{t})}{\partial t_1^2} + \frac{\partial^2 \varphi(\mathbf{t})}{\partial t_2^2} + \frac{\partial^2 \varphi(\mathbf{t})}{\partial t_3^2} = 0. \tag{40}$$

It is linear equation of the form discussed in the previous section. Any solution $\varphi(x, y, z)$ to Eq. (39) can be associated to its tomogram

$$\begin{aligned} w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) &= \frac{1}{8\pi^3 |\nu_1, \nu_2, \nu_3|} \left| \int \varphi(x, y, z) \exp\left(\frac{i\mu_1}{2\nu_1} x^2 - \frac{iX_1 x}{\nu_1}\right.\right. \\ &\quad \left.\left. + \frac{i\mu_2}{2\nu_2} y^2 - \frac{iX_2 y}{\nu_2} + \frac{i\mu_3}{2\nu_3} z^2 - \frac{iX_3 z}{\nu_3}\right) dx\, dy\, dz \right|^2, \end{aligned} \tag{41}$$

where

$$\mathbf{X} = (X_1, X_2, X_3),$$
$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3),$$
$$\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3).$$

Analogous tomograms of magnetic field can be obtained for magnetostatics.

The fundamental solution of the electrostatics equation, which is related to the potential of the unit charge, has the form

$$\varphi(\mathbf{r}) = \frac{1}{|\mathbf{r}|}\,. \tag{42}$$

For this solution, the tomogram of the electrostatic field (41) has the form given by the integral

$$\begin{aligned} w(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) &= \frac{1}{8\pi^3|\nu_1,\nu_2,\nu_3|}\left|\int_0^\infty r\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\varphi\right. \\ &\times\exp\left\{i\left[\frac{r^2}{2}\left(\frac{\mu_1}{\nu_1}\sin^2\theta\cos^2\varphi+\frac{\mu_2}{\nu_2}\sin^2\theta\sin^2\varphi+\frac{\mu_3}{\nu_3}\cos^2\theta\right)\right.\right. \\ &\left.\left.\left.-r\left(\frac{X_1}{\nu_1}\sin\theta\cos\varphi+\frac{X_2}{\nu_2}\sin\theta\sin\varphi+\frac{X_3}{\nu_3}\cos\theta\right)\right]\right\}\right|^2. \end{aligned} \tag{43}$$

The tomogram can be normalized in the case of normalized analytic signal. If the signal's energy is infinite, the tomogram is a generalized function. The example of the fundamental solution of electrostatics considered belongs to the class of nonnormalized tomograms. In quantum mechanics, such nonnormalized tomograms correspond to the states belonging to the continuous energy spectrum of a Hamiltonian.

6 Tomography of Retarded Potentials

It is known that the electromagnetic-wave radiation is described by the retarded potentials given by the solution to Eqs. (3)

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r},t) + \frac{1}{c}\int\frac{\mathbf{j}\,(\mathbf{r}',t-R/c)}{R}\,dx'\,dy'\,dz', \tag{44}$$

where $\mathbf{A}_0$ is a solution to the wave equation and

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}. \tag{45}$$

In the case of the radiation field connected with the current density, one has the solution to Eqs. (3)

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{\mathbf{j}\,(\mathbf{r}',t-R/c)}{R}\,dx'\,dy'\,dz'. \tag{46}$$

The scalar potential has the form

$$\varphi(\mathbf{r},t) = \int \frac{\rho\,(\mathbf{r}',t-R/c)}{R}\,dx'\,dy'\,dz'. \tag{47}$$

We introduce the notation

$$x = t_1, \qquad y = t_2, \qquad z = t_3, \qquad t = t_4.$$

In view of the notation and after the corresponding replacement $\varphi \to f(\mathbf{t})$, we apply the general construction.

The tomogram of the scalar potential $w_\varphi(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu})$ with

$$\mathbf{X} = (X_1, X_2, X_3, X_4),$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4),$$
$$\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4),$$

has the form

$$\begin{aligned} w_\varphi(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) \;=\; & \frac{1}{16\pi^4|\nu_1\nu_2\nu_3\nu_4|}\Bigg|\int \frac{\rho\Big(\mathbf{r}',t-R/c\Big)}{R} \\ & \times \exp\left\{\frac{i}{2}\left(\frac{\mu_1}{\nu_1}\,x^2 + \frac{\mu_2}{\nu_2}\,y^2 + \frac{\mu_3}{\nu_3}\,z^2 + \frac{\mu_4}{\nu_4}\,t^2\right)\right. \\ & \left. -i\left(\frac{X_1}{\nu_1}\,x + \frac{X_2}{\nu_2}\,y + \frac{X_3}{\nu_3}\,z + \frac{X_4}{\nu_4}\,t\right)\right\}\,dx\,dy\,dz\,dt\,dx'\,dy'\,dz'\Bigg|^2. \end{aligned} \tag{48}$$

One has analogous expressions for the tomograms of the vector-potential components. As an example, we write down the tomogram of the A_x-component

$$w_{A_x}(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \frac{1}{16c^2\pi^4|\nu_1\nu_2\nu_3\nu_4|}\left|\int \frac{j_x\Big(\mathbf{r}',t-R/c\Big)}{R} \times \exp\left\{\frac{i}{2}\left(\frac{\mu_1}{\nu_1}\,x^2+\frac{\mu_2}{\nu_2}\,y^2+\frac{\mu_3}{\nu_3}\,z^2+\frac{\mu_4}{\nu_4}\,t^2\right) -i\left(\frac{X_1}{\nu_1}\,x+\frac{X_2}{\nu_2}\,y+\frac{X_3}{\nu_3}\,z+\frac{X_4}{\nu_4}\,t\right)\right\} dx\,dy\,dz\,dt\,dx'\,dy'\,dz'\right|^2. \tag{49}$$

Now we consider the tomogram of the vector-potential $\mathbf{A}(\mathbf{r},t)$ created by a moving electric charge e. Its trajectory is described by the time-dependent vector $\mathbf{r}_0(t)$. The charge's velocity

$$\mathbf{v}(t) = \dot{\mathbf{r}}_0(t)$$

is considered to be given function (as well as the trajectory $\mathbf{r}_0(t)$ itself).

As it is known, in this particular case, the solution to Maxwell equations for the retarded potential reads

$$\mathbf{A}(\mathbf{r},t) = \frac{e\mathbf{v}}{c(R-\frac{\mathbf{vR}}{c})}\,|_{t'} \tag{50}$$

with time t' determined by the relation

$$t'(t,\mathbf{r}) = t - \frac{R(t')}{c}, \tag{51}$$

where the vector

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{r}_0(t)$$

has three components.

The scalar potential is given by

$$\varphi(\mathbf{r},t) = \frac{e}{\Big(R-\frac{\mathbf{vR}}{c}\Big)}\,|_{t'}\,. \tag{52}$$

The tomogram of the scalar potential (52) is given by the relation

$$
\begin{aligned}
w_\varphi(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) \;=\; & \frac{e^2}{16\pi^4|\nu_1\nu_2\nu_3\nu_4|}\left|\int\left[R\Big(x,y,z,t'(x,y,z,t)\Big)\right.\right.\\
& \left.-\frac{\mathbf{v}\Big(x,y,z,t'(x,y,z,t)\Big)\mathbf{R}\Big(x,y,z,t'(x,y,z,t)\Big)}{c}\right]^{-1}\\
& \times\exp\left\{\frac{i}{2}\left(\frac{\mu_1}{\nu_1}\,x^2+\frac{\mu_2}{\nu_2}\,y^2+\frac{\mu_3}{\nu_3}\,z^2+\frac{\mu_4}{\nu_4}\,t^2\right)\right.\\
& \left.\left.-i\left(\frac{X_1}{\nu_1}\,x+\frac{X_2}{\nu_2}\,y+\frac{X_3}{\nu_3}\,z+\frac{X_4}{\nu_4}\,t\right)\right\}\,dx\,dy\,dz\,dt\right|^2, \qquad (53)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{X} &= (X_1,X_2,X_3,X_4),\\
\boldsymbol{\mu} &= (\mu_1,\mu_2,\mu_3,\mu_4),\\
\boldsymbol{\nu} &= (\nu_1,\nu_2,\nu_3,\nu_4).
\end{aligned}
$$

Analogous relation describes the tomogram of the vector-potential (50). For example,

$$
\begin{aligned}
w_{A_x}(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) \;=\; & \frac{e^2}{16c^2\pi^4|\nu_1\nu_2\nu_3\nu_4|}\left|\int v_x\Big(x,y,z,t'(x,y,z,t)\Big)\left[R\Big(x,y,z,t'(x,y,z,t)\Big)\right.\right.\\
& \left.-\frac{\mathbf{v}\Big(x,y,z,t'(x,y,z,t)\Big)\mathbf{R}\Big(x,y,z,t'(x,y,z,t)\Big)}{c}\right]^{-1}\\
& \times\exp\left\{\frac{i}{2}\left(\frac{\mu_1}{\nu_1}\,x^2+\frac{\mu_2}{\nu_2}\,y^2+\frac{\mu_3}{\nu_3}\,z^2+\frac{\mu_4}{\nu_4}\,t^2\right)\right.\\
& \left.\left.-i\left(\frac{X_1}{\nu_1}\,x+\frac{X_2}{\nu_2}\,y+\frac{X_3}{\nu_3}\,z+\frac{X_4}{\nu_4}\,t\right)\right\}\,dx\,dy\,dz\,dt\right|^2, \qquad (54)
\end{aligned}
$$

In the limit $\mathbf{v}(t)=0$, the tomogram of the scalar potential of the moving charge provides the tomogram of the electrostatic potential (43) where one excludes in an appropriate manner the dependence on time.

7 Electromagnetic Signals in Media

Electromagnetic signals in media are described by the field-strength vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ and the field-inductance vectors $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$.

In dispersive media, the vectors are connected by means of the following relationships:

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) &= \int \varepsilon\Big(x,y,z,t,x',y',z',t'\Big)\mathbf{E}\Big(x',y',z',t'\Big)\,dx'\,dy'\,dz'\,dt',\\ \mathbf{B}(\mathbf{r},t) &= \int \mu\Big(x,y,z,t,x',y',z',t'\Big)\mathbf{H}\Big(x',y',z',t'\Big)\,dx'\,dy'\,dz'\,dt'.\end{aligned} \tag{55}$$

In nondispersive media, one has

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) &= \varepsilon(x,y,z,t)\mathbf{E}(\mathbf{r},t),\\ \mathbf{B}(\mathbf{r},t) &= \mu(x,y,z,t)\mathbf{H}(\mathbf{r},t).\end{aligned} \tag{56}$$

In the simplest case of the monochromatic signal (neglecting polarization), Maxwell equations can be reduced to the Helmholtz equation for the potential, e.g.,

$$\Big[\Delta_3 + k^2(x,y,z)\Big]\varphi(x,y,z) = 0. \tag{57}$$

Below we apply the tomographic map to this particular type of the monochromatic signal.

The tomogram of the scalar potential of the monochromatic signal in media reads

$$\begin{aligned}w_\varphi(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) &= \frac{1}{8\pi^3|\nu_1\nu_2\nu_3|}\bigg|\int dx\,dy\,dz\,\varphi(x,y,z)\\ &\times \exp\left\{\frac{i}{2}\left(\frac{\mu_1}{\nu_1}\,x^2 + \frac{\mu_2}{\nu_2}\,y^2 + \frac{\mu_3}{\nu_3}\,z^2\right) - i\left(\frac{X_1}{\nu_1}\,x + \frac{X_2}{\nu_2}\,y + \frac{X_3}{\nu_3}\,z\right)\right\}\bigg|^2.\end{aligned} \tag{58}$$

The tomogram of the monochromatic signal satisfies the equation, which follows from the Helmholtz equation and Eq. (38) where the linear operator takes the form

$$\begin{aligned}\mathcal{L}\left(\mathbf{t},-i\frac{\partial}{\partial\mathbf{t}}\right) &\equiv \mathcal{L}\left(x,y,z,-i\frac{\partial}{\partial x},-i\frac{\partial}{\partial y},-i\frac{\partial}{\partial z}\right)\\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2(x,y,z).\end{aligned} \tag{59}$$

8 Tomograms of Tensorial Signals

In this section, we introduce the Ville–Wigner functions and tomograms of the signals described by functions which are constructed using the tensor product of the analytic signal. Let us consider the partial case of the analytic signal described by a complex function of one variable $f(t)$. Let this function be normalized, i.e.,

$$\int |f(t)|^2 \, dt = 1. \tag{60}$$

An analog of the density matrix can be constructed by means of the product of two functions $f(t_1)f^*(t_2)$. Now we construct the tensor product of the form

$$F(t_1, t_2, \ldots, t_N, t_{N+1}, \ldots, t_{N+n}) = f(t_1)f(t_2)\cdots f(t_N)f^*(t_{N+1})f^*(t_{N+2})\cdots f^*(t_{N+n}). \tag{61}$$

The analytic signal corresponds to a vector in the Hilbert space. The product of analytic signals corresponds to the product of vector components in the Hilbert space of the analytic signal. In view of the mentioned above, the function (61) is an analog of the tensor in which N components are described by the analytic signal at different times and n components are described by complex conjugate functions at different times. If the signal fluctuates one can introduce an analog of the correlation function (or density matrix)

$$R(t_1, t_2, \ldots, t_N, t_{N+1}, \ldots, t_{N+n}) = \langle f(t_1)f(t_2)\cdots f(t_N)f^*(t_{N+1})f^*(t_{N+2})\cdots f^*(t_{N+n})\rangle, \tag{62}$$

where one uses an average of the product of the signals under study.

If one knows the tensor signal (61), the analytic signal can be reconstructed due to the relationship

$$f(t) = F(t, 0, 0, \ldots, 0)K, \tag{63}$$

where K is given by the relationship

$$K = \Big[f(t_2 = 0)f(t_3 = 0)\cdots f(t_N = 0)f^*(t_{N+1} = 0)\cdots f^*(t_{N+n} = 0)\Big]^{-1}. \tag{64}$$

If $N = n$ the signal is reconstructed up to the constant phase factor. For $N \neq n$, the tensorial signal determines the analytic signal completely.

The tensorial signal is normalized function, i.e.,

$$\int |F(t_1, t_2, \ldots, t_N, t_{N+1}, \ldots, t_{N+n})|^2 \; dt_1\, dt_2 \cdots dt_N\, dt_{N+1} \cdots dt_{N+n} = 1. \tag{65}$$

The general construction of the Wigner–Ville and tomographic maps discussed in Sec. 3 can be applied for the particular case of the tensorial signals. To do this, we introduce the vector

$$\mathbf{t} = (t_1, t_2, \ldots, t_N, t_{N+1}, \ldots, t_{N+n})$$

with $(N+n)$ components. The tensorial signal (61) takes the form of a signal which depends on several variables. Using analogs of formulas (17)–(19) and (22) with the replacement $f \to F$, one obtains the Ville–Wigner function and tomogram of the tensorial signal, respectively,

$$W(\mathbf{q}, \mathbf{p}) = \int F\left(\mathbf{q} + \frac{\mathbf{u}}{2}, \mathbf{q} - \frac{\mathbf{u}}{2}\right) e^{-i\mathbf{p}\mathbf{u}}\, d\mathbf{u} \tag{66}$$

and

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int W(\mathbf{q}, \mathbf{p}) \prod_{k=1}^{N+n} \delta(X_k - \mu_k q_k - \nu_k q_k)\, \frac{dq_k\, dp_k}{2\pi}\,. \tag{67}$$

The tomogram of the tensorial signal (67) is the probability distribution function of a random $(N+n)$-vector $\mathbf{X}$ labeled by $2(N+n)$ real parameters μ_k and ν_k, $k = (1, 2, \ldots, N+n)$. The construction of the Wigner–Ville map (66) provides the extension of Wigner functions of higher orders studied recently [29].

9 Conclusions

To summarize, we point out the main results of the paper.

We introduced the tomographic representation for electromagnetic signals obeying to Maxwell equations both in vacuum and media. The tomographic representation gives the possibility to associate to the electromagnetic signal the probability distribution function. This probability distribution function in turn obeys to the Fokker–Planck-type equation determined by means of a linear operator. The form of this operator can be obtained using Maxwell equations for scalar and vector potentials. The solution to Maxwell equations with sources in the form of retarded potentials are mapped onto tomograms of the electromagnetic signal.

We studied the tensorial signal and introduced the Wigner–Ville and tomographic maps of this signal. The maps of the tensorial signal generalize the standard Wigner–Ville and tomographic maps of the analytic signal.

The Wigner–Ville and tomographic maps can be successfully used as additional characteristics of the signals under study.

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