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On Differential Operators in White Noise Analysis

Dedicated to Professor Takeyuki Hida on the occasion of his 70th birthday

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Abstract. White noise analysis is formulated on a general probability space which is such that (1) it admits a standard Brownian motion, and (2) its σ-algebra is generated by this Brownian motion (up to completion). As a special case, the white noise probability space with time parameter being the half-line is worked out in detail. It is shown that the usual differential operators can be defined on the smooth, finitely based functions of at most exponential growth via the chain rule, without supposing the existence of a linear structure (or translations) on the underlying probability space.

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1. Introduction

In [DP97], T. Deck, G. Våge, and the author began the attempt to formulate T. Hida's white noise analysis on a general probability space. One of the reasons that motivated the article [DP97] was that there are important applications – such as in nonlinear filtering – in which a Brownian motion and its noise are *given*, and it seems unnatural, if not wrong, to require that it is realized on a fixed probability space, such as the white noise space. Another reason was the wish to unify the bases of white noise analysis and the Malliavin calculus. It turned out that indeed at least two of the main features of white noise analysis can be formulated within a general framework without any loss: the theory of generalized random variables, and the calculus of differential operators.

In the above-mentioned paper, however, the problem of showing that the differential operators defined there are well-defined was left open, and the main purpose of the present paper is to give a proof of this fact.

Our starting point here will be a general probability space which carries a Brownian motion whose time parameter domain is the half-line $\mathbb{R}_+$. The only additional assumption is that the underlying σ-algebra is (up to completion) generated by the Brownian motion. This entails that the S-transform, which is one of the basic tools of white noise analysis, is injective.

The framework and basic notions will be established in Section 2. In Section 3 one realization of the framework via the white noise space with time parameter domain $\mathbb{R}_+$ is provided in a rather detailed fashion, because this realization seems not to be so well known. Also, I reproduce there a result by S. Albeverio and M. Röckner [AR90] on quasi-invariant measures on Suslin spaces. This result implies that on a Suslin space with a (nontrivial) measure ν which is quasi-invariant with respect to translations of a dense linear subspace, a ν-class of random variables can have at most one continuous representative. This fact will be essential for the proof of the well-definedness of the differential operators which is given in Section 4.

2. Framework

Let $(\Omega, \mathcal{B}, P)$ a complete probability space which satisfies the following conditions:

(H.1) There exists a standard Brownian motion $(B_t,\ t \in \mathbb{R}_+)$ on $(\Omega, \mathcal{B}, P)$;
(H.2) The P-completion of $\sigma(B_t,\ t \in \mathbb{R}_+)$ is equal to $\mathcal{B}$.

It is well known, e.g., [RY91], Lemma V.3.1, that condition (H.2) implies that the algebra generated by the (real, imaginary or complex) exponential functions in the variables $B_{t_1}, \dots, B_{t_n}$, for $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}_+$, is dense in $L^2(P)$. In fact, both statements are equivalent, and it is obvious that the same holds just as well for the algebra of polynomials.

Let us mention two realizations of these assumptions:

(1) Wiener space: $\Omega = C_0(\mathbb{R}_+)$, P is Wiener measure, $\mathcal{B}$ is the completion of the σ-algebra generated by the cylinder sets, and $B_t(\omega) = \omega(t)$ for $\omega \in \Omega$.
(2) White noise over the half line, which is discussed in detail in Section 3.

Let $(\mathcal{F}_t^0,\ t \in \mathbb{R}_+)$ be the filtration generated by $(B_t,\ t \in \mathbb{R}_+)$. Denote by $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$ its standard μ-augmentation: if $\mathcal{N}$ is the ideal of the μ-null-sets in $\mathcal{B}$, $\mathcal{F}_t := \mathcal{F}_t^0 \triangle \mathcal{N}$, $t \in \mathbb{R}_+$. Then $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$ is right-continuous because $(B_t,\ t \in \mathbb{R}_+)$ is strongly Markov (cf. Proposition 2.7.7 in [KS88]). Actually, $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$ is continuous (e.g., [KS88], Corollary 2.7.8), and for all $t, s \in \mathbb{R}_+$ with $t > s$, $B_t - B_s$ is independent of $\mathcal{F}_s$ (e.g., [KS88], Theorem 2.7.9), i.e., $(B_t,\ t \in \mathbb{R}_+)$ is a Brownian motion relative to $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$.

Consider the linear mapping

$$\begin{aligned} X\colon C_c^\infty(\mathbb{R}_+) &\longrightarrow L^2(P) \\ f &\longmapsto X_f, \end{aligned}$$

where $X_f := \int_{\mathbb{R}_+} f(t)\, \mathrm{d}B_t$. A trivial application of the Itô isometry shows that this mapping is continuous, if we give $C_c^\infty(\mathbb{R}_+)$ the norm $|\cdot|_2$ of $L^2(\mathbb{R}_+) \equiv L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda)$, where λ is the Lebesgue measure. Since $C_c^\infty(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, this mapping extends to a continuous linear mapping $\widehat{X}$ from $L^2(\mathbb{R}_+)$ into

$L^2(P)$. If $f \in L^2(\mathbb{R}_+)$ is not in $C_c^\infty(\mathbb{R}_+)$, we mean by X_f any representative of the P-class $\widehat{X}_f$. It is clear, that the family $(X_f,\ f \in L^2(\mathbb{R}_+))$ forms a centered Gaussian family with covariance given by the inner product of $L^2(\mathbb{R}_+)$.

I think it is appropriate to call the mapping $\widehat{X}$ (*Gaussian*) *white noise*, and an essential part of white noise analysis is really the analysis of this mapping and its extensions. Consider the Schwartz space $\mathcal{S}(\mathbb{R}_+)$ over the half-line as defined in [DP97], cf. also Section 3, and the dense continuous embedding $\mathcal{S}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$. $\mathcal{S}(\mathbb{R}_+)$ is by construction nuclear, and therefore we have the canonically associated Gel'fand–Hida triple (e.g., [KL96])

$$(\mathcal{S}_+) \subset L^2(P) \subset (\mathcal{S}_+)^*,$$

where $(\mathcal{S}_+)^*$ is a space of generalized random variables. Therefore, we may consider now $\widehat{X}$ as a mapping from $C_c^\infty(\mathbb{R}_+)$ into $(\mathcal{S}_+)^*$, and it has a continuous linear extension to the Schwartz space of tempered distributions $\mathcal{S}'(\mathbb{R}_+)$ over the half–line (cf. Section 3). In particular, we have for $t \in \mathbb{R}_+$, $\widehat{X}_{\delta_t} \in (\mathcal{S}_+)^*$ (δ_t is the Dirac distribution at t), and this is *white noise at time* t. It is customary to denote this generalized random variable by $\dot{B}_t$, and it is not hard to see that it is indeed the time derivative of Brownian motion at time t, the derivative being taken in the strong topology of $(\mathcal{S}_+)^*$.

Let $Z \in L^2(P)$. Define its S-transform SZ as a function on $C_c^\infty(\mathbb{R}_+)$ by

$$SZ(f) := \mathrm{e}^{-1/2\,|f|_2^2}\,\mathbb{E}\big(Z\,\mathrm{e}^{X_f}\big), \quad f \in C_c^\infty(\mathbb{R}_+).$$

Among other purposes, the factor in front of the expectation serves to normalize the S-transform so that $S1 = 1$.

By what has been said above, it is clear that for $Z \in L^2(P)$, SZ has a continuous extension to $L^2(\mathbb{R}_+)$, and we shall not distinguish the two mappings in the sequel.

The S-transform was first introduced in white noise analysis by I. Kubo and S. Takenaka [KT80], and it is closely related to the Segal–Bargman transform on Fock space, cf., e.g., [KL96].

The S-transform is very useful for a number of reasons. Two are:

(i) It 'diagonalizes' the Itô integral [DP97], and thereby allows to handle and extend 'multiplication by Gaussian (white) noise' very efficiently.
(ii) Spaces of generalized random variables, like $(\mathcal{S}_+)^*$, can be characterized via the S-transform in a way (e.g., [KL96] and literature quoted there) which is quite convenient for theoretical work as well as applications.

3. Schwartz Spaces and White Noise on the Half-Line

In this section we construct a white noise probability space over the half-line $\mathbb{R}_+$, which – according to Section 2 – is to be interpreted as the domain of the time parameter. The construction and properties are very similar to those of the usual white noise probability space (e.g., [Hi80, HK93]).

Let $C^2_\infty(\mathbb{R}_+)$ denote the set of twice continuously differentiable functions f on $(0, +\infty)$, which are such that

(i) f and its first two derivatives vanish rapidly (i.e., faster than any power) at infinity, and
(ii) the first two right derivatives of f exist at 0, and they coincide with $f'(0+)$ and $f''(0+)$, respectively.

Consider the following differential operator

$$Af(t) := -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, t\, \frac{\mathrm{d}}{\mathrm{d}t} f(t) + \frac{1}{4}\, t\, f(t), \quad t \in \mathbb{R}_+,$$

acting on $C^2_\infty(\mathbb{R}_+)$. It is obvious that A is symmetric on the dense subspace $C^2_\infty(\mathbb{R}_+)$ of $L^2(\mathbb{R}_+)$. (As usual, we talk about the elements of this Hilbert space as if they were functions. There will be no danger of confusion.) Introduce the set of Laguerre functions

$$l_k(t) := \mathrm{e}^{-t/2} L_k(t), \quad t \in \mathbb{R}_+,\ k \in \mathbb{N}_0,$$

where L_k is the Laguerre polynomial of order $k \in \mathbb{N}_0$ (e.g., [Sa77]). It is well-known that $(l_k,\ k \in \mathbb{N}_0)$ is a complete orthonormal system in $L^2(\mathbb{R}_+)$. The Laguerre functions l_k obviously belong to $C^2_\infty(\mathbb{R}_+)$, and an elementary calculation shows that they are the eigenfunctions of A:

$$A\, l_k = \left(k + \frac{1}{2}\right) l_k, \quad k \in \mathbb{N}_0.$$

Therefore it is plain (for example by an application of Nelson's analytic vector theorem [RS75]), that A is essentially self-adjoint on $C^2_\infty(\mathbb{R}_+)$, and its unique self-adjoint extension has domain

$$\mathcal{D}(A) := \left\{ f \in L^2(\mathbb{R}_+);\ f = \sum_{k\in\mathbb{N}_0} f_k\, l_k \text{ with } \sum_{k\in\mathbb{N}_0} f_k^2\, k^2 < +\infty \right\}.$$

We see that on the half-line A plays a role which is very similar to the one of the Hamiltonian of the harmonic oscillator on the full line, and the Laguerre functions play the role of the Hermite functions. Now we can proceed as in the case of the full line (e.g., [Hi80, RS72, Si71]). Define

$$\mathcal{S}(\mathbb{R}_+) := \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n),$$

and equip this space as usual with the projective limit topology defined by the Hilbert spaces $\mathcal{D}(A^n)$. Due to the spectral properties of A, it is obvious that $\mathcal{S}(\mathbb{R}_+)$ is a nuclear countable Hilbert space, and as such a Fréchet and a Montel space (e.g., [Tr67]). In [DP97] an elementary argument was given which shows that the functions f in $\mathcal{S}(\mathbb{R}_+)$ are infinitely differentiable on $(0, +\infty)$, and that f together with all its derivatives vanishes rapidly at infinity. It is also straightforward to check that all right derivatives of $f \in \mathcal{S}(\mathbb{R}_+)$ exist at zero, and that they coincide with the right limits of the derivatives of f at 0. That is, $f \in \mathcal{S}(\mathbb{R}_+)$ can be considered as the restriction of a function in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$.

We denote the dual of $\mathcal{S}(\mathbb{R}_+)$ by $\mathcal{S}'(\mathbb{R}_+)$, and call it *Schwartz space of tempered distributions over the half-line*.

Let us equip $\mathcal{S}'(\mathbb{R}_+)$ with some of the various common topologies, and discuss its topological properties. The arguments below are all completely standard, and given here for the convenience of the interested reader.

If we give $\mathcal{S}'(\mathbb{R}_+)$ the strong topology (i.e., the topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}_+)$), then $\mathcal{S}'(\mathbb{R}_+)$ is reflexive since $\mathcal{S}(\mathbb{R}_+)$ is Montel (e.g., [Tr67], p. 376, Corollary). For example, from the Lemma, p. 166, in [RS72] we can conclude that the strong and the Mackey topology coincide. (The Mackey topology is the topology of uniform convergence on the weakly compact, convex subsets of $\mathcal{S}(\mathbb{R}_+)$, and it is the finest dual topology on $\mathcal{S}'(\mathbb{R}_+)$, i.e., the finest locally convex Hausdorff topology on $\mathcal{S}'(\mathbb{R}_+)$ so that $\mathcal{S}(\mathbb{R}_+)$ is its dual in this topology. The weak topology on $\mathcal{S}'(\mathbb{R}_+)$ is the topology of pointwise convergence, and it is the coarsest dual topology on $\mathcal{S}'(\mathbb{R}_+)$. The Mackey–Arens theorem (e.g., [RS72]) states that all locally convex dual topologies are between the weak and the Mackey topology. Furthermore, the strong and the Mackey topology on $\mathcal{S}'(\mathbb{R}_+)$ also coincide with the topology of uniform convergence on compact subsets of $\mathcal{S}(\mathbb{R}_+)$ (e.g., [Tr67], p. 357, Proposition 34.5, but it also follows directly from the definition of a Montel space).

Now we can use Theorem 7 in [S73], p. 112, to conclude that $\mathcal{S}'(\mathbb{R}_+)$ equipped with the strong topology is a Lusin space. (One only has to make the trivial remark that the compact-open topology used for $\mathcal{S}'(\mathbb{R}_+) = \mathcal{L}(E, F)$ there, with $E = \mathcal{S}(\mathbb{R}_+)$ and $F = \mathbb{R}$, is just the topology of uniform convergence on compact subsets of $\mathcal{S}(\mathbb{R}_+)$, which had already been identified with the strong topology on $\mathcal{S}'(\mathbb{R}_+)$ above.)

It is evident from the definition of a Lusin space (e.g., [S73], p. 94), that if we equip $\mathcal{S}'(\mathbb{R}_+)$ with any topology weaker than the strong topology, it remains a Lusin space. In particular, $\mathcal{S}'(\mathbb{R}_+)$ equipped with *any* locally convex dual topology is a Lusin space, and we consider this case from now on. As a Lusin space, $\mathcal{S}'(\mathbb{R}_+)$ is also a Suslin space ([S73], p. 96), and it is strongly Lindelöf ([S73], p. 104, Proposition 3), i.e., every open cover of any open subset of $\mathcal{S}'(\mathbb{R}_+)$ has a countable subcover.

Moreover, since $\mathcal{S}'(\mathbb{R}_+)$ is Suslin for any dual topology, Corollary 2, p. 101, in [S73] entails that they all generate the same Borel σ-algebra, which is the σ-algebra generated by the cylinder sets, because the weak topology is a dual topology on $\mathcal{S}'(\mathbb{R}_+)$. Let us denote this σ-algebra by $\mathcal{B}_0$.

Consider a total subset $\mathcal{T}$ in $\mathcal{S}(\mathbb{R}_+)$. As a Fréchet space $\mathcal{S}(\mathbb{R}_+)$ is barreled. Therefore we have the Banach–Steinhaus theorem which implies that on $\mathcal{S}'(\mathbb{R}_+)$ the topology of pointwise convergence on $\mathcal{T}$ coincides with the weak topology.

Consequently, the σ-algebra $\sigma(\mathcal{T}) := \sigma(\langle\cdot, f\rangle;\ f \in \mathcal{T})$ is equal to $\mathcal{B}_0$. For example, we may choose for $\mathcal{T}$ the set of Laguerre functions.

As usual, we may use Minlos' theorem (e.g., [Hi80]) to introduce the centered Gaussian measure μ on $(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}_0)$ via the relation

$$\int_{\mathcal{S}'(\mathbb{R}_+)} \mathrm{e}^{i\langle\omega, f\rangle}\, \mu(\mathrm{d}\omega) = \mathrm{e}^{-\frac{1}{2}|f|_2^2}, \quad f \in \mathcal{S}(\mathbb{R}_+),$$

where $\langle\cdot,\cdot\rangle$ stands for the dual pairing of $\mathcal{S}'(\mathbb{R}_+)$ and $\mathcal{S}(\mathbb{R}_+)$, and $|\cdot|_2$ for the norm of $L^2(\mathbb{R}_+)$. We shall call μ the *white noise measure on the half line.* It shares almost all properties with the usual white noise measure, and the analysis of $L^2(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}_0, \mu)$ can be done just as in, e.g., [Hi80, HK93]. One aspect, however, should be recorded here for later use, namely that μ is quasi-invariant under the translations by elements in $L^2(\mathbb{R}_+)$ (which we embed in the canonical way into $\mathcal{S}'(\mathbb{R}_+)$ so that it becomes a dense linear subspace). Here is a quick way to see this: give the spaces $\mathcal{D}(A^m)$, $m \in \mathbb{N}$, the norms $|A^m \cdot|_2$ so that they and their duals $\mathcal{D}(A^m)^*$ become Hilbert spaces. Choose $m \in \mathbb{N}$ so that the injection of $\mathcal{D}(A^m)$ into $L^2(\mathbb{R}_+)$ is Hilbert–Schmidt. Notice that Minlos' theorem implies that μ is carried by $\mathcal{D}^*(A^m) \subset \mathcal{S}'(\mathbb{R}_+)$. Thus $(L^2(\mathbb{R}_+), \mathcal{D}^*(A^m), \mu)$ forms an abstract Wiener space, and we can use the well-known Cameron–Martin formula derived, e.g., in [Ku75].

Our next step is to quote a very useful result by S. Albeverio and M. Röckner [AR90].

Consider a Suslin topological vector space E over the reals, equipped with its Borel σ-algebra, denoted again by $\mathcal{B}_0$. Assume that ν is a measure on $(E, \mathcal{B}_0)$ which is not identically zero. Define an open subset V of E as the union of all open subsets which have ν-measure zero. Since by construction V is covered by open subsets of zero measure, the fact that E is strongly Lindelöf (s.a.) tells us, that V has a countable subcover by ν-null sets, and consequently has itself ν-measure zero. Hence V is the largest open subset of E which has zero ν-measure. Albeverio and Röckner define $\operatorname{supp}(\nu) := E \setminus V$.

Let $k \in E$. ν is called k-quasi-invariant if ν is quasi-invariant w.r.t. translations by the elements of the one-dimensional subspace generated by k, i.e., for every $A \in \mathcal{B}_0$, and all $\lambda \in \mathbb{R}$, $\nu(A) = 0$ implies $\nu(A + \lambda k) = 0$. Albeverio and Röckner prove in [AR90] the following

PROPOSITION. *Let $K := \{k \in E;\ \nu$ is k-quasi-invariant$\}$. Then K is a linear space. If K is dense in E then* $\operatorname{supp}(\nu) = E$.

For the convenience of the reader – and with the permission of the authors – I repeat the proof given in [AR90] here:

The fact that K is linear follows directly from the definition of k-quasi-invariance. Assume that K is dense in E, and let $z \in \operatorname{supp}(\nu)$. (Such z exists, otherwise ν would be the zero measure.) We want to show then that every element $z'' \in E$ of the form $z'' = z + \alpha z'$, $z' \in E$, belongs to $\text{supp}(\nu)$, and consequently also every $z'' \in E$. First we show this for $z'' = z + k$ with $k \in K$: Let V_{z+k} be an open neighborhood of $z + k$. Then $V_z := V_{z+k} - k$ is an open neighborhood of z. Thus we must have $\nu(V_z) > 0$, and k-quasi-invariance of ν implies that $\nu(V_{z+k}) > 0$. Hence $z + k \in \text{supp}(\nu)$. Now let $z' \in E$, and choose a net $(k_i,\ i \in I)$ in K which converges to z'. Then $z + \alpha z' = \lim_i (z + \alpha k_i)$. We have $z + \alpha k_i \in \text{supp}(\nu)$ for all $i \in I$. $\text{supp}(\nu)$ is closed, and thus we get $z + \alpha z' \in \text{supp}(\nu)$. □

In other words: if ν is quasi-invariant with respect to the translations of a dense linear subset, then every nonempty open set has nonvanishing ν-measure. Assume now that f and g are continuous mappings from E into a Hausdorff topological space F, which are ν-a.e. equal. If ν is k-quasi-invariant for all k in a dense set K, then they must be equal: The set $\{\omega;\ f(\omega) \neq g(\omega)\}$ is open, and the assumption that it is nonempty leads to a contradiction. For the white noise measure μ on $(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}_0)$ we get therefore the following conclusion.

COROLLARY. *Let f and g be two mappings from $\mathcal{S}'(\mathbb{R}_+)$ into a Hausdorff topological space which are continuous w.r.t. any dual topology on $\mathcal{S}'(\mathbb{R}_+)$, and which are such that $f = g$ μ-a.s. Then $f = g$. In particular, a μ-class of real or complex valued random variables on $(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}_0, \mu)$ can only have one continuous representative.*

Our next task is to introduce a Brownian motion on $(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}_0, \mu)$, and we shall follow the usual constructions. One way to do this is as follows: Consider the linear mapping $\widehat{X}_\cdot\colon \mathcal{S}(\mathbb{R}_+) \to L^2(\mu)$ where $\widehat{X}_f$ is the μ-class of the random variable given by $X_f(\omega) := \langle \omega, f \rangle$ for $f \in \mathcal{S}(\mathbb{R}_+)$, $\omega \in \mathcal{S}'(\mathbb{R}_+)$. It follows directly from the definition of μ that this mapping is continuous, if $\mathcal{S}(\mathbb{R}_+)$ is equipped with the norm of $L^2(\mathbb{R}_+)$. Therefore, it has a unique linear continuous extension to $L^2(\mathbb{R}_+)$, which we continue to denote by $\widehat{X}_\cdot$, and by X_f we mean any representative of $\widehat{X}_f$ if $f \in L^2(\mathbb{R}_+) \setminus \mathcal{S}(\mathbb{R}_+)$. It is clear that $(X_f;\ f \in L^2(\mathbb{R}_+))$ is a centered Gaussian family whose covariance is given by the inner product in $L^2(\mathbb{R}_+)$. Choose for $t \in \mathbb{R}_+$, $f = 1_{[0,t]}$, and set $\widetilde{B}_t := X_{1_{[0,t]}}$. Then it is plain to check, that the process $(\widetilde{B}_t;\ t \in \mathbb{R}_+)$ has the same finite-dimensional distributions as a Brownian motion. Therefore, we can apply the Kolmogorov–Chentsov lemma to find a modification of $(\widetilde{B}_t;\ t \in \mathbb{R}_+)$ with a.s. continuous sample paths. This modification is a standard Brownian motion, and it will be denoted by $(B_t;\ t \in \mathbb{R}_+)$.

Alternatively, we can mimic the Lévy–Ciesielski construction, e.g., [MK69]. To this end, we consider $L^2([0, 1])$ (with Lebesgue measure) as embedded into $L^2(\mathbb{R}_+)$. Let $(s_{n,k};\ n \in \mathbb{N}_0,\ k = 1, 3, \dots, 2^{n-1} - 1)$ be the system of Schauder functions on [0, 1]. That is, for $t \in \mathbb{R}_+$, $s_{n,k}(t) := \int_0^t f_{n,k}(s)\,\mathrm{d}s$, where $f_{n,k}$ is the Haar function of index (n, k), which is $2^{(n-1)/2}$ on the interval $[(k-1)/2^n, k/2^n)$, $-2^{(n-1)/2}$ on $[k/2^n, (k+1)/2^n)$ and zero otherwise. Recall that the system of Haar functions forms a complete orthonormal system on $L^2([0, 1])$ (e.g., [MK69]). It is clear that $s_{n,k}$ is continuous on [0, 1].

Now choose any CONS $(e^0_{n,k};\ n \in \mathbb{N}_0,\ k = 1, 3, \dots, 2^{n-1} - 1)$ of $L^2([0,1])$ in $C^\infty_c([0,1])$. Let $N_{n,k}(\omega) := \langle \omega, e^0_{n,k} \rangle$. Then the family $\big(N_{n,k};\ n \in \mathbb{N}_0,\ k = 1, 3, \dots, 2^{n-1} - 1\big)$ is an i.i.d. system of everywhere defined standard normal random variables. Define

$$B_t := t\, N_{0,1} + \sum_{n,k} s_{n,k}(t)\, N_{n,k}, \quad t \in [0,1].$$

Then we are in the same situation as in [MK69], p. 7, and can use the Borel–Cantelli lemma to prove that this series converges a.s. absolutely and *uniformly in* $t \in [0,1]$. Therefore, the process B_t, $t \in [0,1]$, has a.s. continuous sample paths. It is obvious that B_t is centered Gaussian, and to check that the covariance of B_s, B_t is $s \wedge t$ is an easy exercise. Thus $(B_t,\ t \in [0,1])$ is a standard Brownian motion over $[0,1]$.

For any $N \in \mathbb{N}$ we can now repeat the same construction on the interval $[N, N+1]$ with a CONS $(e^N_{n,k},\ n \in \mathbb{N}_0,\ k = 1, 3, \dots, 2^{n-1} - 1)$ of $L^2([N, N+1])$ in $C^\infty_c([N, N+1])$, and obtain an independent standard Brownian motion indexed by $t \in [N, N+1]$. Gluing these Brownian motions continuously together, we obtain a standard Brownian motion indexed by the half-line.

Let us denote the μ-completion of $\mathcal{B}_0$ by $\mathcal{B}$. (The extension of μ to $\mathcal{B}$ will be denoted by the same symbol.)

Consider our first construction of Brownian motion $(B_t,\ t \in \mathbb{R}_+)$, i.e., by modification of the random variables $\widetilde{B}_t$, $t \in \mathbb{R}_+$. We want to show that the μ-completion $\sigma^\mu(B_t,\ t \in \mathbb{R}_+)$ of $\sigma(B_t,\ t \in \mathbb{R}_+)$ generated by it is equal to $\mathcal{B}$. First we prove the following result which is almost obvious.

LEMMA. *Let $\mathcal{T}$ be any subset of $\mathcal{S}(\mathbb{R}_+)$ which is total in $\mathcal{S}(\mathbb{R}_+)$ with respect to the norm of $L^2(R_+)$ restricted to $\mathcal{S}(\mathbb{R}_+)$. Then the μ-completion of $\sigma(\mathcal{T}) := \sigma(X_f,\ f \in \mathcal{T})$ coincides with $\mathcal{B}$.*

Proof. Since all linear combinations of random variables of the form X_f with $f \in \mathcal{T}$ are $\sigma(\mathcal{T})$-measurable, we may assume without loss of generality that $\mathcal{T}$ is a dense linear subspace of $\mathcal{S}(\mathbb{R}_+)$ with respect to $|\cdot|_2$. Let $f \in \mathcal{S}(\mathbb{R}_+)$, and choose a sequence $(f_n,\ n \in \mathbb{N}_0)$ in $\mathcal{T}$ so that $|f - f_n|_2 \to 0$. Then X_{f_n} converges in mean square to X_f. By choosing a subsequence if necessary, we may assume that X_{f_n} converges to X_f μ-a.s. This implies that X_f has a modification, say

$$\widetilde{X}_f := \limsup_{n \in \mathbb{N}} X_{f_n},$$

which is measurable with respect to the σ-algebra generated by the random variables X_{f_n}, $n \in \mathbb{N}$. In particular, $\widetilde{X}_f$ is $\sigma(\mathcal{T})$-measurable, and therefore X_f is measurable with respect to the μ-completion $\sigma^\mu(\mathcal{T})$ of the σ-algebra $\sigma(\mathcal{T})$. On the other hand, the random variables X_f, $f \in \mathcal{S}(\mathbb{R}_+)$, generate $\mathcal{B}_0$, so that we have the following inclusions

$$\mathcal{B}_0 \subset \sigma^\mu(\mathcal{T}) \subset \mathcal{B}.$$

Consequently $\sigma^{\mu}(\mathcal{T}) = \mathcal{B}$. □

Let $f \in C_c^{\infty}(\mathbb{R}_+)$, and consider the random variable X_f. Define the following sequence

$$X_f^n := \sum_{k=0}^{\infty} f(s_k^n)\left(B_{s_{k+1}^n} - B_{s_k^n}\right), \quad n \in \mathbb{N},$$

where $s_k^n = k/2^n$. Observe that for each $n \in \mathbb{N}$, the series above has only a finite number of terms. It is straightforward to check that X_f^n converges to X_f in mean square sense. By an argument similar to the one in the proof of the lemma, we find that X_f is measurable with respect to $\sigma^{\mu}(B_t,\ t \in \mathbb{R}_+)$. This holds for every f in the subspace $C_c^{\infty}(\mathbb{R}_+)$ of $\mathcal{S}(\mathbb{R}_+)$, which is dense w.r.t. the norm $|\cdot|_2$. Using the lemma with $\mathcal{T} = C_c^{\infty}(\mathbb{R}_+)$ we therefore get the following inclusions:

$$\sigma\left(C_c^{\infty}(\mathbb{R}_+)\right) \subset \sigma^{\mu}(B_t,\ t \in \mathbb{R}_+) \subset \mathcal{B} = \sigma^{\mu}\left(C_c^{\infty}(\mathbb{R}_+)\right),$$

where the last σ-algebra is the μ-completion of $\sigma(C_c^{\infty}(\mathbb{R}_+))$. Hence we can conclude that

$$\sigma^{\mu}(B_t,\ t \in \mathbb{R}_+) = \mathcal{B}.$$

For the Brownian motion defined by the Lévy–Ciesielski construction we obtain the same result – more easily – in a similar way.

Thus we have completed our discussion of a realization of the hypotheses (H.1), (H.2) in the setting of the white noise space over the half-line.

We conclude this section with the following remark. Instead of $\mathcal{S}(\mathbb{R}_+)$ we could as well have chosen the smaller space $\mathcal{S}_0(\mathbb{R}_+)$ which is defined as the closure of $C_c^{\infty}(\mathbb{R}_+)$ in $\mathcal{S}(\mathbb{R})$ (or in $\mathcal{S}(R_+)$, which leads to the same space). It is clear that $\mathcal{S}(\mathbb{R}_+)$ is the subspace of those functions of $\mathcal{S}(\mathbb{R})$ which are supported in $\mathbb{R}_+$ (or those functions in $\mathcal{S}(\mathbb{R}_+)$ which together with all their derivatives vanish at the origin). As a linear subspace of $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}(\mathbb{R}_+)$, $\mathcal{S}_0(\mathbb{R}_+)$ is again nuclear. Therefore, we can repeat the preceding discussion almost word by word, with obvious modifications here and there.

4. Differential Operators

Consider a random variable Y on $(\Omega, \mathcal{B}, P)$. If Ω is a topological vector space (or admits some suitable notion of translations), like in the case of the two realizations mentioned in Section 2, it is obvious how to define directional derivatives. Since in the general case such a structure is not available, we introduce directional derivatives by imitating the chain rule. This has been done in [DP97], but the question whether the derivatives are well-defined had been left open. We start with this question here.

For $n \in \mathbb{N}$, denote by $C_e^\infty(\mathbb{R}^n)$ the space of infinitely often continuously differentiable functions on $\mathbb{R}^n$ which – together with all their derivatives – have at most exponential growth. It will be convenient to denote the ith partial derivative of $f \in C_e^\infty(\mathbb{R}^n)$ at $x \in \mathbb{R}^n$ by $f_{,i}(x)$.

By $C_{\mathrm{fi},e}^\infty(\Omega)$ we mean the space of all random variables Y for which there exist $n \in \mathbb{N}$, $h_1, \ldots, h_n \in C_c^\infty(\mathbb{R}_+)$, and $f \in C_e^\infty(\mathbb{R}^n)$ so that

$$Y = f(X_{h_1}, \ldots, X_{h_n}).$$

Note that $C_{\mathrm{fi},e}^\infty(\Omega) \subset L^p(P)$ for all $p \geqslant 1$, and these inclusions are dense.

LEMMA 1. *Assume that $Y \in C_{\mathrm{fi},e}^\infty(\Omega)$ has two representations of the form*

$$\begin{aligned} Y &= f(X_{h_1}, \ldots, X_{h_n}) \\ &= g(X_{k_1}, \ldots, X_{k_m}) \end{aligned}$$

with $n, m \in \mathbb{N}$, $h_1, \ldots, h_n$, $k_1, \ldots, k_m \in C_c^\infty(\mathbb{R}_+)$, and $f \in C_e^\infty(\mathbb{R}^n)$, $g \in C_e^\infty(\mathbb{R}^m)$. Then for all $u \in L^2(\mathbb{R}_+)$,

$$\begin{aligned} &\sum_{i=1}^n f_{,i}(X_{h_1}, \ldots, X_{h_n})\,(u, h_i)_{L^2(\mathbb{R}_+)} \\ &\quad = \sum_{j=1}^m g_{,j}(X_{k_1}, \ldots, X_{k_m})\,(u, k_j)_{L^2(\mathbb{R}_+)}, \qquad P\text{-a.s.} \end{aligned}$$

Proof. By assumption, we have for all $l \in C_c^\infty(\mathbb{R}_+)$,

$$\big(f(X_{h_1}, \ldots, X_{h_n}) - g(X_{k_1}, \ldots, X_{k_m})\big)\,\mathrm{e}^{X_l} = 0,$$

and in particular the law of this random variable is the Dirac measure ε_0 at 0.

Now consider the Gaussian random variables $Z_{h_1}, \ldots, Z_{h_n}, Z_{k_1}, \ldots, Z_{k_m}, Z_l$ in the white noise realization on $(\mathcal{S}'(\mathbb{R}_+), \mathcal{B}, \mu)$ of Section 3. That is, $Z_\eta(\omega) = \langle \omega, \eta \rangle$ for $\eta \in \mathcal{S}(\mathbb{R}_+)$, $\omega \in \mathcal{S}'(\mathbb{R}_+)$. The random variables $Z_{h_1}, \ldots, Z_l$ have the same joint distribution as $X_{h_1}, \ldots, X_l$. Therefore also the random variable

$$\big(f(Z_{h_1}, \ldots, Z_{h_n}) - g(Z_{k_1}, \ldots, Z_{k_m})\big)\,\mathrm{e}^{Z_l}$$

has ε_0 as its law. Since this is true for every $l \in C_c^\infty(\mathbb{R}_+)$, this implies that the S-transform of

$$f(Z_{h_1}, \ldots, Z_{h_n}) - g(Z_{k_1}, \ldots, Z_{k_m})$$

is zero. Because of the injectivity of the S-transform we conclude that

$$f(Z_{h_1}, \ldots, Z_{h_n}) = g(Z_{k_1}, \ldots, Z_{k_m}), \qquad \mu\text{-a.s.}$$

By construction, both sides of the last equality are weakly continuous on $\mathcal{S}'(\mathbb{R}_+)$. The corollary of Section 3 entails that these two random variables are equal *everywhere* on $\mathcal{S}'(\mathbb{R}_+)$. Hence we may compute the directional derivative of both sides in direction $u \in L^2(\mathbb{R}_+) \subset \mathcal{S}'(\mathbb{R}_+)$ as follows:

$$\begin{aligned} D_u\, f(Z_{h_1}, \ldots, Z_{h_n}) &= \sum_{i=1}^{n} f_{,i}(Z_{h_1}, \ldots, Z_{h_n})\,(u, h_i)_{L^2(\mathbb{R}_+)} \\ &= D_u\, g(Z_{k_1}, \ldots, Z_{k_m}) \\ &= \sum_{j=1}^{m} g_{,j}(Z_{k_1}, \ldots, Z_{k_m})\,(u, k_j)_{L^2(\mathbb{R}_+)}, \end{aligned}$$

because for $\eta \in \mathcal{S}(\mathbb{R}_+)$, $\omega,\ u \in \mathcal{S}'(\mathbb{R}_+)$,

$$\begin{aligned} D_u\, Z_\eta(\omega) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \langle \omega + \lambda u, \eta \rangle \bigg|_{\lambda=0} \\ &= (u, \eta)_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Consequently, we have for all $l \in C_c^\infty(\mathbb{R}_+)$ that

$$\left(\sum_{i=1}^{n} f_{,i}(Z_{h_1}, \ldots, Z_{h_n})\,(u, h_i)_{L^2(\mathbb{R}_+)} - \right. \\ \left. - \sum_{j=1}^{m} g_{,j}(Z_{k_1}, \ldots, Z_{k_m})\,(u, k_j)_{L^2(\mathbb{R}_+)} \right) \mathrm{e}^{Z_l}$$

has ε_0 as its law. Now we go back to the original probability space, i.e., replace the Gaussian random variables Z_{h_1}, etc., by X_{h_1}, etc. Then we obtain that the S-transform of

$$\sum_{i=1}^{n} f_{,i}(X_{h_1}, \ldots, X_{h_n})\,(u, h_i)_{L^2(\mathbb{R}_+)} - \sum_{j=1}^{m} g_{,j}(X_{k_1}, \ldots, X_{k_m})\,(u, k_j)_{L^2(\mathbb{R}_+)}$$

is zero, and hence this random variable is P-a.s. zero. □

The preceding lemma justifies the following

DEFINITION. Let $Y \in C^\infty_{\mathrm{fi},e}(\Omega)$ be given by

$$Y = f(X_{h_1}, \ldots, X_{h_n}),$$

with $n \in \mathbb{N}$, $h_1, \ldots, h_n \in C_c^\infty(\mathbb{R}_+)$, and $f \in C_e^\infty(\mathbb{R}^n)$. For all $u \in L^2(\mathbb{R}_+)$, the random variable in $C^\infty_{\mathrm{fi},e}(\Omega)$ defined by

$$D_u Y := \sum_{i=1}^{n} f_{,i}(X_{h_1}, \ldots, X_{h_n})\,(u, h_i)_{L^2(\mathbb{R}_+)}$$

is called the *derivative of Y in direction u*.

For every $u \in L^2(\mathbb{R}_+)$, D_u is a densely defined operator on $L^2(P)$. Thus it has an adjoint D_u^* with domain $\mathcal{D}(D_u^*)$. It is not very difficult to check that $C^\infty_{\text{fi},e}(\Omega)$ is a subset of $\mathcal{D}(D_u^*)$, and that on $Y \in C^\infty_{\text{fi},e}(\Omega)$ it acts as

$$D_u^* Y = X_u\, Y - D_u Y.$$

In particular, D_u is closable on $L^2(P)$, and we denote its closure by $(D_u,\ \mathcal{D}(D_u))$. At this point we can repeat all arguments, as, e.g., given in [HK93], to show that actually

$$\mathcal{D}(D_u) = \mathcal{D}(D_u^*) = \mathcal{D}(X_u \cdot) = \mathcal{D}(N^{1/2}),$$

where N is the number operator, which can – for example – be defined on the core $C^\infty_{\text{fi},e}(\Omega)$ by the formula

$$N = \sum_{k \in \mathbb{N}} D_{e_k}^*\, D_{e_k},$$

with a CONS $(e_k,\ k \in \mathbb{N})$ of $L^2(\mathbb{R}_+)$.

The following result will be useful.

LEMMA 2. *For every $u \in L^2(\mathbb{R}_+)$, D_u commutes with the S-transform in the sense that for all $Y \in \mathcal{D}(N^{1/2})$, $f \in L^2(\mathbb{R}_+)$,*

$$S(D_u Y)(f) = D_u(SY)(f),$$

where the right-hand side is the usual Gâteaux derivative of a function on $L^2(\mathbb{R}_+)$.

Proof. Compute

$$\begin{aligned} S(D_u Y)(f) &= \mathbb{E}\big(D_u Y : \mathrm{e}^{X_f} :\big) \\ &= \mathbb{E}\big(Y\, D_u^* : \mathrm{e}^{X_f} :\big) \\ &= \mathbb{E}\big(Y\, \big(X_u - (u, f)\big) : \mathrm{e}^{X_f} :\big), \end{aligned}$$

where we have set

$$: \mathrm{e}^{X_f} : = \mathrm{e}^{X_f - 1/2\, |f|_2^2}.$$

On the other hand, we have

$$\begin{aligned} D_u(SY)(f) &= \frac{\partial}{\partial \lambda}\, (SY)(f + \lambda\, u)\Big|_{\lambda=0} \\ &= \frac{\partial}{\partial \lambda}\, \mathbb{E}\big(Y : \mathrm{e}^{X_{f+\lambda u}} :\big)\Big|_{\lambda=0} \\ &= \frac{\partial}{\partial \lambda}\, \mathbb{E}\big(Y\, \mathrm{e}^{X_f + \lambda\, X_u - 1/2\, |f + \lambda\, u|_2^2}\big)\Big|_{\lambda=0} \\ &= \mathbb{E}\big(Y\, \big(X_u - (f, u)\big) : \mathrm{e}^{X_f} :\big), \end{aligned}$$

where the interchange of the derivative w.r.t. λ and the expectation is readily justified, e.g., by an application of the dominated convergence theorem. □

Consider again the formula for $D_u Y$ with $Y \in C^\infty_{\mathrm{fi},e}(\Omega)$. We may write it as follows

$$D_u Y = \int_{\mathbb{R}_+} (\partial_t Y)\, u(t)\, \mathrm{d}t,$$

where we have put

$$\partial_t Y := \sum_{i=1}^n f_{,i}(X_{h_1}, \dots, X_{h_n})\, h_i(t).$$

The integral above could of course be interpreted pointwise on Ω, but we shall take in the sense of an $L^2(P)$-valued Pettis integral (e.g., [HP57]). Therefore we have for all $Z \in L^2(P)$ the relation

$$\mathbb{E}\Big(Z \int_{\mathbb{R}_+} (\partial_t Y)\, u(t)\, \mathrm{d}t\Big) = \int_{\mathbb{R}_+} \mathbb{E}\big(Z\,(\partial_t Y)\big)\, u(t)\, \mathrm{d}t.$$

Choosing in particular $Z = \exp(X_f - \frac{1}{2}|f|_2^2)$ with $f \in C^\infty_c(\mathbb{R}_+)$, we find that

$$S(D_u Y)(f) = \int_{\mathbb{R}_+} S(\partial_t Y)(f)\, u(t)\, \mathrm{d}t.$$

We are interested to compute the S-transform of $\partial_t Y$. To this end, let us prove first the following result

LEMMA 3. *For all $f \in C^\infty_c(\mathbb{R}_+)$, $Y \in C^\infty_{\mathrm{fi},e}(\Omega)$, the mapping*

$$u \mapsto D_u(SY)(f)$$

from $L^2(\mathbb{R}_+)$ into $\mathbb{R}$ is linear and continuous.

Proof. That for all $Y \in C^\infty_{\mathrm{fi},e}(\Omega)$, the mapping $u \mapsto D_u Y$ is linear and continuous from $L^2(\mathbb{R}_+)$ into $L^2(P)$, is obvious from the definition. On the other hand, the S-transform of a random variable is its $L^2(P)$-inner product with $\exp(X_f - \frac{1}{2}|f|_2^2)$, and hence it is clear that this is a continuous linear map from $L^2(P)$ into $\mathbb{R}$. Thus $u \mapsto S(D_u Y)(f)$ is linear and continuous from $L^2(\mathbb{R}_+)$ into $\mathbb{R}$. The proof is finished by an application of Lemma 2. □

From Lemma 3 we conclude that for given $Y \in C^\infty_{\mathrm{fi},e}(\Omega)$, $f \in C^\infty_c(\mathbb{R}_+)$, there exists an element in $L^2(\mathbb{R}_+)$ which we denote by

$$t \mapsto \frac{\delta}{\delta f(t)} SY(f)$$

so that for all $u \in L^2(\mathbb{R}_+)$,

$$D_u(SY)(f) = \int_{\mathbb{R}_+} \frac{\delta}{\delta f(t)} SY(f)\, u(t)\, \mathrm{d}t.$$

$\delta\, SY(f)/\delta f(t)$ is also called the *Fréchet functional derivative of* SY *at* f. If we use again the fact that D_u and S commute, we arrive at the following (slightly informal) intertwining relation for ∂_t and S

$$\partial_t = S^{-1} \frac{\delta}{\delta f(t)} S,$$

which is essentially T. Hida's original definition of ∂_t in the ground-breaking paper [Hi75].

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