Construction of Relativistic Quantum Fields in the Framework of White Noise Analysis

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Abstract

We construct a class of Euclidean invariant distributions Φ_H indexed by a function H holomorphic at zero. These generalized functions can be considered as generalized densities w.r.t. the white noise measure and their moments fulfill all Osterwalder-Schrader axioms except for reflection positivity.

The case where $F(s) = -(H(is) + \frac{1}{2}s^2)$, $s \in \mathbb{R}$, is a Lévy characteristic is considered in [AGW96]. Under this assumption the moments of the Euclidean invariant distributions Φ_H can be represented as moments of a generalized white noise measure P_H.

Here we enlarge this class by convolution with kernels G coming from Euclidean invariant operators $\mathcal{G}$. The moments of the resulting Euclidean invariant distributions Φ_H^G also fulfill all Osterwalder-Schrader axioms except for reflection positivity.

For no nontrivial case we succeeded in proving reflection positivity. Nevertheless, an analytic extension to Wightman functions can be performed. These functions fulfill all Wightman axioms except for the positivity condition. Moreover, we can show that they fulfill the Hilbert space structure condition and therefore the modified Wightman axioms of indefinite metric quantum field theory, see [Str93].

Contents

1 Introduction

This note is motivated by the Euclidean strategy for constructing interacting field theories, see e.g. [Sim74] and the references therein. Formally, the interacting field theory with interaction V lives on the same measure space as the Euclidean free field μ_0 but has measure:

$$d\mu_V = \frac{\exp\Big(- \int_{\mathbb{R}^d} V(\phi(x))\, d^d(x)\Big)\, d\mu_0}{\int \exp\Big(- \int_{\mathbb{R}^d} V(\phi(x))\, d^d(x)\Big)\, d\mu_0}, \tag{1}$$

where $\phi(x)$ is a Gaussian random process at the point $x \in \mathbb{R}^d$, $d \in \mathbb{N}$. Since $\phi(x)$ in general is not an integrable function but rather a generalized function to define $V(\phi(x))$ leads to the problem of defining powers of $\phi(x)$. Furthermore, (1) cannot hope to be anything but formal if $\mu_V \neq \mu_0$ since the only probability measure absolutely continuous w.r.t. μ_0 and invariant under Euclidean translations is μ_0 itself.

A first step in the direction of giving sense to (1) is to construct Wick powers $:\phi(x)^m:$, $m \in \mathbb{N}$, of the Gaussian random process ϕ at the point $x \in \mathbb{R}^d$. For $d \geq 2$ the Wick powers $:\phi(x)^m:$ still are not integrable functions. Additionally, a so called space cut-off is necessary, i.e., the integration in (1) is performed only over a bounded subset of $\mathbb{R}^d$, and sometimes also an ultraviolet cut-off, i.e., the Wick powers $:\phi(x)^m:$ are smeared out with delta sequences. For certain classes of interactions by these renormalizations integrable densities w.r.t. μ_0 have been constructed and then some kind of limit which removes the cut-offs has been taken; a limit which does not require the output to be absolute continuous. Examples are: the $P(\phi)_2$-model (the 2 stands for $d = 2$) where the interaction is given by a Wick ordered polynomial $V = :P:$, semi-bounded from below, see e.g. [Sim74] and the references therein; the Høegh-Krohn model, see [Hø71], in $d = 2$ space-time dimensions where the interaction is given by

$$V(s) = \int :\exp(as):\, d\nu(a),$$

where ν is a finite measure with compact support in the interval $(\sqrt{2\pi}, \sqrt{2\pi})$; and the Albeverio Høegh-Krohn model, see [AH73], in d space-time dimensions where the interaction is given by the Fourier transform of a measure ν

with bounded support on the real line (and $d\nu(s) = \overline{d\nu(-s)}$), i.e.,

$$V(s) = \int \exp(ias)\, d\nu(a).$$

The Schwinger functions associated to the interacting field theory with interaction V are the moments of the measure ν_V. But moments one can also obtain from generalized functions considered as generalized densities w.r.t. a Gaussian measure μ, they only have to have the property that monomials are test functions. This is the basic idea of our approach. Motivated by the Euclidean strategy we consider the following generalized white noise functional

$$\Phi_H = \exp^\diamond \left(- \int_{\mathbb{R}^d} H^\diamond(\phi(x))\, d^d x \right). \tag{2}$$

We assume that the function H is holomorphic at zero and $H(0) = 0$. The Wick analytic function $H^\diamond(\phi(x))$ of the Gaussian process ϕ at the point $x \in \mathbb{R}^d$ coincides with the usual Wick ordered function $:H(\phi(x)):$. It turns out that $H^\diamond(\phi(x))$ is a generalized function from the Kondratiev space $(S)^{-1}$, see Section 2.2, and therefore also its integral, if it exists, is in $(S)^{-1}$. Thus, in general we can not take its exponential. But in the white noise distribution space $(S)^{-1}$ there exists the so called Wick calculus, see Section 2.2, hence we can take its Wick exponential. In the case where H is linear and if we integrate only over $K \subset \mathbb{R}^2$, K compact (space cut-off), the function Φ_H is square-integrable and we have direct correspondence between (1) and (2), i.e.,

$$\Phi_H = \frac{\exp\left(- \int_K H(\phi(x))\, d^2(x) \right)}{\int \exp\left(- \int_K H(\phi(x))\, d^2(x) \right) d\mu},$$

where μ is the Gaussian white noise measure. In general, however, there is no need for the distribution Φ_H to be positive and for a large class of functions H there exits no measure which is representing Φ_H. It turns out that Φ_H can be represented by a measure if and only if the function $F(s) = -H(is) + \frac{1}{2}s^2$, $s \in \mathbb{R}$, is a Lévy characteristic, see Remark 3.7 (ii). The associated measures are called generalized white noise measures.

Generalized white noise measures have been considered in [AGW96]. There the authors constructed Euclidean random fields over $\mathbb{R}^d$ by convoluting generalized white noise with integral kernels G coming from Euclidean invariant operators. The corresponding moments satisfy all Osterwalder-Schrader axioms, see [OS73], except for reflection positivity.

For all convoluted generalized white noise measures such that the Lévy characteristic of the generalized white noise measure has a holomorphic extension at zero we can give an explicit formula for the generalized density w.r.t. the white noise measure, see Theorem 3.9. Furthermore, there exists a large class of generalized function Φ_H as in (2) which do not have an associated measure, see Remark 3.16. In Theorem 3.9 and Theorem 3.15 we prove that the Schwinger functions corresponding to the convoluted generalized functions Φ_H^G also fulfill all Osterwalder-Schrader axioms except for reflection positivity.

For no nontrivial case we succeeded in proving reflection positivity. In [AGW96] the authors present a partial negative result on reflection positivity for the Schwinger functions corresponding to moments of convoluted generalized white noise. More details about their results we quote in Section 4.1.

Without reflection positivity we can not perform the analytic continuation to Wightman functions via the reconstruction theorem proved in [OS73]. Nevertheless, an analytic continuation can be done. Using results from the theory of Laplace transforms in [AGW96] the authors analytically continued the Schwinger functions which are given as moments of convoluted generalized white noise to Wightman functions. In general, these functions only fulfill a part of the Wightman axioms, i.e., positivity (positive definiteness of the set of Wightman functions [SW64], [Jos65], and [BLT75]) is missing. We generalized their idea to our case and in Theorem 4.1 we prove that the Schwinger functions corresponding to convoluted generalized functions Φ_H^G also have an analytic extension to Wightman functions. These Wightman functions fulfill all Wightman axioms except for the positivity property. Furthermore, they fulfill the strong spectral condition with mass gap $m_0 > 0$ and their 2-point functions admit a Källen-Lehmann representation. For the Fourier transform of the truncated Wightman functions in [AGW96] the authors found explicit formulas. Using these formulas and the Jost-Schroer theorem, in Theorem 4.2 we prove a negative result concerning the positivity property, see also Remark 4.3.

Since the appearance of gauge theories it has become natural to consider (local) quantum field theory (QFT) in which not all Wightman axioms are satisfied. Such a consideration has in particular been natural and also necessary for the study of "charged" fields interacting with gauge fields, because their description conflicts either with locality or with positivity. The physical reason for this is that in such theories one must use observables of the charged type which obey a Gaussian law, see e.g. Morchio and Strocchi [MS80], instead of using the usual local observables. Actually, from the study of fields such as e.g. α-gauge type Higgs models which do not satisfy positivity, see e.g. [JS88] and references therein, it turns out that it is better to keep the locality condition and to give up the positivity condition. This leads to the so called modified Wightman axioms of indefinite metric QFT, see [Str93]. The difference between indefinite metric QFT and standard QFT is that the axiom of positivity in the latter is replaced by the so called Hilbert space structure condition in the former which permits the construction of a Hilbert space and a field operator associated to a given collection of functions fulfilling the modified Wightman axioms.

In [AGW97a] the authors proved that the Wightman functions which are analytic continuations of the moments of convoluted generalized white noise fulfill the Hilbert space structure condition and therefore the modified Wightman axioms. Again it was possible to generalize their proof to our case and in Theorem 4.5 we prove that the Wightman functions which are analytic continuation of the moments of convoluted generalized functions Φ_H^G also fulfill the modified Wightman axioms.

The paper is organized as follows. In Section 2 we introduce the concepts of Gaussian and white noise analysis as far as necessary for our considerations. For a detailed exposition we refer to the monographs [Hid80], [BK95], [HKPS93], [Oba94], [HØUZ96], and [Kuo96]. In the framework of white noise analysis various aspects of QFT have been discussed, see [AHP$^+$90a], [AHP$^+$90b], [AHPS89], [PS90], and [HKPS93]. Section 3 of this note is attended to represent Euclidean QFT in the framework of white noise analysis. In Sections 3.1 we show how to check the Osterwalder-Schrader axioms (OS axioms) in terms of the T-transform (the T-transform is an infinite dimensional generalization of the Fourier-transform). The T-transform of a generalized function gives us the generating functionals of the corresponding Schwinger functions. Properties of generating functionals have also been discussed in [Frö74a] and [Frö77]. Having this tool in hands in Section 3.2 we construct the Euclidean invariant distributions Φ_H^G. In Section 4 we discuss the reflection positivity, analytic continuation and QFT with indefinite metric.

2 Gaussian analysis

2.1 Gaussian spaces

We start by considering the Gel'fand triple

$$S(\mathbb{R}^d) \subset \mathcal{H} \subset S'(\mathbb{R}^d),$$

where $S(\mathbb{R}^d)$ is the space of rapidly decreasing, smooth test functions on $\mathbb{R}^d$. We assume $S(\mathbb{R}^d)$ to be equiped with its standard locally convex topology such that it is a nuclear space. $\mathcal{H}$ is a real separable Hilbert space containing $S(\mathbb{R}^d)$ as a dense and topological subspace. For instance, $\mathcal{H}$ can be chosen as the space of real valued square integrable functions w.r.t. the Lebesgue measure on $\mathbb{R}^d$ or as a Sobolev space on $\mathbb{R}^d$. As is well-known, see e.g. [Pie72] and [Sch71], $S = S(\mathbb{R}^d)$ is the projective limit of a family of Hilbert spaces $(\mathcal{H}_p)_{p\in\mathbb{N}_0}$, $\mathcal{H}_0 = \mathcal{H}$, such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $\mathcal{H}_p \subset \mathcal{H}_{p_1}$ and $\mathcal{H}_p \subset \mathcal{H}_{p_2}$ and the embeddings are of Hilbert-Schmidt class. I.e., S is a countably Hilbert space in the sense of [GV68]. The dual space space S' is the space of tempered distributions. It is given as the inductive limit the spaces $(\mathcal{H}_{-p})_{p\in\mathbb{N}_0}$ which are dual to the spaces $(\mathcal{H}_p)_{p\in\mathbb{N}}$ w.r.t. $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle$ the dual pairings between $\mathcal{H}_p$ and $\mathcal{H}_{-p}$ and between S and S' given by the extension of the inner product $(\cdot,\cdot)$ on $\mathcal{H}$. Furthermore, $|\cdot|_p$ denote the norms on $\mathcal{H}_p$ and $\mathcal{H}_{-p}$ respectively and we preserve this notation for the norms on the complexifications $\mathcal{H}_{p,\mathbb{C}}$ and $\mathcal{H}_{-p,\mathbb{C}}$ and tensor powers of these spaces.

Additionally, we introduce the notion of symmetric tensor power of the nuclear space S. The simplest way to do this is to start from usual symmetric tensor powers $\mathcal{H}_p^{\hat{\otimes}n}, n \in \mathbb{N}$, of Hilbert spaces. Using the definition

$$S^{\hat{\otimes}n} := \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p^{\hat{\otimes}n}$$

one can prove, see e.g [Pie72] and [Sch71], that $S^{\hat{\otimes}n}$ is a nuclear space which is called the n-th symmetric tensor power of S. The dual space $S'^{\hat{\otimes}n}$ can be written as

$$S'^{\hat{\otimes}n} = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\hat{\otimes}n}.$$

The space $S'(\mathbb{R}^d)^{\hat{\otimes} n}$ is canonically isomorphic to $\widehat{S'(\mathbb{R}^{nd})}$, the space of symmetric tempered distributions on $\mathbb{R}^{nd}$. All the results quoted above also hold for complex spaces.

In order to introduce a probability measure on the vector space S' we consider the σ-algebra $\mathcal{C}_\sigma(S')$ generated by cylinder sets. The canonical Gaussian measure μ on $(S', \mathcal{C}_\sigma(S'))$ is given by its characteristic function

$$\int_{S'} \exp(i\langle \omega, f\rangle)\, d\mu(\omega) = \exp(-\tfrac{1}{2}|f|^2), \quad f \in S,$$

via Minlos' theorem, see e.g. [BK95], [Hid80] and [HKPS93]. If we chose $\mathcal{H} = L^2(\mathbb{R}^d)$, the space of real valued square-integrable functions w.r.t. the Lebesgue measure on $\mathbb{R}^d$, this is the Gaussian white noise measure. For $\mathcal{H} = H^{-1,2}(\mathbb{R}^d)$, the Sobolev space of order $(-1, 2)$, this is the measure corresponding to the Euclidean free field with mass 1 in d dimensions.

The central space in our setup is the space of complex valued functions which are square-integrable w.r.t. this measure $L^2(\mu) = L^2(S', \mathcal{C}_\sigma(S'), \mu)$. An element of this space is the Wick exponential

$$\begin{aligned} :\exp(\langle \omega, f\rangle): \;&:=\; \frac{\exp(\langle \omega, f\rangle)}{\mathbb{E}_\mu(\exp(\langle \cdot, f\rangle))}, \quad \omega \in S',\ f \in S, \qquad (3)\\ &=\; \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\omega^{\otimes n}:, f^{\otimes n}\rangle. \end{aligned}$$

$\mathbb{E}_\mu$ denotes the expectation w.r.t. μ and the map $S' \ni \omega \mapsto :\omega^{\otimes n}: \in S'^{\hat{\otimes} n}$, $n \in \mathbb{N}$, is called the n-th Wick power of $\omega \in S'$ ($\langle :\omega^{\otimes 0}:, f^{\otimes 0}\rangle := f^{\otimes 0} := 1$), see e.g. [BK95] or [HKPS93]. For any $\varphi^{(n)} \in S_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, $\varphi^{(0)} \in \mathbb{C}$, we define the smooth Wick monomial of order n corresponding to the kernel $\varphi^{(n)}$ as follows:

$$I(\varphi^{(n)})(\omega) \;:=\; \langle :\omega^{\otimes n}:, \varphi^{(n)}\rangle, \quad \omega \in S',\ n \in \mathbb{N}_0.$$

Smooth Wick monomials of different order are orthogonal w.r.t. the standard inner product in $L^2(\mu)$. Furthermore, we can construct Wick monomials $I(f^{(n)})$ with kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ in the sense of measurable functions by using an approximation. More precisely, for any sequence $(\varphi_j^{(n)})_{j\in\mathbb{N}} \subset S_{\mathbb{C}}^{\hat{\otimes} n}$ which converges to $f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ we have the convergence of $I(\varphi^{(n)})$ to $I(f^{(n)})$ in any $L^p(\mu), p \geq 1$, see e.g. [BK95]. We use $I(f^{(n)}) = \langle :\omega^{\otimes n}:, f^{(n)}\rangle$, as a formal notation for the monomial introduced above. For Wick monomials associated to the kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ and $h^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}m}$, $n, m \in \mathbb{N}_0$, we have the following orthogonality property:

$$\begin{aligned}\Big(I(f^{(n)}), I(h^{(m)})\Big)_{L^2(\mu)} &= \int_{S'} \overline{\langle :\omega^{\otimes n}:, f^{(n)}\rangle}\langle :\omega^{\otimes m}:, h^{(m)}\rangle \, d\mu(\omega) \\ &= \delta_{n,m}\, n!\, (\overline{f^{(n)}}, h^{(n)}) \qquad (4)\end{aligned}$$

($\delta_{n,m}$ is the Kronecker delta).

Consider the space $\mathcal{P}(S')$ of smooth Wick polynomial on S':

$$\mathcal{P}(S') = \left\{\varphi \middle| \varphi(\omega) = \sum_{n=0}^{N} \langle :\omega^{\otimes n}:, \varphi^{(n)}\rangle, \varphi^{(n)} \in S_{\mathbb{C}}^{\hat{\otimes}n}, \omega \in S', N \in \mathbb{N}_0\right\}.$$

This space is dense in $L^2(\mu)$ and, as a consequence, for any $f \in L^2(\mu)$ we have Itô-Segal-Wiener chaos decomposition

$$f = \sum_{n=0}^{\infty} I(f^{(n)}), \quad f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}.$$

2.2 Generalized functions

For our considerations the space $L^2(\mu)$ is too small. A convenient way to solve this problem is to introduce a space of test functions in $L^2(\mu)$ and to use its larger dual space. In Gaussian analysis there exist various triples of test and generalized functions with $L^2(\mu)$ as a central space, here we choose the Kondratiev triple

$$(S)^1 \subset L^2(\mu) \subset (S)^{-1},$$

see [KLS96]. In order to construct these spaces of test and generalized functions, we define for any given $p, q \in \mathbb{Z}$ the following Hilbertian norm for the smooth Wick polynomials $\varphi(\omega) = \sum_{n=0}^{N} \langle :\omega^{\otimes n}:, \varphi^{(n)}\rangle$, $\omega \in S'$:

$$\| \varphi \|_{p,q,1}^2 := \sum_{n=0}^{\infty} (n!)^2\, 2^{nq} \left|\varphi^{(n)}\right|_p^2.$$

Then for $p, q \in \mathbb{N}_0$, we define the Hilbert space $(\mathcal{H}_p)_q^1$ as the completion of $\mathcal{P}(S')$ w.r.t. $\| \cdot \|_{p,q,1}$. Or, equivalently,

$$(\mathcal{H}_p)_q^1 = \left\{f \in L^2(\mu) \middle| f(\omega) = \sum_{n=0}^{\infty} \langle :\omega^{\otimes n}:, f^{(n)}\rangle,\ \| f \|_{p,q,1}^2 < \infty\right\}.$$

Finally, the space of test functions $(S)^1$ is defined as the projective limit of the spaces $(\mathcal{H}_p)^1_q$

$$(S)^1 = \bigcap_{p,q\geq 0} (\mathcal{H}_p)^1_q.$$

Let $(\mathcal{H}_{-p})^{-1}_{-q}$ be the dual w.r.t. $L^2(\mu)$ of $(\mathcal{H}_p)^1_q$ and let $(S)^{-1}$ be the dual w.r.t. $L^2(\mu)$ of $(S)^1$. We know from general duality theory that

$$(S)^{-1} = \bigcup_{p,q\geq 0} (\mathcal{H}_{-p})^{-1}_{-q}.$$

The bilinear dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $(S)^1$ and $(S)^{-1}$ is connected to the sesquilinear inner product on $L^2(\mu)$ by

$$\langle\!\langle f, \varphi \rangle\!\rangle = (\overline{f}, \varphi)_{L^2(\mu)}, \quad f \in L^2(\mu),\, \varphi \in (S)^1. \tag{5}$$

Since the constant function 1 is in $(S)^1$ we may extend the notion of expectation from integrable functions to distributions $\Phi \in (S)^{-1}$:

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle.$$

The chaos decomposition introduces the following natural decomposition of $\Phi \in (S)^{-1}$. Let $\Phi^{(n)} \in S'^{\hat{\otimes} n}_{\mathbb{C}}$ be given. Then there exists a distribution $I(\Phi^{(n)})$ acting on test functions $\varphi \in (S)^1$ as

$$\left\langle\!\!\left\langle I(\Phi^{(n)}), \varphi \right\rangle\!\!\right\rangle = n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle.$$

We use $I(\Phi^{(n)}) = \langle : \omega^{\otimes n} :, \Phi^{(n)} \rangle$, as a formal notation for the distribution introduced above. Any $\Phi \in (S)^{-1}$ then has the unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, \Phi^{(n)} \rangle, \tag{6}$$

where the sum converges in $(S)^{-1}$ and we have

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle \Phi^{(n)}, \varphi^{(n)} \rangle, \quad \varphi \in (S)^1,$$

see [KLS96]. Now it is not hard to see that $(\mathcal{H}_{-p})^{-1}_{-q}$ is a Hilbert space which can be described as follows:

$$(\mathcal{H}_{-p})^{-1}_{-q} = \left\{\Phi \in (S)^{-1} \middle| \Phi^{(n)} \in S'^{\hat{\otimes} n}_{\mathbb{C}}, \parallel \Phi \parallel_{-p,-q,-1} < \infty \right\}.$$

An useful tool in order to characterize $(S)^{-1}$ is the S-transform. The S-transform of elements from $(S)^{-1}$ is defined as the dual paring with the Wick exponential, see (3). Since the Wick exponential is not an element of $(S)^1$ the S-transform of an element Φ from $(S)^{-1}$ is defined only locally, i.e.,

$$S\Phi(g) := \langle\!\langle \Phi, : \exp(\langle \cdot, g\rangle) : \rangle\!\rangle, \quad g \in \mathcal{U} \subset S_{\mathbb{C}},$$

where $\mathcal{U}$ is an open neighborhood of zero depending on $\Phi \in (S)^{-1}$.

In order to characterize $(S)^{-1}$ we need to define holomorphic functions.

Definition 2.1 *Let $\mathcal{U} \subset S_{\mathbb{C}}$ be an open neighborhood of zero in $S_{\mathbb{C}}$. We say that*

$$F : \mathcal{U} \to \mathbb{C}$$

is holomorphic in $\mathcal{U}$ if it satisfies the following two properties:
(i) For each $g_0 \in \mathcal{U}$, $g \in S_{\mathbb{C}}$, there exists a neighborhood $V_{g_0,g}$ around zero in $\mathbb{C}$ such that

$$z \mapsto F(g_0 + zg)$$

is holomorphic.
(ii) For each $g \in \mathcal{U}$ there exists an open set $\mathcal{V} \subset \mathcal{U}$ containing g such that $F(\mathcal{V})$ is bounded.

Furthermore, if we identify two functions F_1 and F_2 coinciding on a neighborhood of zero, we can define $Hol_0(S_{\mathbb{C}})$ as the space of germs of functions with the above properties.

The proof of the following characterization theorem is given in [KLS96].

Theorem 2.2 *(i) If $\Phi \in (S)^{-1}$, then $S\Phi \in Hol_0(S_{\mathbb{C}})$.*
(ii) For any $F \in Hol_0(S_{\mathbb{C}})$ there exists a unique $\Phi \in (S)^{-1}$ such that $S\Phi = F$.

As a consequence of this characterization we have the following corollary, for a proof we again refer to [KLS96].

Corollary 2.3 *Let* $(\Lambda, \mathcal{A}, \nu)$ *be a measure space and* $\lambda \mapsto \Phi_\lambda$ *a mapping from* Λ *to* $(S)^{-1}$*. We assume that there exists an open neighborhood* $\mathcal{U} \subset S_{\mathbb{C}}$ *of zero such that*
(i) $S\Phi_\lambda$*,* $\lambda \in \Lambda$*, is holomorphic on* $\mathcal{U}$*,*
(ii) the mapping $\lambda \mapsto S\Phi_\lambda(g)$ *is measurable for every* $g \in \mathcal{U}$*, and*
(iii) there exists $C \in L^1(\Lambda, \nu)$ *such that* $|S\Phi_\lambda(g)| \leq C(\lambda)$ *for all* $g \in \mathcal{U}$ *and for* ν*-almost all* $\lambda \in \Lambda$*.*
Then there are $p, q \in \mathbb{N}_0$ *such that* $\Phi.$ *is Bochner integrable on* $(\mathcal{H}_{-p})^{-1}_{-q}$*. In particular:*

$$\int_\Lambda \Phi_\lambda \, d\nu(\lambda) \in (S)^{-1}.$$

Later on we also use the T-transform of generalized functions. This transform can be defined as

$$T\Phi(g) := \exp(-\tfrac{1}{2}|g|^2) \cdot S\Phi(ig), \quad \Phi \in (S)^{-1}, \, g \in \mathcal{U}. \tag{7}$$

An elementary calculation shows that the T-transform is also given by

$$T\Phi(g) = \langle\!\langle \Phi, i \exp(\langle \cdot, g \rangle) \rangle\!\rangle, \quad g \in \mathcal{U}. \tag{8}$$

The characterization theorem and its corollary are also valid for the T-transform.

For elements from $(S)^{-1}$ we can define the Wick product:

Definition 2.4 *Let* $\Phi, \Psi \in (S)^{-1}$*. Then we define the Wick product by*

$$\Phi \diamond \Psi := S^{-1}(S\Phi \cdot S\Psi).$$

This is well-defined because $Hol_0(S_{\mathbb{C}})$ is an algebra and thus by the characterization theorem in $(S)^{-1}$ there exists an element $\Phi \diamond \Psi$ such that $S(\Phi \diamond \Psi) = S\Phi \cdot S\Psi$. Clearly, this multiplication is associative.

By induction, we can define Wick powers

$$\Phi^{\diamond n} = S^{-1}((S\Phi)^n)$$

in $(S)^{-1}$ and by taking finite linear combinations of them also Wick polynomials of finite order $\sum_{n=1}^{N} a_n \Phi^{\diamond n}$ can be defined in $(S)^{-1}$. Moreover, it is even possible to define Wick analytic functions in $(S)^{-1}$ under very general assumptions.

Theorem 2.5 *Let F be analytic in a neighborhood of the point $z_0 = \mathbb{E}_\mu(\Phi)$ in $\mathbb{C}$, $\Phi \in (S)^{-1}$. Then $F^\diamond(\Phi)$ defined as $F^\diamond(\Phi) := S^{-1}(F(S\Phi))$ exists in $(S)^{-1}$.*

For a proof we refer to [KLS96].

Remark 2.6 *Let F be analytic at $z_0 = \mathbb{E}_\mu(\Phi)$, $\Phi \in (S)^{-1}$, i.e., F has the power series representation $F(z) = \sum_n a_n(z - z_0)^n$, $z, a_n \in \mathbb{C}$. Then the Wick series $\sum_n a_n(\Phi - z_0)^{\diamond n}$ converges in S^{-1} and*

$$F^\diamond(\Phi) = \sum_{n=1}^{\infty} a_n(\Phi - z_0)^{\diamond n}.$$

3 Euclidean QFT in the framework of white noise analysis

3.1 OS axioms in terms of the T-transform

In 1973, E. Nelson [Nel73] showed how to construct a relativistic QFT from a Euclidean Markov field. Inspired by this, in [OS73] and [OS75] Osterwalder and Schrader (see also [Gla74], [Heg74], [Zin95]) gave a set of axioms, where Schwinger functions $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ defined on the Euclidean space-time can be analytically continued to Wightman distributions, i.e., to vacuum expectation values of a relativistic QFT. The OS axioms are:

OS1 (temperedness) The sequence $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ is a sequence of tempered distributions, where $\mathcal{S}_n \in S'_{\mathbb{C}}(\mathbb{R}^{dn})$ and $S_0 = 1$. There exists $p \in \mathbb{N}$ and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of factorial growth such that for all $n \in \mathbb{N}$ the Schwinger functions fulfill the growth condition

$$|\mathcal{S}_n(f_1 \otimes \ldots \otimes f_n)| \leq \sigma_n \prod_{i=1}^{n} |f_i|_p$$

where $f_1, \ldots, f_n \in S_{\mathbb{C}}(\mathbb{R}^d)$. A sequence $(\sigma_n)_{n\in\mathbb{N}}$ of positive numbers is said to be of factorial growth if the exist constants $\alpha, \beta \in \mathbb{R}^+$ such that

$$\sigma_n \leq \alpha(n!)^\beta, \quad \forall n \in \mathbb{N}.$$

OS2 (Euclidean invariance) Each $\mathcal{S}_n$ is Euclidean invariant, i.e.,

$$\mathcal{S}_n(E_{(a,\Lambda)}f) = \mathcal{S}_n(f), \quad \forall f \in S_{\mathbb{C}}(\mathbb{R}^{dn}),$$

for all $(a, \Lambda) \in E^+(\mathbb{R}^d)$, the proper Euclidean group where

$$E_{(a,\Lambda)}f(x_1, \ldots, x_n) = f(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a)),$$

for $a \in \mathbb{R}$, $\Lambda \in SO(d)$.
OS3 (reflection positivity) For each sequence $(f_n)_{n\in\mathbb{N}_0}$ where $f_n \in S_{\mathbb{C}}(\mathbb{R}^{dn}_<)$, $f_0 \in \mathbb{C}$, and each $k \in \mathbb{N}_0$

$$\sum_{n,m=0}^{k} S_{n+m}((\theta f_n)^\star \otimes f_m) \geq 0,$$

where $(\theta f_n)(t_1, \vec{x_1}; \ldots; t_n, \vec{x_n}) = f_n(-t_1, \vec{x_1}; \ldots; -t_n, \vec{x_n})$, $t_i \in \mathbb{R}, \vec{x_i} \in \mathbb{R}^{d-1}$, (time reflection), $f_n^\star(x_1, \ldots, x_n) := \overline{f_n(x_n, \ldots, x_1)}$ and the bar denotes complex conjugation. The space $S_{\mathbb{C}}(\mathbb{R}^{dn}_<)$ is the space of Schwartz test functions having support in $\mathbb{R}^{dn}_< := \{(t_1, \vec{x_1}; \ldots; t_n, \vec{x_n}) \in \mathbb{R}^{dn} | 0 < t_1 < \ldots < t_n\}$.
OS4 (symmetry) For $n \geq 2$ and all $\pi \in \Sigma_n$, the permutation group

$$\mathcal{S}_n(f_1 \otimes \ldots \otimes f_n) = \mathcal{S}_n(f_{\pi(1)} \otimes \ldots \otimes f_{\pi(n)}),$$

where $f_1, \ldots, f_n \in S_{\mathbb{C}}(\mathbb{R}^d)$.
OS5 (cluster property) For all $a \in \mathbb{R}$, $a \neq 0$, and $m, n \geq 1$

$$\lim_{\lambda\to\infty} \Big(\mathcal{S}_{m+n}(f_1 \otimes \ldots \otimes f_m \otimes E_{(\lambda a,0)}(f_{m+1} \otimes \ldots \otimes f_{m+n})) \\ - \mathcal{S}_m(f_1 \otimes \ldots \otimes f_m)\mathcal{S}_n(f_{m+1} \otimes \ldots \otimes f_{m+n}) \Big) = 0,$$

where $f_1, \ldots, f_{m+n} \in S_{\mathbb{C}}(\mathbb{R}^d)$.

Remark 3.1 *The assumptions in axiom (OS1) can be slightly weakened, see [OS75]. For technical reasons this formulation for us is convenient and since the sequences of generalized functions we consider fulfill (OS1) we do not loose anything by this slightly stronger formulation.*

In the case of Euclidean Markov fields and also in the more general case of Euclidean reflection positivity fields [Frö74b], Schwinger functions fulfilling

(OS1)-(OS5) are obtained as the moments of the Euclidean field. In this section we construct Schwinger functions $(\mathcal{S}_n^\Phi)_{n\in\mathbb{N}_0}$ which are moments of generalized functions $\Phi \in (S)^{-1}$ with $\mathbb{E}_\mu(\Phi) = 1$. The moments $(\mathcal{S}_n^\Phi)_{n\in\mathbb{N}_0}$ in general do not satisfy all axioms (OS1)-(OS5). Nevertheless, we call them Schwinger functions because our aim is to work out a class of generalized functions $\Phi \in (S)^{-1}$ such that their moments fulfill all or a part of the OS axioms.

Definition 3.2 *Let $f_1, \ldots, f_n \in S(\mathbb{R}^d)$, $n \in \mathbb{N}$. We define the n-th Schwinger function corresponding to $\Phi \in (S)^{-1}$, $\mathbb{E}_\mu(\Phi) = 1$, by*

$$\mathcal{S}_n^\Phi(f_1 \otimes \ldots \otimes f_n) = \langle\!\langle \Phi, \langle \omega, f_1\rangle \cdot \ldots \cdot \langle \omega, f_n\rangle \rangle\!\rangle,$$

and $\mathcal{S}_0^\Phi = \mathbb{E}_\mu(\Phi) = 1$.

Since $\mathcal{P}(S') \subset (S)^1$ the dual paring in the above definition is well-defined.

The Schwinger functions corresponding to $\Phi \in (S)^{-1}$ can be calculated under use of their T-transform, see (8).

Proposition 3.3 (Wick theorem) *Let $f_1, \ldots, f_n \in S(\mathbb{R}^d)$, $n \in \mathbb{N}$. Then the n-th Schwinger functions corresponding to $\Phi \in (S)^{-1}$ is given by*

$$\mathcal{S}_n^\Phi(f_1 \otimes \ldots \otimes f_n) = (-i)^n \frac{\partial^n}{\partial t_1 \ldots \partial t_n} T\Phi(t_1 f_1 + \ldots + t_n f_n)\Big|_{t_1 = \ldots t_n = 0}.$$

Proof: By construction every distribution $\Phi \in (S)^{-1}$ is of finite order, i.e., for each $\Phi \in (S)^{-1}$ there exist $p, q \in \mathbb{N}_0$ such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q}$. Furthermore, a straight forward calculation shows that for each $f \in S(\mathbb{R}^d)$ there exists $t_0 > 0$ such that $\exp(it\langle \cdot, f\rangle) \in (\mathcal{H}_p)^1_q$ for all $0 \le t < t_0$, and

$$-i\frac{d}{dt}\exp(it\langle \cdot, f\rangle)\Big|_{t=0} = \langle \cdot, f\rangle$$

w.r.t. the Hilbert space norm in $(\mathcal{H}_p)^1_q$. From this we can conclude that

$$\mathcal{S}_1^\Phi(f) = -i\frac{d}{dt}T\Phi(tf)\Big|_{t=0} = -i\frac{d}{dt}\langle\!\langle \Phi, \exp(it\langle \cdot, f\rangle)\rangle\!\rangle\Big|_{t=0}.$$

Since $(S)^1$ is an algebra under multiplication and this multiplication is continuous we can define the point-wise product $\Phi \cdot \varphi \in (S)^{-1}$ of a distribution $\Phi \in (S)^{-1}$ with a test function $\varphi \in (S)^1$ via the dual paring. Utilizing this product the proposition follows by an induction argument. ■

Proposition 3.4 *For each generalized function $\Phi \in (S)^{-1}$ with $\mathbb{E}_\mu(\Phi) = 1$ the Schwinger functions $(\mathcal{S}_n^\Phi)_{n\in\mathbb{N}_0}$ fulfill the axioms (OS1) and (OS4). Furthermore, (OS2) is fulfilled if $T\Phi$ is Euclidean invariant.*

Proof: The Schwinger functions $(\mathcal{S}_n^\Phi)_{n\in\mathbb{N}_0}$ are symmetric by definition. Temperedness and factorial growth follows immediately from the fact that $\Phi \in (S)^{-1}$. Thus, (OS1) and (OS4) are fulfilled.

Assume that $T\Phi$ is Euclidean invariant. Then we apply Proposition 3.3 to calculate the n-th Schwinger function corresponding to Φ, of course, it is also Euclidean invariant. ■

3.2 Euclidean invariant distributions

In this section we construct a class of Euclidean invariant generalized functions. Generalized functions from $(S)^{-1}$ we call Euclidean invariant if their T-transform is Euclidean invariant. Our construction is motivated by the Euclidean strategy for constructing interacting field theories, see e.g. [Sim74] and the references therein. In the framework of white noise analysis we can define the Gaussian random process indexed by $\mathcal{H} = L^2(\mathbb{R}^d)$ as

$$\phi(h) := \langle \cdot, h \rangle, \quad h \in L^2(\mathbb{R}^d).$$

As discussed in Section 2.1 $\phi(h)$ is an element of $L^2(\mu)$ for all $h \in L^2(\mathbb{R}^d)$. We are interested in the Gaussian random process ϕ at time the $t \in \mathbb{R}$ and at the point $\vec{x} \in \mathbb{R}^{d-1}$, here we write $x \in \mathbb{R}^d$ as $x = (t, \vec{x})$. $\phi(t, \vec{x})$ does not exist as a square-integrable function but we can define

$$\phi(t, \vec{x}) := \langle \cdot, \delta_{t,\vec{x}} \rangle, \quad \in (S)^{-1},$$

see (6), where $\delta_{t,\vec{x}} \in S'(\mathbb{R}^d)$ is the Dirac delta function at point $(t, \vec{x}) \in \mathbb{R}^d$.

Assume that $H(z) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k z^k$, $z \in U \subset \mathbb{C}$, is a holomorphic function in U where U is an open neighborhood of $\mathbb{E}_\mu(\phi(t, \vec{x})) = 0$. Then by using Theorem 2.5 we can define

$$\begin{aligned} H^\diamond(\phi(t, \vec{x})) &= \sum_{k=0}^{\infty} \frac{1}{k!} H_k \, \phi(t, \vec{x})^{\diamond k} \quad \in (S)^{-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} H_k \, \langle : \omega^{\otimes k} :, {\delta_{t,\vec{x}}}^{\otimes k} \rangle, \end{aligned}$$

see also Remark 2.6.

Next we want to define the integral

$$\int_{\mathbb{R}^d} H^\diamond(\phi(x))\, d^d x. \tag{9}$$

This can only be possible if we assume $H_0 = 0$.

Theorem 3.5 *Let H be holomorphic at zero such that $H(0) = 0$, then (9) exists as a Bochner integral in a subspace of $(S)^{-1}$.*

Proof: Our aim is to apply Corollary 2.3. Let $a > 0$ be in the radius of convergence of H. We define

$$\mathcal{U} = \{g \in S_{\mathbb{C}} |\, \sup_{x\in\mathbb{R}^d} \{(1+|x|^2)^d |g(x)| + |g(x)|\} < a\}.$$

It is easy to check that $\mathcal{U}$ is an open neighborhood of zero. For $g \in \mathcal{U}$ we have

$$\begin{aligned} S(H^\diamond(\phi(x)))(g) &= \sum_{k=1}^\infty \frac{1}{k!} H_k \left\langle g^{\otimes k}, {\delta_x}^{\otimes k}\right\rangle \\ = & \sum_{k=1}^\infty \frac{1}{k!} H_k g(x)^k \leq \sum_{k=1}^\infty |a^{-1} g(x)|^k |H_k| \frac{a^k}{k!} \\ \leq & \frac{1}{(1+|x|^2)^d} (1+|x|^2)^d |g(x)| a^{-1} \sum_{k=1}^\infty |H_k| \frac{a^k}{k!} \\ \leq & \frac{1}{(1+|x|^2)^d} \sum_{k=1}^\infty |H_k| \frac{a^k}{k!}. \end{aligned} \tag{10}$$

Obviously, $S(H^\diamond(\phi(\cdot)))(g)$ is measurable for all $g \in \mathcal{U}$. Having the estimate (10) in hands holomorphy of $S(H^\diamond(\phi(x)))$ is clear. Since $(1+|x|^2)^{-d} \in L^1(\mathbb{R}^d)$ all assumptions required in Corollary 2.3 are fulfilled and the theorem is proved. ■

Corollary 3.6 *Let the function H be as in Theorem 3.5. Then the generalized function*

$$\Phi_H := \exp^\diamond \left(- \int_{\mathbb{R}^d} H^\diamond(\phi(x))\, d^d x \right)$$

is an element of $(S)^{-1}$. *Its* T*-transform is given by*

$$T\Phi_{H}(g) = \exp\left(- \int_{\mathbb{R}^d} H(ig(x)) + \tfrac{1}{2} g(x)^2 \, d^d x \right)$$

for all g *in a neighborhood* $\mathcal{U} \subset S_{\mathbb{C}}$ *of zero. In particular,* $\mathbb{E}_\mu(\Phi_H) = 1$.

Proof: This corollary is an immediate consequence of Theorem 2.5. For the calculation of the T-transform we used (7). Observe, that $\mathbb{E}_\mu(\Phi_H) = T\Phi_H(0)$. ■

Remark 3.7 *(i) Since the Lebesgue measure on* $\mathbb{R}^d$ *is Euclidean invariant* $T\Phi_H$ *is Euclidean invariant.*
(ii) Consider the case in which the function $F(s) := -(H(is) + \frac{1}{2}s^2)$, $s \in \mathbb{R}$, *is a Lévy characteristic, i.e.,*

$$F(s) = ias - \frac{\sigma^2 s^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(\exp(irs) - 1 - \frac{irs}{1+r^2} \right) d\nu(r), \quad s \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ *and the measure* ν *satisfies the following condition:*

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, r^2\} \, d\nu(r) < \infty.$$

Then by the Lévy-Khinchine theorem, see e.g. [Luk70], we know that there exists a probability measure P_H *on* $S'(\mathbb{R}^d)$ *such that*

$$T\Phi_H(f) = \int_{S'(\mathbb{R}^d)} \exp(i\langle \omega, f\rangle) \, dP_H(\omega), \quad f \in S(\mathbb{R}^d).$$

This implies that the Schwinger functions $(\mathcal{S}_n^{\Phi_H})_{n\in\mathbb{N}_0}$ *are the moments of the measure* P_H. *These measures are called generalized white noise measures.*

Next we enlarge the class of Euclidean invariant distributions. We do this by convolution with kernels associated to Euclidean invariant operators. This idea is inspired by the method used in [AGW96]. There the authors started with Euclidean invariant measures from the Lévy-Khinchine class and then they constructed image measures by convoluting the corresponding generalized white noise with kernels associated to Euclidean invariant operators. These image measures are called convoluted generalized white noise measures.

Let $\mathcal{G} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ be a linear continuous mapping. Then by the well-known kernel theorem there exists a distribution $K \in S'(\mathbb{R}^{2d})$, hereafter called the kernel of $\mathcal{G}$, such that

$$\mathcal{G}f(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\, dy, \quad f \in S(\mathbb{R}^d),\, x \in \mathbb{R}^d,$$

in the distributional sense. It is clear that the adjoint operator $\mathcal{G}^\star : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ is a measurable transformation from $(S'(\mathbb{R}^d), \mathcal{C}_\sigma(S'(\mathbb{R}^d)))$ into itself. Furthermore, we assume that $\mathcal{G}$ is Euclidean invariant, i.e., $\mathcal{G}E_{(a,\Lambda)} = E_{(a,\Lambda)}\mathcal{G}$ for all $E_{(a,\Lambda)} \in E^+(\mathbb{R}^d)$. This implies that $\mathcal{G}$ is translation invariant, thus, its kernel K has the form $K(x,y) = G(x-y)$, see [SW64]. The action of $\mathcal{G}$ on test functions from $S(\mathbb{R}^d)$ (and by duality on $S'(\mathbb{R}^d)$) is by convolution

$$\mathcal{G}f(x) = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy, \quad f \in S(\mathbb{R}^d),\, x \in \mathbb{R}^d.$$

We can also write $\mathcal{G}f$ as $G * f$. From now on we assume $\mathcal{G}$ to be essentially self-adjoint in $L^2(\mathbb{R}^d)$ with $S(\mathbb{R}^d)$ as a core. Then the convolution of the Gaussian random process ϕ with G we define as

$$(G * \phi)(h)(\omega) := \langle G * \omega, f\rangle = \langle \omega, G * f\rangle, \quad \omega \in S'(\mathbb{R}^d),\, f \in S(\mathbb{R}^d).$$

This definition can also be generalized from test functions f to tempered distributions. Then the process is in $(S)^{-1}$.

Example 3.8 *Let Δ be the Laplace operator on $\mathbb{R}^d$. Let $K(x,y) = G_\alpha(x-y)$ be the Green function of the pseudo differential operator $\mathcal{G}_\alpha = (-\Delta + m_0^2)^{-\alpha}$ for some arbitrary $m_0 > 0$ and $0 < \alpha$. It is given by*

$$G_\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\exp(ikx)}{(|k|^2 + m_0^2)^\alpha}\, dk, \quad x \in \mathbb{R}^d,$$

where the integral has to be understood in the sense of a Fourier transform of a tempered distribution. Easily on proves that $\mathcal{G}_\alpha : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is continuous, essentially self-adjoint in $L^2(\mathbb{R}^d)$ with $S(\mathbb{R}^d)$ as a core, and Euclidean invariant.

Let $\Phi \in (S)^{-1}$ be a generalized function. We define its convolution with an Euclidean invariant kernel G by

$$T\Phi^G(g) := T\Phi(G * g), \quad g \in \mathcal{U} \subset S_{\mathbb{C}}(\mathbb{R}^d),$$

where $\mathcal{U}$ is an open neighborhood of zero. Since the operator $\mathcal{G}$ is linear and continuous, by characterization, Φ^G is a well-defined and unique element in $(S)^{-1}$.

Theorem 3.9 *Let H be as in Theorem 3.5 and the operator $\mathcal{G}$ continuous in $S(\mathbb{R}^d)$, essentially self-adjoint in $L^2(\mathbb{R}^d)$ with $S(\mathbb{R}^d)$ as a core, and Euclidean invariant. Then the generalized function $\Phi_H^G \in (S)^{-1}$ is Euclidean invariant and can be written as*

$$\Phi_H^G = \exp^\diamond\left(- \int_{\mathbb{R}^d} H^\diamond(G * \phi(x))\, d^d x + \tfrac{1}{2}\langle :\cdot^{\otimes 2}:, (\mathcal{G}^{\otimes 2} - 1)Tr\rangle \right), \qquad (11)$$

here $Tr \in S'(\mathbb{R}^d)^{\hat{\otimes}2}$ denotes the trace kernel defined by $\langle Tr, f \otimes g\rangle = (f, g)$, $f, g \in S(\mathbb{R}^d)$. The Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ (from now on we use the abbreviation $\mathcal{S}^{H,G} = \mathcal{S}^{\Phi_H^G}$) fulfill the axioms (OS1), (OS2), and (OS4).

Remark 3.10 *In the case where F is a Lévy characteristic, see Remark 3.7(ii), the measure P_H^G corresponding to the distribution Φ_H^G is a image measures of the measure P_H, more concretely, $P_H^G(A) = P_H(\mathcal{G}^{-1}A)$, $A \in \mathcal{C}_\sigma(S'(\mathbb{R}^d))$.*

Proof of Theorem 3.9: The formula (11) is clear among taking its T-transform. Euclidean invariance follows from the Euclidean invariance of $\mathcal{G}$ and Φ_H. Obviously, $\mathbb{E}_\mu(\Phi_H) = 1$, thus the theorem follows by an application of Proposition 3.3. ■

Example 3.11 *The choice $H \equiv 0$ and $\mathcal{G}_{1/2} = (-\Delta + m_0^2)^{-1/2}$ gives us the free Euclidean field with mass $m_0 > 0$, see Example 3.8. Theorem 3.9 implies that the corresponding measure $P_0^{G_{1/2}}$ has the generalized density*

$$\Phi_0^{G_{1/2}} = \exp^\diamond(\tfrac{1}{2}\langle :\cdot^{\otimes 2}:, (\mathcal{G}_{1/2}^{\otimes 2} - 1)Tr\rangle),$$

w.r.t. the Gaussian white noise measure.

Remark 3.12 *Consider the Hilbert space N_{m_0} which is defined as the closure of $S(\mathbb{R}^d)$ w.r.t. the Hilbert space norm $|\cdot|_{m_0}$ given by the scalar product*

$$(f, g)_{m_0} := \int_{\mathbb{R}^d} f(x)(-\Delta + m_0^2)^{-1} g(x)\, dx, \quad f, g \in S(\mathbb{R}^d),\ m_0 > 0.$$

The random process indexed by $\mathcal{H} = N_{m_0}$:

$$\phi(h) := \langle \cdot, h\rangle_{m_0}, \quad h \in N_{m_0},$$

is the free Euclidean field with mass m_0. Since N_{m_0} fulfills the assumptions on the Hilbert space $\mathcal{H}$ required in Section 2.1, it is also possible to take the measure corresponding to the free Euclidean field as reference measure for the Euclidean invariant distributions constructed above (this is the usual choice in constructive Euclidean QFT). Here we have chosen the white noise measure because it has the identity operator as covariance operator. This for our approach by convolution with operators kernels G is a reasonable choice.

In [AGW96] the authors studied Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ which are moments of the measures P_H^G corresponding to the generalized functions Φ_H^G where $F(s) = -(H(is)+\frac{1}{2}s^2)$, $s \in \mathbb{R}$, is a Lévy characteristic, see Remark 3.7(ii) and Remark 3.10. For the truncated Schwinger functions the authors worked out explicit formulas. Before we give them lets us recall the definition of truncated Schwinger functions.

A partition of the ordered set $\{1, \dots, n\}$ is a family of ordered subsets $I_1 = \{i_1, \dots i_{k(1)}\}, \dots, I_l = \{i'_1, \dots i'_{k(l)}\}$ so that $i_1 < \dots < i_{k(1)}, \dots, i'_1 < \dots < i'_{k(l)}$ and so that $\cup_{1\leq j\leq l} I_j = \{1, \dots, n\}$ and $I_j \cap I_q = \emptyset$, $j \neq q$. The set of all partitions of $\{1, \dots, n\}$ we denote $P^{(n)}$.

Definition 3.13 *The truncated Schwinger functions $(\mathcal{S}_{n,T})_{n\in\mathbb{N}_0}$ corresponding to a given sequence of Schwinger functions $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ are defined recursively by the relation*

$$\begin{aligned} &\mathcal{S}_n(f_1 \otimes \dots \otimes f_n) \\ = &\sum_{P^{(n)}} \mathcal{S}_{k(1),T}(f_{i_1} \otimes \dots \otimes f_{i_{k(1)}}) \cdot \ldots \cdot \mathcal{S}_{k(l),T}(f_{i'_1} \otimes \dots \otimes f_{i'_{k(l)}}), \end{aligned}$$

where $f_1, \dots, f_n \in S(\mathbb{R}^d)$, $n \geq 1$.

Proposition 3.14 *Let $H(z) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n z^n$, $z \in U \subset \mathbb{C}$, and $\mathcal{G}$ be as in Theorem 3.9 and $f_1, \dots, f_n \in S(\mathbb{R}^d)$, $n \geq 1$. Then the truncated Schwinger functions $(\mathcal{S}_{n,T}^{H,G})_{n\in\mathbb{N}}$ are given by*

$$\begin{aligned} \mathcal{S}_{n,T}^{H,G}(f_1 \otimes \dots \otimes f_n) &= -H_n \int_{\mathbb{R}^d} G * f_1(x) \cdot \ldots \cdot G * f_n(x)\, d^d x, \quad n \neq 2, \\ \mathcal{S}_{2,T}^{H,G}(f_1 \otimes f_2) &= (-H_2 + 1) \int_{\mathbb{R}^d} G * f_1(x) \cdot G * f_2(x)\, d^d x. \qquad (12) \end{aligned}$$

Proof: In the case where $F(s) = -(H(is) + \frac{1}{2}s^2)$, $s \in \mathbb{R}$, is a Lévy characteristic this follows from Proposition 3.9. in [AGW96] and the uniqueness of the truncated Schwinger functions. The coefficients in front of the integrals corresponding to the n-th truncated Schwinger function in (12) are just the n-th derivative of the Lévy characteristic divided by i^n. Hence, for a general H as in Theorem 3.9 these coefficients are given by the n-th derivative of $-(H(iz) + \frac{1}{2}z^2)$, $z \in U$. ■

In Corollary 4.7. of [AGW96] the authors have proved the cluster property of the Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ coming from measures. The proof given there easy generalizes to our case.

Theorem 3.15 *Let $\Phi_H^G \in (S)^{-1}$ be as in Theorem 3.9. Then the corresponding Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ fulfill the cluster property (OS5), i.e., for all $a \in \mathbb{R}$, $a \neq 0$, and $m, n \geq 1$*

$$\lim_{\lambda\to\infty} \Big(\mathcal{S}_{m+n}^{H,G}(f_1 \otimes \ldots \otimes f_m \otimes E_{(\lambda a,0)}(f_{m+1} \otimes \ldots \otimes f_{m+n})) \\ -\mathcal{S}_m^{H,G}(f_1 \otimes \ldots \otimes f_m)\mathcal{S}_n^{H,G}(f_{m+1} \otimes \ldots \otimes f_{m+n})\Big) = 0,$$

where $f_1, \ldots, f_{m+n} \in S(\mathbb{R}^d)$.

Proof: See the proof of Corollary 4.7. in [AGW96]. There the authors proved the cluster property in the case where $F(s) = -(H(is) + \frac{1}{2}s^2)$, $s \in \mathbb{R}$, is a Lévy characteristic. The idea is to express the cluster property of the Schwinger functions as an equivalent property of the truncated Schwinger functions. Since their proof works independent of the coefficients in front of the integrals corresponding to the n-th truncated Schwinger function, see (12), it easy generalizes to our case. ■

Remark 3.16 *The class of Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ corresponding to the distributions $\Phi_H^G \in (S)^{-1}$ as in Theorem 3.9 differs from the class of Schwinger functions corresponding to the convoluted generalized white noise measures in [AGW96]. Let us compare the properties of the Lévy characteristics F have been used in [AGW96] with the properties of the functions H we utilize, where $F(s) = -(H(is) + \frac{1}{2}s^2)$, $s \in \mathcal{O} \subset \mathbb{R}$.*

We need that the function H is holomorphic at zero and $H(0) = 0$. This is our restriction in choosing the coefficients in front of the integrals corresponding to the n-th truncated Schwinger function, see (12).

In [AGW96] the authors need that the measure ν in the representation of the Lévy characteristic, see Remark 3.7(ii), has all moments. This implies that $F \in C^\infty(\mathbb{R})$, but F does not have to have a holomorphic extension. Furthermore, also $F(0) = 0$ and F can not be a polynomial of order larger than 2. I.e., if only finite many H_n, $n \in \mathbb{N}$, different from zero than all H_n, $n \geq 3$, have to be zero. Furthermore, the constant $-H_n$ is the n-th moment of the measure ν for $n \geq 3$.

4 On reflection positivity, analytic continuation, and QFT with indefinite metric

4.1 Reflection positivity

In Section 3.2 we proved all OS axioms for Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ corresponding to the distributions $\Phi_H^G \in (S)^{-1}$, H, G as in Theorem 3.9, except for reflection positivity.

In [AGW96] the authors present a partial negative result on reflection positivity of Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ which are moments of convoluted generalized white noise P_H^G. Consider a Lévy characteristic represented as in Remark 3.7(ii). The part arising from the measure ν is called Poisson part and the other part is called Gaussian part (the reason for these names and decomposition lies in the properties of the corresponding measures). For the Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ which are moments of convoluted generalized white noises P_H^G with nonzero Poisson in part in [AGW96] some examples have been constructed which do not have the reflection positivity property. Roughly speaking, the Schwinger functions $(\mathcal{S}_n^{H,G})_{n\in\mathbb{N}_0}$ do not have the reflection positivity property, if the terms in $\mathcal{S}_n^{H,G}$ emerging from the "interaction" (Poisson part) are large in comparison with the "free" terms (Gaussian part). More details on this considerations can be found in [AGW96], Remark 5.12.

The question, whether reflection positivity holds or does not hold we discuss in the next section in terms of the Wightman functions, see Theorem 4.2 and Remark 4.3.

4.2 Analytic continuation to Wightman functions

If a sequence of Schwinger functions fulfills all OS axioms one can perform the analytic continuation to Wightman functions via the reconstruction theorem proved in [OS73]. These Wightman functions fulfill the Wightman axioms:

W1 (temperedness) The sequence $(W_n)_{n\in\mathbb{N}_0}$ is a sequence of tempered distributions, where $W_n \in S'_{\mathbb{C}}(\mathbb{R}^{dn})$ and $W_0 = 1$. These functions fulfill the Hermiticity condition

$$\overline{W_n(f)} = W_n(f^\star).$$

W2 (Poincaré invariance) Each W_n is Poincaré invariant, i.e.,

$$W_n(P_{(a,\Lambda)}f) = W_n(f), \quad \forall f \in S_{\mathbb{C}}(\mathbb{R}^{dn}),$$

for all $(a,\Lambda) \in P_+^\uparrow(\mathbb{R}^d)$ where $P_+^\uparrow(\mathbb{R}^d)$ is the proper, orthochronous Poincaré group. The definition of $P_{(a,\Lambda)}f$ is analog to the definition of Euclidean transformations in $S_{\mathbb{C}}(\mathbb{R}^{dn})$, see (OS2).

W3 (positivity) For each sequence $(f_n)_{n\in\mathbb{N}_0}$ where $f_n \in S_{\mathbb{C}}(\mathbb{R}^{dn})$, $f_0 \in \mathbb{C}$, and each $k \in \mathbb{N}_0$

$$\sum_{n,m=0}^{k} W_{n+m}(f_n^\star \otimes f_m) \geq 0.$$

W4 (locality) If for $n \geq 2$ for some $1 \leq j \leq n-1 : (x_{j+1} - x_j)^2 < 0$, then

$$W_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = W_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n),$$

where $x^2 = \langle x, x\rangle_M = t^2 - |\vec{x}|^2$, $x = (t, \vec{x}) \in \mathbb{R}^d$, is the Minkowski inner product.

We remark that by (W2) every W_n is actually a distribution in the difference variables, i.e., there is a tempered distribution $w_n \in S'_{\mathbb{C}}(\mathbb{R}^{d(n-1)})$ defined as

$$w_n(x_1 - x_2, \ldots, x_{n-1} - x_n) := W_n(x_1, \ldots, x_n).$$

The Fourier transform on $S_{\mathbb{C}}(\mathbb{R}^{dn})$ and $S'_{\mathbb{C}}(\mathbb{R}^{dn})$, respectively, we denote by $\mathcal{F}$ or $\hat{}$ and is taken w.r.t. the Minkowski inner product. The forward mass cone of mass m_0 is defined as

$$V_{m_0}^+ := \{p \in \mathbb{R}^d | p^2 > m_0^2,\, p^0 = \langle p, e_0\rangle_M < 0\}, \quad m_0 \geq 0,$$

where $e_0 = (1, 0, 0, 0)$. By $V_{m_0}^{\star,+}$ we denote its closure and V_0^+ is called forward light cone. The backward mass cone is defined by $V_{m_0}^- := \theta V_{m_0}^+$ where θ again denotes the time reflection.

W5 (spectral condition) For any $n \in \mathbb{N}$ the Fourier transform $\hat{w}_n$ is supported in the backward light cones $(V_0^{\star -})^{n-1}$. (A different sign convention on the Fourier transform in a part of the physical literature leads to the interchange of forward and backward cones.)
W6 (cluster property) For any $n, m \in \mathbb{N}$ and any space like $a \in \mathbb{R}^d$, i.e., $a^2 < 0$

$$\lim_{\lambda \to \infty} \Big(W_{m+n}(f_1 \otimes \ldots \otimes f_m \otimes T_{\lambda a}(f_{m+1} \otimes \ldots \otimes f_{m+n})) \\ - W_m(f_1 \otimes \ldots \otimes f_m) W_n(f_{m+1} \otimes \ldots \otimes f_{m+n}) \Big) = 0,$$

for $f_1, \ldots, f_{m+n} \in S_{\mathbb{C}}(\mathbb{R}^d)$, where $T_{\lambda a}$ denotes the translation by λa.

Without reflection positivity we can not perform the analytic continuation to Wightman functions via the reconstruction theorem. Nevertheless, an analytic continuation can be done. Using results from the theory of Laplace transforms in [AGW96] the authors analytically continued the truncated Schwinger functions $(\mathcal{S}_{n,T}^{H,G_\alpha})_{n \in \mathbb{N}_0}$, which are moments of convoluted generalized white noise, to truncated Wightman functions $(W_{n,T}^{H,G_\alpha})_{n \in \mathbb{N}_0}$ for $\alpha \in (0, 1/2]$, see Example 3.8. The truncated Wightman functions are related to the Wightman functions in the same way as truncated Schwinger functions are related to Schwinger functions, see Definition 3.13. In particular, the authors found an explicit formula for $\hat{W}_{n,T}^{H,G_\alpha}$, the Fourier transform of the n-th truncated Wightman function. Before we can give these formulas we have to introduce the notations

$$\begin{aligned} \mu_\alpha^+(p) &= (2\pi)^{-d/2} \sin(\pi\alpha)\, \mathbf{1}_{\{p^2 > m_0^2, p^0 > 0\}}(p) \frac{1}{(p^2 - m_0^2)^\alpha}, \quad p \in \mathbb{R}^d,\ m_0 > 0, \\ \mu_\alpha^-(p) &= (2\pi)^{-d/2} \sin(\pi\alpha)\, \mathbf{1}_{\{p^2 > m_0^2, p^0 < 0\}}(p) \frac{1}{(p^2 - m_0^2)^\alpha}, \quad \alpha \in (0, 1/2], \\ \mu_\alpha(p) &= (2\pi)^{-d/2} \Big(\cos(\pi\alpha)\, \mathbf{1}_{\{p^2 > m_0^2\}}(p) + \mathbf{1}_{\{p^2 < m_0^2\}}(p) \Big) \frac{1}{|p^2 - m_0^2|^\alpha}, \end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of the subset $A \subset R^d$. In Proposition 7.12. and Corollary 7.13. in [AGW96] it is proved that in the case when $F(s) = -(H(is) + \frac{1}{2}s^2)$, $s \in \mathbb{R}$, is a Lévy characteristic and $\alpha \in (0, 1/2]$ the Fourier transform of the n-th truncated Wightman function for $n \geq 3$ is given by

$$\hat{W}_{n,T}^{H,G_\alpha} = -H_n (2\pi)^d 2^{(n-1)} \Big(\sum_{j=1}^{n} \prod_{l=1}^{j-1} \mu_\alpha^-(p_l) \mu_\alpha(p_j) \prod_{l=j+1}^{n} \mu_\alpha^+(p_l) \Big) \delta\Big(\sum_{l=1}^{n} p_l \Big). \quad (13)$$

In the case $n = 2$ one has to distinguish between the case $\alpha \in (0, 1/2)$ and $\alpha = 1/2$. For $\alpha \in (0, 1/2)$ the 2-point function is given by

$$\hat{W}_{2,T}^{H,G_\alpha} = (-H_2 + 1)(2\pi)^d \, 2 \left(\mu_\alpha(p_1)\mu_\alpha^+(p_2) + \mu_\alpha^-(p_1)\mu_\alpha(p_2)\right) \delta(p_1 + p_2) \quad (14)$$

and

$$\hat{W}_{2,T}^{H,G_{1/2}} = (-H_2 + 1)(2\pi)^{d+1} \mathbf{1}_{\{p_1^0 < 0\}}(p_1)\delta(p_1^2 - m_0^2)\delta(p_1 + p_2) \quad (15)$$

is the Fourier transform of the well-known 2-point function of the relativistic free field.

The truncated Wightman function $W_{n,T}^{H,G_\alpha}$ is an analytic continuation of the truncated Schwinger function $\mathcal{S}_{n,T}^{H,G_\alpha}$ in the sense that

$$\mathcal{S}_{n,T}^{H,G_\alpha}(\Im(z_1^0), \Re(\vec{z}_1), \ldots, \Im(z_n^0), \Re(\vec{z}_n)) = \mathcal{L}\hat{W}_{n,T}^{H,G_\alpha}(z), \quad z \in \mathbb{C}_<^{dn}, \quad (16)$$

where $\mathcal{L}$ denotes the Laplace transform and

$$\mathbb{C}_<^{dn} := \{(z_1^0, \vec{z}_1; \ldots; z_n^0, \vec{z}_n) \in \mathbb{C}^{dn} | \Im(z_j^0 - z_{j+1}^0) < 0, \, j = 1, \ldots, n-1, \\ \Im(\vec{z}_j) = 0, \, \Re(z_j^0) = 0, \, j = 1, \ldots, n\}$$

($\Re(z)$ is the real part and $\Im(z)$ is the imaginary part of a (vector valued) complex variable z). The function $\hat{W}_{n,T}^{H,G_\alpha}$ is determined uniquely by this requirement. Furthermore, $W_{n,T}^{H,G_\alpha}(\Re(z))$ is the boundary value of $\mathcal{L}\hat{W}_{n,T}^{H,G_\alpha}(z)$ for $\Im(z_j - z_{j+1}) \to 0$ inside T^n, i.e., the relation

$$\lim_{\Gamma \ni \Im(z_j - z_{j+1}) \to 0} \mathcal{L}\hat{W}_{n,T}^{H,G_\alpha}(z) = W_{n,T}^{H,G_\alpha}(\Re(z)) \quad (17)$$

holds in the sense of tempered distributions in the argument $\Re(z) \in \mathbb{R}^d$. Here T^n is the tubular domain in $\mathbb{C}^{dn}$ with base V_0^-, i.e.,

$$T^n := \{(z_1, \ldots, z_n) \in \mathbb{C}^{dn} | z_j - z_{j+1} \in \mathbb{R}^d + iV_0^-, \, j = 1, \ldots, n-1\}$$

and $\Gamma \subset V_0^-$ is a sub-cone of V_0^- such that $\Gamma \cup \{0\}$ is closed in $\mathbb{R}^d$.

Theorem 4.1 *Let H be as in Theorem 3.5 and $\mathcal{G}_\alpha$ as in Example 3.8, $\alpha \in (0, 1/2]$.*
(i) The Schwinger functions $(\mathcal{S}_n^{H,G_\alpha})_{n \in \mathbb{N}_0}$ can be analytically extended to Wightman functions $(W_n^{H,G_\alpha})_{n \in \mathbb{N}_0}$ in the sense of (16) and (17).

(ii) The sequence $(W_n^{H,G_\alpha})_{n\in\mathbb{N}_0}$ *satisfies the axioms* $(W1)$, $(W2)$, *and* $(W4)$-$(W6)$.
(iii) The Fourier transform of the truncated Wightman functions are given by the formulas (13), (14), and (15), respectively.
(iv) For $0 < \alpha < 1/2$, $H_1 = 0$, $H_2 < 1$, $\hat{W}_2^{H,G_\alpha} = \hat{W}_{2,T}^{H,G_\alpha}$ *admits a Källen-Lehmann representation. Therefore, the corresponding Gaussian Euclidean field with covariance function* $\mathcal{S}_2^{H,G_\alpha}$ *is reflection positive but not Markov. For* $\alpha = 1/2$ *the corresponding Gaussian Euclidean field is the Markov free field of mass* m_0.
(v) $\hat{W}_n^{H,G_\alpha}$ *fulfills the strong spectral condition with mass gap* m_0*, i.e.,* $\hat{w}_n^{H,G_\alpha}$ *is supported in the backward mass cones* $(V_{m_0}^{\star -})^{n-1}$.

Proof: (i): Let us consider the Fourier transformed truncated Wightman functions in (13), (14), and (15). If we now chose coefficients H_n corresponding to a general function H as in the theorem assumed then the corresponding truncated Wightman functions $W_{n,T}^{H,G_\alpha}$ are analytic continuations of the truncated Schwinger functions $\mathcal{S}_{n,T}^{H,G_\alpha}$ in the sense of (16) and (17). Of course, the corresponding Wightman functions W_n^{H,G_α} are analytic continuations of the truncated Schwinger functions $\mathcal{S}_n^{H,G_\alpha}$ in the same sense.
(ii)-(v): In the case where the Wightman functions correspond to Schwinger functions obtained from convoluted generalized white noise this was proved in [AGW96], Section 7.5. Since in our case we only have different coefficient H_n the same is true for a general function H as in the theorem assumed. ■

Now let us return to the question whether positivity holds or not. In terms of the Schwinger functions this question has been discussed in [AGW96], see Section 4.1. The following theorem is an immediate consequence of the Jost-Schroer theorem, see e.g. [Jos61], [FJ60], [Poh69], and [Ste82].

Theorem 4.2 *Let* H *be as in Theorem 3.5,* $H_1 = 0$, $H_2 < 1$. *Then the following statements are equivalent:*
(i) The sequence of Wightman functions $(W_n^{H,G_{1/2}})_{n\in\mathbb{N}_0}$ *fulfills the positivity condition* $(W3)$.
(ii) For $n \geq 3$ *vanish the truncated Wightman functions, i.e.,*

$$W_{n,T}^{H,G_{1/2}} = 0, \quad n \geq 3.$$

Proof: Since the 2-point function $W_2^{H,G_{1/2}}$ is the 2-point function of the relativistic free field with mass m_0 and the sequence of Wightman functions

$(W_n^{H,G_{1/2}})_{n\in\mathbb{N}_0}$ fulfills $(W1)$, $(W2)$, and $(W4)$-$(W6)$, see Theorem 4.1, the statement of Theorem 4.2 is just the statement of the Jost-Schroer theorem. ■

Remark 4.3 *(i) Theorem 4.2 implies together with the explicit formulas for the Fourier transform of the truncated Wightman functions, see (13), that in the case* $\alpha = 1/2$ *positivity holds if and only if* $H_n = 0$, $n \geq 3$.
(ii) For $\alpha = 1/2$ *Theorem 4.2 implies the negative result on reflection positivity of the Schwinger functions derived in [AGW96], see Section 4.1.*
(iii) Under certain assumptions on the measure in the Källen-Lehmann representation one can also prove a Jost-Schroer theorem for generalized free fields. It is still an open question, whether this generalization of the Jost-Schroer theorem can be applied to the sequence of Wightman functions $(W_n^{H,G_\alpha})_{n\in\mathbb{N}_0}$, $\alpha \in (0, 1/2)$, *see Theorem 4.1. One has to check, whether one can prove a Jost-Schroer theorem for generalized free fields having a Källen-Lehmann representation as the 2-point functions* $\hat{W}_{2,T}^{H,G_\alpha}$, $\alpha \in (0, 1/2)$, *see Theorem 4.1(iv).*

4.3 QFT with indefinite metric

In Section 4.2 we performed the analytic continuation from Schwinger functions to Wightman functions. The main interesting object, however, is the underlying quantum field theory. Given a family $(W_n)_{n\in\mathbb{N}}$ obeying $(W1)$ - $(W6)$, by the Wightman reconstruction theorem, see [Wig56], there exists an essentially unique field theory obeying the Gårding-Wightman axioms for single Hermitian scalar fields. Since for no nontrivial cases we proved positivity of the sequence of tempered distributions $(W_n^{H,G_\alpha})_{n\in\mathbb{N}_0}$ as in Theorem 4.1, we can not reconstruct the field theory by the Wightman reconstruction theorem. The positivity conditions is used in order to construct a the physical Hilbert space as the closure of the Borchers algebra. This is not possible without the positivity condition.

Morchio and Strocchi, see e.g. [MS80] and [Str76], considered quantum field theories in which not all Wightman axioms are satisfied. For Wightman functions not fulfilling the positivity condition Morchio and Strocchi introduced the so called modified Wightman axioms of indefinite metric QFT. In their set of axioms the positivity condition is substituted by the weaker Hilbert space structure condition (HSSC):

W′3 (HSSC) There exists a sequence $(p_n)_{n\in\mathbb{N}}$, where for all $n \in \mathbb{N}$, $p_n : S(\mathbb{R}^{nd}) \to [0,\infty)$ is a Hilbert semi-norm, such that

$$|W_{m+n}(f^\star \otimes g)| \leq p_m(f)p_n(g)$$

for all $f \in S_{\mathbb{C}}(\mathbb{R}^{dm})$ and $g \in S_{\mathbb{C}}(\mathbb{R}^{dn})$, $n, m \in \mathbb{N}$.

The HSSC permits the construction of a Hilbert space $\mathcal{K}$ and a quantum field ϕ associated to a given collection of tempered distributions $(W_n)_{n\in\mathbb{N}_0}$ fulfilling the modified Wightman axioms $(W1)$, $(W2)$, $(W'3)$, $(W4)$, and $(W5)$. Moreover, in [MS80] the following theorem is proved.

Theorem 4.4 *Let $(W_n)_{n\in\mathbb{N}_0}$ be a sequence of Wightman functions which fulfill $(W1)$, $(W2)$, $(W'3)$, $(W4)$, and $(W5)$. Then there exists*
(i) a Hilbert space $\mathcal{K}$ with scalar product $(\cdot,\cdot)_{\mathcal{K}}$, a distinguished vacuum vector $\Omega \in \mathcal{K}$ and an indefinite inner product $(\cdot,\cdot)_T$ which differs from $(\cdot,\cdot)_{\mathcal{K}}$ only by a self-adjoint metric operator T with $T^2 = 1$, i.e., $(\cdot,\cdot)_T = (\cdot, T\cdot)_{\mathcal{K}}$;
(ii) a T-symmetric and local quantum field ϕ, which is a distribution valued field operator $\phi(x)$ acting on a dense core $\mathcal{D} \subset \mathcal{K}$ with adjoint $\phi^\star(x) = T\phi(x)T$ and commutator

$$[\phi(x), \phi(y)] = 0$$

for x and y space-like separated. Furthermore, ϕ is connected with the Wightman functions of the theory by

$$W_n(x_1,\ldots,x_n) = (\Omega, \phi(x_1)\ldots\phi(x_n)\Omega)_T;$$

(iii) a T unitary representation $\mathcal{U}$ of the orthochonous Poincaré group on $\mathcal{K}$, i.e., a representation with $T\mathcal{U}^\star T = \mathcal{U}^{-1}$, such that Ω is invariant under $\mathcal{U}$ and $\phi(x)$ transforms covariantely

$$\mathcal{U}(a,\Lambda)\phi(x)\mathcal{U}(a,\Lambda)^{-1} = \phi(\Lambda^{-1}(x-a)), \quad (a,\Lambda) \in P_+^\uparrow(\mathbb{R}^d).$$

Furthermore, $\mathcal{U}$ fulfills the following spectral condition:

$$\int_{\mathbb{R}^d} (\Psi_1, \mathcal{U}(a,1)\Psi_2)_T \exp(iqa)\, da \quad \forall \Psi_1, \Psi_2 \in \mathcal{D},$$

if $q \notin V_0^{\star -}$.

A quadruple $((\mathcal{K}, (\cdot,\cdot)_{\mathcal{K}}, \Omega), T, \phi, \mathcal{U})$ is called a QFT in indefinite metric.

Theorem 4.5 *The Wightman functions* $(W_n^{H,G_\alpha})_{n\in\mathbb{N}_0}$ *as obtained in Theorem 4.1 fulfill the modified Wightman axioms* $(W1)$, $(W2)$, $(W'3)$, $(W4)$, *and* $(W5)$ *(of Morchio and Strocchi).*

Proof: In [AGW97a], Theorem 4.1, this is proved for the Wightman functions corresponding to the moments of convoluted generalized white noise. The proof is done under use of explicit formulas for the Fourier transform of the truncated Wightman functions and works for an arbitrary sequence of coefficients $(H_n)_{n\in\mathbb{N}}$, see (13), (14), and (15). Thus, also in our case. ■

In general the semi-norms in the HSSC are not invariant under transformations of the orthochonous Poincaré group. Hence, in general the metric operator T does not commute with $\mathcal{U}(a,\Lambda)$, $(a,\Lambda) \in P_+^\uparrow(\mathbb{R}^d)$, and the representation of the orthochonous Poincaré group on $\mathcal{H}$ is not unitary. In our case, however, the semi-norms in the HSSC at least can be chosen translations invariant.

Theorem 4.6 *For the sequence of Wightman functions* $(W_n^{H,G_\alpha})_{n\in\mathbb{N}_0}$ *as in Theorem 4.5 the Hilbert semi-norms in the HSSC can be chosen translations invariant. Thus, there exists a Hilbert space structure such that* $[\mathcal{U}(a,1),T] = 0$, $a \in \mathbb{R}^d$, *and the representation of the translation group* $\mathcal{U}(a,1)$ *is unitary. If* P *denotes the generator of* $\mathcal{U}(a,1)$ *then* $spec(P) \subset V_0^{\star+}$.

Proof: In [AGW97b], Theorem 4.3, this is proved for the Wightman functions corresponding to the moments of convoluted generalized white noise. By the same arguments as in the proof of Theorem 4.5 their proof generalizes to our case. ■

Remark 4.7 *(i) We remark that the cluster property of Wightman functions is not an item of the modified Wightman axioms, since, in general, it does not imply the uniqueness of the vacuum and irreducibility of the field algebra as it does in the standard QFT.*
(ii) The uniqueness of the vacuum cannot hold if T *commutes with* $\mathcal{U}(a,1)$, $a \in \mathbb{R}^d$, *and* $T\Omega \notin \mathbb{C}\Omega$, *since in this case* $T\Omega$ *is translations invariant.*
(iii) We observe that there exist sequences of Wightman functions, associated to sequences of Schwinger functions which are not moments of measures, fulfilling the modified Wightman axioms of Morchio and Strocchi.

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