A two space dimensional semilinear heat equation perturbed by (Gaussian) white noise

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ABSTRACT

A two-space dimensional heat equation perturbed by a white noise driven in a bounded volume is considered. The equation is perturbed by a non-linearity of the type $\lambda : f(AU) :$, where $: \, :$ means Wick (re)ordering with respect to the free solution; λ, A are small parameters, U denotes a solution, f is the Fourier transform of a complex measure with compact support.

Existence and uniqueness of the solution in a class of Colombeau-Oberguggenberger generalized functions is proven. An explicit construction of the solution is given and it is shown that each term of the expansion in a power series in λ is associated with an L^2-valued measure when A is a small enough.

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Introduction

The motivation of this article is to consider semi-linear stochastic partial differential equations of parabolic type which are perturbed by a space-time Gaussian white noise with space-dimension $d \geq 2$. Moreover we are interested in strong (probabilistic) solutions in the sense that given a probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian space time white noise $(\dot{W}(t,x),\, t \geq 0,\, x \in I\!R^d)$ we would like to construct a solution to

$$
(1) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + LU = \lambda\, f(AU)\, \chi + \dot{W} \\ U(0,\cdot) = U_0 \end{cases}
$$

where χ is a smooth indicator function, L is a symmetric uniformly elliptic operator, $f \in C_b^\infty$, that is to say f is smooth and all its derivatives are bounded, $U_0 \in \mathcal{S}'(I\!R^d)$, $\lambda \in I\!R$; A is a small real number.

The first difficulty we encounter is that the sample paths of $\dot{W}$ are tempered distributions ; the reader can consult [W] for a more precise location of $\dot{W}$ depending on the dimensions.

To solve problem (1), it means

- to give a reasonable sense to the equation
- to provide non trivial solutions ; if $\lambda \neq 0$, and (1) has a solution, this should be for instance non Gaussian.

Remaining at a formal level, we can smear (1) with respect to a test function in space $\varphi \in \mathcal{S}(I\!R^d)$ and replace problem (1) with

$$
(2) \qquad \begin{aligned} U(t,\varphi) &= U_0(\varphi) - \int_0^t ds\, U(s, L\varphi) \\ &\quad + \lambda \int_0^t ds\, f(AU)\,(s, L\varphi) + \int_{[0,t]\times I\!R^d} dW(s,y)\, \varphi(y). \end{aligned}
$$

If we are interested in solutions being random classical fields $(U(t,x), t \geq 0, x \in I\!R^d)$ then $U(t,\varphi) = \displaystyle\int_{[0,t]\times I\!R^d} ds\, dy\, U(s,y)\, \varphi(y)$, and

$$
f(AU)\,(s,\psi) = \int dy\, f(AU)\,(s,y)\, \psi(y).
$$

But, even if U_0 is a continuous functions, by our choice of noise, it will almost never happen that solutions of (2) are classical random fields; so we will need to give a precise meaning to $f(AU)$.

We concentrate for a moment on the case $\lambda = 0$. It is possible to show that equation (2) has a unique solution with paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ which is given by

$$X(t,\varphi) = (P_t U_0)(\varphi) + \int_{[0,t]\times \mathbb{R}^d} dW(s,y)\,(P_s\varphi)(y), \quad \varphi \in \mathcal{S}(\mathbb{R}^d) \tag{3}$$

where $(P_t)_{t\geq 0}$ is the semi-group of generator L on $\mathcal{S}'(\mathbb{R}^d)$. It has a density $(p_t(x,y), t \geq 0, x, y \in \mathbb{R}^d)$ which means that

$$P_t\, f(x) = \int p_t(x,y)\, f(y)\, dy.$$

Properties of this semi-group will be stated in section 2.

If $U_0 = 0$ then X is the distributional random field which is associated formally with the field

$$X(t,x) = \int_{[0,t]\times \mathbb{R}^d} p_{t-s}(x,y)\, dW(s,y). \tag{4}$$

Unfortunately the latter integral only makes sense if

$$\int_{[0,t]\times \mathbb{R}^d} p^2_{t-s}(x,y) < \infty, \quad \forall t \geq 0,\ x \in \mathbb{R}^d. \tag{5}$$

This holds if and only if $d = 1$.

Let us suppose that U_0 is a continuous function (with polynomial growth) in the case of space dimension $d = 1$. Then a continuous random field $(U(t,x), t \geq 0, x \in \mathbb{R})$ solves (2) if and only if

$$\begin{aligned} U(t,x) =& (P_t U_0)(x) + \lambda \int_{[0,t]\times \mathbb{R}^d} ds\, dy\, f(AU(s,y))\, \chi(y) \\ & p_{t-s}(x,y) + \int_{[0,t]\times \mathbb{R}^d} p_{t-s}(x,y)\, dW(s,y). \end{aligned} \tag{6}$$

Using classical fixed point methods, it is possible to show the existence and uniqueness of a solution for equation (6).

If $d \geq 2$ the equivalence between (2) and (6) will only hold at the formal level. For this reason, we introduce the class of random Colombeau-Oberguggenberger generalized functions which extends the family of random Schwarz distributions. In this general framework it is possible to state problems (2) and (6) and to show they are equivalent. The non-linearity is understood in the sense of Colombeau generalized functions, see [C1], [C2], [O1], [O2], [B], [R], [Sc].

In the case $d = 1$, when U_0 is a continuous function, standard generalized functions techniques allow to show that the unique solution to (1) is $L^2(\Omega)$-associated, see section 1, to the solution of equation (6).

Coming back to the case $d \geq 2$, similar techniques used in [AHR2] can show that the unique solution of (6) in the framework of random generalized functions is trivial because associated with the solution X of the free equation ($\lambda = 0$). This concretely means that, if regularize the white noise $\dot{W}$ in space by convoluting with a mollifier ϕ^ε converging to Dirac measure, then (6) has a solution U^ε and this solution converges in the sense of the $L^2(\Omega)$-valued distributions to X.

This is not astonishing. In order to find a non-trivial solution we need to modify the non-linearity f by multiplying with an infinite function.

In section 3 we are interested in the following semi-linear equation

$$(7) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + LU = \lambda : f(AU) : \chi + \dot{W} \\ U(0, \cdot) = U_0 \end{cases}$$

for which we construct a solution in the framework of random generalized functions. The Wick reordering is understood with respect to the free (Gaussian) solution ($\lambda = 0$), see [GJ], [HKPS].

f will be supposed to be the Fourier transform of a complex measure μ with compact support such that $\mu(\{0\}) = 0$. If $\mu = \dfrac{\delta_{-1} + \delta_1}{2}$ we obtain $f(u) = \cos u$ and (7) is of Sine-Gordon type. Of course we need a redefinition of Wick reordering of $f(U)$ when f is a function of this nature and U a random generalized function.

However the most important part of the paper is constituted by section 4. For A small we obtain there an asymptotic expansion in powers of λ of the solution constructed in section 3 and we show that the first term of the expansion is the free solution and all the others are classical square integrable processes which are explicitely given linear combination of itereated integrals involving Wick-reordered functions of the free solutions.

We discuss now the connection with the literature. Other papers related to random Colombeau generalized functions and the stochastic wave equation are [OR1;OR2;OR3]. In particular the first two articles discuss different classes of non linearities producing trivial solutions.

The model studied here is very close to the stochastic quantization equation of quantum fields in a finite volume. Our method gives a constructive way to build a strong solution of the stochastic quantization equation of the Sine-Gordon field.

Given a quantum field Φ (which is a Borel probability) living on $\mathcal{S}'(\mathbb{R}^2)$, the problem of stochastic quantization consists in constructing a stationary process $(\xi(t), t \geq 0)$ in law with value in $\mathcal{S}'(\mathbb{R}^2)$ whose invariant measure is Φ. All the authors have looked for $(\xi(t), t \geq 0)$ as solution of a pseudo-differential stochastic equation of the following type

$$(8) \qquad \frac{\partial U}{\partial t} = -\frac{1}{2} L^{1-\varepsilon} U + \lambda L^{-\varepsilon} : f(U) : + L^{-\frac{\varepsilon}{2}} \dot{W},$$

with $L = I - \Delta$, Δ being the Laplacian for the two-dimensional space, $\varepsilon \in]0,1]$. f is generally of polynomial type (stochastic quantization of $P(\phi)_2$) or trigonometric (stochastic quantization of Sine-Gordon) : the most famous example is ϕ_2^4 which provides

$$f(u) = -4 : u^3 :$$

The regularization in (8) is introduced in such a way that the invariant measure does not depend on ε.

The case of polynomial interactions has been most extensively studied. Among the contributions we mention [J-LM], [BCM], [AR], [Do], [BJ-LP], [HK], [GG], [DT]. These papers give weak solutions (in the sense of probability).

[J-LM] have the restriction $\varepsilon \in \left]\frac{9}{10}, 1\right[$, [GG], [BJ-LP] (which in fact considers a slightly different equation) and [BCM] have the restriction $\varepsilon \in]0,1[$.

[Do] treats the case of a suitable coloured noise in any dimension. The methods of [BCM] and [AR] are based on the theory of Dirichlet forms. [J-LM], [BCM] and [GG] treat the finite volume case, [AR] the finite and the infinite volume case. For partial results in the latter case, see also [BCM].

[AR] also treat the general $P(\phi)_2$ case (general polynomial interaction in two-space dimension) and in fact it is the only paper which treats the (local) case $\varepsilon = 0$. However this solution is not constructive.

In this paper we are able

1 to construct a strong probabilistic solution (in the pathwise sense);
2 to treat the Sine-Gordon case for which there are only some remarks in [AHPS], [AHR1] and [ARu];
3 to treat, constructively as well, the SPDE case ($\varepsilon = 0$) instead of the pseudo SPDE situation $\varepsilon > 0$;
4 to obtain on explicit expression of the solution in terms of iterated integrals of Wick reordered functions of the free solution.

Concerning other approaches to strong solutions to SPDE's involving non-linear functions of distributions, let us mention the white noise Wiener distribution approach, see for instance [HLOUZ, HOUZ, BDP, De]; the second reference concerns elliptic bilinear equations, the third for the parabolic case, the fourth is related to Burger's equation. Other interesting approaches have been performed for the Navier-Stokes equation: see [AC] for the stochastic flows approach and [CC] for the non-standard framework.

One Problem however remains open. We are still interested to know if such a solution is a random distribution; for the moment we only know that the asymptotic expansion in powers of λ is a classical random field.

1. On a class of random tempered generalized functions

This section will introduce a special class of generalized functions which we will call of Colombeau-Oberguggenberger type. The main difference with the classical Colombeau framework is that they are only generalized in space but ordinary in time. The correct term should be: algebra of *simplified Fréchet-valued random tempered generalized functions*. The following description is partly inspired by ch. 4 of [C1] and §12 of [O1].

We first recall, see for instance [DS], ch. II, that a Fréchet space F is a linear topological space, complete, equipped with a homogeneous metric. Examples of such spaces are $\mathcal{S}(\mathbb{R}^d)$, $C^k(\mathbb{R}^d)$, for $k \geq 0$. Given a closed subset B of $\mathbb{R}^d$, $\mathcal{S}(B)$ will be the Fréchet space of the functions in $\mathcal{S}(\mathbb{R}^d)$ restricted to B. $\mathcal{C}(B)$ is the Fréchet space of continuous complex valued functions on B, equipped with a metric which is equivalent to the sup norm on each compact. $\mathcal{C}(B)$ is also an algebra. $\mathcal{C}(B;\mathbb{F})$ is the Fréchet space of $\mathbb{F}$-valued continuous functions on B.

In this paper T will be a positive number. As far as differentiability is concerned $\mathbb{C}$ will always be identified with $\mathbb{R}^2$, const will be a generic constant.

The class of generalized functions we introduce, will so consist of objects which are generalized in space and ordinary functions in time. The starting point is the differential algebra $\mathcal{E}(\mathbb{R}^2;\mathcal{C}(B))$ of functions $\{(t;x,\varepsilon) \to R(t;x,\varepsilon), t \in B, x \in \mathbb{R}^2, \varepsilon > 0\}$ which are complex valued and such that $R_\varepsilon : \mathbb{R}^2 \to \mathcal{C}(B)$ is C^∞ for any $\varepsilon > 0$, where

$$R_\varepsilon(x)(t) = R(t;x,\varepsilon), \ \forall t \geq 0, \ x \in \mathbb{R}^2, \ \varepsilon > 0 \ .$$

A function $R \in \mathcal{E}(\mathbb{R}^2;\mathcal{C}(B))$ will be said to have *a tempered moderate bound* if it fulfills the following property:

For all compacts $B_{\text{loc}} \subset B \subset \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$\text{(1.1)} \qquad \sup_{t\in B_{\text{loc}}, x\in\mathbb{R}^2} \frac{|R(t;x,\varepsilon)|}{1+|x|^n} = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0 \ .$$

In particular the left hand side quantity in the previous expression is finite for small $\varepsilon > 0$. O is the usual Landau symbol.

The family of $R \in \mathcal{E}(\mathbb{R}^2;\mathcal{C}(B))$ satisfying (1.1) constitutes a (differentiable) subalgebra.

Next we consider the ideal of functions R such that

For each compact $B_{\text{loc}} \subset B$, for all $q \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that

$$\text{(1.2)} \qquad \sup_{t\in B_{\text{loc}}, x\in\mathbb{R}^2} \frac{|R(t;x,\varepsilon)|}{1+|x|^n} = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0 \ .$$

Such R are said to have a *tempered null bound.*

$R \in \mathcal{E}(\mathbb{R}^2;\mathcal{C}(B))$ will be said to be *moderate* if for any partial derivation operator D, DR has a temperate moderate bound. The family of moderate R constitutes a differential subalgebra of $\mathcal{E}(\mathbb{R}^2;\mathcal{C}(B))$ and will be denoted by $\mathcal{E}_M(\mathbb{R}^2;\mathcal{C}(B))$.

$R \in \mathcal{E}_M(I\!R^2;\mathcal{C}(B))$ will be said to be *null* if for any partial derivation operator D, DR has a null tempered bound. The family of null elements R constitutes an ideal of $\mathcal{E}_M(I\!R^2;\mathcal{C}(B))$, denoted by $\mathcal{N}(I\!R^2;\mathcal{C}(B))$.

We define $\mathcal{G}(I\!R^2;\mathcal{C}(B))$ as the quotient $\mathcal{E}_M(I\!R^2;\mathcal{C}(B))/\mathcal{N}(I\!R^2;\mathcal{C}(B))$. Any element of $\mathcal{G}(I\!R^2;\mathcal{C}(B))$ will be called *a Colombeau-Oberguggenberger tempered $\mathcal{C}(B)$-valued generalized function* or simply a *$\mathcal{C}(B)$-valued generalized function.*

$\mathcal{S}'(I\!R^2)$ is the space of classical tempered distributions. We denote by $\mathcal{S}'(I\!R^2;\mathcal{C}(B))$ the space of linear continuous functionals from $\mathcal{S}(I\!R^2)$ to $\mathcal{C}(B)$, in other words the $\mathcal{C}(B)-$ valued classical tempered distributions. For the concept of vector valued distributions we mention, e. g., the early reference [S].

We recall that $\tau \in \mathcal{S}'(I\!R^2)$ has the following property. There is a continuous function $f \in \mathcal{C}(I\!R^2)$ with at most polynomial increase and a partial derivation operator D so that $\tau = Df$.

At the same way, if $\tau \in \mathcal{S}'(I\!R^2;\mathcal{C}(B))$ there is a function $F \in \mathcal{C}(B \times I\!R^2)$, a derivation operator D such that $\tau(t,\cdot) = DF(t,\cdot)$, D acting on x and the following property is verified:

for every compact $B_{\rm loc} \subset B$, there is $n \in I\!N$ with

$$\sup_{t\in B_{\rm loc},x\in I\!R^2} \frac{|F(t,x)|}{1+|x|^n} < \infty \tag{1.3}$$

Let $F : B \times I\!R^2 \to \mathbb{C}$ so that $x \to F(\cdot,x) \in C^\infty(I\!R^2;\mathcal{C}(B))$ and F fulfills (1.3). Then F can be identified with the $\mathcal{C}(B)$-valued generalized function

$$R_F(t;x,\varepsilon) = F(t,x) \ .$$

However, if F is not smooth in x, there is no canonical identification. We will however provide a particular one.

Let $\phi \in \mathcal{S}(I\!R^2)$ be fixed with

$$\int \phi(x)dx = 1 \ , \ \int x^m \phi(x)dx = 0$$

for any multi-index m over $I\!N^2$ so that $|m| \geq 1$, as stated for instance in [O1].

In fact, for such a purpose, it is enough to find $\phi_1 \in \mathcal{S}(I\!R)$ with the same property and then to take the tensor product with itself. This particular function ϕ_1 can be obtained as the Fourier transform of a function $\psi \in \mathcal{S}(I\!R)$ so that $\psi(\xi)$ is identically 1 in a neighborhhood of $\xi = 0$; then $\psi(0) = 1$ and $\psi^{(k)}(0) = 0$ for every positive integer k; this gives the desired result for ϕ_1.

For $\varepsilon > 0$, we set

$$\phi^\varepsilon(x) = \frac{1}{\varepsilon^2}\phi\left(\frac{x}{\varepsilon}\right) \ .$$

This will help us to embed in the space of $\mathcal{C}(B)-$ valued generalized functions, non-smooth functions with values in $\mathcal{C}(B)$, or more generally elements of *trace type* $\eta : B \times \mathcal{S}(I\!\!R^2) \to \mathbb{C}$, that it to say such that

(1.4) $\alpha \to (t \to \eta(t,\alpha))$ is a linear continuous functional, i.e. a vector valued distribution from $\mathcal{S}(I\!\!R^2)$ to $\mathcal{C}(B)$.

The space of such η is a Fréchet space (F-space in the sense of [DS], ch. 2). We remark in particular that for any compact $B_{\rm loc} \subset B$ and bounded subset S of $\mathcal{S}(I\!\!R^2)$

$$\sup_{\alpha\in S} \sup_{t\in B_{\rm loc}} |\eta(t,\alpha)| < \infty$$

A tempered distribution $\bar{\eta} \in \mathcal{S}'(I\!\!R^{d+2})$ is said to have the *trace property* with respect to the closure of an open subset B of $I\!\!R^d$ if

$$\forall t \in B,\ \alpha \in \mathcal{S}(I\!\!R^2),\ \lim_{\varepsilon\to 0}\langle \bar{\eta}, \phi^\varepsilon(t-\cdot)\otimes\alpha\rangle \tag{1.5}$$

exists and it is a continuous function.

There is a one-to-one correspondence between elements of trace type $\eta : \mathcal{S}(I\!\!R^2) \to \mathcal{C}(B)$ and distributions $\bar{\eta}$ having the trace property. The relations are given by

$$\langle \bar{\eta}, \beta\rangle = \int_B dt \eta(t, \beta(t,\cdot))$$
$$\eta(t,\alpha) = \lim_{\varepsilon\to 0}\langle \bar{\eta}, \phi^\varepsilon(t-\cdot)\otimes\alpha\rangle \ .$$

Given $\bar{\eta} \in \mathcal{S}'(I\!\!R^{d+2})$ fulfilling (1.5), the limit η will automatically fulfill (1.4). In fact, for any $\varepsilon > 0$, we can consider

$$\alpha \to (t \to \langle \eta, \phi^\varepsilon(t-\cdot)\otimes\alpha\rangle)$$

as a linear map from $\mathcal{S}(I\!\!R^2)$ to $C(I\!\!R^d)$. For any compact $B_{\rm loc} \subset I\!\!R^d$, $s,t \in B_{\rm loc}$ we have

$$||[\phi^\varepsilon(t-\cdot) - \phi^\varepsilon(s-\cdot)]\otimes\alpha||_{p,s} \le |t-s| C_\varepsilon ||\alpha||_{p,s} \ ,$$

where the first $||\cdot||_{p,s}$ norm is a classical Sobolev norm with respect to $I\!\!R^{d+2}$, the second $||\cdot||_{p,s}$ is such a norm with respect to $I\!\!R^2$ and C_ε is a Lipschitz constant for ϕ^ε . Now, the generalization of the Banach-Steinaus theorem for Fréchet spaces, see [DS], ch. 2 and property (1.5) imply that the limit is also a linear continous functional from $\mathcal{S}(I\!\!R)$ to $\mathcal{C}(B)$.

Given an element $\eta : B \times \mathcal{S}(I\!\!R^2) \to \mathbb{C}$ of trace type, the corresponding $\mathcal{C}(B)$-valued generalized function will be given by

$$R_\eta(t;x,\varepsilon) = \eta(t;\phi^\varepsilon(\cdot - x)) \ . \tag{1.6}$$

In particular a continuous function on $B \times I\!\!R^2$ can be identified with an element of the trace type.

The equality in $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ is quite strong; it is just compatible with the algebra of functions $F \in C^\infty(I\!\!R;\mathcal{C}(B))$ fulfilling (1.3). If the objects are not smooth in x, the ordinary product is *different* from the one in $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$.

There is a weaker concept of equality, which generalizes the notion of a.e. equality or equality among distributions having the trace property.

Two generalized functions G_1 and G_2 in $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ are said to be *associated* if for any compact $B_{\text{loc}} \subset B$, $\alpha \in \mathcal{S}(I\!\!R^2)$,

$$\lim_{\varepsilon\to 0} \sup_{t\in B_{\text{loc}}} \int_{I\!\!R^2} \left(R_{G_1} - R_{G_2}\right)(t;x,\varepsilon)\alpha(x)dx = 0 \ .$$

The association defines an equivalence relation in $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$. The respective quotient is a linear space; an important subspace is given by the $\mathcal{C}(B)$-valued generalized functions which are associated with an element of trace type. We remark that there are non null elements in $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ which are associated with zero, see [C1] p. 64. An element of $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ is at most associated with one element of trace type.

We also remark that $\mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ is an algebra in the following sense:
if $G_1, G_2 \in \mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ then $G_1 \cdot G_2$ is the class given by $R_{G_1}R_{G_2}$. The zero element is $\mathcal{N}(I\!\!R^2;\mathcal{C}(B))$ and it is absorbing for the product. We observe on the other hand that it may happen that the product of a generalized function associated with zero and another generalized function is not associated with zero. However the product of two generalized functions associated with ordinary functions is associated with the product of ordinary functions.

Moreover if $f \in C^\infty(\mathbb{C})$, with almost polynomial increase at infinity, then $f(G)$ is represented by $(t;x,\varepsilon) \to f(R_G(t;x,\varepsilon))$.

In (1.6) we have defined one embedding of elements of trace type in an algebra of generalized functions. Other embeddings are however possible. Consider for instance

$$\tilde{R}_\eta(t;x,\varepsilon) = \eta(t, \phi^{\psi(\varepsilon)}(\cdot - x)) \ ,$$

so that $\psi : I\!\!R_+ \to I\!\!R_+$ is an increasing continuous function so that $\psi(\varepsilon) \geq \text{const} \quad \varepsilon$. Then $\tilde{R}_\eta$ is associated with R_η. We will say that the class of R_η and the class of $\tilde{R}_\eta$ are *indistinguishible* or also that they are indistinguishible from η.

More generally, G_1 and $G_2 \in \mathcal{G}(I\!\!R^2;\mathcal{C}(B))$ are *indistinguishible* if there are ψ_1, ψ_2 increasing continuous such that $\psi_1(0) = \psi_2(0) = 0$ and $R_{G_1}(\cdot, \psi_1(\varepsilon)) - R_{G_2}(\cdot, \psi_2(\varepsilon))$ is null. In particular two indistinguishible $\mathcal{C}(B)$-valued generalized functions are associated.

We also denote by $\mathcal{S}'(I\!\!R^2;\mathcal{C}(B))$ the family of $\mathcal{C}(B)$ valued generalized functions which are indistinguishible from an element having the trace property. In particular for G in this space, it is possible to show (similarly as in [C1]) that there is a function $F \in C(B \times I\!\!R^2)$ and a derivation operator D on x so that

$$G(t,x) = DF(t,x)$$

and G satisfies condition (1.3).

We introduce now the concept of *random $\mathcal{C}(B)$-valued generalized function.*
Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A map $Y : \Omega \to \mathcal{G}(I\!R^2; \mathcal{C}(B))$ will be said to be a *random* (Colombeau- Oberguggenberger) $\mathcal{C}(B)$ valued generalized function if there is $R : \Omega \times B \times I\!R^2 \times]0, \infty[$ so that

i) $R(\omega, \cdot)$ represents $Y(\omega)$ a. s. $\omega \in \Omega$

ii) For every $\varepsilon > 0$, $(\omega, t, x) \to R(\omega, t; x, \varepsilon)$ is a measurable process

We denote by $\mathcal{G}_\Omega(I\!R^2; \mathcal{C}(B))$ the algebra of random generalized functions. We say that two random $\mathcal{C}(B)$-valued generalized functions Y_1, Y_2 are *indistinguishible* if there are ψ_1, ψ_2 increasing continuous so that

$$R_{Y_1}(\cdot, \psi_1(\varepsilon)) - R_{Y_2}(\cdot, \psi_2(\varepsilon))$$

is null a.s.

The class of such random representatives will be denoted by $\mathcal{E}_{M,\Omega}(I\!R^2, \mathcal{C}(B))$. The random representatives which are null a.s. will be denoted by $\mathcal{N}_\Omega(I\!R^2, \mathcal{C}(B))$.

This theory of generalized functions can be generalized to the vector valued case. In our case the vector space will always be $I\!F = L^2$. We can easily define the linear space of moderate vector valued elements $\mathcal{E}(I\!R^2; \mathcal{C}(B; I\!F))$, the linear subspace of null elements $\mathcal{N}(I\!R^2; \mathcal{C}(B; I\!F))$ and the quotient $\mathcal{G}(I\!R^2; \mathcal{C}(B; I\!F))$. It is enough to replace the $\mathbb{C}$-modulus with the $I\!F$ norm or $I\!F$ homogeneous distance if we have a Fréchet space like $I\!F = L^{\infty-} \equiv \bigcap_{p>1} L^p$. In fact $L^{\infty-}$ is a Fréchet algebra; $\mathcal{E}_M(I\!R^2, \mathcal{C}(B; I\!F))$ is an algebra and $\mathcal{N}(I\!R^2; \mathcal{C}(B; I\!F))$ is an ideal.

In the case $I\!F = L^2$, $\mathcal{E}_M$ and $\mathcal{N}$ are not algebras but this does not play any role in our further considerations. $\mathcal{G}(I\!R^2; \mathcal{C}(B; I\!F))$ is called the space of $\mathcal{C}(B; I\!F)$ valued generalized functions and it is the quotient of $\mathcal{E}(I\!R^2; \mathcal{C}(B; F))$ by $\mathcal{N}(I\!R^2; \mathcal{C}(B; F))$.

Here, a map $\eta : B \times \mathcal{S}(I\!R^2) \to I\!F$ will be said to be of *$I\!F$-trace type* if η is such that $\alpha \to \eta(\cdot, \alpha) \in \mathcal{S}'(I\!R^2; \mathcal{C}(B; I\!F))$, that is to say, it is a linear continuous operator from $\mathcal{S}(I\!R^2)$ to $\mathcal{C}(B; I\!F)$. In particular, for any compact $\subset B$ and any bounded subset S of $\mathcal{S}(I\!R^2)$

$$\sup_{\alpha \in S} \sup_{t \in B_{\rm loc}} \|\eta(t, \alpha)\|_{I\!F} < \infty \ , \tag{1.7}$$

where $\|\cdot\|_{I\!F}$ is the homogeneous distance from zero in F.

In fact, an element η of $I\!F$-trace type can be identified with a $\mathcal{C}(B; I\!F)$-valued generalized function through relation (1.6). (1.5) allows to relate $I\!F$-vector valued distributions having the trace property and elements of F-trace type.

The concept of association can easily be extended to this situation. Strong derivatives and Bochner integrals are in such case involved.

An example of a random generalized field is the *time integrated white noise* (cylindrical Wiener process). We first recall the concept of (Gaussian) white noise. Formally it is a Gaussian random field $(\dot W(t, x), t \in I\!R, x \in I\!R^2)$ with covariance $E(\dot W(t, x)\dot W(s, y)) =$

$\delta(t-s)\delta(x_1-y_1)\delta(x_2-y_2)$, δ being the Dirac measure. Rigorously, it can be realised as a Hilbert valued orthogonally scattered measure. Formally $W(B)=\int_B \dot{W}(s,y)dsdy$. W is a Gaussian field such that $E(W(B_1)W(B_2))$ is the Lebesgue measure of $B_1\cap B_2$. If $g\in L^2(I\!\!R^3)$, $\int g\, dW$ is the integral of g with respect to the orthogonal measure W.

The time integrated white noise is defined first as the following martingale measure $(t,\alpha)\to \displaystyle\int_{[0,t]\times I\!\!R^2} \alpha(y)dW(s,y)$, see [W]. So defined, it is an element $I\!\!R_+\times I\!\!R^2\to L^2(\Omega)$ having the $I\!\!F$-trace property. However, there is a unique version

$$\hat{W}:\Omega\to C(I\!\!R_+;H_{-n}(I\!\!R^2))$$

for $n\geq 2$, see [W]. Therefore $(t,\alpha)\to\hat{W}(\omega)(t)(\alpha)$, is a.s. of trace type. We observe that

$$\hat{W}(\omega)(t,\alpha)=\int_{[0,t]\times I\!\!R^2}\alpha(y)dW(s,y) \text{ a.s. } \forall t\in I\!\!R_+,\ \alpha\in\mathcal{S}(I\!\!R^2)\ .$$

$\hat{W}$ can be identified with the random generalized function represented by

$$R_{\hat{W}}(\omega,t;x,\varepsilon)=\hat{W}(\omega)(t;\phi^\varepsilon(\cdot-x))\ . \tag{1.8}$$

W can be identified with the vector valued generalized function represented by

$$R_W(t;x,\varepsilon)=\int_{[0,t]\times I\!\!R^2} dW(s,y)\phi^\varepsilon(y-x)\ ,$$

$\hat{W}\in\mathcal{G}_\Omega(I\!\!R^2;\mathcal{C}(I\!\!R_+))$, $W\in\mathcal{G}(I\!\!R^2;\mathcal{C}(I\!\!R_+;L^2(\Omega)))$. However, we observe that nguishible random (resp. vector valued) generalized functions of W and $\hat{W}$ can be produced, for instance by substituting ε with $\psi(\varepsilon)$, so that $c_1\left(\log\frac{1}{\varepsilon}\right)^{c_2}=\frac{1}{\psi(\varepsilon)}$, where c_1,c_2 are positive constants.

2. The free case. Estimates on the semigroup

The equation (2) in the case $\lambda=0, U_0=0$ will be called the free equation. In this section, we will make some comments on such equation and obtain estimates on the heat semigroup.

Let L be a second order differential operator with smooth coefficients with bounded derivatives of each order. We suppose moreover L to be uniformly elliptic and symmetric in the $L^2(I\!\!R^2,dx)$ space i. e. $L\subset L^*$.

We consider the semigroup $(P_t,t\geq 0)$ related to the problem

$$\begin{cases}\partial_t v=Lv & \text{on }]0,\infty[\times I\!\!R^2\\ v(0,\cdot)=g\end{cases} \tag{2.1}$$

where $g \in C^0(\mathbb{R}^2) \cap \mathcal{S}'(\mathbb{R}^2)$.

Remark 2.1 *It is well known that (2.1) has a unique solution* $v(t,y) = (P_t g)(y)$. *Moreover the semigroup has a smooth density* $p_t(x,y)$, *so that* $(P_t g)(y) = \int p_t(x,y) g(x) dx$.
Since L *is symmetric, we have* $p_t(x,y) = p_t(y,x)$ *for any* $x, y \in \mathbb{R}^2$. *Moreover* $P_t \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$.

In this section, we are interested in the solution of the free equation

$$\begin{cases} \partial_t U = LU + \dot{W} & \text{on }]0,\infty[\times \mathbb{R}^2 \\ U(0,\cdot) \equiv 0 \end{cases} \tag{2.2}$$

Remark 2.2 *Let* $\eta \in \mathcal{S}'(\mathbb{R}^3)$ *with support in* $]0,\infty[\times\mathbb{R}^2$ *having the trace property with respect to* $\mathbb{R}$, *see (1.5). Let* $g \in \mathcal{S}'(\mathbb{R}^2)$. *The problem*

$$\begin{cases} \partial_t U = LU + \eta \\ U(0,\cdot) = g \end{cases} \tag{2.3}$$

is understood in the distributional sense, as in [W], chapter 5. We look for $\bar{U} \in \mathcal{S}'(\mathbb{R}^3)$ *with support in* $\mathbb{R}_+ \times \mathbb{R}^2$ *with the trace property such that for every* $\bar{\alpha} \in \mathcal{S}(\mathbb{R}^3)$

$$\begin{cases} -\langle \bar{U}, \frac{\partial \bar{\alpha}}{\partial t} \rangle = \langle \bar{U}, L\bar{\alpha} \rangle + \langle \eta, \bar{\alpha} \rangle \\ U(0,\cdot) \equiv g \end{cases} , \tag{2.4}$$

where $U(0,\alpha)$ *is the* $\lim_{\varepsilon\to 0} \langle \bar{U}, \phi^\varepsilon(\cdot - 0) \otimes \alpha \rangle$.

Proposition 2.1 *Let* η *be a Radon measure on* $\mathbb{R} \times \mathbb{R}^2$ *without atoms. Let* $U : \mathbb{R} \times \mathcal{S}(\mathbb{R}^2) \to \mathbb{R}$ *vanishing for* $t \le 0$ *and such that* $\alpha \mapsto U(\cdot,\alpha)$ *is of trace type. Then* $\bar{U}$ *solves (2.4) if and only if*

$$U(t,\alpha) = \int_0^t ds U(s, L\alpha) + \int_{[0,t]\times\mathbb{R}^2} \alpha(x) d\eta(s,x) + \langle g, \alpha \rangle \ . \tag{2.5}$$

We recall that U *and* $\bar{U}$ *are related through Remark 1.1.*

Proof Let $\beta \in \mathcal{S}(\mathbb{R})$, $\alpha \in \mathcal{S}(\mathbb{R}^2)$. Let us suppose that $\bar{U}$ solves (2.4), in particular it fulfills (1.5). Then we can evaluate both members of (2.4) with respect to $\beta \otimes \alpha$. The left member gives

$$-\int_{-\infty}^{\infty} ds \beta'(s) U(s,\alpha). \tag{2.6}$$

The right member equals

$$(2.7) \qquad \langle \bar U, \beta\otimes L\alpha\rangle + \int_{I\!R_+\times I\!R^2} \beta(s)\alpha(x) d\eta(s,x)\ .$$

For $\varepsilon > 0$, $t\geq 0, s\in I\!R$, we choose

$$\beta_{\varepsilon,t}(s) = \int_s^\infty \frac{1}{\varepsilon}\left[\phi\left(\frac{u-t}{\varepsilon}\right) - \phi\left(\frac{u}{\varepsilon}\right)\right] du = \int_{\frac{s-t}{\varepsilon}}^{\frac{s}{\varepsilon}} dv\phi(v)\ ,$$

where ϕ has been defined in section 1. We have

$$\beta'_{\varepsilon,t}(s) = -\frac{1}{\varepsilon}\phi\left(\frac{s-t}{\varepsilon}\right) + \frac{1}{\varepsilon}\phi\left(\frac{s}{\varepsilon}\right)\ .$$

Then (2.6) gives

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{ds}{\varepsilon}\phi\left(\frac{s-t}{\varepsilon}\right) U(s,\alpha) - \int_{-\infty}^\infty \frac{ds}{\varepsilon}\phi\left(\frac{s}{\varepsilon}\right) U(s,\alpha)\\
&= \int dv\phi(v)U(t-\varepsilon v,\alpha) - \langle \bar U, \phi^\varepsilon(\cdot - 0)\otimes\alpha\rangle\\
&\stackrel{\varepsilon\to 0}{\longrightarrow} U(t,\alpha) - \langle g,\alpha\rangle\ .
\end{aligned}$$

On the other hand, (2.7) gives

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{ds}{\varepsilon}\int_s^\infty du\left[\phi\left(\frac{u-t}{\varepsilon}\right) - \phi\left(\frac{u}{\varepsilon}\right)\right] U(s,L\alpha)\\
&+\int_{[0,\infty[\times I\!R^2} d\eta(s,x)\frac{\alpha(x)}{\varepsilon}\int_s^\infty \left[\phi\left(\frac{u-t}{\varepsilon}\right) - \phi\left(\frac{u}{\varepsilon}\right)\right] du\\
&= \int_0^\infty dv\phi(v)\int_0^{t-v\varepsilon} dsU(s,L\alpha) + \int_{[0,\infty[\times I\!R^2} d\eta(s,x)\frac{\alpha(x)}{\varepsilon}\int_{\frac{s-t}{\varepsilon}}^\infty dv\phi(v) + 0(\varepsilon)\\
&\stackrel{\varepsilon\to 0}{\longrightarrow}\int_0^t dsU(s,L\alpha) + \int_{[0,t]\times I\!R^2} d\eta(s,x)\alpha(x).
\end{aligned}$$

Therefore U solves (2.5). Viceversa if U solves (2.5) then by inspection and integration by parts, $\bar U$ solves (2.4). ■

Remark 2.3 *Proposition 2.1 also holds for vector valued equations. Consider for instance a Fréchet space $I\!F$ with homogeneous metric d. Let η be a continuous linear functional from $\mathcal{C}(I\!R^3)$ into $I\!F$ so that $d(\eta(\varphi),0)\leq\int\varphi d\nu$ for any $\varphi\in\mathcal{C}(I\!R^3)$ where ν is a non-atomic Borel measure on $I\!R^3$. This is the case for a white noise η and $I\!F = L^2(\Omega)$.*

The previous proposition allows us to study the form (2.5) instead of (2.4).

Proposition 2.2 *Let $g\in\mathcal{S}'(I\!R^2)$ be a Radon measure on $I\!R_+\times I\!R^2$ without atoms. There is a unique solution U to (2.5) of trace type and it is given by*

$$(2.8) \qquad U(t,\alpha) = \int_{[0,t]\times \mathbb{R}^2} d\eta(s,y) \int_{\mathbb{R}^2} dx p_{t-s}(x,y)\alpha(x) + \langle g, P_t\alpha\rangle \ .$$

Remark 2.4 *Remark 2.3 can be extended to the case of Proposition 2.2. Let us suppose that η is as in Remark 2.3. Then, there is a solution to (2.5) $U : B \times \mathcal{S}(\mathbb{R}^2) \to \mathbb{F}$ such that $\alpha \mapsto U(\cdot,\alpha)$ is of $\mathbb{F}$ trace type and it is given by (2.8).*

Proof: (of the proposition) Let U be given by formula (2.8). Then

$$\int_0^t dsU(s,L\alpha) = \int_0^t ds \left(\int_{[0,t]\times \mathbb{R}^2} d\eta(u,y) P_{s-u} L\alpha(y) + \langle g, P_s L\alpha\rangle \right) \ ,$$

where

$$P_s g(x) = \int p_s(x,y)g(y)dy \ .$$

Using Fubini theorem, this equals to

$$\int_{[0,t]\times \mathbb{R}^2} d\eta(u,y) \int_0^t ds P_{s-u} L\alpha(y) + \int_0^t ds\langle g, P_s L\alpha\rangle \ .$$

But $P_s L = LP_s = P_s'$ so that the previous expression is equal to

$$\int_{[0,t]\times \mathbb{R}^2} d\eta(u,y)\left\{P_{t-u}\alpha(y) - \alpha(y)\right\} + \langle g, P_t\alpha\rangle - \langle g,\alpha\rangle$$

$$= U(t,\alpha) - \int_{[0,t]\times \mathbb{R}^2} d\eta(u,y)\alpha(y) - \langle g,\alpha\rangle \ .$$

This implies existence. Uniqueness is a consequence of the following lemma.

Lemma 2.1 *There is only one function $V : \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ (resp. $\mathbb{F}$) such that $\alpha \mapsto V(\cdot,\alpha)$ is of trace (resp. $\mathbb{F}$-trace) type and*

$$(2.9) \qquad V(t,\alpha) = \int_0^t V(s,L\alpha)ds \ , \ \forall \alpha \in \mathcal{S}(\mathbb{R}^2), \ \forall t \geq 0 \ :$$

it is the zero function, i.e. $V \equiv 0$.

Proof: Let V be a solution to (2.9). We set $\|\cdot\|$ to be the distance from zero in F. Let S be a bounded subset in the Fréchet space $\mathcal{S}(\mathbb{R}^3)$, see [DS], ch 2.

$$(2.10) \qquad \|V(t,\alpha)\| \leq \int_0^t \|V(s,L\alpha)\|ds \leq \int_0^t \sup_{\beta\in S}\|V(s,\beta)\|ds \sup_{\alpha\in S}\|L\alpha\| \ .$$

Since $L : \mathcal{S}(I\!R^2) \to \mathcal{S}(I\!R^2)$ is continuous, the previous supremum is finite. (2.10) and Gronwall lemma imply $V \equiv 0$. ■

Lemma 2.2 *Let* $f : \mathbb{C} \times I\!R \times I\!R^2 \to \mathbb{C}$ *be a continuous function,* $g \in C^0(I\!R^2) \cap \mathcal{S}'(I\!R^2)$*. We suppose that* $f(y; t, \cdot)$ *has support in a fixed compact subset* $K \subset I\!R^2$, $\forall y \in \mathbb{C}$, $t \geq 0$.

Let u *be a function in* $\mathcal{C}(I\!R_+ \times I\!R^2)$ *with polynomial growth in* $x \in R^2$*. Let* ν *be a Radon measure without atoms,* $h \in \mathcal{C}(I\!R^2)$ *with compact support. Then* u *is solution to*

$$u(t,x) = \int_0^t \{Lu(s,x) + f(u(s,x); s, x)\} ds + g(x) + \int_{[0,t]\times I\!R^2} d\nu(s,y) h(x-y) \tag{2.11}$$

if and only if

$$\begin{aligned} u(t,x) = & \int_{I\!R^2} dy p_t(x,y) g(y) + \int_0^t ds \int_{I\!R^2} p_{t-s}(x,y) f(u(s,y); s, y) \\ & + \int d\nu(s,z) \int p_{t-s}(x,y) h(y-z) dy \ . \end{aligned} \tag{2.12}$$

Remark 2.5

a) *Taking Fourier transform in space and applying Picard's method, (2.11) can be shown to have a unique solution if* f *is Lipschitz continous with respect to the first argument uniformly with respect to* (s,x).

b) *Lemma 2.2 has a Fréchet vector valued* $I\!F$ *version.*

Proof (of the lemma 2.2) Let u be a solution to (2.11). We consider the Radon measure η defined by

$$d\eta(s,x) = f(u(s,x); s, x) ds dx + \{\int d\nu(s,y) h(x-y)\} dx.$$

Then

$$\eta([0,t] \times B) = \int_{[0,t]\times B} f(u(s,y); s, y) ds dy + \int_{[0,t]\times I\!R^2 \times B} d\nu(s,z) dy h(y-z) \ .$$

Consequently, $U(t,\alpha) = \int u(t,x)\alpha(x)dx$ is of the form (2.5). Proposition 2.2 implies then that U solves (2.8). Since u is a function, it must have the form (2.12).

Conversely, if v is a function satisfying (2.12), then by symmetric arguments as before, v solves (2.11). ■

We recall that the unique solution $X : I\!R_+ \times \mathcal{S}(I\!R^2) \to L^2$ to

$$X(t,\alpha) = \int_0^t X(s, L\alpha) ds + \int_{[0,t]\times I\!R^2} dW(s,y) \alpha(y) \tag{2.13}$$

is given by

$$(2.14)\qquad X(t,\alpha)=\int_{[0,t]\times I\!R^2} dW(s,y)\int_{I\!R^2} dx\alpha(x)p_{t-s}(x,y)\ .$$

This is the approach proposed for instance by [W], ch. 6.

We go on now with some important estimates on the density semigroup $(p_t(x,y))$.

Proposition 2.3 ([D], ch. 3)
There are constants $\delta_1,\delta_2,C_1,C_2>0$ such that

$$\frac{C_1}{t}\exp\left(\frac{-|x-y|^2}{\delta_1 t}\right)\le p_t(x,y)\le \frac{C_2}{t}\exp\left(\frac{-|x-y|^2}{\delta_2 t}\right)\ .$$

We recall that the free solution X can formally be written as

$$X(t,x)=\int_{[0,t]\times I\!R^2} dW(s,y)p_{t-s}(x,y)\ .$$

Therefore Proposition 2.3 says immediately that the variance at each point $t>0$, $x\in I\!R^2$

$$\int_{[0,t]\times I\!R^2} dsdyp^2_{t-s}(x,y)$$

is infinite. However the covariance

$$\mathrm{Cov}(t_1,x_1;t_2,x_2)=\int_0^{t_1\wedge t_2}\int_{I\!R^2} dyp_{t_1-s}(x_1,y)p_{t_2-s}(x_2,y)$$

is finite. It represents $E(X(t_1,x_1)\ X(t_2,x_2))$.

Remark 2.6 *The symmetry of the kernel and Chapman-Kolmogorov equation give*

$$\mathrm{Cov}(t_1,x_1;t_2,x_2)=\int_0^{t_1\wedge t_2} ds\ p_{t_1+t_2-2s}(x_1,x_2)\ .$$

The covariance will be estimated in the following proposition.

Proposition 2.4 *Let $K\subset I\!R^2$ be a compact. There are real constants a_1,a_2,a_3 and positive constants D_1,D_2,D_3 such that*

i)

$$0\le \mathrm{Cov}\ (t_1,x_1;t_2,x_2)\le D_1\log\left|\frac{t_2+t_1}{t_2-t_1}\right|+a_1$$

for every $t_1,t_2\in]0,T]$ so that $t_1\neq t_2$ and $x_1,x_2\in K$.

ii)

$$a_2+D_2\log\frac{t}{|x_2-x_1|^2}\le \mathrm{Cov}\ (t,x_1;t,x_2)\le D_3\log\frac{1}{|x_2-x_1|^2}+a_3$$

where $t = t_1 = t_2 > 0$, $x_1, x_2 \in K$.

Proof We will suppose $t_1 \le t_2$. By the change of variables $v = t_1 + t_2 - 2s$, Remark 2.6 gives

$$\text{Cov }(t_1, x_1; t_2, x_2) = \int_{t_2-t_1}^{t_2+t_1} \frac{dv}{2} p_v(x_1, x_2). \tag{2.16}$$

i) Proposition 2.3 allows to dominate the right member of (2.16) so that

$$\text{Cov }(t_1, x_1; t_2, x_2) \le C_2 \int_{t_2-t_1}^{t_2+t_1} \frac{dv}{v} \exp\left(\frac{-\left|x_1 - x_2\right|^2}{\delta_2 v}\right)$$

for positive constants $C_2, \delta_2 > 0$. The change of variables $s = \dfrac{\left|x_2 - x_1\right|^2}{\delta_2 v}$ says that the right member equals

$$C_2\left(Ei\left(\frac{\left|x_2 - x_1\right|^2}{(t_2 + t_1)\delta_2}\right) - Ei\left(\frac{\left|x_2 - x_1\right|^2}{(t_2 - t_1)\delta_2}\right)\right) \tag{2.17}$$

where

$$Ei(y) = \int_y^\infty \frac{e^{-v}}{v} dv, y > 0. \tag{2.18}$$

We observe that

- Ei is positive and decreasing.
- $Ei(y) = \psi(y) - log(y)$, where

$$\psi(y) = \int_y^1 \frac{e^{-v} - 1}{v} dv + Ei(1)$$

defines a locally bounded positive function.
- $Ei(y)$ behaves as $log\frac{1}{y}$ when y goes to zero.

We set $\tilde{\psi}(y) = \sup_{z \le y} \psi(z)$ We will show that

$$\text{Cov }(t_1, x_1; t_2, x_2) \le D_1 \log \frac{t_2 + t_1}{t_2 - t_1} + a_1. \tag{2.19}$$

Let $R > 0$.

a) If $\frac{|x_2-x_1|^2}{\delta_2(t_2-t_1)} \leq R$, then $\frac{|x_2-x_1|^2}{\delta_2(t_2+t_1)} \leq R$ and

$$\frac{1}{C_2} Cov(t_1, x_1; t_2, x_2) \leq log\left(\frac{t_2+t_1}{t_2-t_1}\right) + \psi\left(\frac{|x_2-x_1|^2}{(t_2+t_1)\delta_2}\right) - \psi\left(\frac{|x_2-x_1|^2}{(t_2-t_1)\delta_2}\right)$$

$$\leq log\left(\frac{t_2+t_1}{t_2-t_1}\right) + 2\tilde{\psi}(R)$$

b) If $\frac{|x_2-x_1|^2}{\delta_2(t_2-t_1)} > R$ and $\frac{|x_2-x_1|^2}{\delta_2(t_2+t_1)} \leq R$ then

$$\frac{1}{C_2} Cov(t_1, x_1; t_2, x_2) \leq log\left(\frac{(t_2+t_1)\delta_2}{|x_2-x_1|^2}\right) + \psi\left(\frac{|x_2-x_1|^2}{(t_2+t_1)\delta_2}\right) - Ei\left(\frac{|x_2-x_1|^2}{(t_2-t_1)\delta_2}\right)$$

$$\leq log\left(\frac{t_2+t_1}{R(t_2-t_1)}\right) + \tilde{\psi}(R)$$

$$= log\left(\frac{t_2+t_1}{t_2-t_1}\right) + \tilde{\psi}(R) - logR$$

c) If

$$\frac{|x_2-x_1|^2}{(t_2-t_1)} > \frac{|x_2-x_1|^2}{(t_2+t_1)} > R$$

then $Cov(t_1, x_1; t_2, x_2) \leq 2Ei(R)$ and the result follows.

ii) (2.16) and Proposition 2.3 say that

$$\begin{aligned} \text{Cov }(t, x_1; t, x_2) &= \int_0^{2t} \frac{dv}{2} p_v(x_1, x_2) \\ &\leq C_2 \int_0^{2t} \frac{dv}{v} \exp\left(\frac{-|x_1-x_2|^2}{\delta_2 v}\right) \\ &= C_2 Ei\left(\frac{|x_2-x_1|^2}{2\delta_2 t}\right) \end{aligned}$$

where Ei has been defined before. Therefore, using the fact that Ei is decreasing, we get

$$\text{Cov }(t, x_1; t, x_2) \leq C_2 Ei\left(\frac{|x_1-x_2|^2}{2\delta_2 t}\right) \leq D_3 \log \frac{1}{|x_2-x_1|^2} + a_3.$$

This proves the upper bound. The lower bound is obtained similarly. ■

Let $\phi \in \mathcal{S}(I\!R^2)$ be as defined in section 1. We consider the object X defined in (2.14). We recall that it is the solution to (2) with $U_0 = 0, \lambda = 0$ in the sense of distributions.

R_X is the representative of the $\mathcal{C}(I\!R_+; L^2)$-valued generalized function identified with X through (1.6). We compute some covariance terms related to R_X. We set

$$Cov_{\varepsilon,\delta}\ (t_1, x_1; t_2, x_2) = E(R_X(t_1; x_1, \varepsilon) R_X(t_2; x_2, \delta))$$

where $\varepsilon, \delta > 0$, $t_1, t_2 \geq 0$, $x_1, x_2 \in I\!R^2$.

We recall that

$$R_X(t; x, \varepsilon) = \int_{[0,t]\times I\!R^2} dW(s,y) \int p_{t-s}(z,y) \phi^\varepsilon(x-z) dz$$

Proposition 2.5 *Let $b \in I\!R$ and $K \subset I\!R^2$ a compact. There are $C_1, C_2, D_1, D_2 > 0$ such that*

i)

$$\exp(b\ Cov_{\varepsilon,\delta}\ (t_1, x_1; t_2, x_2)) \leq D_1 \left| \frac{t_2 + t_1}{t_2 - t_1} \right|^{bC_1}$$

for every $t_1, t_2 \in]0, T]$ so that $t_1 \neq t_2$ and $x_1, x_2 \in K$.

ii)

$$\exp(b\ Cov_{\varepsilon,\delta}\ (t, x_1; t, x_2)) \leq D_2 \int \frac{dz_1 dz_2 \phi(z_1)\phi(z_2)}{|x_2 - x_1 + \varepsilon z_1 - \delta z_2|^{C_2 b}} ,$$

where $t = t_1 = t_2 \in]0, T]$, $x_1, x_2 \in K$.

Proof We suppose $t_1 \leq t_2$. Using (2.8) and (1.6) we can write

$$R_X(t_1; x_1, \varepsilon) = \int_{[0,t_1]\times I\!R^2} dW(s,y) \int_{I\!R^2} p_{t_1-s}(x_1 - \varepsilon z, y)\phi(z) dz$$

$$R_X(t_2; x_2, \delta) = \int_{[0,t_2]\times I\!R^2} dW(s,y) \int_{I\!R^2} p_{t_2-s}(x_2 - \delta z, y)\phi(z) dz\ .$$

Therefore

$$\begin{aligned} & Cov_{\varepsilon,\delta}\ (t_1, x_1; t_2, x_2)) \\ &= \int_{[0,t_1]\times I\!R^2} dsdy \int\int dz_1 dz_2 \phi(z_1)\phi(z_2) p_{t_1-s}(x_1 - \varepsilon z_1, y) p_{t_2-s}(x_2 - \delta z_2, y) \\ &= \int_{I\!R^2\times I\!R^2} \phi(z_1)\phi(z_2) dz_1 dz_2 \int_{[0,t_1]} ds \int_{I\!R^2} dy p_{t_1-s}(x_1 - \varepsilon z_1, y) p_{t_2-s}(x_2 - \delta z_2, y) \\ &= \int_{I\!R^2\times I\!R^2} dz_1 dz_2 \,\mathrm{Cov}\ (t_1, x_1 - \varepsilon z_1; t_2, x_2 - \delta z_2)\phi(z_1)\phi(z_2)\ . \end{aligned} \tag{2.20}$$

i) is now a direct consequence of Proposition 2.4 i).

Concerning ii), Jensen's inequality and (2.20) say that for $t = t_1 = t_2$

$$\exp(bCov_{\varepsilon,\delta}\ (t,x_1;t,x_2)) \leq \int_{I\!R^2\times I\!R^2} dz_1dz_2\phi(z_1)\phi(z_2)\exp(b\,\mathrm{Cov}\ (t,x_1-\varepsilon z_1;t,x_2-\delta z_2))\ .$$

By Proposition 2.4 ii), it follows that this is smaller than

$$\begin{aligned} &\text{const}\ \int_{I\!R^2\times I\!R^2} dz_1dz_2\phi\otimes\phi(z_1,z_2)\left(\frac{1}{|x_2-x_1+\varepsilon z_1-\delta z_2|^{bC_2}}\right) \\ &\leq \text{const}\ \int_{I\!R^2\times I\!R^2} dz_1dz_2\frac{\phi\otimes\phi(z_1,z_2)}{|x_2-x_1+\varepsilon z_1-\delta z_2|^{bC_2}}\ . \end{aligned}$$

■

We discuss now some linear equations in Colombeau sense. First of all we start with two lemmas.

Lemma 2.3 *Let $g\in C^\infty(I\!R^2;\mathcal{C}(I\!R_+))$ such that every derivative fulfills property (1.3). Then*

$$u(t,x)=\int_0^t ds\int_{I\!R^2} dyp_{t-s}(x,y)g(s,y)$$

has the same property. Moreover, for any given derivation operator D acting on the space argument x we have

$$Du(t,x)=\int_0^t ds\int_{I\!R^2} dyp_{t-s}(x,y)D_yg(s,y)\ .$$

Proof By classical arguments of [F], ch. 1, it is possible to prove the existence of a polynomial $P\left(\frac{1}{t},x-y\right)$ and a constant $\delta>0$ so that

$$Dp_t(x,y)\leq P\left(\frac{1}{t},x-y\right)\exp\left(\frac{-|x-y|^2}{\delta t}\right)\ . \tag{2.21}$$

Clearly (2.21) implies that

$$\int dy\,|Dp_t(x,y)|\,(1+|y|^n)<\infty\ ,\ \forall n\in I\!N,\ x\in I\!R^2,\ t>0\ . \tag{2.22}$$

For fixed $t>0$ we set

$$v(s,x)=\int_{I\!R^2} dyp_{t-s}(x,y)g(s,y),\quad s\leq t\ .$$

Using (2.22) we get

$$\int dy\,|Dp_{t-s}(x,y)|\ \ |g(s,y)| < \infty, \quad s \le t\ .$$

Using the definition of partial derivatives and Lebesgue dominated convergence theorem, one gets that Dv exists and

$$Dv(s,x) = \int dy Dp_{t-s}(x,y)g(s,y)\ .$$

By integration by parts and using the symmetry of the kernel we get

$$Dv(s,x) = \int dy Dp_{t-s}(y,x)g(s,y) = \int dy p_{t-s}(x,y)D_y g(s,y)\ .$$

On the other hand, there is $n \in I\!N$ such that

$$\sup_{s\le T, y\in I\!R^2} \frac{|D_y g(s,y)|}{1+|y|^n} < \infty\ .$$

Therefore, using also Proposition 2.3, it follows that

$$Du(t,x) \le \int_0^t ds\,|Dv(s,x)| = \int_0^t ds \int dy p_{t-s}(x,y)\,|D_y g(s,y)|$$

$$\int_0^t ds \int dy p_{t-s}(x,y)(1+|y|^n) \le const(1+|x|^n)$$

for any $t \in]0,T]$.

Proposition 2.6 *Let $R \in \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+))$ (resp. $\mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+))$). Then*

i) $LR(t;x,\varepsilon)$

ii) $\int_{[0,t]} dsR(s;x,\varepsilon)$

iii) $R_1(t;x,\varepsilon) = \int_{[0,t]\times I\!R^2} dsdyp_{t-s}(x,y)R(s;y,\varepsilon)$

all belong to the same space $\mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+))$ (resp. $\mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+))$).

Remark 2.7

a) The proposition above says in particular that for $U \in \mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+))$

$$U_1(t,x) = LU(t,x)\ ,$$

$$U_2(t,x) = \int_{[0,t]\times I\!R^2} dsdyp_{t-s}(x,y)U(s,y)$$

are well defined.

b) We can replace $\mathcal{C}(I\!R_+)$ by $\mathcal{C}(I\!R_+;F)$.

Proof (of the proposition) i) If R belongs to $\mathcal{E}_M$ (resp. $\mathcal{N}$) then all derivatives have a tempered moderate (resp. null) bound. Since the coefficients of L have a tempered moderate bound then LR belongs to $\mathcal{E}_M$ (resp. $\mathcal{N}$). ii) is straightforward and iii) is a consequence of Lemma 2.3. ∎

Proposition 2.7 *The unique solution $U \in \mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+))$ to the equation*

$$U(t,x) = \int_{[0,t]\times I\!R^2} LU(s,x)dx \tag{2.23}$$

is the zero generalized function.

Proof If $U \in \mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+))$ solves (2.23), there are $R_U \in \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+))$, $R_0 \in \mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+))$ so that

$$R_U(t;x,\varepsilon) = \int_{[0,t]} dsLR_U(s;x,\varepsilon) + R_0(t;x,\varepsilon) \ .$$

By Proposition 2.6

$$R_1(t;x,\varepsilon) = \int_0^t dsLR_0(s;x,\varepsilon)$$

still belongs to $\mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+))$. We now set

$$R_2(t;x,\varepsilon) = R_U(t;x,\varepsilon) - R_0(t;x,\varepsilon),$$

getting

$$R_2(t;x,\varepsilon) = \int_0^t dsLR_2(s;x,\varepsilon) + R_1(t;x,\varepsilon) \ .$$

According to Lemma 2.2

$$R_2(t;x,\varepsilon) = \int_0^t ds \int_{I\!R^2} dyp_{t-s}(x,y)R_1(s;x,\varepsilon) \ . \tag{2.24}$$

R_2 belongs to $\mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+))$ because of Proposition 2.6, therefore the same holds for R_U. ∎

Next section will be devoted to the study of the existence and uniqueness of a random $\mathcal{C}(I\!R_+)$-valued generalized function for a non-linear heat equation involving Wick reordering.

We need first to define some new generalized functions which incorporate "renormalizing coupling constants". First of all we observe that X defined by (2.13) is an element of $L^2(\Omega)$-trace type and through (1.6) it defines a generalized function in $\mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. On the other hand it is not difficult to show that there is a random element a.s. of trace type $\hat{X}$ such that

$$\hat{X}(t,\alpha) = X(t,\alpha) \text{ a.s. } \forall t \geq 0,\ \alpha \in \mathcal{S}(I\!R^2) .$$

To $\hat{X}$ we can also relate a random generalized function (still denoted by $\hat{X}$) in $\mathcal{G}_\Omega(I\!R^2;\mathcal{C}(I\!R_+))$, operating according to (1.6).

For a given constant $c > 0$, we can define another random generalized function $\hat{X}_c$ through the representative

$$R_{\hat{X}_c}(t;x,\varepsilon) = R_{\hat{X}}(t;x,\psi_c(\varepsilon)) ,$$

where

$$\psi_c(\varepsilon) = \left(\frac{-1}{\log\varepsilon}\right)^{\frac{1}{c}} \text{ so that } \left(\frac{1}{\psi_c(\varepsilon)}\right)^c = \log\frac{1}{\varepsilon} . \tag{2.25}$$

In particular

$$R_{\hat{X}_c}(t;x,\varepsilon) = \int_{[0,t]\times I\!R^2} dW(s,y) \int_{I\!R^2} p_{t-s}(z,y)\phi^{\psi_c(\varepsilon)}(x-z)dz \ a.s. \tag{2.26}$$

Also we can replace $R_{\hat{X}_c}$ by R_{X_c} and define X_c, as a $\mathcal{C}(I\!R_+;L^2)$-valued generalized function.

Remark 2.8 *$\hat{X}$ and $\hat{X}_c$ (resp. X and X_c) are indistinguishible random $\mathcal{C}(I\!R_+)$-valued generalized functions (resp. $\mathcal{C}(I\!R_+;L^2)$-valued generalized functions).*

Lemma 2.4 *The elements*

$$R_G(a,t;x,\varepsilon) = \exp(a^2 E(R^2_X(t;x,\varepsilon)))$$

$$R_{G_c}(a,t;x,\varepsilon) = \exp(a^2 E(R^2_{X_c}(t;x,\varepsilon)))$$

belong to $\mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R\times I\!R_+))$.
Moreover, for $T > 0$ and any compact $I \subset I\!R$, there is $c_0 > 0$ such that for $c \geq c_0$

$$\sup_{a\in I, t\leq T, x\in I\!R^2} |R_{G_c}(a,t;x,\varepsilon)| = O\left(\log\frac{1}{\varepsilon}\right) .$$

Remark 2.9 *The class of G and G_c define indistinguishible $\mathcal{C}(I\!R\times I\!R_+)$-valued generalized functions.*

Proof (of the lemma) The calculation (2.20) and Proposition 2.4 ii) say that there is a constant $C_1 > 0$ such that

$$
\begin{aligned}
&\exp(a^2 E(R_X^2(t;x,\varepsilon))) \\
&\quad = \exp\left(a^2 \int_{\mathbb{R}^2\times\mathbb{R}^2} dz_1 dz_2 \phi\otimes\phi(z_1,z_2)\ \text{Cov}\ (t, x-\varepsilon z_1; t, x-\varepsilon z_2)\right) \\
&\quad \le \exp\left(\int dz_1 dz_2 a^2 C_1 \log\left(\frac{1}{\varepsilon^2 |z_1-z_2|^2}\right)\phi(z_1)\phi(z_2)\right) \\
&\quad = \exp\left(a^2 C_1 \log\frac{1}{\varepsilon^2}\right)\ \exp\left(a^2 C_1 \int dz_1 dz_2 \phi(z_1)\phi(z_2)\log\frac{1}{|z_1-z_2|^2}\right)
\end{aligned}
$$

The second exponential is finite for every $a \in I$ and will be denoted by C_2. Therefore

$$
\exp(a^2 E(R_X^2(t;x,\varepsilon))) \le C_2 \left(\frac{1}{\varepsilon^2}\right)^{a^2 C_1} = \text{const}\ (T,I)\ \varepsilon^{-M(I)};\ \forall a \in I\ . \tag{2.27}
$$

$M(I)$ is a positive constant depending on I and $const(T,I)$ is a constant depending on T and I. This shows in particular that

$$
\exp(a^2 E(R_X^2(t;x,\varepsilon))
$$

has a tempered moderate bound with $n = M(I)$.

Let us now discuss the derivatives. Using (2.20) we have

$$
E(R_X^2(t;x,\varepsilon)) = \int_{\mathbb{R}^2\times\mathbb{R}^2} dz_1 dz_2 \phi^\varepsilon\otimes\phi^\varepsilon(z_1-x, z_2-x)\ \text{Cov}\ (t,z_1;t,z_2)\ . \tag{2.28}
$$

Again by Proposition 2.4 ii)

$$
\int dz_1 dz_2\ \text{Cov}\ (t,z_1;t,z_2) \le \text{const}\ \left(\int dz_1 dz_2 \log\frac{1}{|z_1-z_2|^2} + 1\right)\ ;
$$

this integral is finite. On the other hand for any given partial derivation operator D with respect to x we have:

$$
|D[\phi^\varepsilon\otimes\phi^\varepsilon](z_1-x, z_2-x)| \le \frac{1}{\varepsilon^{n_1}}\,\text{const}(\phi)\ ,
$$

for some $n_1 > 0$. Therefore

$$
DE(R_X^2(t;x,\varepsilon)) \le \frac{\text{const}}{\varepsilon^{n_1}}\ .
$$

Now

$$D \exp(a^2 E(R_X^2(t;x,\varepsilon)) = \exp(a^2 E(R_X^2(t;x,\varepsilon))Q \ ,$$

where Q is a polynomial in the derivatives of $E(R_X^2(t;x,\varepsilon))$. It follows that

$$\sup_{a\in I, t\leq T, x\in \mathbb{R}^2} |DR_{G_c}(a,t;x,\varepsilon)| \leq \text{const } \frac{1}{\varepsilon^{M(I)}}$$

for suitable $M(I) > 0$.

Remark 2.10 *When L has constant coefficients, $p_t(x,y)$ is of convolution type, that is to say there is $\tilde{p}_t$ such that $p_t(x,y) = \tilde{p}_t(x-y)$. In this case*

$$\text{Cov } (t, x-\varepsilon z_1; t, x-\varepsilon z_2) = \int_0^t ds \tilde{p}_{2t-2s}(\varepsilon z_1 - \varepsilon z_2) \ ,$$

so R_G does not depend on x; so all derivatives with respect to x vanish and the situation is in this case simpler.

We discuss the case R_{G_c}. In this case

$$R_{G_c}(a,t;x,\varepsilon) = R_G(a,t;x,\tilde{\varepsilon}) \ .$$

where $\tilde{\varepsilon} = \psi_c(\varepsilon)$. So, by (2.27), there is another constant $const(T,I)$ such that

$$\sup_{a\in I, t\leq T, x\in \mathbb{R}^2} |R_{G_c}(a,t;x,\varepsilon)| = O\left(\text{const } (T,I)\tilde{\varepsilon}^{-M(I)}\right) = \text{const } O\left(\left(\tilde{\varepsilon}^{-c}\right)^{\frac{M(I)}{c}}\right)$$
$$= O\left(\left(\log\frac{1}{\varepsilon}\right)^{\frac{M(I)}{c}}\right) = O\left(\left(\log\frac{1}{\varepsilon}\right)\right) \text{ if } M(I) \leq c \ .$$

We set then $c_0 = M(I)$. ■

X (resp. $\hat{X}$) solve the (Gaussian) free equation in distributional sense, see [W], chap. 5. Another $\mathcal{C}(\mathbb{R}_+; L^2)$-valued (resp. random $\mathcal{C}(\mathbb{R}_+)$-valued) generalized function of interest will be the solution in Colombeau sense of the free heat equation (2.2), which is driven by an integrated white noise W (resp. $\hat{W}$). It will be denoted by Y (resp. $\hat{Y}$). In general we cannot expect X and Y (resp. $\hat{X}$ and $\hat{Y}$) to coincide in the (strong) generalized functions sense; in principle they are only equal in the sense of the association: this phenomenon has been observed in the case of the two-space dimensional white noise driven free wave equation: see [AHR2] Proposition 3.3.

Y is a $\mathcal{C}(\mathbb{R}_+; L^2)$-valued generalized function represented by

$$R_Y(t;x,\varepsilon) = \int_{[0,t]\times\mathbb{R}^2} dW(s,z) \int_{\mathbb{R}^2} dy p_{t-s}(x,y)\phi^\varepsilon(y-z) \ .$$

$\hat{Y}$ is a $\mathcal{C}(\mathbb{R}_+)$-valued generalized function such that

$$R_{\hat{Y}}(t;x,\varepsilon) = R_Y(t;x,\varepsilon) \text{ a.s.}$$

We can also consider Y_c (resp. $\hat{Y}_c$) as for X (resp. $\hat{X}$). We define $K, K_c \in \mathcal{G}(\mathbb{R}^2; \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}))$, $K = K(a,t;x), K_c = K_c(a,t;x)$ as

$$R_K(a,t;x,\varepsilon) = \exp(a^2 E^2(R_Y(t;x,\varepsilon)))$$

$$R_{K_c}(a,t;x,\varepsilon) = \exp(a^2 E^2(R_{Y_c}(t;x,\varepsilon)))$$

This definition is possible because of the following lemma:

Lemma 2.5 *R_K, R_{K_c} belong to $\mathcal{E}_M(\mathbb{R}^2; \mathcal{C}(\mathbb{R} \times \mathbb{R}_+))$. Moreover for any compact real interval I, $T > 0$, there is c_0 such that for $c \geq c_0$ we have*

$$\sup_{a \in I, t \leq T, x \in \mathbb{R}^2} |R_{K_c}(a,t;x,\varepsilon)| = O\left(\log \frac{1}{\varepsilon}\right) .$$

Proof We observe that

$$R_Y(t;x,\varepsilon) = \int_{[0,t]\times\mathbb{R}^2} dW(s,z) \int_{\mathbb{R}^2} dy p_{t-s}(x, z+\varepsilon y)\phi(y) .$$

Therefore

$$E(R_Y^2(t;x,\varepsilon)) =$$

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} dy_1 dy_2 \phi \otimes \phi(y_1,y_2) \int_{[0,t]\times\mathbb{R}^2} dsdz p_{t-s}(x, z+\varepsilon y_1) p_{t-s}(x, z+\varepsilon y_2) dz . \quad (2.29)$$

Using the upper bound of Proposition 2.3, the same proof as for Lemma 2.4 applies. ■

Remark 2.11 *If the kernel $(p_t(x,y))$ is of convolution type then*

$$E(R_Y^2(t;x,\varepsilon)) = E(R_X^2(t;x,\varepsilon))$$

In this case $G \equiv K$ and $G_c \equiv K_c$ if they are defined.

Proposition 2.8 *If the kernel $p_t(x,y)$ is of convolution type then $X \equiv Y$ (resp. $X_c \equiv Y_c$) in the sense of $\mathcal{C}(\mathbb{R}_+; L^2)$-valued generalized functions.*

Proof Similar to Proposition 3.3 in [AHR2]. ■

3. The non-linear heat equation

From now on μ will be called **complex measure** if it is a measure on the Borel real $\sigma-$ algebra with values in $\mathbb{C}$ such that $\mu\{0\} = 0$ and it has compact support.

Let μ be a complex measure, $\chi \in C_0^\infty(\mathbb{R}^2)$, $f^o : \mathbb{C} \to \mathbb{C}$ smooth such that all the derivatives are bounded for each compact of $\mathbb{R}_+ \times \mathbb{R}^2, \lambda \in \mathbb{C}$.

This section will be devoted to the study of equation (7) in the framework of random generalized functions. We can see that (3.1) is a generalization of such an equation. For simplicity all along this section, we will set $A = \lambda = 1, U_0 = 0$.

For a given $c > 0$ we are interested in the following equation in the Colombeau sense

$$U(t,x) =$$

(3.1)
$$\int_0^t dsLU(s,x) + \chi(x)\left\{\int d\mu(a)f^o(aU(s,x))K_c(a,s;x)\right\} + \hat{W}_c(t,x) \ ,$$

where $\hat{W}_c$ is the indistinguishible random generalized function of the integrated white noise of section 1, defined by

(3.2)
$$R_{\hat{W}_c}(t;x,\varepsilon) = R_{\hat{W}}(t;x,\psi_c(\varepsilon))$$

and

$$\psi_c(\varepsilon) = \left(\frac{-1}{\log\varepsilon}\right)^{\frac{1}{c}} \ .$$

By a solution to (3.1) we mean a random generalized function $U \in \mathcal{G}_\Omega(\mathbb{R}^2;\mathcal{C}(\mathbb{R}_+)$ which solves (3.1) and such that $U \in \mathcal{G}(\mathbb{R}^2;\mathcal{C}(\mathbb{R}_+;L^2))$.

Remark 3.1

From now on, Rey will stand for the real part of a complex number y. If $f^o(y) = \exp(iRey)$, $\int d\mu(a)f^o(aU(s,x))K_c(a,s;x)$ corresponds to : $\int d\mu(a)f^o(aU) : (s,y)$. Therefore, if we set $f(y) = \int d\mu(a)f^o(ay)$, equation (3.1) corresponds actually to

$$U(t,x) = \int_0^t dsLU(s,x) + \chi(x)\left\{: f(aU) : (t,x)\right\} + \hat{W}_c(t,x) \ .$$

In fact $(K_c(a,s;x))$ can be seen as a $\mathcal{C}(\mathbb{R}\times\mathbb{R}_+)$-valued generalized function represented by some $R_G = R_{K_c}$ fulfilling

(3.3)
$$\sup_{a\in\mathbb{R}_{\text{loc}},t\le T,x\in\text{supp }\chi} |R_G(a,t;x,\varepsilon)| = O\left(\log\frac{1}{\varepsilon}\right) \ .$$

for every compact subset $I\!R_{\rm loc}$ of $I\!R$.

From now on, to avoid overcharge of notations, we will replace f by f^o.

Remark 3.2 *We recall that $\hat{W}$ (resp. $\hat{W}_c$) is the pathwise version of the $L^2(\Omega)$-valued object W (resp. W_c).*

Remark 3.3 *When $(p_t(x,y))$ is of convolution type then we can replace K_c by G_c.*

Lemma 3.1 *Let $U \in \mathcal{G}_\Omega(I\!R^2;\mathcal{C}(I\!R_+)) \cap \mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. U solves (3.1) in L^2 if and only if U solves*

$$U(t,x) = \int_0^t ds \int_{I\!R^2} dy p_{t-s}(x,y) \int d\mu(a) f(aU(s,y)) K_c(a,s;y)\chi(y) + \hat{Y}_c(t,x) \ , \tag{3.4}$$

where $\hat{Y}$ is a random $\mathcal{C}(I\!R_+)$-valued generalized function represented by

$$R_{\hat{Y}}(t;x,\varepsilon) = \int_{[0,t]\times I\!R^2} dW(s,z) \int p_{t-s}(x,y)\phi^\varepsilon(y-z)dy \text{ a.s.} \tag{3.5}$$

and $\hat{Y}_c$ is defined in such a way that

$$R_{\hat{Y}_c}(t;x,\varepsilon) = R_{\hat{Y}}(t;x,\psi_c(\varepsilon)) \ .$$

Remark 3.4 *$\hat{Y}_c$ solves (3.1) when $f \equiv 0$.*

Remark 3.5 *The notation $U \in \mathcal{G}_\Omega(I\!R^2;\mathcal{C}(I\!R_+)) \cap \mathcal{G}(I\!R^2;\mathcal{C}(I\!R_+;L^2))$ means that U is a random generalized function represented by $R_U \in \mathcal{E}_{M,\Omega}(I\!R^2,\mathcal{C}(I\!R_+))$ which belongs to*

$$\mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+;L^2))$$

as an L^2-valued function.
We remark that such $R_U \in \mathcal{N}_\Omega(I\!R^2;\mathcal{C}(I\!R_+))$ may not belong to $\mathcal{N}(I\!R^2;\mathcal{C}(I\!R_+;L^2))$.

Proof (of Lemma 3.1) If U solves (3.4), then there is a representative $R_U \in \mathcal{E}_{M,\Omega}(I\!R^2;\mathcal{C}(I\!R_+)) \cap \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+;L^2))$ so that

$$\begin{aligned} R_U(t;x,\varepsilon) = & \\ & \int_0^t ds \int_{I\!R^2} dy p_{t-s}(x,y) \int d\mu(a) f(aR_U(s;y,\varepsilon)) R_{K_c}(a,s;y,\varepsilon)\chi(y) \\ & + R_{\hat{Y}_c}(t;x,\varepsilon) + R_0(t;x,\varepsilon) \ , \end{aligned} \tag{3.6}$$

where R_0 belongs to $\mathcal{N}_\Omega(I\!R^2;\mathcal{C}(I\!R_+)) \cap \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. We set

$$R_1(t;x,\varepsilon) = R_U(t;x,\varepsilon) - R_0(t;x,\varepsilon)\ .$$

Now R_1 solves

$$R_1(t;x,\varepsilon) = \int_0^t ds \int_{I\!R^2} dy p_{t-s}(x,y) \int d\mu(a) f(aR_U(s;y,\varepsilon) + aR_0(s;y,\varepsilon)) R_{K_c}(a,s;y,\varepsilon)\chi(y) + R_{\hat{Y}_c}(t;x,\varepsilon).$$

$R_0(\cdot,\varepsilon)$ belongs to $C^\infty(I\!R^2;\mathcal{C}(I\!R_+;L^2))$ and it has polynomial increase; in particular it belongs to $C(I\!R_+ \times I\!R; L^2) \cap \mathcal{S}'(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. The extension of Lemma 2.2 to the case of $F = L^2(\Omega)$- valued functions and distributions implies that

$$R_1(t;x,\varepsilon) = \int_0^t dsLR_1(s;y,\varepsilon) + \int_0^t ds \int d\mu(a) f(aR_1(s;y,\varepsilon) + aR_0(s;y,\varepsilon))\chi(x) + R_{\hat{W}_c}(t;x,\varepsilon) \text{ a. s.}$$

Since R_1 and $R_1 + R_0$ are both random representatives of U, then U solves (3.1).

Conversely, let us suppose that U is a solution of (3.1). Let R_U be a representative, i. e. R_U is a solution of

$$R_U(t;x,\varepsilon) = \int_0^t dsLR_U(s;x,\varepsilon) + \int d\mu(a) f(aR_U(s;x,\varepsilon)) R_{K_c}(a,s;x,\varepsilon)\chi(x) + R_{\hat{W}_c}(t;x,\varepsilon) + R_0(t;x,\varepsilon)$$

where $R_0 \in \mathcal{N}_\Omega(I\!R^2;\mathcal{C}(I\!R_+)) \cap \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. We set

$$R(t;x,\varepsilon) = \int_0^t ds \int_{I\!R^2} dy p_{t-s}(x,y) \int d\mu(a) f(aR_U(s;y,\varepsilon)) R_{K_c}(a,s;y,\varepsilon)\chi(y) + R_{\hat{Y}_c}(t;x,\varepsilon)\ .$$

According to Lemma 2.3, R belongs to $\mathcal{E}_{M,\Omega}(I\!R^2, \mathcal{C}(R_+)) \cap \mathcal{E}_M(I\!R^2;\mathcal{C}(I\!R_+;L^2))$. Of course by additivity and again by Lemma 2.2 we have

$$R_1(t;x,\varepsilon) = \int_0^t dsLR_1(s;x,\varepsilon) + R_0(t;x,\varepsilon)$$

for $R_1(t;x,\varepsilon) = R_U(t;x,\varepsilon) - R(t;x,\varepsilon)$. Therefore the class of R_1 solves (2.23); by Proposition 2.7 the class of R is null and so $R_U = R$ up to a null element. ∎

The next step will be the proof of the (pathwise) existence and uniqueness for equation (3.4). We recall that this will solve in L^2-sense the initial equation (3.1). For this we will need some deterministic estimates. Let us suppose f as before (even if for the present argument, we really just need f to be Lipschitz continuous). $i=1,2$, k_i, g, r_i are continuous functions on $I\!\!R \times I\!\!R_+ \times I\!\!R^2$, whose support in y is included in a compact subset K of $I\!\!R^2$ for any a and s; μ is still a complex measure. h_i are any continuous functions on $I\!\!R_+ \times I\!\!R^2$ with at most polynomial growth.

We consider the following equations

$$
\text{(3.7i)} \quad u_i(t,x) = \int_{[0,t]\times I\!\!R^2} ds\, dy p_{t-s}(x,y) \{f(au_i(s,y))r_i(a,s,y)g(a,s,y) + k_i(a,s,y)\} + h_i(t,x)
$$

A suitable contraction principle allows to show existence and uniqueness of solutions $u_i \in C(I\!\!R_+ \times I\!\!R^2)$ having at most polynomial growth in $x \in I\!\!R^2$.

Lemma 3.2 *Let u_i be solutions of (3.7i), $i=1,2$. Let K be a compact of $I\!\!R^2$, $T>0$. There are constants $a_1,\dots,a_5$ such that, for $x \in I\!\!R^2$*

1)

$$
\sup_{t\le T, x\in K} |u_i(t,x)| \le \exp\left(a_1 \int_0^t \sup_{y\in K, a\in \text{supp } \mu} |g(a,s,y)|\ |r_i(a,s,y)|\, ds\right)
\cdot a_2 \left\{ \sup_{y\in K, t\le T} |g(a,t,y)| \sup_{y\in K, t\le T, a\in\text{supp } \mu} |r_i(a,t,y)| + \sup_{a\in\text{supp } \mu, t\le T, y\in K} |k_i(a,t,y)| + \sup_{t\le T, x\in K} |h_i(t,x)| \right\}
$$

2)

$$
\sup_{t\le T, x\in K} |u_1-u_2|(t,x) \le \exp\left(a_3 \int_0^t \sup_{a\in\text{supp } \mu, y\in K} |g(a,s,y)|\ |r_1(a,s,y)|\, ds\right)
\cdot \left[a_4 \sup_{y\in K, t\le T, a\in\text{supp } \mu} \left\{ |r_1(a,t,y)| + |u_2(t,y)| + a_5 |r_1-r_2|(a,t,y) + |k_1-k_2|(a,t,y) \right\} + \sup_{t\le T, x\in K} |h_1-h_2|(t,x) \right].
$$

Proof

1) Since f is Lipschitz, there are constants $C_1, C_2 > 0$ so that for $t\le T, x\in K$

$$
|u_i(t,x)| \le \int_0^t ds \int_{I\!\!R^2} dy p_{t-s}(x,y) \left\{ C_1 \sup_{y\in K} |u_i(s,y)| + C_2 \right\}
$$

$$\left(\sup_{a\in\text{supp }\mu, s\le T, y\in K}\{|g(a,s,y)\; r_i(a,s,y)|\}+|k_i(a,s,y)|\right)+|h_i(t,x)|$$

Remark 3.6 *The Markov character of the semigroup* (P_t) *implies that*

$$\sup_{t\in[0,T]}\sup_{x\in I\!R^2}\int_{I\!R^2}p_t(x,y)dy=\sup_{t\in[0,T]}P_t1=1<\infty\;.$$

Gronwall lemma and Remark 3.6 give result 1).

2) For $t\le T$, $x\in K$, we have

$$\begin{aligned}|u_1-u_2|\,(t,x)\le&\int_0^t ds\int dy p_{t-s}(x,y)\\ &\left\{\int d\mu(a)g(a,s,y)|f(au_1(s,y))-f(au_2(s,y))|r_1(a,s,y)\right.\\ &\left.+f(au_2(s,y))|r_1-r_2|(a,s,y)+|k_1-k_2|\,(a,s,y)\right\}+|h_1-h_2|\,(t,x)\;.\end{aligned}$$

Again the Lipschitz property of f and Gronwall lemma give the result. ■

Proposition 3.1 *Let μ be a complex measure, $f:\mathbb{C}\to\mathbb{C}$ be smooth so that all the derivatives are bounded, $H\in\mathcal{G}(I\!R^2;\mathcal{C}(B))$, $\chi\in C^\infty(I\!R^2)$ support included in some compact $K\in I\!R^2$. Let G be a real $\mathcal{C}(B)$-valued generalized function fulfilling (3.3).*

Then the following integral equation has a unique solution in $\mathcal{G}(I\!R^2;\mathcal{C}(B))$:

$$\begin{aligned}&U(t,x)=H(t,x)\\ (3.8)\qquad&+\int_{[0,t]\times I\!R^2}ds\,dyp_{t-s}(x,y)\int d\mu(a)f(aU(s,y))G(a,s;y)\chi(y)\;.\end{aligned}$$

Proof At the level of representatives, (3.8) can be expressed as follows:

$$\begin{aligned}&R_U(t;x,\varepsilon)=R_H(t;x,\varepsilon)\\ (3.9)\qquad&+\int_{[0,t]\times I\!R^2}ds\,dyp_{t-s}(x,y)\int d\mu(a)f(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon)\chi(y)\;.\end{aligned}$$

We now prove the following.

Lemma 3.3 *Let D be an $l\ge 1$ order derivation operator and $R(t;x,\varepsilon)=DR_U(t;x,\varepsilon)$. Then*

$$
(3.10)\qquad
\begin{aligned}
R(t;x,\varepsilon) = &\int_{[0,t]\times I\!R^2} ds\, dy\, p_{t-s}(x,y) \int d\mu(a) \\
&\Big[\chi(y)R(s;y,\varepsilon)\tilde{f}(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon) + P(a,s;y,\varepsilon)\Big] \\
&+ DR_H(t;x,\varepsilon) \ ,
\end{aligned}
$$

where P is a polynomial in derivatives of R_U up to order $l-1$, as well as in derivatives of R_G and $\tilde{f}_j(aR_U(s;y,\varepsilon))$, where $\tilde{f}_j$ is bounded such that all derivatives are bounded, and $\tilde{\chi}_j \in C^\infty(I\!R^2)$ with compact support in K; moreover for every a,s, $P(a,s;\cdot,\varepsilon)$ has its support included in K.

Proof (of lemma 3.3). We start with $l=1$. We apply $D = \dfrac{\partial}{\partial x_i}$, $i=1,2$ to (3.9), setting $R = DR_U$. Since p_t is symmetric, similarly to the proof of Lemma 2.3 we have

$$
\begin{aligned}
R(t;x,\varepsilon) = &\int_{[0,t]\times I\!R^2} dsdyp_{t-s}(x,y) \int d\mu(a) \\
&D\Big\{ f(aR_U(s;y,\varepsilon))\chi(y)R_G(a,s;y,\varepsilon)\Big\} + DH(t;x,\varepsilon) \ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
R(t;x,\varepsilon) = &\int_{[0,t]\times I\!R^2} dsdyp_{t-s}(x,y) \int d\mu(a)\chi(y)\Big\{af'(aR_U(s;y,\varepsilon))R(s;y,\varepsilon)R_G(a,s;y,\varepsilon) \\
&+ f(aR_U(s;y,\varepsilon))DR_G(a,s;y,\varepsilon)\Big\} + D\chi(y)f(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon) \\
&+ DR_H(t;x,\varepsilon) \ .
\end{aligned}
$$

Therefore (3.10) holds true for $l=1$.

We suppose now (3.10) is valid for an integer $l \geq 1$. Let D^1 be a first order operator. Then

$$
\begin{aligned}
D^1R(t;x,\varepsilon) = &\int_{[0,t]\times I\!R^2} dsdyp_{t-s}(x,y) \int d\mu(a) \\
&\Big[\chi(y)D^1R(s;y,\varepsilon)\tilde{f}(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon) \\
&+ \chi(y)R(s;y,\varepsilon)\tilde{f}(aR_U(s;y,\varepsilon))D^1R_G(a,s;y,\varepsilon) \\
&+ \chi(y)R(s;y,\varepsilon)\tilde{f}'(aR_U(s;y,\varepsilon))aD^1R_U(s;y,\varepsilon)R_G(a,s;y,\varepsilon) \\
&+ D^1\chi(y)R(s;y,\varepsilon)\tilde{f}(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon) \\
&+ D^1P(a,s;y,\varepsilon)\Big] + D^1DR_H(t;x,\varepsilon) \ .
\end{aligned}
$$

Therefore (3.10) is now established for $l+1$. R is replaced by D^1R; $P(a,s;y,\varepsilon)$ is replaced by

$$R(s;y,\varepsilon)\Big\{\tilde{f}(aR_U(s;y,\varepsilon))D^1R_G(a,s;y,\varepsilon)\chi(y)$$
$$+\tilde{f}'(aR_U(s;y,\varepsilon))aD^1R_U(s;y,\varepsilon)R_G(a,s;y,\varepsilon)\chi(y)$$
$$+\tilde{f}(aR_U(s;y,\varepsilon))R_G(a,s;y,\varepsilon)D^1\chi(y)\Big\}+D^1P(a,s;y,\varepsilon)\ .$$

This is a polynomial in the derivatives of R_U up to order l, as well as in derivatives of R_G and $\tilde{f}_j(aR_U(s;y,\varepsilon))$, where $\tilde{f}_j$ are bounded such that all derivatives are bounded and smooth functions $\tilde{\chi}_j$ with support in K; moreover the support of previous expression with respect to the variable y is included in K. ■

We are able now to prove that the unique solution of (3.8) exists. For this we need to prove that the unique solution to (3.9) (whose existence is guaranteed by Remark 2.5) is moderate. In other words, we will prove that every derivative of any order $l \geq 0$ has a moderate tempered bound.

We will prove this by induction on $l \geq 0$ by starting with $l=0$. Let $n \in I\!N$ such that

$$\sup_{t\leq T, x\in I\!R^2} \frac{R_H(t;x,\varepsilon)}{1+|x|^n} = O\left(\varepsilon^{-n}\right)\ .$$

We can apply Lemma 3.2 1) with $u_i = R_U$, $g = R_G$, $f \equiv f$, $r_i = 1$, $k_i = 0$, $h = R_H$, where R_U is the solution of (3.9). Using assumption (3.3), we get the following estimate

$$\sup_{t\leq T, x\in I\!R^2} \frac{|R_U(t;x,\varepsilon)|}{1+|x|^n} \leq C_0 \left(\frac{1}{\varepsilon}\right)^{TC_1} \left(log\frac{1}{\varepsilon}+\frac{1}{\varepsilon^n}\right)$$

for suitable positive constants $C_0, C_1, n \in I\!N$ and $\varepsilon > 0$ is small enough.

We suppose that every derivative of R_U up to order $l-1 \geq 1$ has a temperate moderate bound. Let $R = DR_U$ where D is a order ℓ partial derivation. Now, Lemma 3.3 and assumption (3.3) for G say that $\int d\mu(a)P(a,s;y,\varepsilon)$ has a temperate moderate bound. By definition DR_H has a temperate moderate bound. We recall that there is a constant C_2 such that

$$\sup_{a\in \text{supp }(\mu), s\leq T, y\in K} |R_G(a,s;y,\varepsilon)| \leq C_2 \log\frac{1}{\varepsilon}\ . \qquad (3.11)$$

We apply again Lemma 3.2 1) with $g = R_G$, $f(y) = y$, $r_i(a,s,y) = \int d\mu(a)\tilde{f}(aR_U(s;y,\varepsilon))$, $k_i(a,s,y) = P_i(a,s;y,\varepsilon)$, $h(t,x) = DR_H(t;x,\varepsilon)$.

Let us fix $n \in I\!N$ such that

$$\sup_{t\leq T, x\in I\!R^2} \frac{DR_H(t;x,\varepsilon)}{1+|x|^n} = O\left(\varepsilon^{-n}\right)\ .$$

Using recurrence assumptions, $P_i(a,s;y,\varepsilon)$ can be suitably bounded. Therefore there are constants C_3, C_4, C_5 such that

$$\sup_{t\le T, x\in \mathbb{R}^2} \frac{|R(t;x,\varepsilon)|}{1+|x|^n}$$

$$\le \exp\left(C_3 \int_0^t \sup_{a\in \mathrm{supp}(\mu), s\le t, y\in K} |R_G(a,s;y,\varepsilon)|\, ds \right) \left(C_4 \log\frac{1}{\varepsilon} + C_5 \varepsilon^{-n} \right)$$

and because of (3.11)

$$\sup_{t\le T, x\in \mathbb{R}^2} \frac{|R(t;x,\varepsilon)|}{1+|x|^n} \le \left(\frac{1}{\varepsilon}\right)^{C_2C_3T} \left(C_4 \log\frac{1}{\varepsilon} + C_5 \varepsilon^{-n} \right) \le \text{const } \varepsilon^{-M}$$

for some suitable $M > 0$.

This proves the existence for (3.9) and therefore for (3.8). In order to show uniqueness, we start with two solutions R_U^1, R_U^2 of (3.9).

We have to show that $R_U^1 - R_U^2$ and all their derivatives have a tempered null bound. For this we apply Lemma 3.3 and Lemma 3.2 2) and we implement a similar procedure as for the existence part. To achieve this we have to take into account the fact that if $R_U^1 - R_U^2$ is null then $\tilde{f}_j(R_U^1) - \tilde{f}_j(R_U^2)$ is also null; on the other hand we recall that the family of generalized functions with temperate moderate bounds is an algebra and the family of generalized functions having a tempered null bound is an ideal. ∎

An interesting application covers the following mild type equation which includes the stochastic quantization equation with Sine-Gordon interaction.

Theorem 3.1 *Let μ be a complex measure with compact support, $f^o \in C_b^\infty(\mathbb{C})$, χ a smooth function with compact support. G_c has been defined in the lemma 2.4 and let $\hat{X}_c$ be the Gaussian generalized process defined in (2.26) coming from the Walsh [W] solution.*

Then, there is a unique solution in $\mathcal{G}_\Omega(\mathbb{R}^2; \mathcal{C}(\mathbb{R}_+))$ of

$$(3.12)\quad U(t,x) = \int_{[0,t]\times\mathbb{R}^2} ds\, dy\, p_{t-s}(x,y) \int d\mu(a) f^o(aU(s,y)) G_c(a,s;y)\chi(y) + \hat{X}_c(t,x).$$

Remark 3.7 *If $f^o(y) = \lambda \exp(iARey)$, $f(y) = \int \exp(iaRey) d\mu(a)$, equation (3.12) becomes*

$$(3.13)\ .\qquad U(t,x) = \lambda \int_{[0,t]\times\mathbb{R}^2} ds\, dy\, p_{t-s}(x,y) : f(AU) : (s,y)\chi(y) + \hat{X}_c(t,x)$$

Proof (of the theorem) We choose c_0 according to Lemma 2.4; this gives us a logarithmic type estimates (3.3) for G_c.

Let U_1, U_2 be two solutions of (3.12). In particular there are two measurable representatives R_{U_1}, R_{U_2} solving ω a. s. (3.8) with $H(t,x) = \hat{X}_c(t,x)$. Proposition 3.1 gives pathwise uniqueness and therefore $R_{U_1} - R_{U_2} \in \mathcal{N}_\Omega(I\!R^2; \mathcal{C}(I\!R_+))$.

In order to get existence we proceed again by Proposition 3.1. Setting $H(t,x) = \hat{Y}_c(t,x)$, the existence part of this proposition provides $R_U \in \mathcal{E}_M(I\!R^2; \mathcal{C}(I\!R_+))$, ω a. s. We observe that R_U is measurable because it solves (3.9); in fact the contraction principle says that the solution of (3.9) with $R_H = R_{\hat{Y}_c}$ is limit of Picard iterations which are measurable. ∎

Remark 3.8 *By using the same procedure as in Proposition 3.1 we can show that the unique solution of (3.12) also belongs to* $\mathcal{G}(I\!R^2; \mathcal{C}(I\!R_+; L^2))$. *This is so because* $Y_c \in \mathcal{G}(I\!R^2; \mathcal{C}(I\!R_+; L^2))$.

Remark 3.9 *Let us consider an indistinguishible random generalized function* $\tilde{G}_c$ *from* G_c *and* $\tilde{X}$ *from* X. *It is not difficult to prove that the two corresponding solutions are indistinguishible.*

Remark 3.10 *In fact, the object of this section is equation (3.1). This is equivalent to (3.12) where we have replaced* G_c *with* K_c *and* $\hat{X}_c$ *with* $\hat{Y}_c$. *We recall that* K *and* K_c *have been defined just before lemma 2.5. In such a case we can of course get the same existence and uniqueness result as for Theorem 3.1. A priori we cannot compare the two solutions obtained in these different ways. We know however that in case* $p_t(x,y)$ *is of convolution type the two solutions are strongly identical (in the generalized functions sense).*

In the next section we will proceed to a Taylor expansion of the solution U of (3.13) with respect to λ. Each term of the expansion will be shown to be associated in the L^2-sense with a classical object; the zero-term is the free solution ($\lambda = 0$); all the others will be connected with classical suare integrable processes.

4. Association with random distributions

Let $T > 0$, $g \in \bigcup_{q<2} L^q\left([0,T]^N \times I\!R^{2N}\right)$, A be a positive constant, $\chi \in C_0^\infty(I\!R^2)$, $\Delta = \operatorname{supp} \chi$. For given complex measures μ_j, we define $f_j(\mu_j) : \mathbb{C} \to \mathbb{C}$, $f_j = f_j(\mu_j)$ by

$$f_j(y) = \int_{I\!R} \exp(iaRey) d\mu_j(a) \ .$$

$\underline{s} = (s_1, \ldots, s_N)$ and $\underline{y} = (y_1, \ldots, y_N)$ will be generic elements of $[0,T]^N$ resp. $I\!R^N$.

In this last section X will be again the L^2-trace type element $X : I\!R_+ \times \mathcal{S}(I\!R^2) \to L^2$, as in (2.14) and its related $\mathcal{C}(I\!R_+, L^2)$-valued generalized function through formulas as (1.6); on the other hand, we keep in mind the random generalized function $\hat{X}$ (resp. $\hat{X}_c$) defined before Remark 2.8.

We set $\chi^N(\underline{y}) = \chi(y_1) \ldots \chi(y_N)$. Given the L^2-valued distribution X an important object to be defined is

$$(4.1) \qquad \int_{\mathbb{R}^{2N}} : f_1(AX) : (s_1, y_1) \ldots : f_N(AX) : (s_N, y_N)\chi(\underline{y})g(\underline{s}, \underline{y})d\underline{y}$$

for a.e. $(s_1, \ldots s_N) \in [0,T]^N$.
This quantity will be shown to belong to $L^2(\Omega \times [0,T]^N)$ and the map

$$g \to \int_{[0,T]^N} d\underline{s} \int_{\mathbb{R}^{2N}} : f_1(AX) : (s_1, y_1) \ldots : f_N(AX) : (s_N, y_N)\chi(\underline{y})g(\underline{s}, \underline{y})d\underline{y}$$

will be shown to be a continuous linear map from $\bigcap_{q<2} L^q\left([0,T]^N \times \mathbb{R}^{2N}\right)$ to $L^2(\Omega)$.

For R_X being a representative of X (see (1.6)), we set

$$(4.2) \qquad \begin{aligned} : exp(iaAR_X) : (s; y, \varepsilon) &= \exp\left(iaAR_X(s; y, \varepsilon) + \frac{a^2A^2}{2}E(R_X(s; y, \varepsilon))^2\right) \\ : f_j(AR_X) : (s; y, \varepsilon) &= \int d\mu_j(a) : \exp\left(iaAR_X\right) : (s; y, \varepsilon) \ . \end{aligned}$$

The object (4.2) does not define in principle a $\mathcal{C}(\mathbb{R}_+, L^2)$-valued generalized function. But, for $0 < c \leq c_0$ small enough, then $: f_j(AR_{X_c}) :$ defined as in (4.2) by replacing X with the indistinguishible X_c, is moderate (with values in $\mathcal{C}(\mathbb{R}_+, L^2)$) and so it introduces a $\mathcal{C}(\mathbb{R}_+, L^2)$-valued generalized function.

Furthermore, at least for small A, (4.1) will be a square integrable random variable, introduced as the L^2 limit of

$$(4.3) \qquad \int_{\mathbb{R}^{2N}} \chi^N(\underline{y}) : f_1(AR_X) : (s_1; y_1, \varepsilon) \ldots : f_N(AR_X) : (s_N; y_N, \varepsilon)g(\underline{s}, \underline{y})d\underline{y} \ .$$

Now (4.3) equals

$$\int_{\mathbb{R}^{2N}} d\underline{\mu}(a) \int_{\mathbb{R}^{2N}} \chi^N(\underline{y}) : \exp(iAa_1R_X) : (s_1; y_1, \varepsilon) \ldots : \exp(iAa_NR_X) : (s_N; y_N, \varepsilon) \ ,$$

where $\underline{\mu} = \mu_1 \otimes \ldots \otimes \mu_N$. We set

$$H^\varepsilon(\underline{s}, \underline{y}, \underline{b}) = \chi^N(\underline{y}) : \exp(ib_1R_X) : (s_1; y_1, \varepsilon) \ldots : \exp(ib_NR_X) : (s_N; y_N, \varepsilon) \ ,$$

$\underline{b} = (b_1, \ldots, b_N) \in \mathbb{R}^N$. Moreover, we set

$$F^\varepsilon(g)(\underline{s},\underline{b}) = \int H^\varepsilon(\underline{s};\underline{y},\underline{b})g(\underline{s},\underline{y})d\underline{y} \ .$$

Proposition 4.1 *Let $p \geq 2$. There is $b_0 > 0$ small enough such that if $0 < |b| \leq b_0$ we have the following:*

a) For any real function $g \in \bigcap_{q<2} L^q\left([0,T]^N \times \Delta^N\right)$, $F^\varepsilon(g)(\underline{s},\underline{b})$ is $L^2(\Omega)$-Cauchy, $\underline{s} \in [0,T]^N$, $\underline{b} \in I\!R^N$, $|b| \leq b_0$.

b) The following quantity

$$\int_{[0,T]^N} d\underline{s} \sup_{\varepsilon\leq 1, |b|\leq b_0} E\left\{\int_{\Delta^N\times\Delta^N} d\underline{y}d\underline{x}E\left\{\left|H^\varepsilon(\underline{s};\underline{y},\underline{b})\overline{H^\varepsilon(\underline{s};\underline{y},\underline{b})}\right|^p\right\}\right\}$$

is finite.

Proof We start with a).

Let $\underline{s} \in [0,T]^N$, $\underline{b} \in I\!R^N$. For $\varepsilon, \delta > 0$ we have

$$E\left|F^\varepsilon(g)(\underline{s},\underline{b}) - F^\delta(g)(\underline{s},\underline{b})\right|^2 = q^{\varepsilon,\varepsilon} - 2q^{\varepsilon,\delta} + q^{\delta,\delta} \ , \tag{4.4}$$

where

$$\begin{aligned} q^{\varepsilon,\delta} &= E\left\{\int_{\Delta^{2N}} H^\varepsilon(\underline{s};\underline{y},\underline{b})\overline{H^\delta(\underline{s};\underline{x},\underline{b})}g(\underline{s},\underline{y})g(\underline{s},\underline{x})d\underline{y}d\underline{x}\right\} \\ &= \int d\underline{y}d\underline{x}g(\underline{s},\underline{y})g(\underline{s},\underline{x})\chi^N(\underline{y})\chi^N(\underline{x}) \\ &E\left\{\exp\left(i\sum_{j=1}^N b_j\left[R_X(s_j;y_j,\varepsilon) - R_X(s_j;x_j,\delta)\right]\right)\right\} \\ &\exp\left(\sum_{j=1}^N \frac{b_j^2}{2}\left[E(R_X(s_j;y_j,\varepsilon))^2 + E(R_X(s_j;x_j,\delta))^2\right]\right) \end{aligned}$$

(4.5)

$$\begin{aligned} &= \int d\underline{y}d\underline{x}g(\underline{s},\underline{y})g(\underline{s},\underline{x})\chi^N(\underline{y})\chi^N(\underline{x}) \\ &\exp\left\{-2\sum_{j,l=1,j<l}^N b_jb_lE(R_X(s_j;y_j,\varepsilon)R_X(s_l;y_l,\varepsilon))\right. \\ &-2\sum_{j,l=1,j<l}^N b_jb_lE(R_X(s_j;x_j,\delta)R_X(s_l;x_l,\delta)) \\ &\left.+2\sum_{j,l=1}^N b_jb_lE(R_X(s_j;y_j,\varepsilon)R_X(s_l;x_l,\delta))\right\} \end{aligned}$$

Let $p > 2$ and $q < 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's inequality says that the previous expression is smaller than

$$\left(\int_{\Delta^{2N}} d\underline{y}\, g^q\left(\underline{s}, \underline{y}\right) \right)^{\frac{2}{q}} \tag{4.6}$$

times

$$\begin{aligned} \int_{\Delta^{2N}} \exp \Bigg\{ &-2p \sum_{j,l=1,j<l}^{N} b_j b_l \operatorname{Cov}\ _{\varepsilon,\varepsilon}(s_j, y_j; s_l, y_l) \\ &- 2p \sum_{j,l=1,j<l}^{N} b_j b_l \operatorname{Cov}\ _{\delta,\delta}(s_j, x_j; s_l, x_l) \\ &+ 2p \sum_{j,l=1}^{N} b_j b_l \operatorname{Cov}\ _{\varepsilon,\delta}(s_j, y_j; s_l, x_l) \Bigg\} d\underline{y} d\underline{x} \end{aligned} \tag{4.7}$$

(4.6) is finite. (4.7) can be rewritten as

$$\begin{aligned} \int_{\Delta^{2N}} d\underline{y} d\underline{x} \exp &\left\{ 2p \sum_{j,l=1,j<l}^{N} b_j b_l (\operatorname{Cov}\ _{\varepsilon,\delta}(s_j, y_j; s_l, x_l) - \operatorname{Cov}\ _{\varepsilon,\varepsilon}(s_j, y_j; s_l, y_l)) \right\} \\ \exp &\left\{ 2p \sum_{j,l=1,j<l}^{N} b_j b_l (\operatorname{Cov}\ _{\delta,\varepsilon}(s_j, x_j; s_l, y_l) - \operatorname{Cov}\ _{\delta,\delta}(s_j, x_j; s_l, x_l)) \right\} \\ \exp &\left\{ 2p \sum_{j=1}^{N} b_j^2 \operatorname{Cov}\ _{\varepsilon,\delta}(s_j, y_j; s_j, x_j) \right\} \end{aligned} \tag{4.8}$$

Taking in account the fact that the covariance expressions are non-negative and using Proposition 2.5, there are constants $C_1, C_2, C_3 > 0$ so that the previous expression is bounded by

$$\text{const} \int_{\Delta^{2N}} d\underline{y} d\underline{x} \prod_{j,l=1,j\neq l}^{N} \left| \frac{s_j + s_l}{s_j - s_l} \right|^{2|b_j b_l| p C_1}$$

times

$$\int dz_1 dz_2 \phi(z_1) \phi(z_2) \prod_{j=1}^{N} \frac{1}{|x_j - y_j + \varepsilon z_1 - \delta z_2|^{b_j^2 C_2}}$$

The first contribution comes from the product of the two first exponential terms in (4.8); the second term from the third exponential term. This gives a bound of the form

$$\text{const} \int_{\Delta^{2N}} d\underline{y} d\underline{x} \prod_{j,l=1, j\neq l}^{N} \left| \frac{s_j + s_l}{s_j - s_l} \right|^{|b_j b_l| p C_1}$$

(4.9) *times*

$$\int dz_1 dz_2 \phi(z_1)\phi(z_2) \prod_{j=1}^{N} \frac{1}{|x_j - y_j + \varepsilon z_1 - \delta z_2|^{b_j^2 C_3}}$$

We need here a technical lemma.

Lemma 4.1

$$\prod_{j,l=1, j\neq l}^{N} \left| \frac{s_j - s_l}{s_j + s_l} \right| = \left| \sum_{P=1}^{N!} \delta(P) \prod_{j=1}^{N} \frac{2 s_j}{s_j + s_{P(j)}} \right| \tag{4.10}$$

where $P : (1, \ldots, N) \to (P(1), \ldots P(N))$ *is a permutation and* $\delta(P)$ *is the signature.*

Proof We set $t_j = -s_j$ and we rewrite the first member of (4.10) as

$$\left| \frac{\prod_{j,l=1, j<l}^{N} (s_j - s_l)(t_j - t_l)}{\prod_{j,l=1}^{N} (s_j - t_l)} \right| \left| \prod_{j=1}^{N} (s_j - t_j) \right| . \tag{4.11}$$

By [DL], p. 2600, (2.2), the first absolute value of (4.11), equals

$$\left| \det_{j,l=1,\ldots,N} \left(\frac{1}{s_l - t_j} \right) \right| .$$

On the other hand

$$\prod_{j=1}^{N} (s_j - t_j) = \det\{\text{Diag } (s_j - t_j)\} .$$

By the property of determinants products, (4.10) equals

$$\left| \det \left\{ \left(\frac{1}{s_l - t_j} \right)_{l,j} \cdot \text{Diag } (s_j - t_j) \right\} \right| = \left| \det \left(\frac{s_j - t_j}{s_l - t_j} \right)_{l,j} \right|$$
$$= \left| \sum_{P=1}^{N!} \delta(P) \prod_{j=1}^{N} \left(\frac{s_j - t_j}{s_{P(j)} - t_j} \right) \right|$$

This establishes (4.10). ■

We are in fact interested in bounding

$$\prod_{j,l=1,j\neq l}^{N}\left|\frac{s_j+s_l}{s_j-s_l}\right|^{|b_jb_l|pC_1} \tag{4.12}$$

(4.12) is smaller than

$$\left|\prod_{j,l=1,j\neq l}^{N}\left(\frac{s_j+s_l}{s_j-s_l}\right)\right|^{b_0^2pC_1}=\left|\prod_{j\neq l}\left(\frac{s_j-s_l}{s_j+s_l}\right)\right|^{-b_0^2pC_1}$$

where $b_0=\sup_j|b_j|$. By lemma 4.1 this equals

$$\left|\sum_{P=1}^{N!}\delta(P)\prod_{j=1}^{N}\frac{2s_j}{s_j+s_{P(j)}}\right|^{-b_0^2pC_1}\ . \tag{4.13}$$

But $x\to x^{-b_0^2pC_1}$ is a convex function on a $\mathbb{R}_+$. So by Jensen's inequality applied to the uniform probability measure on $\{1,\ldots,N!\}$, (4.13) is equal to

$$\begin{aligned}
&\left|\sum_{P=1}^{N!}\frac{1}{N!}\delta(P)2^NN!\prod_{j=1}^{N}\left(\frac{s_j}{s_j+s_{P(j)}}\right)\right|^{-b_0^2pC_1}\\
&\leq\sum_{P=1}^{N!}\frac{1}{N!}\left|\delta(P)2^NN!\prod_{j=1}^{N}\left(\frac{s_j}{s_j+s_{P(j)}}\right)\right|^{-b_0^2pC_1}\\
&=\sum_{P=1}^{N!}\frac{1}{(N!)^{1+b_0^2pC_1}}2^{-Nb_0^2pC_1}\prod_{j=1}^{N}\left(\frac{s_j+s_{P(j)}}{s_j}\right)^{b_0^2pC_1}\\
&\leq\text{const }(b_0,N,T)\prod_{j=1}^{N}\frac{1}{s_j^{b_0^2pC_1}}\ .
\end{aligned}$$

This function is integrable with respect to $ds_1,\ldots,ds_N$ provided b_0 is small enough. Therefore (4.9) is bounded by

$$\frac{\text{const }(b_0,N,T)}{\prod_{j=1}^{N}s_j^{b_0^2pC_1}}\cdot\int dz_1dz_2\phi(z_1)\phi(z_2)\int_{\Delta^{2N}}d\underline{y}d\underline{x}\prod_{j=1}^{N}\frac{1}{|x_j-y_j+\varepsilon z_1-\delta z_2|^{b_j^2pC_3}}\ . \tag{4.14}$$

This integral is bounded by

$$\int_{\Delta_{loc}^{2N}} d\underline{y}d\underline{x} \prod_{j=1}^{N} \frac{1}{|y_j - x_j|^{b_j^2 p C_3}} \leq \text{const } (b_0, p, N, T)$$

where Δ_{loc} is a suitable compact subset of $I\!R^2$ containing Δ. Finally, we have been able to show that

$$(4.7) \leq \text{const } (b_0, N, T) \frac{1}{\left(\prod_{j=1}^{N} s_j\right)^{b_0^2 p C_1}} .$$

This shows that

i) The exponential term in (4.7) is bounded in L^p for some $p > 2$ with respect to the measure $d\underline{y}d\underline{x}g(\underline{s}, \underline{y})g(\underline{s}, \underline{x})$ uniformly with respect to $\varepsilon, \delta > 0$.
In particular this shows the uniform integrability of the exponential term in (4.7) with respect to $d\underline{y}d\underline{x}g(\underline{s}, \underline{y})g(\underline{s}, \underline{x})$.

ii) On the other hand

$$\text{Cov }_{\varepsilon,\delta}(t_1, x_1; t_2, x_2) \stackrel{\varepsilon,\delta \to 0}{\longrightarrow} \text{Cov } (t_1, x_1; t_2, x_2)$$

so that for a.e. $\underline{y}, \underline{x}$ and for fixed $\underline{s} \in [0, T]^N$, the exponential term in (4.7) converges to

$$\exp\left(-2 \sum_{j,l,j<l} b_j b_l \text{ Cov } (s_j, y_j; s_l, y_l) \right.$$
$$- 2 \sum_{j<l} b_j b_l \text{ Cov } (s_j, x_j; s_l, x_l)$$
$$\left. + 2 \sum_{j,l=1}^{N} b_j b_l \text{ Cov } (s_j, y_j; s_l, x_l) \right)$$

i) and ii) allow to say that

$$\lim_{\varepsilon,\delta \to 0} q^{\varepsilon,\delta}$$

exists and part a) of Proposition 4.1 is established. Property b) follows easily from the inequality

$$\sup_{|\varepsilon| \leq 1, |b| \leq b_0} E \left\{ \int_{\Delta^N \times \Delta^N} d\underline{y}d\underline{x} E \left| H^\varepsilon(\underline{s}, \underline{y}, \underline{b}) \overline{H^\varepsilon(\underline{s}, \underline{y}, \underline{b})} \right|^p \right\} \leq \frac{\text{const } (b_0, N, T)}{\prod_{j=1}^{N} s_j^{b_0^2 p C_1}} .$$

which we have already given after (4.7). ■

We denote for the moment by $F(g)(\underline{s},\underline{b})$ the limit obtained in a). It exists for $\underline{b}$ sufficiently small and a.e. $\underline{s} \in [0,T]^N$.

Remark 4.1 *For a sufficiently small real number A, the iterated integral in (4.1) can be defined as*

$$\int_{\mathbb{R}^N} d\underline{\mu}(\underline{a}) F(\underline{g})(\underline{s}, A\underline{a}) \ .$$

Taking into account Proposition 4.1 b), the quantity

$$\int_{[0,T]^N} d\underline{s} \int_{\mathbb{R}^N} d\underline{\mu}(\underline{a}) F(\underline{g})(\underline{s}, A\underline{a})$$

can be understood as a $d\underline{s}\, L^2(\Omega)$-valued Bochner integral.

Proposition 4.2 *For A small enough,*
a)

$$\int_{[0,T]^N} d\underline{s} \int d\mu(\underline{a}) F^\varepsilon(g)(\underline{s}, A\underline{a}) \xrightarrow{L^2} \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F(g)(\underline{s}, A\underline{a})$$

$\forall g \in \bigcap_{q<2} L^q\left([0,T]^N \times \Delta^N\right).$

b) The map $g \to \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F(g)(\underline{s}, A\underline{a})$ is continuous from the Fréchet space $\bigcap_{q<2} L^q\left([0,T]^N \times \Delta^N\right)$ to $L^2(\Omega)$.

Proof For fixed $\varepsilon > 0$, (4.6) and (4.7) tell that

$$g \to \int d\underline{s} \int d\underline{\mu}(\underline{a}) F^\varepsilon(g)(\underline{s}, A\underline{a})$$

is continuous. This, a) and the extension of the Banach-Steinaus theorem to the case of Fréchet spaces (see e.g. [DS], ch.2) imply the result b).

It remains to check a). We have to check that

$$\text{(4.15)} \qquad \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) E\left(F^\varepsilon(g)(\underline{s}, A\underline{a}) - F(g)(\underline{s}, A\underline{a})\right)^2 \xrightarrow{\varepsilon \to 0} 0 \ .$$

According to part a) of Proposition 4.1, the expectation in (4.15) converges to zero, for any $\underline{s}, \underline{a}$ a. e. Moreover it is bounded by

$$\text{(4.16)} \qquad 2E\left|F^\varepsilon(g)(\underline{s}, A\underline{a})\right|^2 + 2E\left|F(g)(\underline{s}, A\underline{a})\right|^2 \ .$$

In order to obtain (4.15) with the help of the dominated convergence theorem it is useful to check the existence of $A_0 > 0$ such that for $|A| < A_0$

i)

$$\sup_{|\varepsilon|\leq 1, |\underline{a}|\leq a_0} E\left|F^\varepsilon(g)(\cdot, A\underline{a})\right|^2$$

is integrable on $[0,T]^N$ for small $a_0 > 0$.

ii)

$$\int d\underline{s} d\underline{\mu}(\underline{a}) E\left|F(g)(\underline{s}, A\underline{a})\right|^2 < \infty.$$

We start with i). Let $p > 2$ and $q < 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For A small enough, and almost every $\underline{s}, \underline{a}$, Hölder's inequality gives

$$\begin{aligned} E\left|F^\varepsilon(g)(\underline{s}, A\underline{a})\right|^2 &= E\left\{\int_{\Delta^N} d\underline{y} \int_{\Delta^N} d\underline{x} g(\underline{s}, \underline{y}) g(\underline{s}, \underline{x}) H^\varepsilon(\underline{s}, \underline{y}, A\underline{a}) \overline{H^\varepsilon(\underline{s}, \underline{x}, A\underline{a})}\right\} \\ &\leq \left\{\int_{\Delta^{2N}} d\underline{y} d\underline{x} \left|g\right|^q(\underline{s}, \underline{y}) \left|g\right|^q(\underline{s}, \underline{x})\right\}^{\frac{1}{q}} \\ &\quad \left[E\left\{\int_{\Delta^{2N}} d\underline{y} d\underline{x} \left|H^\varepsilon(s, \underline{y}, A\underline{a}) \overline{H^\varepsilon(\underline{s}, \underline{x}, A\underline{a})}\right|^p\right\}\right]^{\frac{1}{p}} \end{aligned} \tag{4.17}$$

Therefore

$$\int_{[0,T]^{2N}} d\underline{s} \sup_{|\varepsilon|\leq 1, |\underline{a}|\leq a_0} E\left|F^\varepsilon(g)(\underline{s}, A\underline{a})\right|^2$$

is finite because of Proposition 4.1 b). This proves a).

Concerning ii), Proposition 4.1 a) implies that

$$E\left|F^\varepsilon(g)(\underline{s}, A\underline{a})\right|^2 \to E\left|F(g)(\underline{s}, A\underline{a})\right|^2 . \tag{4.18}$$

On the other hand, for $p > 2$, A_0 small enough, and $|A| \leq A_0$

(4.19)

$$\int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) \left(E\left|F^\varepsilon(g)(\underline{s}, A\underline{a})\right|^2\right)^{\frac{p}{2}} \leq \text{const} \int_{[0,T]^N} d\underline{s} \sup_{|\varepsilon|\leq 1, |\underline{b}|\leq b_0} E\left|F^\varepsilon(g)(\underline{s}, \underline{b})\right|^p$$

where $b_0 = A_0 \,\text{diam}\,(\text{supp}\,(\underline{\mu}))$. By a similar argument as in (4.17) and a Hölder's inequality, we can bound the latter expression by

$$\text{const} \int_{[0,T]^N} d\underline{s} \sup_{\varepsilon\leq 1, |\underline{b}|\leq b_0} E\left\{\int_{\Delta^{2N}} d\underline{y} d\underline{x} \left|H^\varepsilon(\underline{s}, \underline{y}, \underline{b}) \overline{H^\varepsilon(\underline{s}, \underline{y}, \underline{b})}\right|^{p'}\right\} \tag{4.20}$$

where $p > p' > 2$. Again Proposition 4.1 b) implies that (4.20) and therefore (4.19) is bounded for $\varepsilon \in]0,1[$.

This, a uniform integrability argument and (4.18) allow to conclude the proof of statement a) of Proposition 4.2. ■

We go back to the stochastic non-linear heat equation in its mild form. Let $c > 0$, $\lambda \in \mathbb{C}$. We denote by $U = U(\lambda, \cdot)$ the solution in the sense of Colombeau of

$$\begin{aligned} &U(t,x) = \\ (4.21) \qquad &\lambda \int_{[0,t]\times\Delta} dsdy\chi(y)p_{t-s}(x,y) \int d\mu(a) \exp\left(iaU(s,y)\right) G_c(a,t;x,\varepsilon) + \hat{X}_c(t,x) \end{aligned}$$

where $t \geq 0$, $x \in \mathbb{R}^2$ and μ is again a complex measure defined on the Borel sets of $\mathbb{R}$ and having compact support. We recall that $\hat{X}_c$ has been defined in (2.26) and G_c at lemma 2.4.

Remark 4.2 *If $p_t(x,y)$ is of convolution type we recall that U solves in the L^2-sense the following equation*

$$U(t,x) = \int_0^t ds \left\{ LU(s,x) + \lambda \int d\mu(a) \exp\left(iaU(s,x)\right) G_c(a,s;x,\varepsilon) \right\} + \hat{W}_c(t,x)$$

U can be represented by the solution R_U of the following equation

$$\begin{aligned} &R_U(t;x,\varepsilon) = \int_{[0,t]\times D} dsdy\chi(y)p_{t-s}(x,y) \\ (4.22) \qquad & \int d\mu(a) \exp\left(iaReR_U(s;y,\varepsilon) + \frac{a^2}{2} E(R_{X_c}(s;y,\varepsilon))^2 \right) + R_{\hat{X}_c}(t;x,\varepsilon). \end{aligned}$$

Using classical arguments and the fact that $\int d\mu(a) \exp(iaRe.)$ is real analytical, it is possible to prove that $R_U = R_U(\lambda, t; x, \varepsilon)$ is real analytical in λ. Therefore it admits the following convergent expansion

$$R_U(\lambda, t; x, \varepsilon) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} R_n(t;x,\varepsilon)$$

where

$$R_n(t;x,\varepsilon) = \frac{\partial^n ReR_U}{\partial \lambda^n}(\lambda, t; x, \varepsilon)\big|_{\lambda=0}.$$

According to [C1], definition 3.3.3.

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} \text{ class } R_n(t;x,\varepsilon)$$

defines an approximate expansion.

The most important result of the paper is the following.

Theorem 4.1 *The asymptotic expansion of U is $L^2(\Omega)$- associated with elements of $L^2(\Omega)$-trace type. This means the following: let $\lambda_0 \in \mathbb{C}$, $m \in \mathbb{N}$,*

$$\sum_{n=0}^{m} \frac{\lambda_0^n}{n!} \text{ class } R_n$$

is L^2-associated with an $L^2(\Omega)$-trace type element.

Remark 4.3 *We recall that an $L^2(\Omega)$-trace type element is a continuous linear functional $\mathcal{S}(\mathbb{R}^2) \to C(\mathbb{R}_+; L^2(\Omega))$. Following the proof we can even show that this element is a linear continuous map from $C_b(\mathbb{R}^2) \to C(\mathbb{R}_+; L^2(\Omega))$, that is to say it is a vector valued measure.*

Remark 4.4 *If $n = 0$, $R_0 = R_{\hat{X}_c}$ and it is one representative of the free solution. This is obviously associated with an $L^2(\Omega)$- trace type element, given by*

$$\alpha \to \left(t \to \int_{[0,t]\times \mathbb{R}^2} dW(s,y) \int_{\mathbb{R}^2} p_{t-s}(x,y)\alpha(x)dx \right) .$$

Proof We will partially follow the scheme of [AHR2]. It is enough to show that for any $n \in \mathbb{N}$, the class R_n is L^2-associated with an L^2-trace type element. Given a complex measure μ with compact support and a real valued A we denote by $f : \mathbb{C} \to \mathbb{C}$, the ($\mu$ - dependent) function

$$f(y) = \int d\mu(a) \exp(iaRey) .$$

Given a representative of a random generalized function R_U, for $c > 0$, $A \in \mathbb{R}$, we consider the Wick reordering $: f(AR_U) :$ with respect to X_c defined by

$$(4.23) \quad : f(AR_U) : (t;x,\varepsilon) = \int d\mu(a) \exp\left\{ iaAReR_U(t;x,\varepsilon) + \frac{a^2}{2} A^2 E(R_{X_c}(t;x,\varepsilon))^2 \right\} .$$

We need a technical lemma.

Lemma 4.2 *Let $\varepsilon > 0$ and $n \geq 1$, then there are polynomials P_n^n and P_{n-1}^n in the variables*

$$ARe\frac{\partial R_U}{\partial \lambda} , Re\frac{\partial^2 R_U}{\partial \lambda^2} , \ldots , Re\frac{\partial^n R_U}{\partial \lambda^n}$$

and in the variables

$$ARe\frac{\partial R_U}{\partial \lambda} , Re\frac{\partial^2 R_U}{\partial \lambda^2} , \ldots , Re\frac{\partial^{n-1} R_U}{\partial \lambda^{n-1}}$$

such that each monomial multiplies a term $: f_n(AR_U) :$ *(with dependence on* μ*) and*

$$\frac{\partial^n}{\partial \lambda^n} R_U(t;x,\varepsilon) = \int_{[0,t]\times\Delta} dsdyp_{t-s}(x,y)\left(\lambda P_n^n + P_{n-1}^n\right)(s,y)\chi(y) \tag{4.24}$$

Proof (of the lemma) For simplicity of the notation, we will write $Re_U := ReR_U$. We first observe that

$$\frac{\partial}{\partial\lambda} : f_n(AR_U) :=: f_{n+1}(AR_U) : A\frac{\partial}{\partial\lambda}Re_U \ . \tag{4.25}$$

We operate now by induction. For $n = 1$, the derivation of (4.21) gives

$$\begin{aligned} &\frac{\partial}{\partial\lambda}R_U(t;x,\varepsilon) = \\ &\int_{[0,t]\times\Delta} ds\, dyp_{t-s}(x,y) : f(AR_U) : (s;y,\varepsilon)\chi(y)\left(\lambda A\frac{\partial}{\partial\lambda}Re_U(s;y,\varepsilon)+1\right) \ , \end{aligned} \tag{4.26}$$

so that (4.24) is confirmed.

Let us suppose by induction that (4.24) holds for some integer $n-1 \geq 1$. Then

$$\frac{\partial^n}{\partial\lambda^n}R_U = \frac{\partial}{\partial\lambda}\left\{\int_{[0,t]\times\Delta} ds\, dyp_{t-s}(x,y)\left(\lambda P_{n-1}^{n-1} + P_{n-2}^{n-1}\right)(s,y)\chi(y)\right\} \tag{4.27}$$

Using (4.25), we remark that the derivative of each monomial with respect to λ gives a term

$$: f_n(AR_U) : A\frac{\partial}{\partial\lambda}Re_U$$

times the monomial plus a term $: f_{n-1}(AR_U) :$ times the derivative of the monomial; in this latter derivative, the maximal order of derivation with respect to λ increases by one unit.

Therefore

$$\frac{\partial}{\partial\lambda}P_{n-1}^{n-1} = P_n^n \ , \frac{\partial}{\partial\lambda}P_{n-2}^{n-1} = P_{n-1}^n$$

Consequently, (4.26) gives

$$\begin{aligned} \frac{\partial^n}{\partial\lambda^n}R_U &= \int_{[0,T]\times\Delta} ds\, dyp_{t-s}(x,y)\left(\lambda\frac{\partial P_{n-1}^{n-1}}{\partial\lambda} + P_{n-1}^{n-1} + \frac{\partial P_{n-2}^{n-1}}{\partial\lambda}\right)(s,y)\chi(y)dy \\ &= \int_{[0,T]\times\Delta} ds\, dyp_{t-s}(x,y)\left(\lambda P_n^n + P_{n-1}^n\right)(s,y) \ . \end{aligned}$$

■

We consider the following family $\mathcal{F}^\varepsilon$ of functions $I\!R_+ \times \Delta \to L^2$ which is constituted by linear combinations of $H : I\!R_+ \times \Delta \to L^2$ such there are f_i (depending on μ_i), $i = 1, \cdots, N$, with

$$\begin{aligned} H(t;x,\varepsilon) = & \int_{[0,t]\times\Delta} ds_N\, dy_N p_{t-s_N}(x,y_N) : f_N(AR_{X_c}) : (s_N; y_N, \varepsilon) \\ & \int_{[0,s_{j(N-1)}]\times\Delta} ds_{N-1}\, dy_{N-1} p_{s_{j(N-1)}-s_{N-1}}(y_{j(N-1)}, y_{N-1}) \\ & : f_{N-1}(AR_{X_c}) : (s_{N-1}; y_{N-1}, \varepsilon) \\ & \cdots \int_{[0,s_{j(1)}]\times\Delta} ds_1\, dy_1 p_{s_{j(1)}-s_1}(y_{j(1)}, y_1) : f_1(AR_{X_c}) : (s_1; y_1, \varepsilon) \chi^N(\underline{y}) \end{aligned} \tag{4.28}$$

where $t = s_{j(N)}$, $y = y_{j(N)}$ and $j(l) \in \{l+1, \ldots, N\}$.

Remark 4.5 *$\mathcal{F}^\varepsilon$ is a vector algebra.*

Proposition 4.3 *Every R_n belongs to $\mathcal{F}^\varepsilon$, $n \geq 1$.*

Proof (of the proposition) The case $n = 1$ is a direct consequence of (4.26) and Remark 4.4. Let us suppose that Proposition 4.1 holds for $1 \leq k \leq n-1$, $n-1 \geq 1$. Then $R_k \in \mathcal{F}^\varepsilon$, for any $1 \leq k \leq n-1$. Lemma 4.2 says that there is a polynomial P^n_{n-1} in the variables

$$A \frac{\partial Re_U}{\partial \lambda}\bigg|_{\lambda=0}, \ldots, \frac{\partial^{n-1} Re_U}{\partial \lambda^{n-1}}\bigg|_{\lambda=0},$$

such that each monomial multiplies a term of the type $: f_n(AR_U) :$ and

$$R_n(t;x,\varepsilon) = \frac{\partial^n Re_U}{\partial \lambda^n}(t;x,\varepsilon)\bigg|_{\lambda=0} = \int_{[0,T]\times\Delta} ds\, dy \chi(y) p_{t-s}(x,y) P^n_{n-1}(s,y) \; .$$

By the induction hypothesis and by Remark 4.5 all monomials belong to $\mathcal{F}^\varepsilon$. Therefore, R_n is a linear combination of terms of the following type

$$\int_{[0,t]\times\Delta} ds\, dy \chi(y) p_{t-s}(x,y) : f(AR_X) : (s;y,\varepsilon) \Phi^\varepsilon(s;y,\varepsilon) \; ,$$

where $\Phi^\varepsilon \in \mathcal{F}^\varepsilon$. By definition, the latter integral still belongs to $\mathcal{F}^\varepsilon$. ■

The proof of the theorem will be completed after proving the following

Proposition 4.4 *Let $\Phi^\varepsilon \in \mathcal{F}^\varepsilon$. Then for small A the $L^2 - \lim_{\varepsilon\to 0} \Phi^\varepsilon$ exists and it is an element Φ of $L^2(\Omega)$- trace type. Moreover, Φ is a linear combination of elements of the form*

$$(4.29)\quad H(t,x) = \int_{[0,T]\times\Delta} d\underline{s}d\underline{y} g_{t,x}(\underline{s},\underline{y})\chi^N(\underline{y}) : f_N(AX) : (s_N, y_N) \ldots : f_1(AX) : (s_1, y_1)$$

where

$$\begin{aligned} g_{t,x}(\underline{s},\underline{y}) &= 1_{[0,t]\times\Delta}(s_N, y_N) p_{t-s_N}(x, y_N) 1_{[0,s_{j(N-1)}]\times\Delta}(s_{N-1}, y_{N-1}) \\ &\quad p_{s_{j(N-1)}-s_{N-1}}(y_{j(N-1)}, y_{N-1}) \ldots 1_{[0,s_{j(1)}]\times\Delta}(s_1, y_1)\; p_{s_{j(1)}-s_1}(y_{j(1)}, y_1) \end{aligned} \quad (4.30)$$

with $t = s_{j(N)}$, $x = y_{j(N-1)}$ *and* $j(l) \in \{l, l+1, \ldots, N\}$.

Remark 4.6 *It will be enough to show that an object of the form (4.28) type converges to an element of the form (4.29).*

Since X and X_c are indistinguishible, we can replace R_{X_c} by R_X in (4.28).

Before proceeding to the proof of Proposition 4.4, we need to state the following lemma.

Lemma 4.3 *Let* $q < 2$*, then*

$$\sup_{t\le T, x\in D} \|g_{t,x}\|_{L^q([0,T]^N\times\Delta^N)} < \infty\ .$$

Proof For $t \le T$, $x \in \Delta$ we set $s_{j(N)} = t$, $y_{j(N)} = x$. We have

$$\int_{\Delta^N} d\underline{y} \int_{[0,T]^N} d\underline{s} \left|g_{t,x}(\underline{s},\underline{y})\right|^q = \prod_{i=1}^N \int_\Delta dy_i \int_0^{s_{j(i)}} ds_i \left\{p_{s_{j(i)}-s_i}(y_{s_{j(i)}}, y_i)\right\}^q\ .$$

According to Proposition 2.3 this is dominated by

$$C \prod_{i=1}^N \int_\Delta dy_i \int_0^{s_{j(i)}} ds_i \frac{1}{\left|s_{j(i)} - s_i\right|^q} \exp\left(-\frac{\left|y_{s_{j(i)}} - y_i\right|^2 q}{\delta(s_{j(i)} - s_i)}\right)$$

where C, δ are positive constants. Let $R > 0$ such that Δ is included in a zero centered ball with radius R. The latter expression is dominated by

$$
\begin{aligned}
&C\prod_{i=1}^{N}\int_{B(0,R)} dy_i \int_0^{s_{j(i)}} \frac{ds_i}{(s_{j(i)}-s_i)^q} \exp\left(-\frac{\left|y_{s_{j(i)}}-y_i\right|^2 q}{\delta(s_{j(i)}-s_i)}\right) \\
&\le C\prod_{i=1}^{N}\int_0^{s_{j(i)}} \frac{ds_i}{(s_{j(i)}-s_i)^q}\int_0^{2R} d\rho\rho\exp\left(-\frac{\rho^2 q}{\delta(s_{j(i)}-s_i)}\right) \\
&= C\prod_{i=1}^{N}\int_0^{s_{j(i)}} \frac{ds_i}{(s_{j(i)}-s_i)^q}\frac{\delta(s_{j(i)}-s_i)}{2q}\underbrace{\left(1-\exp\left(-\frac{4R^2 q}{\delta(s_{j(i)}-s_i)}\right)\right)}_{\le 1} \\
&\le C\left(\frac{\delta}{2q}\right)^N \prod_{i=1}^{N}\int_0^{s_{j(i)}} ds_i(s_{j(i)}-s_i)^{1-q} \le C\left(\frac{\delta T^{2-q}}{2q(2-q)}\right)^N ,
\end{aligned}
$$

which establishes the lemma. ■

Proof (of Proposition 4.4) For $\varepsilon > 0$ an element H of the form (4.28), defines a L^2-trace type element $\mathcal{S}(I\!R^2) \to C(I\!R_+; L^2(\Omega))$ by

$$
\alpha \to \left(t \to \int \alpha(x) H(t; x, \varepsilon) dx\right) .
$$

For $t \le T$, $\alpha \in \mathcal{S}(I\!R^2)$, according to the notation of the beginning of the section we have

$$
\begin{aligned}
\int_{I\!R^2} dx\alpha(x) H(t; x, \varepsilon) &= \int_{I\!R^2} dx\alpha(x) \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F^\varepsilon(g_{t,x})(\underline{s}, A\underline{a}) \\
&= \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F^\varepsilon\left(\int \alpha(x) g_{t,x} dx\right)(\underline{s}, A\underline{a})
\end{aligned}
$$

where $g_{t,x}$ has been defined in (4.30). Using Proposition 4.2 a), the latter expression converges in L^2 to

$$
Z(t,\alpha) \equiv \int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F\left(\int \alpha(x) g_{t,x} dx\right)(\underline{s}, A\underline{a}) .
$$

It remains to check the fact that Z is of the L^2-trace type.

Let $\alpha \in C(I\!R^2)$, $t \le T$, $t_0 \le T$, $q < 2$. Using Proposition 4.2 b) we get

$$
\begin{aligned}
&\|Z(t,\alpha) - Z(t_0,\alpha)\|_{L^2(\Omega)} = \\
&\qquad \|\int_{[0,T]^N} d\underline{s} \int d\underline{\mu}(\underline{a}) F\left(\int \alpha(x)(g_{t,x} - g_{t_0,x}) dx\right)(\underline{s}, A\underline{a})\|_{L^2} \\
&\le \text{const }(q) \int dx\alpha(x) \|g_{t,x} - g_{t_0,x}\|_{L^q([0,T]^N \times \Delta^N)} .
\end{aligned}
\tag{4.31}
$$

Now $\lim\limits_{t\to t_0} g_{t,x}(\underline{s},\underline{y})$ converges $(x,\underline{s},\underline{y})$ a.e. Lemma 4.3 and uniform integrability arguments imply that the right hand side of (4.31), is bounded by

$$\text{const } (q)\|\alpha\|_\infty \int dx \|g_{t,x} - g_{t_0,x}\|_{L^q([0,T]^N\times\Delta^N)} \overset{t\to t_0}{\longrightarrow} 0 \ . \tag{4.32}$$

This proves Proposition 4.4 and finally Theorem 4.1. ■

Remark 4.7 *From the proof of Theorem 4.1, we observe that the asymptotic expansion of U in powers of λ is associated with an L^2-trace type element which is the sum of the free solution and a classical process $\mathbb{R}_+ \times \Delta \to L^2$ which is a linear combination of iterated integrals of type (4.29) which involves non-linearities of the solution X of the free equation.*

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