MARTINGALE APPROXIMATION FOR SELF-INTERSECTION LOCAL TIME OF BROWNIAN MOTION

MARGARIDA DE FARIA, ANIS REZGUI, AND LUDWIG STREIT

Abstract. Given that each term of the multiple Wiener integral expansion for the renormalized self-intersection local time of higher dimensional Brownian motion converges in law to another, independent Brownian motion we resum the leading, martingale parts of these terms in closed form and also represent this sum as a stochastic integral.

1. Introduction

An informal definition of the self-intersection local time of d-dimensional Brownian motion is given in terms of an integral over Donsker's $\delta-$function.

$$L_T = \int_{\Delta_T} dt_2 dt_1 \delta(B_{t_2} - B_{t_1}),$$

where B_t is a d-dimensional Brownian motion and $\Delta_T = \{(t_1, t_2) : 0 < t_1 < t_2 < T\}$.
To make sense of this integral one can invoke a regularization such as

$$L^{\epsilon}_T = \int_{\Delta_T} dt_2 dt_1 \delta_{\epsilon}(B_{t_2} - B_{t_1}) \tag{1.1}$$

where $\delta_\epsilon(x) = \frac{1}{(2\pi\epsilon)^{d/2}} e^{-\frac{x^2}{2\epsilon}}$ for $x \in I\!R^d$, and define L_T as the limit when ϵ goes to zero.
If the dimension is $d \geq 2$, $\lim_{\epsilon\to 0+^+} I\!E(L^\epsilon_T) = +\infty$; Varadhan, in [20] renormalizes L^ϵ_T by substracting its expectation and proves for $d = 2$ that the limit exists in mean square.
Hence we consider the centered self-intersection local time defined by

$$L^{\epsilon}_{T,c} = \int_{\Delta_T} dt_2 dt_1 \delta_{\epsilon,c}(B_{t_2} - B_{t_1})$$

where $\delta_{\epsilon,c}(B_{t_2} - B_{t_1}) = \delta_\epsilon(B_{t_2} - B_{t_1}) - I\!E\Big(\delta_\epsilon(B_{t_2} - B_{t_1})\Big)$, this is the so called Varadhan renormalization.
For $d \geq 3$, a further multiplicative renormalization $r_d(\epsilon)$ is required for the existence of a limiting process. M. Yor in [19] shows, using the near passage regularization

$$\delta(B_{t_2} - B_{t_1}) \to \delta(B_{t_2} - B_{t_1} + \epsilon)$$

for $d = 3$, that

$$r_3(\epsilon)(L^\epsilon - I\!E(L^\epsilon)) \stackrel{\mathcal{L}}{\to} c\beta$$

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with β a Brownian motion independent of B.
This can be understood in the light of the fact that each term in the (renormalized) chaos expansion or multiple Wiener integral expansion for the self-intersection local time converges in law to a Brownian motion, for any $d > 2$ as shown in [4], using the fact that the dominant part of each of these multiple Wiener integrals is in fact a martingale. With a view towards convergence results for the renormalized local time in dimensions $d > 3$, it is desirable to resum these dominant terms, i.e. to split the local time into a dominant martingale part and a subdominat remainder. We do this in Theorems 1 and 2, in Theoreom 3 we express the two terms as stochastic integrals via the Clark-Ocone formula.
Recent investigations of self-intersection local time have used white noise analysis [1][4][5][21]. So, before announcing the main results of this paper, let us briefly recall some tools from white noise analysis and some of these results on self-intersection local time.

1.1. **Tools from White Noise Analysis.** We quote some white noise analysis concepts as introduced in [4], referring to [9] for a systematic presentation.

Consider a white noise space $(S'(\mathbb{R})^d, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the weak Borel σ-algebra of $S'(\mathbb{R})^d$, and μ is the centered Gaussian measure whose covariance is given by the inner product of $L^2(\mathbb{R})^d$, in the sense that the vector valued white noise has the characteristic function

$$C(\mathbf{f}) = \mathbb{E}(e^{i\langle\omega,\mathbf{f}\rangle}) = \int_{S'(\mathbb{R})^d} d\mu[\omega] e^{i\langle\omega,\mathbf{f}\rangle} = e^{-\frac{1}{2}\langle\mathbf{f},\mathbf{f}\rangle},$$

where $\langle\omega,\mathbf{f}\rangle = \sum_{i=1}^d \langle\omega_i, f_i\rangle$ and $f_i \in S(\mathbb{R},\mathbb{R})$.

Then a realization of a vector of independent Brownian motions B_i, $i = 1, \cdots, d$, is given by

$$B_i(t) = \langle\omega_i, \mathbf{1}_{[0,t]}\rangle = \int_0^t \omega_i(s)ds.$$

Hence we consider independent d-tuples of Gaussian white noise $\omega = (w_1, \cdots, \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, \cdots, f_d) \in S(\mathbb{R}, \mathbb{R}^d)$, and use the following multi-index notation:

$$\vec{n}! = \prod_1^d n_i!$$

$$\langle\mathbf{f},\mathbf{f}\rangle = \sum_{i=1}^d \int dt f_i^2(t)$$

$$\langle F_{\vec{n}}, \mathbf{f}^{\otimes\vec{n}}\rangle = \int d^n t F_{\vec{n}}(t_1, \cdots, t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \cdots, t_n)$$

and similarly for $\langle :\omega^{\otimes\vec{n}}:, F_{\vec{n}}\rangle$ where for d-tuples of white noise the Wick product $:\cdots:$ (see [9]) generalizes to

$$:\omega^{\otimes\vec{n}} := \bigotimes_{i=1}^d :\omega_i^{\otimes n_i}: .$$

The Hilbert space

$$(L^2) = L^2(d\mu)$$

is canonically isomorphic to the d-fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq \Big(\bigoplus_{k=0}^{\infty} Sym L^2(\mathbb{R}^k, k! d^k t)\Big)^{\otimes d} = \mathcal{F}.$$

For a general element φ of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} \langle : \omega^{\otimes \vec{n}} :, F_{\vec{n}} \rangle,$$

the norm of φ is given by

$$\|\varphi\|^2_{(L^2)} = \sum_{\vec{n}}^{\infty} \vec{n}! |F_{\vec{n}}|^2_{2,n}$$

with kernel functions F in $\mathcal{F}$ and where $|\ |_{2,n}$ is the norm in $L^2(\mathbb{R}^n, dt)$.
Given $\xi \in S(\mathbb{R})^d$, let us consider the Wick exponential

$$\begin{aligned} : \exp\langle \omega, \xi \rangle : \quad &\equiv \quad \exp\left(\langle \omega, \xi \rangle - \frac{1}{2}(\xi, \xi) \right) \\ &= \quad \sum_{\vec{n}} \frac{1}{\vec{n}!} \langle : \omega^{\otimes \vec{n}} :, \xi^{\otimes \vec{n}} \rangle, \qquad \omega \in S'(\mathbb{R})^d. \end{aligned}$$

The S-transform plays an important role in the study of stochastic processes in the white noise framework, see for example [3][9]; we define the S-transform of φ in (L^2) as

$$S\varphi(\xi) \equiv \ll \varphi, : \exp\langle ., \xi \rangle : \gg = \sum_{\vec{n}} (\varphi_{\vec{n}}, \xi^{\otimes \vec{n}})_{2,n}.$$

In particular, for Hermitean operators A in $L^2(\mathbb{R})$, we can define the "second quantization" of A as an operator $\Gamma(A)$ in $\left(L^2\right)$ given by

$$S\Gamma(A)\Phi(\cdot) = S\Phi(A\cdot) \tag{1.2}$$

for $\Phi \in (L^2)$.

1.2. **Self-Intersection Local Time in Terms of White Noise.** The regularization of the self-intersection local time given by (1) has the following chaos expansion, for $d \geq 3$.

Proposition 1. [1] *For any t, $\epsilon > 0$, $L^{\epsilon}_{t,c}$ has kernel functions $F \in \mathcal{F}$ given by*

$$F_{\epsilon,\vec{n}}(s_1, \cdots, s_n) =$$

$$(-1)^{\frac{n}{2}} \left(\chi(\chi+1)(2\pi)^{d/2} 2^{\frac{n}{2}} \frac{\vec{n}}{2}! \right)^{-1} \theta(u)\theta(t-v) \cdot$$

$$\left((v-u+\epsilon)^{-\chi} + (t+\epsilon)^{-\chi} - (v+\epsilon)^{-\chi} - (t-u+\epsilon)^{-\chi} \right)$$

if all n_i are even, and zero otherwise, with $v(s_1, \cdots, s_n) \equiv \max(s_1, \cdots, s_n)$, $u(s_1, \cdots, s_n) \equiv \min(s_1, \cdots, s_n)$, and $\chi \equiv \frac{n+2}{2} - 2$. θ is the Heaviside function.

One can then divide each chaos into a martingale part

$$M^{\epsilon}_{\vec{n}} = \int_{[0,t]^n} d^n s (v-u+\epsilon)^{-\chi} : w^{\otimes \vec{n}}(s) :$$

and a remainder

$$N_{\vec{n}}^{\epsilon} = \int_{[0,t]^n} d^n s \left((t+\epsilon)^{-\chi} - (v+\epsilon)^{-\chi} - (t-u+\epsilon)^{-\chi} \right) : w^{\otimes \vec{n}}(s) : \ .$$

which is less singular in the limit $\varepsilon \to 0$. We note that one could have included the part $N_{\vec{n}}^{\epsilon}$ arising from the second term in the integrand, i.e. from $(v+\epsilon)^{-\chi}$, without losing the martingale property:

$$M_{\vec{n}}^{\prime\epsilon} = \int_{[0,t]^n} d^n s \left((v-u+\epsilon)^{-\chi} - (v+\epsilon)^{-\chi} \right) : w^{\otimes \vec{n}}(s) :$$

is also a martingale since the varable t does not appear in the integrand. After renormalization the difference vanishes in (L^2) as $\varepsilon \to 0$.

For the $\vec{n}$th order chaos,

$$K_{\vec{n}}^{\epsilon} = \alpha_{\vec{n}} \left(M_{\vec{n}}^{\epsilon} + N_{\vec{n}}^{\epsilon} \right) ,$$

$\vec{n} \in \mathbb{N}^d - \{0\}$, it was shown in [4], that for any $d \geq 3$,

$$r_d(\epsilon) K_{\vec{n}}^{\epsilon} \stackrel{\mathcal{L}}{\to} c_{\vec{n}} \beta_{\vec{n}} \tag{1.3}$$

where $\beta_{\vec{n}}$ are one dimensional Brownian motions independent of the initial one and among each other, with

$$c_{\vec{n}}^2 = k_n^2 \alpha_{\vec{n}}^2 ,$$

$$k_n^2 = \begin{cases} n(n-1) & d=3 \\ \frac{n!(d-4)!}{(n+d-5)!} & d>3 , \end{cases}$$

$$\alpha_{\vec{n}} = (-1)^{n/2} \left(\chi(\chi+1)(2\pi)^{d/2} 2^{n/2} \frac{\vec{n}}{2}! \right)^{-1} ,$$

$$r_d(\epsilon) = \begin{cases} |\log \epsilon|^{-1/2} & d=3 \\ \epsilon^{\frac{d-3}{2}} & d>3 . \end{cases}$$

It has been proved in [5] that the variances of these Brownian motions sum up to the variance of the renormalized self-intersection local time,

$$T \sum_{\vec{n}, n \neq 0} c_{\vec{n}}^2 = \lim_{\epsilon \to 0} \mathbb{E} \left((r_d(\epsilon) L_{T,c}^{\epsilon})^2 \right)$$

and that the right hand side can be calculated in closed form for all $d \geq 3$, using a Clark-Ocone formula for L^{ϵ}. For higher order self-intersections chaos decompositions and their renormalized limits are discussed in [14] and [11].

2. Statement of the results

If we inspect the kernels given in the above proposition we can see that the dominant terms of the chaos expansion for the local time are martingales. Hence it is desirable to sum up these dominant terms. This is indeed possible.

Theorem 1. *Let $\epsilon, T > 0$ and denote by θ_T the multiplication operator by $1_{[0,T]}$ Then*

$$L_{T,c}^{\epsilon} = M_{T,c}^{\epsilon} + R_{T,c}^{\epsilon}$$

where

$$M_{T,c}^{\epsilon} = \int_0^{\infty} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T) \delta_{\epsilon,c}(B_{t_2} - B_{t_1}) \tag{2.1}$$

is a martingale, and

$$R^{\epsilon}_{T,c} = -\int_{\epsilon}^{+\infty} dt_2 \int_0^T dt_1 \delta_{t_2,c}(B_T - B_{t_1}).$$

The next theorem gives us the corresponding chaos expansions, for $d \geq 3$.

Theorem 2. *Let $d > 2$, for any $\epsilon > 0$ and $\vec{n} \in I\!N^d$, $\vec{n} \neq \vec{0}$, $M^{\epsilon}_{T,c}$ and $R^{\epsilon}_{T,c}$ have respectively kernel functions $\mathcal{M}$ and $\mathcal{R}$ in $\mathcal{F}$ given by*

$$\begin{aligned}\mathcal{M}^{\epsilon}_{\vec{n}} &= (-1)^{n/2}\left(\kappa(\kappa+1)(2\pi)^{d/2}2^{n/2}\frac{\vec{n}}{2}!\right)^{-1} \\ &\cdot\theta(u)\theta(T-v)\left((v-u+\epsilon)^{-\kappa} - (v+\epsilon)^{-\kappa}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{N}^{\epsilon}_{\vec{n}} &= (-1)^{n/2}\left(\kappa(\kappa+1)(2\pi)^{d/2}2^{n/2}\frac{\vec{n}}{2}!\right)^{-1} \\ &\cdot\theta(u)\theta(T-v)\left((T+\epsilon)^{-\kappa} - (T-u+\epsilon)^{-\kappa}\right)\end{aligned}$$

if all n_i are even, and zero otherwise, with v=$v(s_1,\cdots,s_n) = \max_i s_i$, $u(s_1,\cdots,s_n) = \min_i s_i$, $\kappa = n + d/2 - 2$ and θ the Heaviside function.

We recall that if $(\Phi_t)_{t\geq 0}$ is a stochastic process or even a generalized one (such as $(\Phi_t)_{t\geq 0} \subset \mathcal{G}^{-1}$, the Potthoff-Timpel space [16]), it is shown in [2] that for each $t \geq 0$, Φ_t can be written as

$$\Phi_t = I\!E\Phi_t + I(m_{\Phi_t}) \tag{2.2}$$

This is the so called generalized Clark-Ocone formula, where $m_{\Phi_t} \in I\!R^d \otimes L^2(I\!R) \otimes \mathcal{G}^{-1}$ such that

$$m^i_{\Phi_t} = \Gamma(\theta_.)\partial^i_. \Phi \tag{2.3}$$

and

$$I(m_{\Phi_t}) = \sum_{i=1}^d I_i(m^i_{\Phi_t})$$

where $I_i(m^i_{\Phi_t})$ is in $\mathcal{G}^{-1}$, defined by

$$\ll I_i(m^i_{\Phi_t}), \varphi \gg = \ll m^i_{\Phi_t}, \partial^i_. \varphi \gg \qquad \text{for every } \varphi \in \mathcal{G}^{+1}.$$

$\partial^i_. \varphi \in L^2(I\!R) \otimes \mathcal{G}^{+1}$is the Hida-gradient of φ, see also [2] for an explicit formula. In our next theorem, we give the explicit formula of the integrand by applying the Clark-Ocone formula **??**.

Theorem 3. *For T, $\epsilon > 0$ we have*

$$M^{\epsilon}_{T,c} = \frac{1}{(2\pi)^{d/2}} \int_0^{+\infty} dt_2 \int_0^{T\wedge t_2} dt_1 \int_{t_1}^{T\wedge t_2} dB_\tau \frac{\left(B_\tau - B_{t_1}\right)}{\left(\epsilon + t_2 - \tau\right)^{\frac{d}{2}+1}}\, e^{\frac{-(B_\tau - B_{t_1})^2}{2(\epsilon+t_2-\tau)}}$$

$$R^{\epsilon}_{T,c} = -\frac{1}{(2\pi)^{d/2}} \int_T^{+\infty} dt_2 \int_0^T dt_1 \int_{t_1}^T dB_\tau \frac{\left(B_\tau - B_{t_1}\right)}{\left(\epsilon + t_2 - \tau\right)^{\frac{d}{2}+1}}\, e^{\frac{-(B_\tau - B_{t_1})^2}{2(\epsilon+t_2-\tau)}}$$

3. Proof of the results

Proof of Theorem 1.
Let $(\mathcal{F}_T)_{T\geq 0}$ be the Brownian filtration. Recall that if $\Gamma(\theta_T)\Phi \in (L^2)\ \forall T>0$, then $I\!E(\Phi|\mathcal{F}_T) = \Gamma(\theta_T)\Phi$ is a martingale with respect to the filtration $(\mathcal{F}_T)_{T\geq 0}$, thus

$$M^{\epsilon}_{T,c} = \int_0^{+\infty} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1})$$

is a martingale with respect to the Brownian filtration. Let us decompose $M^{\epsilon}_{T,c}$ as:

$$\begin{aligned} M^{\epsilon}_{T,c} &= \int_0^{T} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1}) \\ &\quad + \int_T^{+\infty} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1}). \end{aligned}$$

Note that if $\Phi \in (L^2)$ we have

$$S\Gamma(\theta_T)\Phi(\cdot) = S\Phi(\theta_T\cdot). \tag{3.1}$$

To compute the S-transform of $\delta_\epsilon(B_{t_2}-B_{t_1})$ we note that

$$\delta_\epsilon(B_{t_2}-B_{t_1}) = (2\pi\epsilon)^{-d/2}\prod_{i=1}^{d} e^{-\frac{1}{2}\langle\omega_i, K\omega_i\rangle}$$

with $K=\epsilon^{-1}|t_2-t_1|P$, where P is the projector onto the indicator function of the interval $[t_1,t_2]$. It is well known [9] that the S-transform of such a Gauss kernels is given by

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You are using the "gather" environment in a style in which it is not defined.
S\delta _{\epsilon }(B_{t_{2}}-B_{t_{1}})(\xi )=(2\pi \epsilon
det(1+K))^{-d/2}e^{-\frac{1}{2}(\xi ,\frac{K}{1+K}\xi )} \\
=\frac{1}{(2\pi (\epsilon +t_{2}-t_{1}))^{d/2}}\exp \left\{ -\frac{%
\sum_{i=1}^{d}\Big(\int_{t_{1}}^{t_{2}}\,\xi _{i}(u)du\,\Big)^{2}}{%
2(\epsilon +t_{2}-t_{1})}\right\}  \notag \\
=\frac{1}{(2\pi (\epsilon +t_{2}-t_{1}))^{d/2}}\exp \left\{ -\frac{\Big(%
\int_{t_{1}}^{t_{2}}\,\xi (u)du\,\Big)^{2}}{2(\epsilon +t_{2}-t_{1})}%
\right\} .  \label{stransform}
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As a result we obtain

$$\Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1}) = \begin{cases} \delta_{\epsilon,c}(B_{t_2}-B_{t_1}) & T>t_2 \\ 0 & T<t_1 \\ \delta_{t_2-T+\epsilon,c}(B_T-B_{t_1}) & t_1<T<t_2. \end{cases}$$

So

$$\begin{aligned} L^{\epsilon}_{T,c} &= \int_0^{T} dt_2 \int_0^{t_2} dt_1 \delta_{\epsilon,c}(B_{t_2}-B_{t_1}) \\ &= \int_0^{T} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1}) \\ &= \int_0^{+\infty} dt_2 \int_0^{t_2} dt_1 \Gamma(\theta_T)\delta_{\epsilon,c}(B_{t_2}-B_{t_1}) \\ &\quad - \int_T^{+\infty} dt_2 \int_0^{T} dt_1 \delta_{t_2-T+\epsilon,c}(B_T-B_{t_1}) \end{aligned}$$

and theorem 2 is proved.
Proof of theorem 2.
Using the formula (**??**) we obtain the following expansion in powers of ξ

$$\begin{aligned} & S\delta_{\epsilon,c}(B_{t_2}-B_{t_1})(\xi) \\ = & \frac{1}{(2\pi(\epsilon+t_2-t_1))^{\frac{d}{2}}} \sum_{\vec{n}\in \mathbb{N}^d, n\neq 0} \frac{1}{\vec{n}!}\left(\frac{-1}{2(\epsilon+t_2-t_1)}\right)^n \langle \xi^{\otimes 2\vec{n}}; \mathbf{1}_{[t_1,t_2]}^{\otimes 2n}\rangle \end{aligned}$$

then in view of (**??**) and (**??**)

$$\begin{aligned} \mathcal{M}^{\epsilon}_{2\vec{n}} &= (2\pi)^{-\frac{d}{2}} \frac{1}{\vec{n}!}(\frac{-1}{2})^n \, \theta_T^{\otimes 2n} \int_0^{+\infty} dt_2 \int_0^{t_2} dt_1 \, (\epsilon+t_2-t_1)^{-n-d/2} \mathbf{1}_{[t_1,t_2]}^{\otimes 2n} \\ &= (2\pi)^{-\frac{d}{2}} \frac{1}{\vec{n}!}(\frac{-1}{2})^n \theta_T^{\otimes 2n} \int_v^{+\infty} dt_2 \int_0^{u} dt_1 \, (\epsilon+t_2-t_1)^{-n-d/2} \mathbf{1}_{[t_1,t_2]}^{\otimes 2n} \end{aligned}$$

So we obtain

$$\begin{aligned} \mathcal{M}^{\epsilon}_{2\vec{n}} &= (2\pi)^{-d/2} \frac{1}{\vec{n}!}(\frac{-1}{2})^n \frac{1}{(n+d/2-1)(n+d/2-2)} \\ & \cdot \theta_T^{\otimes 2n} \left\{ (\epsilon+v-u)^{-(n+d/2-2)} - (\epsilon+v)^{-(n+d/2-2)} \right\} \end{aligned}$$

where $v = v(s_1,\ldots,s_{2n}) = \max_i s_i$ and $u = \min_i s_i$.
An analogous argument produces the kernels of R^{ϵ}_T.
Proof of theorem 3.
We first find a representation as in (**??**) for $\Gamma(\theta_T)\delta_\epsilon$

$$\Gamma(\theta_T)\delta_\epsilon(B_{t_2}-B_{t_1}) = \mathbb{E}\Big(\Gamma(\theta_T)\delta_\epsilon(B_{t_2}-B_{t_1})\Big) + I(m^\epsilon)$$

note that $m^\epsilon = (m^\epsilon_i)_{1\le i\le d}$ is in $\mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes (L^2)$ and that of course it depends of t_1, t_2.
By theorem 4.1 in [2] we have

$$m_i^\epsilon(\tau) = S^{-1}\frac{\delta}{\delta\xi_i(\tau)} S\Big(\Gamma(\theta_T)\delta_\epsilon(B_{t_2}-B_{t_1})\Big)(\theta_\tau\xi)$$

or

$$\begin{aligned} S\Big(\Gamma(\theta_T)\delta_\epsilon(B_{t_2}-B_{t_1})\Big)(\xi) &= [2\pi(t_2-t_1+\epsilon)]^{-d/2} \exp\left\{ -\frac{\mathbf{1}_{[0,T]}(t_1)\sum_{i=1}^d (\int_{t_1}^{t_2\wedge T}\xi_i du)^2}{2(t_2-t_1+\epsilon)} \right\} \\ &= [2\pi(t_2-t_1+\epsilon)]^{-d/2} \exp\left\{ -\frac{\mathbf{1}_{[0,T]}(t_1)(\int_{t_1}^{t_2\wedge T}\xi du)^2}{2(t_2-t_1+\epsilon)} \right\}. \end{aligned}$$

It is known from [2] that if $g(x_1,...,x_d)$ is a smooth function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, $(h_i)_{1\le i\le d} \in \mathbb{R}^d \otimes L^2(\mathbb{R})$ and $E(\xi) = g(\langle\xi_1,h_1\rangle,...,\langle\xi_d,h_d\rangle)$ then

$$\frac{\delta E}{\delta\xi_i(\tau)}(\xi) = \frac{\partial g}{\partial x_i}(\langle\xi_1,h_1\rangle,...,\langle\xi_d,h_d\rangle)h_i(\tau).$$

Using this formula we obtain for fixed $i = 1,...,d$

$$\begin{aligned} \frac{\delta}{\delta\xi_i(\tau)}\Big(\Gamma(\theta_T)\delta_\epsilon(B_{t_2}-B_{t_1})\Big)(\theta_\tau\xi) &= \frac{-\mathbf{1}_{[0,T]}(t_1)\mathbf{1}_{[t_1,t_2\wedge T]}(\tau)}{(2\pi)^{d/2}(t_2-t_1+\epsilon)^{d/2+1}} \quad \times \\ & \int_{t_1}^{\tau}\xi_i du \ \exp\left\{ -\frac{(\int_{t_1}^{\tau}\xi du)^2}{2(t_2-t_1+\epsilon)} \right\} \end{aligned}$$

we have now to compute the inverse of the S-transform of the last expression. For each $1 \leq i \leq d$, denote by

$$G_i(\xi) = \int_{t_1}^{\tau} \xi_i du \ \exp\left\{ -\frac{(\int_{t_1}^{\tau} \xi du)^2}{2(t_2 - t_1 + \epsilon)} \right\}$$

so

$$G_i(\xi) = \sqrt{\tau - t_1} \langle h, \xi_i \rangle \exp\left\{ -\frac{(\tau - t_1)\langle h, \xi_i \rangle^2}{2(t_2 - t_1 + \epsilon)} \right\} \prod_{j \neq i} \exp\left\{ -\frac{(\tau - t_1)\langle h, \xi_j \rangle^2}{2(t_2 - t_1 + \epsilon)} \right\}$$

where $h = \frac{\mathbf{1}_{[t_1,\tau]}}{\sqrt{\tau - t_1}}$, then $S^{-1}G_i$ is also a product of functions, each of them depend only of $\langle \omega_j, h \rangle$.

Lemma 1. *Let* $\|h\|_{L^2(I\!R)} = 1,\ \eta \in S(I\!R)$ *and* $0 < c < 1/2$. *Let* E *and* F *be two function of* η *defined by*

$$E(\eta) = \exp\Big(- c\langle h, \eta \rangle^2 \Big)$$

and

$$F(\eta) = \langle h, \eta \rangle \exp\Big(- c\langle h, \eta \rangle^2 \Big)$$

where $\langle h, \eta \rangle = \int h(t)\eta(t)dt$. *Then*

$$S^{-1}E(\omega) = \frac{1}{\sqrt{1 - 2c}} \exp\Big(\frac{c}{2c - 1} \langle \omega, h \rangle^2 \Big)$$

and

$$S^{-1}F(\omega) = \frac{\langle \omega, h \rangle}{(1 - 2c)^{3/2}} \exp\Big(\frac{c}{2c - 1} \langle \omega, h \rangle^2 \Big).$$

Using this lemma, we obtain for $i = 1, ..., d$

$$m_i^{\epsilon}(\tau) = \frac{\mathbf{1}_{[0,T]}(t_1)\mathbf{1}_{[t_1, t_2 \wedge T]}(\tau)}{(2\pi)^{d/2}\big(\epsilon + t_2 - \tau\big)^{d/2+1}} \Big(B_\tau^i - B_{t_1}^i \Big) e^{-\frac{(B_\tau - B_{t_1})^2}{2(\epsilon + t_2 - \tau)}}$$

and finally we obtain the desired formula for $M_{T,c}^{\epsilon}$

$$M_{T,c}^{\epsilon} = \frac{1}{(2\pi)^{d/2}} \int_0^{+\infty} dt_2 \int_0^{T \wedge t_2} dt_1 \int_{t_1}^{T \wedge t_2} dB_\tau \frac{\Big(B_\tau - B_{t_1} \Big)}{\big(\epsilon + t_2 - \tau \big)^{d/2+1}} \exp\left\{ -\frac{(B_\tau - B_{t_1})^2}{2(\epsilon + t_2 - \tau)} \right\}.$$

We proceed analogously for R_T^{ϵ}, and the theorem is proved.

References

[1] **De Faria M., Hida T., Streit L. and Watanabe H.:** *Intersection local times as generalized white noise functionals, Acta Appl. Math. 46 (1997), 351-362.*

[2] **De Faria M., Oliveira M. J. and Streit L.:** *Generalized Clark-Ocone formula. ROSE. 8 (2000) 163-174.*

[3] **Deck Th., Potthoff J. and Vage G.:** *A Review of White Noise Analysis from a Probabilistic Standpoint. Acta Appl. Math. 48 (1997) 91-112.*

[4] **Drumond C., de Faria M. and Streit L.:** *The renormalization of self intersection local times I: The chaos expansion. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000) 223-236.*

[5] **Drumond C., de Faria M. and Streit L.:** *The square of self intersection local time of Brownian motion. Stochastic processes, physics and geometry: new interplays, I (F. Gesztesy et al., eds.). CMS Conference Proceedings 28 (2000) 115-122.*
[6] **Dynkin E. B.:** *Polynomials of the occupation field and related random fields, J. Funct. Anal. 58 (1984) 20-52.*
[7] **Dynkin E. B.:** *Regularized self-intersection local times of planar Brownian motion. Ann. Prob. 16 (1988) 58-74.*
[8] **Hida T.:** *"Brownian Motion", Springer, Berlin-Heidelberg- New York. 1980.*
[9] **Hida T., Kuo H-H., Potthoff J. and Streit L.:** *White Noise - An Infinite Dimensional Calculus. Kluwer-Academic. 1993.*
[10] **Imkeller P. and Yan J-A.:** *Multiple intersection local time of planar Brownian motion as a particular Hida distribution. J. Funct. Anal. 140 (1996) 256-273.*
[11] **Jenane Gannoun R.:** *PhD thesis, Tunis, 2001.*
[12] **Kuo H-H.:** *Donker's delta function as a generalized Brownian functional and its application Lecture Notes in Control and Information Sciences. 49 (1983) 167-178.*
[13] **Le Gall J.F.:** *Sur le temps local d'intersection du mouvement Brownian plan et la méthode de renormalisation de Varadhan. Springer L.N.M. 1123. (1985) 314-331.*
[14] **Mendonça S. and Streit L.:** *Multiple Intersection Local Time in Terms of White Noise. IDAQP 4 (2001) 533-543,*
[15] **Nualart D. and Vives J.:** *Chaos expansion and local times. Publications Mathematiques 36 (1992) 827-836.*
[16] **Potthoff J. and Timpel M.:** *On dual pair of spaces of smooth and generalized random variables. Potential Analysis 4 (1995) 637-654.*
[17] **Rosen J.:** *A local time approach to the self-intersections of Brownian paths in space. Comm. Math. Phys. 88 (1983) 327-338.*
[18] **Rosen J.:** *Tanaka's formula and renormalization for intersections of planar Brownian motion. Ann. Prob. 14 (1986) 1245-1251.*
[19] **Yor M.:** *Renormalisation et convergence en loi pour les temps locaux d'intersections du mouvement brownien dans* $\mathbb{R}^3$*. Springer L.N.M. 1123 (1985) 350-365.*
[20] **Varadhan S.R.S.:** *Appendix to "Euclidian quantum field theory" by Szymanzik K., in: R. Jost ed., Local Quantum Theory. Academic Press, New York. 1969.*
[21] **Watanabe H.:** *The local time of self-intersections of Brownian motions as generalized Brownian functionals. Lett. Math. Phys. 23 (1991) 1-9.*

M. de Faria, CCM, Universidade da Madeira, P 9000 Funchal
E-mail address: ccm@uma.pt

A. Rezgui, ZiF, Univ. Bielefeld, D 33615 Bielefeld
E-mail address: rezgui@physik.uni-bielefeld.de

L. Streit, CCM, Universidade da Madeira, P 9000 Funchal, BiBoS, Univ. Bielefeld, D 33615 Bielefeld
E-mail address: streit@physik.uni-bielefeld.de