Marked Gibbs measures via cluster expansion

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Abstract

We give a sufficiently detailed account on the construction of marked Gibbs measures in the high temperature and low fugacity regime. This is proved for a wide class of underlying spaces and potentials such that stability and integrability conditions are satisfied. That is, for state space we take a locally compact separable metric space X and a separable metric space S for the mark space. This framework allowed us to cover several models of classical and quantum statistical physics. Furthermore, we also show how to extend the construction for more general spaces as e.g., separable standard Borel spaces. The construction of the marked Gibbs measures is based on the method of cluster expansion.

Contents

1 Introduction

The purpose of this paper is to give a detailed and comprehensive account on the construction of marked Gibbs measures in the high temperature and low fugacity regime for general underlying spaces using the method of cluster expansion. Our motivation for this general framework is on the one hand related to the examples in statistical physics we would like to cover, see Examples 3.5 - 3.8 below and also Subsection 5.4. On the other hand, in recent papers [AKR98], [AKR97a] (see also lecture notes [Röc98]) the authors put special emphasis in the construction of differential geometry on the simple configuration space Γ_X over a manifold X, i.e.,

$$\Gamma_X := \{\gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X\},$$

(cf. (2.1)) via a lifting of the geometry from the underlying manifold X (see as well [KSS97a] for an extension for compound Poisson spaces). In [AKR97b] the authors applied the aforementioned differential geometry to construct representations of current algebras and hence non-relativistic quantum field theories. This provides a scheme of canonical quantizations which uses a Gibbs measure on the configuration space as a ground state measure of the considered models. Having in mind the study of quantum models with internal degrees of freedom we are interested to extend the corresponding analysis to marked configurations and non flat underlying spaces. It gives an additional motivation to develop analysis, geometry, etc. on marked configuration spaces. In all applications mentioned above marked Gibbs measures are playing a fundamental role. At present moment any general results about the existence and uniqueness of marked Gibbs measures are absent. The aim of our paper is to describe a construction of such kind of measures in the case of general underlying and marked space.

The results of this paper (which we will give an account below) are based on the so-called cluster expansion method, see e.g., [MM91], [Pen63], [Rue64], and [Rue69], and we follow closely the scheme of V. A. Malyshev and R. A. Minlos (cf. [MM91, Chap. 3 and 4]), which the authors realized for the configuration space over $\mathbb{R}^d$. Let us explain this more precisely. Let X be a locally compact separable metric space (the space describing the position of particles) and S a separable metric space (the mark space) describing some internal degrees of freedom, e.g., spin, momentum, or different types of particles. We construct a marked Poisson measure π_σ^τ (σ is an intensity

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measure on X and τ a transition kernel on S) over the marked configuration space, i.e.,

$$\Omega_X(S) := \{\omega = \{(x,s)\} \in \Gamma_{X \times S} | \{x\} =: \gamma_\omega \in \Gamma_X\},$$

via Kolmogorov's theorem, see Subsections 2.1 and 2.2 below. The desired measure μ on $\Omega_X(S)$ is obtained as a limit (in a sense to be specified later) of a family of measures $\Pi_\Lambda^{\sigma^\tau,\phi}$, cf. Subsection 5.1. Here σ^τ is the measure defined on $(X \times S, \mathcal{B}(X \times S))$ by $\sigma^\tau(dx,ds) = \tau(x,ds)\sigma(dx)$, see (2.7) for details. For finite volume $\Lambda \subset X$ (i.e., bounded Borel set) the measure $\Pi_\Lambda^{\sigma^\tau,\phi}$ is defined as a Gibbs type perturbation of the marked Poisson measure π_σ^τ, i.e.,

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega,F) := \frac{\mathbb{1}_{\{Z_\Lambda^{\sigma^\tau,\phi}<\infty\}}(\omega)}{Z_\Lambda^{\sigma^\tau,\phi}(\omega)} \int_\Omega \mathbb{1}_F(\omega_{X\setminus\Lambda} \cup \omega'_\Lambda)e^{-E_\Lambda^\phi(\omega_{X\setminus\Lambda}\cup\omega'_\Lambda)}\pi_\sigma^\tau(d\omega'),$$

(cf. Definition 3.1 in Section 3). It is well-known that $\Pi_\Lambda^{\sigma^\tau,\phi}$ is a specification in the sense of [Pre76, Section 6] (see also [Pre79] and [Pre80]) for the given pair potential ϕ. Shortly speaking, a marked Gibbs measure is defined as a probability measure which has as conditional expectation the specification $\Pi_\Lambda^{\sigma^\tau,\phi}$. The aforementioned limit measure μ is locally absolutely continuous with respect to the marked Poisson measure π_σ^τ (cf. Theorem 5.3). If we assume additionally that the potential ϕ has finite range, then we give a direct proof that the limit measure μ fulfils the DLR equation, see Subsection 5.2, Theorem 5.6, and hence it is a Gibbs measure. Let us mention that using further consequences of the cluster expansion developed in [Kun98] and the general results from [KK98] it is possible to show that the limit measure μ is a Gibbs measure for a much wider class of potentials.

We would like to emphasize that the above results (specially the one of Theorem 5.3) are strongly related with the procedure of cluster expansion and the estimates obtained there. As usual, this procedure is possible under some conditions on the potential ϕ and other parameters of the system.

Thus the contents of Sections 3, 4, and 5 has been described. It remains to add that Section 2 consists of the necessary preliminaries for the further sections. Namely, we give a sketch of the construction of the marked configuration space $\Omega_X(S)$ and its measurable structure, (cf. Subsection 2.1) as well as the marked Poisson measures π_σ^τ, see Subsection 2.2. In the remainder of Section 2 we introduce some algebraic structures in order to perform

easier calculations and combinatorics involved in cluster expansion. This is the contents of Subsection 2.3 and 2.4. For the clarity of the presentation we moved some proofs to the Appendix.

Finally, we would like to remark that all our results extends to underlying spaces more general than we discuss in the main body of the work, namely, separable standard Borel spaces. The necessary modifications are described in Subsection 5.3. In a second paper, see [Kun98], we collect further results for Gibbs measures in the high temperature regime.

2 Marked configurations spaces

In this section we describe the framework to be used in the rest of the paper. Hence in Subsection 2.1 we introduce the measurable structure of the space on which the marked Gibbs measure will be defined, see Section 3. Let us mention that such measures are called *states* in statistical physics of continuous systems and in probability theory they are known as *marked point random fields*, cf. e.g. [AGL78], [GZ93], [Kin93], and [MM91].

The marked Poisson measures are constructed in Subsection 2.2. Finally, in Subsection 2.3 (resp. Subsection 2.4) we introduce some facts from graph theory (resp. *-calculus) which will simplify our calculations later on, namely in Section 4.

Let X be a locally compact separable metric space (which fulfils the second axiom of countability, i.e., the topology is countably generated). It describes the *position space* of the particles. Denote by $\mathcal{B}(X)$ the Borel σ-algebra on X and by $\mathcal{B}_c(X)$ the set of all elements in $\mathcal{B}(X)$ which have compact closures (sets from $\mathcal{B}_c(X)$ we call finite volumes). Additionally, we suppose given a complete separable metric space S. The corresponding Borel σ-algebra we denote by $\mathcal{B}(S)$. The elements of this space we call *marks* (they can describe e.g., internal degrees of freedom).

2.1 The marked configuration space over a manifold

We briefly recall the basic definitions of the simple configuration space over a manifold X for the reader's convenience. The presentation is very much based along the lines of the works by S. Albeverio et al. [AKR98].

The *simple configuration space* $\Gamma := \Gamma_X$ over the space X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}. \qquad (2.1)$$

Here (and below) $|A|$ denotes the cardinality of a set A. For any $Y \subset X$ we define

$$\Gamma_Y := \{\gamma \in \Gamma \,|\, |\gamma \cap (X \backslash Y)| = 0\}.$$

In this paper we are interested in a bigger space of configurations, the so-called marked configuration space, thus we proceed giving its abstract definition. For concrete examples we refer to Subsection 3.2.

The *marked configuration space* $\Omega_X(S) := \Omega_X := \Omega$ is defined by

$$\Omega := \{\omega = \{(x,s)\} \in \Gamma_{X \times S} | \{x\} =: \gamma_\omega \in \Gamma_X, s \in S\}. \qquad (2.2)$$

Equivalently Ω can be described as follows

$$\Omega := \{\omega = (\gamma_\omega, s) | \gamma_\omega \in \Gamma_X, s \in S^{\gamma_\omega}\},$$

where S^{γ_ω} stands for the set of all maps $\gamma_\omega \ni x \mapsto s_x \in S$. For any $Y \in \mathcal{B}(X)$ we define in a similar way the space $\Omega_Y(S) := \Omega_Y$. We sometimes use the shorthand ω_Y (resp. γ_Y) for $\omega \cap (Y \times S)$, $Y \subset X$ (resp. $\gamma \cap Y$) and $\hat{x} := (x, s_x) \in X \times S$.

In order to define a measurable structure on Ω we use the following family of sets $\mathfrak{I}$, the "local" sets

$$\mathfrak{I} := \{B \in \mathcal{B}(X) \times \mathcal{B}(S) | \exists \Lambda \in \mathcal{B}_c(X) \text{ with } B \subset \Lambda \times S\}. \qquad (2.3)$$

For any $A \in \mathfrak{I}$ define the mapping $N_A : \Omega \to \mathbb{N}_0$ by

$$N_A(\omega) := |\omega \cap A|, \ \omega \in \Omega,$$

then

$$\mathcal{B}(\Omega) := \sigma(\{N_A | A \in \mathfrak{I}\}).$$

For any $Y \in \mathcal{B}(X)$ we define the following σ-algebra on Ω

$$\mathcal{B}_Y(\Omega) := \sigma(\{N_A | A \in \mathfrak{I}, \ A \subset Y \times S\}).$$

For any $Y \in \mathcal{B}(X)$ the σ-algebra $\mathcal{B}_Y(\Omega)$ is isomorphic to $\mathcal{B}(\Omega_Y)$. The "filtration" $(\mathcal{B}_\Lambda(\Omega))_{\Lambda \in \mathcal{B}_c(X)}$ is one of the basic structures in the definition of the Gibbs measures, see Section 3 and 5. Moreover, if $Y_1, Y_2 \in \mathcal{B}(X)$ such that $Y_1 \cap Y_2 \neq \emptyset$, then $\Omega_{Y_1 \sqcup Y_2}$ is isomorphic to $\Omega_{Y_1} \times \Omega_{Y_2}$.

Finally we want to give another useful description of the marked configuration space Ω. For any $n \in \mathbb{N}_0$ and any $Y \in \mathcal{B}(X)$ we define the *n-point configuration space* $\Omega_Y^{(n)}$ as a subset of Ω_Y by

$$\Omega_Y^{(n)} := \Omega_Y^{(n)}(S) := \{\omega \in \Omega_Y | |\omega| = n\}, \ \Omega_Y^{(0)} := \{\emptyset\},$$

and denote the corresponding σ-algebra by $\mathcal{B}(\Omega_Y^{(n)})$.

There is a bijection

$$(\widetilde{Y \times S})^n / S_n \to \Omega_Y^{(n)}, \ n \in \mathbb{N}, \ Y \in \mathcal{B}(X), \qquad (2.4)$$

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where

$$(\widetilde{Y \times S})^n := \{((x_1, s_{x_1}), \ldots, (x_n, s_{x_n})) | x_i \in Y, s_{x_i} \in S, x_i \neq x_j, \text{ for } i \neq j\},$$

and S_n denotes the permutation group over $\{1, \ldots, n\}$. Since this bijection is measurable in both directions the natural σ-algebra on $(\widetilde{Y \times S})^n / S_n$ is isomorphic to $\mathcal{B}(\Omega_Y^{(n)})$.

One can reconstruct Ω from the sets $\Omega_\Lambda^{(n)}$ using the following scheme. First notice that we can write for any $\Lambda \in \mathcal{B}_c(X)$

$$\Omega_\Lambda = \bigsqcup_{n=0}^{\infty} \Omega_\Lambda^{(n)},$$

hence the σ-algebra $\mathcal{B}(\Omega_\Lambda)$ is the disjoint union of the σ-algebras $\mathcal{B}(\Omega_\Lambda^{(n)})$.

For any $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(X)$ with $\Lambda_1 \subset \Lambda_2$ there are natural maps

$$p_{\Lambda_2, \Lambda_1} : \Omega_{\Lambda_2} \longrightarrow \Omega_{\Lambda_1},$$

$$p_{\Lambda_1} : \Omega \longrightarrow \Omega_{\Lambda_1}$$

defined by $p_{\Lambda_2, \Lambda_1}(\omega) := \omega_{\Lambda_1}$, $\omega \in \Omega_{\Lambda_2}$ (resp. $p_{\Lambda_1}(\omega) = \omega_{\Lambda_1}$, $\omega \in \Omega$). It can be shown that $(\Omega, \mathcal{B}(\Omega))$ coincides with the projective limit of the measurable spaces $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$, $\Lambda \in \mathcal{B}_c(X)$.

Finally, we would like to introduce one more subspace of Ω which plays a fundamental role in our calculations below, the *finite configuration space* $\Omega_{X,fin} := \Omega_{fin}$. It is defined by

$$\Omega_{fin} := \{\omega \in \Omega | \, |\omega| < \infty\}.$$

The finite configuration space Ω_{fin} has the following useful representation in terms of the n-point configuration spaces

$$\Omega_{fin} = \bigsqcup_{n=0}^{\infty} \Omega_X^{(n)}. \tag{2.5}$$

Analogously for $\Omega_{Y,fin}$, $Y \in \mathcal{B}(X)$. The space Ω_{fin} (resp. $\Omega_{Y,fin}$) is equipped with the σ-algebra $\mathcal{B}(\Omega_{fin})$ (resp. $\mathcal{B}(\Omega_{Y,fin})$) of the disjoint unions of measurable spaces $(\Omega_X^{(n)}, \mathcal{B}(\Omega_X^{(n)}))$ (resp. $(\Omega_Y^{(n)}, \mathcal{B}(\Omega_Y^{(n)}))$).

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2.2 Marked Poisson measures

For constructing the marked Poisson measure on Ω we need, first of all, to fix an intensity measure σ on the underlying space X. Thus, let us assume that σ is a non-atomic Radon measure on X. Additionally, we define a kernel $\tau : X \times \mathcal{B}(S) \to \mathbb{R}$, i.e., $\forall x \in X$ $\tau(x, \cdot)$ is a finite measure on $(S, \mathcal{B}(S))$ and $\tau(\cdot, A)$ is $\mathcal{B}(X)$-measurable for all $A \in \mathcal{B}(S)$. Moreover we assume that the following condition is fulfilled for any $\Lambda \in \mathcal{B}_c(X)$

$$\int_{\Lambda} \tau(x, S)\sigma(dx) < \infty. \tag{2.6}$$

This condition will be essential in the estimates later on (cf. proof of Proposition 4.13).

In the product space $X \times S$ we define a σ-finite measure σ^τ by

$$\sigma^\tau(dx, ds) := \tau(x, ds)\sigma(dx),$$

that means for $A \times B \in \mathcal{B}(X \times S)$

$$\sigma^\tau(A \times B) = \int_A \tau(x, B)\sigma(dx), \tag{2.7}$$

which is a non-atomic Radon measure.

For any $Y \in \mathcal{B}(X)$ and $n \in \mathbb{N}$ the product measure $\sigma^{\tau \otimes n}$ can be considered as a measure on $(\widetilde{Y \times S})^n$, cf. Lemma A.1 in the Appendix. Let

$$\sigma_n^\tau \restriction \Omega_Y^{(n)} := \sigma^{\tau \otimes n} \circ (\mathrm{sym}_Y^n)^{-1},$$

be the corresponding measure on $\Omega_Y^{(n)}$, where

$$\mathrm{sym}_Y^n : (\widetilde{Y \times S})^n \to \Omega_Y^{(n)},$$

given by
$$\mathrm{sym}_Y^n((\hat{x}_1, \dots, \hat{x}_n)) := \{\hat{x}_1, \dots, \hat{x}_n\} \in \Omega_Y^{(n)}.$$

Then we consider the so-called *Lebesgue-Poisson measure* $\nu_{z\sigma^\tau}$ on $\mathcal{B}(\Omega_{fin})$, which coincides on each $\Omega_X^{(n)}$ with the measure $\frac{z^n}{n!}\sigma_n^\tau \restriction \Omega_X^{(n)}$, as follows

$$\nu_{z\sigma^\tau} := \sum_{n=0}^{\infty} \frac{z^n}{n!}(\sigma_n^\tau \restriction \Omega_X^{(n)}), \tag{2.8}$$

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and $\sigma_0^\tau(\emptyset) := 1$. As a result $\nu_{z\sigma\tau}$ is σ-finite. $z > 0$ is the so called activity parameter.

Considered as a measure on Ω_Λ, $\Lambda \in \mathcal{B}_c(X)$, the measure $\nu_{z\sigma\tau}$ is finite with $\nu_{z\sigma\tau}(\Omega_\Lambda) = e^{z\sigma^\tau(\Lambda \times S)}$. Therefore, we can define a probability measure $\pi_{z\sigma}^{\tau,\Lambda}$ on Ω_Λ putting

$$\pi_{z\sigma}^{\tau,\Lambda} := e^{-z\sigma^\tau(\Lambda \times S)} \nu_{z\sigma\tau}.$$

The measure $\pi_{z\sigma}^{\tau,\Lambda}$ has the following property

$$\pi_{z\sigma}^{\tau,\Lambda}(\Omega_\Lambda^{(n)}) = \frac{z^n}{n!}(\sigma^\tau(\Lambda \times S))^n e^{-z\sigma^\tau(\Lambda \times S)},$$

which gives the probability of the occurrence of exactly n points of the marked Poisson process (with arbitrary values of marks) inside of the volume Λ.

In order to obtain the existence of a unique probability measure $\pi_{z\sigma}^\tau$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1}, \ \Lambda \in \mathcal{B}_c(X),$$

we notice that the family $\{\pi_{z\sigma}^{\tau,\Lambda} | \Lambda \in \mathcal{B}_c(X)\}$ is consistent, i.e.,

$$\pi_{z\sigma}^{\tau,\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1} = \pi_{z\sigma}^{\tau,\Lambda_1}, \ \Lambda_1, \Lambda_2 \in \mathcal{B}_c(X), \Lambda_1 \subset \Lambda_2,$$

and thus, by a version of Kolmogorov's theorem for the projective limit space Ω (cf. [Par67, Chap. V Theorem 3.2] or Theorem 5.12 below) any such family determines uniquely a measure $\pi_{z\sigma}^\tau$ on $\mathcal{B}(\Omega)$ such that $\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1}$. The measure $\pi_{z\sigma}^\tau$ is called *marked Poisson measure*.

2.3 Basic concepts in graph theory

Now we are going to introduce some standard concepts of graph theory, see e.g., [Ore67] for more details.

Let X be a non empty set. A *partition* of X is a family of non empty subsets $(X_i)_{i \in I}$ of X, called parts, such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_i X_i = X$. The set of all partitions of X where all parts are non-empty is denoted by $\mathfrak{P}(X)$ and by $\mathfrak{P}^n(X)$ we denote the subset of partitions of $\mathfrak{P}(X)$ consisting of n parts. $\mathfrak{P}_\emptyset^n(X)$ stands for the set of all partitions of n parts which might be empty.

We now give the notion of a graph as well as some of its properties. We note here and henceforth that the graphs under consideration are undirected, see [Ore67, Chap. 1] for this notion.

Definition 2.1   *1. A graph $G := G(X)$ is a subset of*

$$\{\{x, y\} \subset X | x \neq y\}.$$

One calls $\{x, y\} \in G$ the edges of the graph and $V(G) := X$ the vertices of the graph. The collection of all such graphs on X is denoted by $\mathfrak{G}(X)$.

2. Given two graphs $G_1 \in \mathfrak{G}(X_1)$, $G_2 \in \mathfrak{G}(X_2)$ with $G_1 \cap G_2 = \emptyset$, their sum graph is the graph given by

$$G_1 \sqcup G_2 = \{\{x, y\} \subset X_1 \cup X_2 | \{x, y\} \in G_1 \sqcup G_2\}.$$

If G_1 and G_2 have no common vertices, then the sum graph is denoted by $G_1 \oplus G_2$. This procedure extends to an arbitrary family $\{G_i\}$ of graphs.

3. A graph G is called connected iff any pair of vertices is connected. The set of all connected graphs in X is denoted by $\mathfrak{G}^c(X)$. We assume that the single point is a connected graph.

Proposition 2.2 *(cf. [Ore67, Theorem 2.2.1]) Let $G \in \mathfrak{G}$ be given. Then G decomposes uniquely into a disjoint sum $\oplus_i G_i$ of its connected components.*

Definition 2.3 *A connected graph G is called a tree iff it has no loops. The set of all trees on X is denoted by $\mathfrak{T}(X)$.*

Proposition 2.4 *(cf. [Ore67, Theorem 4.1.3]) The number of different trees which can be constructed on n given vertices is n^{n-2}.*

Remark 2.5   *1. Since for any $n \geq 0$ we have*

$$\sqrt{2\pi n} n^n e^{-n} \leq n! \leq \sqrt{2\pi n} n^n e^{-n} \exp\left(\frac{1}{12(n-1)}\right),$$

it is not hard to see that $n^{n-2} < e^n n!$.

2. We use the shorthand $[n]$ for $\{1, \ldots, n\}$ and thus the symbol $\mathfrak{T}([n])$ denotes the trees in $\{1, \ldots, n\}$.

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2.4   *-calculus

In this subsection we point out an algebraic structure (see, e.g.[Rue64], [Rue69], and [MM91],) which turns out to simplify our notation and calculations later on. It will be very interesting to clarify more the related analytic and algebraic structure of this calculus.

Let $\mathcal{A}$ be the set of all measurable (complex-valued) functions ψ on Ω_{fin}, i.e.,

$$\mathcal{A} := \{\psi : \Omega_{fin} \to \mathbb{C}, \psi \text{ is } \mathcal{B}(\Omega_{fin})-\text{measurable}\}.$$

In $\mathcal{A}$ we introduce the following operation: for any $\psi_1, \psi_2 \in \mathcal{A}$ and $\omega \in \Omega_{fin}$ we define $\psi_1 * \psi_2$ by

$$(\psi_1 * \psi_2)(\omega) := \sum_{(\omega_1, \omega_2) \in \mathfrak{P}^2_\emptyset(\omega)} \psi_1(\omega_1)\psi_2(\omega_2), \quad \omega \in \Omega_{fin},$$

which is $\mathcal{B}(\Omega_{fin})$-measurable because the restriction to $\Omega_X^{(n)}$ is of the form

$$(\psi_1 * \psi_2)(\{\hat{x}_1, \ldots, \hat{x}_n\}) = \sum_{(I,J) \in \mathfrak{P}^2_\emptyset([n])} \psi_1(\{\hat{x}_i | i \in I\})\psi_2(\{\hat{x}_j | j \in J\}).$$

The set $\mathcal{A}$ equipped with $*$ and the natural vector space structure forms a commutative algebra with unit element

$$1^*(\omega) = \begin{cases} 1, & \omega = \emptyset \\ 0, & \omega \neq \emptyset \end{cases}. \tag{2.9}$$

Notice that for any $\psi_1, \ldots \psi_n \in \mathcal{A}$ we have

$$(\psi_1 * \ldots * \psi_n)(\omega) = \sum_{(\omega_1, \ldots, \omega_n) \in \mathfrak{P}^n_\emptyset(\omega)} \psi_1(\omega_1) \ldots \psi_n(\omega_n), \quad \omega \in \Omega_{fin}. \tag{2.10}$$

Let us define $\mathcal{A}_0$ as a subset of $\mathcal{A}$ by

$$\mathcal{A}_0 := \{\psi \in \mathcal{A} | \psi(\emptyset) = 0\}.$$

For any $\psi \in \mathcal{A}$ and $\varphi \in \mathcal{A}_0$ we have

$$\begin{aligned} (\psi * \varphi)(\emptyset) &= \sum_{(\omega_1, \omega_2) \in \mathfrak{P}^2_\emptyset(\emptyset)} \psi(\omega_1)\varphi(\omega_2) \\ &= \psi(\emptyset)\varphi(\emptyset) = 0, \end{aligned}$$

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thus it follows that $\mathcal{A}_0$ is an ideal in $\mathcal{A}$.

Let us introduce the mapping $\exp^* : \mathcal{A}_0 \to 1^* + \mathcal{A}_0$ defined by

$$\exp^* \psi := \sum_{n=0}^{\infty} \frac{1}{n!} \psi^{*n} = 1^* + \psi + \frac{1}{2!} \psi^{*2} + \ldots + \frac{1}{n!} \psi^{*n} + \ldots. \qquad (2.11)$$

It follows from (2.10) that for any $\psi \in \mathcal{A}_0$

$$(\exp^* \psi)(\emptyset) = 1^*,$$
$$(\exp^* \psi)(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1, \ldots, \omega_n) \in \mathfrak{P}_\emptyset^n(\omega)} \psi(\omega_1) \ldots \psi(\omega_n), \; \omega \in \Omega_{fin} \backslash \{\emptyset\}.$$

Moreover, if we define the mapping $\ln^* : 1^* + \mathcal{A}_0 \to \mathcal{A}_0$ by

$$\ln^*(1^* + \psi) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \psi^{*n},$$

then $\exp^*$ and $\ln^*$ are inverse one each other.

For simplicity, in what follows we introduce some notation: $\{\hat{x}\}_1^n$ denotes $\{\hat{x}_1, \ldots, \hat{x}_n\}$ and $\sigma^\tau(d\hat{x})_1^n$ denotes $\sigma^\tau(dx_1, ds_{x_1}) \ldots \sigma^\tau(dx_n, ds_{x_n})$ (analogous for $\nu_{z\sigma^\tau}(d\omega)_1^n$).

Next we prove some lemmas which will be useful later on.

Lemma 2.6 *Let* $F, \psi_1, \ldots, \psi_n$ *be* $\mathcal{B}(\Omega_{fin})$-*measurable functions. Then the following equality holds:*

$$\int_{\Omega_{fin}} F(\omega)(\psi_1 * \ldots * \psi_n)(\omega) \nu_{z\sigma^\tau}(d\omega) \qquad (2.12)$$

$$= \int_{\Omega_{fin}} \ldots \int_{\Omega_{fin}} F(\omega_1 \cup \ldots \cup \omega_n) \psi_1(\omega_1) \ldots \psi_n(\omega_2) \nu_{z\sigma^\tau}(d\omega)_1^n,$$

whenever all functions are positive or one side make sense for the modulus of the functions.

Proof. Let $F, \psi_1, \ldots \psi_p$ be as aforementioned. Then the definition of $\nu_{z\sigma^\tau}$ on the right hand side of (2.12) gives

$$\sum_{n_1, \ldots, n_p = 0}^{\infty} \frac{z^{n_1 + \ldots + n_p}}{n_1! \ldots n_p!} \int_{X^{n_1}} \int_{S^{n_1}} \ldots \int_{X^{n_p}} \int_{S^{n_p}} F(\{\hat{x}\}_1^{n_1 + \ldots + n_p})$$

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$$\times \psi_1(\{\hat{x}\}_1^{n_1}) \ldots \psi_p(\{\hat{x}\}_{n_1+\ldots+n_{p-1}+1}^{n_1+\ldots+n_p}) \sigma^\tau(d\hat{x})_1^{n_1+\ldots+n_p}$$

$$= \sum_{n=0}^\infty \frac{z^n}{n!} \sum_{n_1+\ldots+n_p=n} \frac{n!}{n_1! \ldots n_p!} \int_{X^n} \int_{S^n} F(\{\hat{x}\}_1^n)$$

$$\times \psi_1(\{\hat{x}\}_1^{n_1}) \ldots \psi_p(\{\hat{x}\}_{n_1+\ldots+n_{p-1}+1}^n) \sigma^\tau(dx, ds)_1^n.$$

Then interchanging the second sum with the integrals and using the definition of $\nu_{z\sigma\tau}$ we derive the desired result. ∎

Corollary 2.7 *For any $Y \in \mathcal{B}(X)$ and $\psi \in \mathcal{A}$ such that either ψ positive or $\psi \in L^1(\Omega_{Y,fin}, \nu_{z\sigma\tau})$ the following equality holds*

$$\int_{\Omega_{Y,fin}} (\exp^* \psi)(\omega) \nu_{z\sigma\tau}(d\omega) = \exp\left(\int_{\Omega_{Y,fin}} \psi(\omega) \nu_{z\sigma\tau}(d\omega)\right). \qquad (2.13)$$

Lemma 2.8 *Let $\psi \in \mathcal{A}$ and $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ be given such that $\Lambda' \subset \Lambda$, suppose that $\psi \in L^1(\Omega_\Lambda, \nu_{z\sigma\tau})$. Then the following equality holds*

$$\int_{\Omega_{\Lambda\setminus\Lambda'}} (\exp^* \psi)(\omega \cup \omega') \nu_{z\sigma\tau}(d\omega) \qquad (2.14)$$

$$= \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega) \nu_{z\sigma\tau}(d\omega)\right) \exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot \cup \omega)\nu_{z\sigma\tau}(d\omega)\right)(\omega'),$$

for $\nu_{z\sigma\tau}$-a.e. $\omega' \in \Omega_{\Lambda'}$.

The details of the proof are given in Appendix A.1.

Definition 2.9 *Let $\psi \in \mathcal{A}$. We define a "differential" operator D by setting for $\omega, \omega' \in \Omega_{fin}$*

$$(D_{\omega'}\psi)(\omega) := \psi(\omega \cup \omega'), \text{ if } \gamma_\omega \cap \gamma_{\omega'} = \emptyset, \qquad (2.15)$$

and $(D_{\omega'}\psi)(\omega) = 0$ otherwise.

Remark 2.10 *Let us mention that the operator D is related with the Poissonian gradient ∇^P (see e.g., [KSS97b] and [NV95]) by*

$$(\nabla^P \psi)(\omega, \hat{x}) = (D_{\{\hat{x}\}}\psi)(\omega) - \psi(\omega).$$

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Finally, we state some properties of the operator D, which can be easily checked using the Definition 2.9.

Proposition 2.11 *Let $\psi, \psi_1, \psi_2 \in \mathcal{A}$, $\omega \in \Omega_{fin}$, $\hat{x}, \hat{y} \in X \times S$, with $x \neq y$ and $x \notin \gamma_\omega$ then*

1. $D_{\{\hat{x}\}} D_{\{\hat{y}\}} = D_{\{\hat{y}\}} D_{\{\hat{x}\}}$.

2. $\left[D_{\{\hat{x}\}} (\psi_1 * \psi_2) \right] (\omega) = \left[(D_{\{\hat{x}\}} \psi_1) * \psi_2 + \psi_1 * (D_{\{\hat{x}\}} \psi_2) \right] (\omega)$.

3. $\left[D_{\{\hat{x}\}} \exp^* \psi \right] (\omega) = \left[(\exp^* \psi) * (D_{\{\hat{x}\}} \psi) \right] (\omega)$.

3   Marked Gibbs measures

In the previous section we introduced the probability measure π_σ^τ on $(\Omega, \mathcal{B}(\Omega))$, the so-called marked Poisson measure, cf. Subsection 2.2. Now we will describe a more wide class of probability measures on $(\Omega, \mathcal{B}(\Omega))$, namely, the *marked Gibbs measures*. In Subsection 3.2 we state various examples and the associated marked Gibbs measures will be considered in Subsection 5.4.

3.1   Specifications, Gibbs measures, and global conditions

A symmetric measurable function $\phi : (X \times S) \times (X \times S) \to \mathbb{R} \cup \{+\infty\}$ is called a *pair potential*. For a given pair potential we define the *energy functional* $E^\phi : \Omega_{fin} \to \mathbb{R} \cup \{+\infty\}$ by

$$E^\phi(\omega) := \sum_{\{\hat{x}, \hat{y}\} \subset \omega} \phi(\hat{x}, \hat{y}), \qquad (3.1)$$

with $E^\phi(\emptyset) := 0$.

Let $\omega \in \Omega_{fin}$ and $\zeta \in \Omega$ be given, then the *interaction energy* between ω and ζ is given by

$$W^\phi(\omega, \zeta) := \begin{cases} \displaystyle\sum_{\hat{x} \in \omega, \hat{y} \in \zeta} \phi(\hat{x}, \hat{y}), & \text{if } \displaystyle\sum_{\hat{x} \in \omega, \hat{y} \in \zeta} |\phi(\hat{x}, \hat{y})| < \infty \\ \\ +\infty, & \text{otherwise} \end{cases} . \qquad (3.2)$$

For any $\Lambda \in \mathcal{B}_c(X)$ the *conditional energy* $E_\Lambda^\phi : \Omega \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E_\Lambda^\Phi(\omega) = E^\phi(\omega_\Lambda) + W^\phi(\omega_\Lambda, \omega_{X \setminus \Lambda}).$$

Notice that the energy E^ϕ may be expressed for any $\omega, \omega' \in \Omega_{fin} \setminus \{\emptyset\}$ such that $\gamma_\omega \cap \gamma_{\omega'} = \emptyset$ as

$$E^\phi(\omega \cup \omega') = E^\phi(\omega) + E^\phi(\omega') + W(\omega, \omega'). \qquad (3.3)$$

Now we can define grand canonical marked Gibbs measures.

Definition 3.1 *For any $\Lambda \in \mathcal{B}_c(X)$ the marked specification $\Pi_\Lambda^{\sigma^\tau,\phi}$ is defined for any $\omega \in \Omega$, $F \in \mathcal{B}(\Omega)$ by (see [Pre76])*

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega,F) := \mathbb{1}_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi} < \infty\}}(\omega)[\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega)]^{-1} \int_{\Omega_\Lambda} \mathbb{1}_F(\omega_{X\setminus\Lambda} \cup \omega') \quad (3.4)$$

$$\times \exp[-\beta E_\Lambda^\phi(\omega_{X\setminus\Lambda} \cup \omega')]\nu_{z\sigma^\tau}(d\omega'), \quad (3.5)$$

where $\beta > 0$ is the inverse temperature. $\tilde{Z}_\Lambda^{\sigma^\tau,\phi}$ is called partition function:

$$\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega) := \int_{\Omega_\Lambda} \exp[-\beta E_\Lambda^\phi(\omega_{X\setminus\Lambda} \cup \omega')]\nu_{z\sigma^\tau}(d\omega'). \quad (3.6)$$

A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called a grand canonical marked Gibbs measure with interaction potential ϕ iff

$$\mu\Pi_\Lambda^{\sigma^\tau,\phi} = \mu, \text{ for all } \Lambda \in \mathcal{B}_c(X), \quad (3.7)$$

where for any $F \in \mathcal{B}(\Omega)$ the measure $\mu\Pi_\Lambda^{\sigma^\tau,\phi}$ is defined by

$$(\mu\Pi_\Lambda^{\sigma^\tau,\phi})(F) := \int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega,F)d\mu(\omega),$$

and (3.7) above are called Dobrushin-Landford-Ruelle (DLR) equations.
  Let $\mathcal{G}_{gc}(\sigma^\tau,\phi)$ denote the set of all such probability measures μ.

Remark 3.2 *1. It is well-known that $\{\Pi_\Lambda^{\sigma^\tau,\phi}\}_{\Lambda \in \mathcal{B}_c(X)}$ is a $\{\mathcal{B}_{X\setminus\Lambda}(\Gamma)\}_{\Lambda \in \mathcal{B}_c(X)}$-specification in the following sense (see e.g., [Föl75], [Pre76], [Pre79]), for all $\Lambda, \Lambda' \in \mathcal{B}_c(X)$.*

(S1) $\Pi_\Lambda^{\sigma^\tau,\phi}(\omega,\Omega) \in \{0,1\}$ for all $\omega \in \Omega$.

(S2) $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot,Y)$ is $\mathcal{B}_{X\setminus\Lambda}(\Omega)$-measurable for all $Y \in \mathcal{B}(\Omega)$.

(S3) $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y \cap Y') = \mathbb{1}_{Y'}\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot,Y)$ for all $Y \in \mathcal{B}(\Omega)$, $Y' \in \mathcal{B}_{X\setminus\Lambda}(\Omega)$.

(S4) $\Pi_{\Lambda'}^{\sigma^\tau,\phi} = \Pi_{\Lambda'}^{\sigma^\tau,\phi}\Pi_\Lambda^{\sigma^\tau,\phi}$ if $\Lambda \subset \Lambda'$. Here for any $\omega \in \Omega$, $Y \in \mathcal{B}(\Omega)$

$$(\Pi_{\Lambda'}^{\sigma^\tau,\phi}\Pi_\Lambda^{\sigma^\tau,\phi})(\omega,Y) := \int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega',Y)\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\omega,d\omega').$$

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2. *It can be easily shown that because of (2.8) for all $\Lambda \in \mathcal{B}_c(X)$, $\omega \in \Omega$, $F \in \mathcal{B}(\Omega)$*

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) = \mathbb{1}_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi} < \infty\}}(\omega)[\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega)]^{-1}\{\mathbb{1}_F(\omega_{X\setminus\Lambda})$$
$$+ \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{(\Lambda\times S)^n} \mathbb{1}_F(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n) \exp[-\beta E^\phi(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n)]\sigma^\tau(d\hat{x})_1^n\},$$

where

$$\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{(\Lambda\times S)^n} \exp[-\beta E^\phi(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n)]\sigma^\tau(d\hat{x})_1^n.$$

3. *From properties (S2) and (S3) a probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is a grand canonical Gibbs measure iff for all $\Lambda \in \mathcal{B}_c(X)$ and all $Y \in \mathcal{B}(\Omega)$*

$$\mathbb{E}_\mu[\mathbb{1}_Y|\mathcal{B}_{X\setminus\Lambda}(\Omega)] = \Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y) \; \mu - a.e.,$$

where for a sub-σ-algebra $\Sigma \subset \mathcal{B}(\Omega)$, $\mathbb{E}_\mu[\cdot|\Sigma]$ denotes the conditional expectation with respect to μ given Σ.

Furthermore, we want to state the notion of correlation functions.

Definition 3.3 *For any $m \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$ we define the m-point correlation function $\rho_\Lambda^{(m)} : \Omega_\Lambda^{(m)} \to \mathbb{R}$ (with empty boundary condition) by*

$$\rho_\Lambda^{(m)}(\{\hat{x}\}_1^m; \emptyset) := \frac{1}{\tilde{Z}_\Lambda^{z\sigma^\tau,\phi}(\emptyset)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(\Lambda\times S)^n} e^{-\beta E^\phi(\{\hat{x}\}_1^m \cup \{\hat{y}\}_1^n)} \sigma^{\tau\otimes n}(d\hat{y})_1^n.$$

We now formulate the conditions on the interaction which will be used in the next section.

(S) (Stability) There exists $B \geq 0$ such that

$$E^\phi(\omega) \geq -B|\omega|, \text{ for any } \omega \in \Omega_{fin}. \tag{3.8}$$

(I) (Integrability) We assume the following integrability condition:

$$C(\beta) := \operatorname*{ess\,sup}_{(y,t)\in X\times S} \int_X \int_S |e^{-\beta\phi((x,s),(y,t))} - 1|\tau(x, ds)\sigma(dx) < \infty. \tag{3.9}$$

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Only in Theorem 5.6 we also need the following notion

(F) (Finite range) There exists $R > 0$ such that

$$\phi((x,s),(y,t)) = 0, \quad \text{if } d(x,y) \geq R \tag{3.10}$$

where d denotes the Riemannian distance on X.

Remark 3.4   *1. In the case $X = \mathbb{R}^d$, $S = \{s\}$, σ Lebesgue measure on $\mathbb{R}^d$, $\tau(x, \cdot) = \delta_s$ Dirac-measure, and for translation invariant potential ϕ the above integrability condition (I) reduces to the standard integrability condition, see e.g., [Rue69].*

2. The stability condition (3.8) implies that for every $\omega \in \Omega_{fin}$ there is $\hat{x}_0 \in \omega$ such that

$$\sum_{\hat{x} \in \omega \setminus \{\hat{x}_0\}} \phi(\hat{x}_0, \hat{x}) \geq -2B, \tag{3.11}$$

in particular, ϕ is bounded below.

3.2   Examples

Below we give some examples which illustrate different kinds of marked specifications arising in models of statistical physics. These examples can be handled in our framework and we will give more details on the construction of the marked Gibbs measures corresponding to them in Subsection 5.4.

Example 3.5 *Let $X = \mathbb{R}^d$, $S = \mathbb{R}^l$ be given. As intensity measure σ we take the Lebesgue measure on $\mathbb{R}^d$ and the kernel τ is independent of the position and has support in a compact set.*

The potential ϕ is given by

$$\phi((x,s_x),(y,s_y)) := \Phi(|x - y|) + J(|x - y|)s_x s_y,$$

where Φ, J are measurable functions on $\mathbb{R}$, such that exist a $R, \varepsilon, C_1, C_2 > 0$ with $\Phi(r) \geq C_1 r^{-d}$ for all $r \leq R$ and $|\Phi| \leq C_2 r^{-d-\varepsilon}$ for all $r > R$. J is positive, decreasing with the distance and for some $a > 0$

$$\sum_{x \in \mathbb{Z}^d} J(|ax|) < \infty. \tag{3.12}$$

This model describes a ferromagnetic interaction in fluids, cf. [RZ98, Sect. I]. The authors showed the breaking of the discrete symmetry corresponding to the spin.

Example 3.6 *Let $X = \mathbb{R}^d$ be endowed with Lebesgue measure dx and $S = \mathbb{T}$ be the one dimensional torus with measure τ given by $\tau(x, d\theta_x) = \frac{1}{2\pi} d\theta_x$ (we parametrize the torus by $\theta_x \in [0, 2\pi)$). The potential is of the following form*

$$\phi((x, \theta_x), (y, \theta_y)) := \Phi(|x - y|) - J(|x - y|) \cos(\theta_x - \theta_y),$$

where Φ, J are measurable functions on $\mathbb{R}^d$, $J \geq 0$ which fullfil the following conditions: there exist $R, \varepsilon, C_1, C_2, C_3 > 0$ such that

1. *$\Phi(r) - |J(r)| \geq C_1 r^{-d}$ for all $r \leq R$.*

2. *$|\Phi(r)| \leq C_2 r^{-d-\varepsilon}$ for all $r > R$.*

3. *$|J(r)| \leq C_3 r^{-d-\varepsilon}$ for all $r > R$.*

This model describes a classical gas of planar rotators.

Example 3.7 *We consider as X the Euclidean space $\mathbb{R}^d$ with Lebesgue measure and the space of marks $S = \{1, \ldots, q\}$. The potential is given by*

$$\phi((x, s_x), (y, s_y)) := \varphi(x - y)(1 - \delta_{s_x, s_y}) + \psi(x - y),$$

where δ_{s_x, s_y} is the Kronecker symbol and $\varphi, \psi : \mathbb{R}^d \rightarrow]-\infty, \infty]$ are measurable functions. We assume that there exist $0 \leq r_1 \leq r_2$ such that

(A1) (repulsion of φ) $\varphi \geq 0$.

(A2) (finite range of φ) $\varphi(x) = 0$ when $|x| \geq r_2$.

(A3) (strong stability and regularity of ψ) either $\psi \geq 0$, or ψ is superstable and lower regular in the sense of [Rue70].

(A4) the positive part ψ_+ of ψ satisfies

$$\int_{\{x \mid |x| \geq r_1\}} \psi_+(x) dx < \infty.$$

This model is known as continuum Potts model, cf. [GH96].

Example 3.8 *Let $\mathcal{L}^\theta(\mathbb{R}^d)$ be the Banach space of all continuous functions $s : [0, \theta] \to \mathbb{R}^d$ with $s(0) = s(\theta)$ and $\theta = \frac{1}{k_B T}$, k_B denotes the Boltzmann constant and T the temperature. On $\mathcal{L}^\theta(\mathbb{R}^d)$ we consider the measure $W_{x,x}(ds)dx$, where dx denotes the Lebesgue measure on $\mathbb{R}^d$ and $W_{x,x}(ds)$ the conditional Wiener measure, which is concentrated on the trajectories starting and ending in $x \in \mathbb{R}^d$. In this framework the potential is of the form*

$$\phi(s_1, s_2) := \int_0^\theta V(s_1(t) - s_2(t))dt,$$

where $V \in L^1(\mathbb{R}^d)$ and satisfies

$$\sum_{i=1}^n \sum_{j=i+1}^n V(x_i - x_j) \geq -Bn, \ \forall x_1, \ldots x_n \in \mathbb{R}^d.$$

Our aim is to handle the loop space as a marked configuration space putting $X = \mathbb{R}^d$ equipped with the Lebesgue measure. It would be natural to consider as mark space at the point $x \in X$ the space $\mathcal{L}_x^\theta(X)$ of all loops starting and ending in x. In our setting we are forced to put $S = \mathcal{L}^\theta(X)$. A point in $s \in \mathcal{L}^\theta(X)$ is then interpreted as a pair $(s(0), s)$ and denoted by (x, s_x). The kernel τ is given by $\tau(x, ds_x) := W_{x,x}(ds_x)$. This implies, in particular, that the space $\mathcal{L}_x^\theta(X)$ has full $\tau(x, \cdot)$-measure. In Subsection 5.4 we will consider this in more details.

This model is related to the path space representation of the states in quantum statistical mechanics for Maxwell-Boltzmann statistics. A beautiful description of this connection for the standard density matrices is given in [Gin71]. Ginibre also considers the cases of the Bose-Einstein and Fermi-Dirac statistics. Ginibre does not use any concept of Euclidean Gibbs measure in his considerations, rather he introduce special version of correlations functions for which he constructed cluster expansion, etc. The concept of Euclidean Gibbs measures in quantum statistics was introduced in the paper [KLR$^+$97]. This example shows that it is natural to interpret such objects as marked Gibbs measures.

4 Cluster expansion

In this section we derive the cluster expansion of the Gibbs factor $e^{-\beta E^\phi(\omega)}$, see (4.6) below. Moreover we perform some estimates which will be used in Section 5 to prove the existence of the marked Gibbs measures, cf. Theorem 5.3.

4.1 Cluster decomposition property

Definition 4.1 *For any $\omega \in \Omega_{fin}$ we define the function k by*

$$k(\omega) := \ln^*(e^{-\beta E^\phi})(\omega), \tag{4.1}$$

or equivalently

$$(\exp^* k)(\omega) = e^{-\beta E^\phi(\omega)},$$

where E^ϕ is defined in (3.1). k is called Ursell function see e.g., [Rue69].

Proposition 4.2 *The partition function $\tilde{Z}_\Lambda^{\sigma^\tau, \phi}(\emptyset)$, $\Lambda \in \mathcal{B}_c(X)$ has the following representation*

$$\tilde{Z}_\Lambda^{\sigma^\tau, \phi}(\emptyset) = \exp\left(\int_{\Omega_\Lambda \setminus \{\emptyset\}} k(\omega)\nu_{z\sigma^\tau}(d\omega)\right), \tag{4.2}$$

if $k \in L^1(\Omega_\Lambda, \nu_{z\sigma^\tau})$, c.f. Corollary 4.14 below.

Proof. This result follows from the fact that $e^{-\beta E^\phi(\omega)} = (\exp^* k)(\omega)$ and Corollary 2.7. ■

Proposition 4.3 *The Ursell function k allows the following representation*

$$k(\omega) = \sum_{G \in \mathfrak{G}^c(\omega)} \prod_{\{\hat{x}, \hat{y}\} \in G} (e^{-\beta\phi(\hat{x}, \hat{y})} - 1), \ \omega \in \Omega_{fin} \setminus \{\emptyset\},$$

and $k(\emptyset) = 0$.

Proof. According to the definition of the energy E^ϕ (cf. (3.1)) we have

$$
\begin{aligned}
e^{-\beta E^\phi(\omega)} &= \prod_{\{\hat{x}, \hat{y}\} \subset \omega} e^{-\beta\phi(\hat{x}, \hat{y})} \\
&= \sum_{G \in \mathfrak{G}(\omega)} \prod_{\{\hat{x}, \hat{y}\} \in G} (e^{-\beta\phi(\hat{x}, \hat{y})} - 1). \tag{4.3}
\end{aligned}
$$

Recall that every graph G can be decomposed into the direct sum of its connected components, i.e., $G = \bigoplus_{i=1}^{n} G_i$, where G_i is a connected subgraph and $\{V(G_i)\}$ is a partition of ω, (cf. Subsection 2.3). This yields

$$e^{-\beta E^\phi(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n) \in \mathfrak{P}^n(\omega)} \prod_{l=1}^{n} \sum_{G_l \in \mathfrak{G}^c(\omega_l)} \prod_{\{\hat{x},\hat{y}\} \in G_l} (e^{-\beta\phi(\hat{x},\hat{y})} - 1). \quad (4.4)$$

Define $\tilde{k}$ for any $\omega \in \Omega_{fin}\backslash\{\emptyset\}$ by

$$\tilde{k}(\omega) := \sum_{G \in \mathfrak{G}^c(\omega)} \prod_{\{\hat{x},\hat{y}\} \in G} (e^{-\beta\phi(\hat{x},\hat{y})} - 1), \quad (4.5)$$

and $\tilde{k}(\emptyset) = 0$. The expression for $e^{-\beta E^\phi(\omega)}$ can be written as

$$e^{-\beta E^\phi(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n) \in \mathfrak{P}^n(\omega)} \tilde{k}(\omega_1)\ldots\tilde{k}(\omega_n),$$

which is nothing but $\exp^* \tilde{k}$. Thus $\tilde{k} = k$ and the result follows. ∎

Remark 4.4 *The equality*

$$e^{-\beta E^\phi(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n) \in \mathfrak{P}^n(\omega)} k(\omega_1)\ldots k(\omega_n), \quad (4.6)$$

is known as the cluster decomposition of the Gibbs factor $e^{-\beta E^\phi(\omega)}$. We also notice that the function k is $\mathcal{B}(\Omega_{fin})$-measurable.

Proposition 4.3 allows the following representation for the specification

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, F) = \frac{1}{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\emptyset)} \int_{\Omega_\Lambda} \mathbb{1}_F(\omega) \exp^*(k)(\omega) \nu_{z\sigma^\tau}(d\omega), \quad F \in \mathcal{B}(\Omega)$$

where $\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\emptyset)$ is given by (4.2).

The next proposition gives a relation between correlations functions (see Definition 3.3) and Ursell functions.

Proposition 4.5 *Let $\omega \in \Omega_\Lambda^{(m)}$, $\Lambda \in \mathcal{B}_c(X)$ be given. Define*

$$\bar{k}(\omega, \omega') := (\exp^*(-k) * D_\omega e^{-\beta E^\phi})(\omega'),$$

for $\gamma_\omega \cap \gamma_{\omega'} \neq \emptyset$. If $k \in L^1(\Omega_\Lambda, \nu_{z\sigma\tau})$ (cf. (4.20) below), then

$$\rho_\Lambda^{(m)}(\omega; \emptyset) = \int_{\Omega_\Lambda} \bar{k}(\omega, \omega') \nu_{z\sigma\tau}(d\omega').$$

Proof. It follows from the definition of $\rho_\Lambda^{(m)}$ and (4.2) that

$$\rho_\Lambda^{(m)}(\omega; \emptyset) = \exp\left(-\int_{\Omega_\Lambda} k(\omega') \nu_{z\sigma\tau}(d\omega')\right) \int_{\Omega_\Lambda} (D_\omega e^{-\beta E^\phi})(\omega') \nu_{z\sigma\tau}(d\omega').$$

Now taking into account Lemma 2.6 with $F = 1$ and Corollary 2.7 the above equality gives

$$\int_{\Omega_\Lambda} (\exp^*(-k) * D_\omega e^{-\beta E^\phi})(\omega') \nu_{z\sigma\tau}(d\omega').$$

∎

Let us now derive an explicit relation between k and $\bar{k}$.

Lemma 4.6 *Let $\omega \in \Omega_{fin}\backslash\{\emptyset\}$ be given and suppose that $\hat{x} \in \omega$. Then k and $\bar{k}$ are related by the equation*

$$\bar{k}(\{\hat{x}\}, \omega\backslash\{\hat{x}\}) = k(\omega). \tag{4.7}$$

Proof. By definition of $\bar{k}$ and Proposition 2.11-3 we have

$$
\begin{aligned}
\bar{k}(\{\hat{x}\}, \omega\backslash\{\hat{x}\}) &= (\exp^*(-k) * D_{\{\hat{x}\}} \exp^*(k))(\omega\backslash\{\hat{x}\}) \\
&= (\exp^*(-k) * \exp^*(k) * D_{\{\hat{x}\}} k)(\omega\backslash\{\hat{x}\}) \\
&= (D_{\{\hat{x}\}} k)(\omega\backslash\{\hat{x}\}) \\
&= k(\omega).
\end{aligned}
$$

Hence the result is proved. ∎

Remark 4.7 *Notice that for any $\omega, \omega' \in \Omega_{fin}\backslash\{\emptyset\}$ with $\gamma_\omega \cap \gamma_{\omega'} = \emptyset$, $\bar{k}$ may be written as*

$$\bar{k}(\omega, \omega') = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{(\omega_1,\ldots,\omega_l) \in \mathfrak{P}^l(\omega)} \sum_{(\omega_1',\ldots,\omega_l') \in \mathfrak{P}_\emptyset^l(\omega')} k(\omega_1 \cup \omega_1') \ldots k(\omega_l \cup \omega_l'),$$

which is the same as the sum of all graphs where each connected component has at least one vertex in the points of ω, cf. [MM91, Chap. 4].

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Our aim now is to find a bound for $\bar{k}$. First we derive an equation of recursive type for $\bar{k}$. Let $\omega, \zeta \in \Omega_{fin}$ be such that $\gamma_\omega \cap \gamma_\zeta = \emptyset$ and $\hat{x}_0$ an arbitrary fixed element in ω. To this end we look again into the definition of $\bar{k}$ which can be expressed as

$$\bar{k}(\omega, \zeta) = \sum_{\omega' \subset \zeta} (\exp^*(-k))(\zeta \backslash \omega') D_{\{\hat{x}_0\}} e^{-\beta E^\phi(\omega \backslash \{\hat{x}_0\} \cup \omega')}. \qquad (4.8)$$

Having in mind the decomposition (3.3) for E^ϕ one obtains (taking into account (3.2))

$$
\begin{aligned}
e^{-\beta W(\{\hat{x}_0\}, \omega')} &= \prod_{\hat{x} \in \omega'} e^{-\beta \phi(\hat{x}_0, \hat{x})} \\
&= \sum_{\omega'' \subset \omega'} \prod_{\hat{x} \in \omega''} (e^{-\beta \phi(\hat{x}_0, \hat{x})} - 1) \\
&= \sum_{\omega'' \subset \omega'} k_{\omega''}(\hat{x}_0),
\end{aligned}
$$

where

$$k_{\omega''}(\hat{x}_0) := \prod_{\hat{x} \in \omega''} (e^{-\beta \phi(\hat{x}_0, \hat{x})} - 1).$$

According to equation (4.8) $\bar{k}$ can be formulated as

$$
\begin{aligned}
\bar{k}(\omega, \zeta) = \; & e^{-\beta W(\{\hat{x}_0\}, \omega \backslash \{\hat{x}_0\})} \sum_{\omega' \subset \zeta} (\exp^*(-k))(\zeta \backslash \omega') \\
& \times \sum_{\omega'' \subset \omega'} k_{\omega''}(\hat{x}_0) e^{-\beta E^\phi(\omega \backslash \{\hat{x}_0\} \cup \omega')}.
\end{aligned}
$$

Interchanging the two sums the right hand side becomes

$$e^{-\beta W(\{\hat{x}_0\}, \omega \backslash \{\hat{x}_0\})} \sum_{\omega'' \subset \zeta} k_{\omega''}(\hat{x}_0) \sum_{\substack{\omega' \\ \omega'' \subset \omega' \subset \xi}} (\exp^*(-k))(\zeta \backslash \omega') e^{-\beta E^\phi(\omega \backslash \{\hat{x}_0\} \cup \omega')},$$

and the second sum may be rewritten as

$$
\begin{aligned}
& \sum_{\breve{\omega} \subset \zeta \backslash \omega''} (\exp^*(-k))(\zeta \backslash (\breve{\omega} \cup \omega'')) e^{-\beta E^\phi(\omega \backslash \{\hat{x}_0\} \cup \breve{\omega} \cup \omega'')} \\
= \; & \sum_{\breve{\omega} \subset \zeta \backslash \omega''} (\exp^*(-k))((\zeta \backslash \omega'') \backslash \breve{\omega}) (D_{\omega \backslash \{\hat{x}_0\} \cup \omega''} e^{-\beta E^\phi})(\breve{\omega}).
\end{aligned}
$$

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Therefore $\bar{k}$ can be expressed as

$$
\begin{aligned}
\bar{k}(\omega, \zeta) \;=\; & e^{-\beta W(\{\hat{x}_0\}, \omega \setminus \{\hat{x}_0\})} \sum_{\omega'' \subset \zeta} k_{\omega''}(\hat{x}_0) \\
& \times (\exp^*(-k) * (D_{\omega \setminus \{\hat{x}_0\} \cup \omega''} e^{-\beta E^\phi}))(\zeta \setminus \omega'').
\end{aligned}
$$

Finally, taking into account the definition of $\bar{k}$ we arrive at

$$
\bar{k}(\omega, \zeta) = e^{-\beta W(\{\hat{x}_0\}, \omega \setminus \{\hat{x}_0\})} \sum_{\omega' \subset \zeta} k_{\omega'}(\hat{x}_0) \bar{k}(\omega \setminus \{\hat{x}_0\} \cup \omega', \zeta \setminus \omega'). \tag{4.9}
$$

According to the definition of $\bar{k}$ we have for the case $\omega = \emptyset$ that $\bar{k}(\emptyset, \zeta) = 1^*(\zeta)$, where 1^* is defined by (2.9).

4.2 Convergence of cluster expansion

We want to derive now a bound for $|\bar{k}(\omega, \zeta)|$ which will be used later on in the main estimation in this section (cf. 4.17). The idea is to define a function Q dominating $\bar{k}$ which fulfills an equation similar to (4.9) which can be solved, see Proposition 4.10 below.

Let us choose a mapping $I : \Omega_{fin} \to X \times S$, $\tilde{\omega} \mapsto I(\tilde{\omega}) \in \tilde{\omega}$ such that the following equation is fulfilled

$$
\sum_{\hat{x} \in \tilde{\omega} \setminus I(\tilde{\omega})} \phi(\hat{x}, I(\tilde{\omega})) > -2B. \tag{4.10}
$$

Such a mapping exists by the stability condition, see (3.11).

Of course given I and $\bar{k}$, (4.9) implies

$$
\bar{k}(\omega, \zeta) = \exp\left(-\beta \sum_{\hat{x} \in \omega \setminus I(\omega)} \phi(\hat{x}, I(\omega)) \right) \sum_{\omega' \subset \zeta} k_{\omega'}(I(\omega)) \bar{k}(\omega \setminus I(\omega) \cup \omega', \zeta \setminus \omega').
$$

$$\tag{4.11}$$

Now we can start defining Q_I inductively. For $\omega = \emptyset$ we define

$$
Q(\emptyset, \zeta) := 1^*(\zeta), \tag{4.12}
$$

and by definition of $\bar{k}(\emptyset, \zeta)$ we have $|\bar{k}(\emptyset, \zeta)| \leq Q_I(\emptyset, \zeta)$.

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Assume we already have defined Q_I for all $\omega, \zeta \in \Omega_{fin}$, $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, and $|\omega \cup \zeta| = n$ such that

$$|\bar{k}(\omega, \zeta)| \leq Q_I(\omega, \zeta)$$

is satisfied. Then if $\omega, \zeta \in \Omega_{fin}$, are such that $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, and $|\omega \cup \zeta| = n + 1$, we have, applying (4.10) and (4.11)

$$|\bar{k}(\omega, \zeta)| \leq e^{2\beta B} \sum_{\omega' \subset \zeta} |k_{\omega'}(I(\omega))| Q_I(\omega \backslash I(\omega) \cup \omega', \zeta \backslash \omega').$$

Thus we define

$$Q_I(\omega, \zeta) := e^{2\beta B} \sum_{\omega' \subset \zeta} |k_{\omega'}(I(\omega))| Q_I(\omega \backslash I(\omega) \cup \omega', \zeta \backslash \omega'), \qquad (4.13)$$

Remark 4.8 *The solutions of the equations (4.9) and (4.13) exist and are unique. Let us explain this in more details. On the one hand the equations are linear, on the other hand the value at the point (ω, ζ) for $|\omega| + |\zeta| = n$ only depends on the values at points $(\tilde{\omega}, \tilde{\zeta})$ with $|\tilde{\omega}| + |\tilde{\zeta}| = n - 1$, thus the corresponding matrix is an strict upper triangle matrix for a suitable choice of the bases.*

Hence we have the following proposition.

Proposition 4.9 *For I and $\bar{k}$ as above, there exists a unique solution Q_I of the equation (4.13) with the initial condition (4.12) which dominates $\bar{k}$, i.e., for any $\omega, \zeta \in \Omega_{fin}$, such that $\gamma_\omega \cap \gamma_\zeta = \emptyset$ we have $|\bar{k}(\omega, \zeta)| \leq Q_I(\omega, \zeta)$.*

The next proposition gives a solution for the equation (4.13) which does not depend on the choice of I.

Proposition 4.10 *Let $\omega, \zeta \in \Omega_{fin}$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$. The solution of (4.13) for $\omega = \{\hat{x}_1, \ldots, \hat{x}_l\}$, $l \geq 1$ has the form*

$$Q(\{\hat{x}_1, \ldots, \hat{x}_l\}, \zeta) = \sum_{(\omega_1, \ldots, \omega_l) \in \mathfrak{P}_\emptyset^l(\zeta)} Q(\{\hat{x}_1\}, \omega_1) \cdots Q(\{\hat{x}_l\}, \omega_l), \qquad (4.14)$$

where

$$Q(\{\hat{x}\}, \zeta) := (e^{2\beta B})^{|\zeta|+1} \sum_{T \in \mathfrak{T}(\{\hat{x}\} \cup \zeta)} \prod_{\{\hat{y}, \hat{y}'\} \in T} |e^{-\beta\phi(\hat{y}, \hat{y}')} - 1|, \qquad (4.15)$$

for $\zeta \neq \emptyset$ and $Q(\{\hat{x}\}, \emptyset) := e^{2\beta B}$. In the case $\omega = \emptyset$ we define $Q(\emptyset, \zeta)$ as in (4.12).

The proof of this proposition is notationally quite involved because of the "reordering" of graphs, therefore we give the details in Appendix A.2.

As a result we have the following proposition.

Proposition 4.11 *For any $\omega, \zeta \in \Omega_{fin}$ such that $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, and $\omega = \{\hat{x}_1, \ldots, \hat{x}_l\}$, $l \geq 1$ we have*

$$
\begin{aligned}
|\bar{k}(\omega, \zeta)| &\leq Q(\{\hat{x}_1, \ldots, \hat{x}_l\}, \zeta) \\
&= \sum_{(\zeta_1, \ldots, \zeta_l) \in \mathcal{P}_\emptyset^l(\zeta)} Q(\{\hat{x}_1\}, \zeta_1) \cdots Q(\{\hat{x}_l\}, \zeta_l),
\end{aligned}
$$

and

$$
|k(\omega)| \leq e^{2\beta B |\omega|} \sum_{T \in \mathfrak{T}(\omega)} \prod_{\{\hat{x}, \hat{x}'\} \in T} |e^{-\beta \phi(\hat{x}, \hat{x}')} - 1|. \tag{4.16}
$$

Proof. The first part follows from the previous proposition and the second part follows from the relation between k and $\bar{k}$ (cf. (4.7)) and (4.15). ∎

Using the fact that the sum in the function Q is only over trees we can give also estimates for integrals.

Lemma 4.12 *For every $\hat{x} \in X \times S$, $Y \in \mathcal{B}(X)$, and $n \geq 1$ we have*

$$
\int_{(Y \times S)^n} Q(\{\hat{x}\}, \{\hat{y}\}_1^n) \sigma^\tau (d\hat{y})_1^n \tag{4.17}
$$

$$
\leq e^{2\beta B(n+1)} C(\beta)^{n-1} (n+1)^{n-1} \int_{Y \times S} |e^{-\beta \phi(\hat{x}, \hat{y})} - 1| \sigma^\tau (d\hat{y}).
$$

We refer to the Appendix A.3 for the proof of this lemma.

Proposition 4.13 *Let $\Lambda \in \mathcal{B}_c(\Omega)$ be given. Then for any z such that*

$$
|z| < \frac{1}{2e} (e^{2\beta B} C(\beta))^{-1},
$$

where $C(\beta)$ is given by integrability condition (3.9), we have

$$
\int_{\Omega_\Lambda \setminus \{\emptyset\}} \int_{\Omega_{fin}} |k(\omega \cup \omega')| \nu_{z\sigma^\tau}(d\omega) \nu_{z\sigma^\tau}(d\omega') < \infty. \tag{4.18}
$$

Proof: Using the definition of $\nu_{z\sigma\tau}$ and the relation between k and $\overline{k}$ (cf. (4.7)) we may write (4.18) as

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{z^{n+m}}{n!m!}\int_{(\Lambda\times S)^n}\int_{(X\times S)^m}|\overline{k}(\{x_n\},\{\hat{x}\}_1^{n-1}\cup\{\hat{y}\}_1^m)|\sigma^{\tau}(d\hat{x})_1^n\sigma^{\tau}(d\hat{y})_1^m.$$

According to Proposition 4.11 and Lemma 4.12 we can bound the above term by

$$\leq\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{z^{n+m}}{n!m!}e^{2\beta B(n+m)}C(\beta)^{n+m-2}(n+m)^{n+m-2}C(\beta)\int_{(\Lambda\times S)}\sigma^{\tau}(d\hat{x}_n).\tag{4.19}$$

Using the fact that $(n+m)^{n+m-2}\leq e^{m+n}(m+n)!$ (c.f. Remark 2.5.1.) we estimate (4.19) by

$$\frac{\int_{\Lambda}\tau(x,S)\sigma(dx)}{C(\beta)}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!}(zeC(\beta)e^{2\beta B})^{m+n}$$

$$=\quad\frac{\int_{\Lambda}\tau(x,S)\sigma(dx)}{C(\beta)}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}\frac{l!}{m!(l-m)!}\right)(zeC(\beta)e^{2\beta B})^l$$

$$\leq\quad\frac{\int_{\Lambda}\tau(x,S)\sigma(dx)}{C(\beta)}\sum_{l=0}^{\infty}(2zeC(\beta)e^{2\beta B})^l,$$

from which the result follows. ∎

As a consequence of the last proposition and Fubini's theorem we have the following corollary.

Corollary 4.14 *For any $\Lambda\in\mathcal{B}_c(X)$ we have (notice that $k(\emptyset)=0$, see Proposition 4.3)*

$$\int_{\Omega_{\Lambda}}|k(\omega)|\nu_{z\sigma\tau}(d\omega)<\infty,\tag{4.20}$$

and for $\nu_{z\sigma\tau}$-a.a. $\omega'\in\Omega_{fin}\backslash\{\emptyset\}$

$$\int_{\Omega_{fin}}|k(\omega\cup\omega')|\nu_{z\sigma\tau}(d\omega)<\infty.\tag{4.21}$$

5 Construction of marked Gibbs measure

5.1 Limiting measures from cluster expansion

Below we construct the marked measure μ on Ω as a limiting measure of the specification $\Pi_\Lambda^{\sigma^\tau,\phi}$ for the empty boundary condition in the weak local sense (cf. Theorem 5.3). Some sets of full μ-measure are considered (cf. Proposition 5.4). For the case that the potential ϕ has finite range we also give an easy proof that the resulting limiting measure satisfies the DLR equations, cf. Theorem 5.6. All results extend to separable standard Borel spaces. This is explained in some details in Subsection 5.3. Finally, we show in Subsection 5.4 how to apply the abstract results to the examples given in Subsection 3.2.

Lemma 5.1 *For any $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ such that $\Lambda' \subset \Lambda$ and $F \in \mathcal{B}(\Omega_{\Lambda'})$ the specification $\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, p_{\Lambda'}^{-1}(F))$ has the following representation*

$$
\begin{aligned}
\Pi_\Lambda^{\Lambda'}(\emptyset, F) &:= \Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, p_{\Lambda'}^{-1}(F)) \\
&= \frac{1}{\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)} \int_{\Omega_{\Lambda'}} \mathbb{1}_F(\omega) \exp^* \left(\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot) \int_{\Omega_{\Lambda\setminus\Lambda'}} k(\omega' \cup \cdot) \nu_{z\sigma^\tau}(d\omega') \right)(\omega) \nu_{z\sigma^\tau}(d\omega),
\end{aligned}
$$

where

$$
\tilde{Z}_\Lambda^{\Lambda'}(\emptyset) := \exp\left(\int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} \int_{\Omega_{\Lambda\setminus\Lambda'}} k(\omega \cup \omega') \nu_{z\sigma^\tau}(d\omega') \nu_{z\sigma^\tau}(d\omega) \right).
$$

Proof. This is a result of the following direct calculation

$$
\begin{aligned}
\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, p_{\Lambda'}^{-1}(F)) &= [\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\emptyset)]^{-1} \int_{\Omega_\Lambda} \mathbb{1}_{p_{\Lambda'}^{-1}(F)}(\omega') \exp[-\beta E^\phi(\omega')] \nu_{z\sigma^\tau}(d\omega') \\
&= [\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\emptyset)]^{-1} \int_{\Omega_{\Lambda'}} \mathbb{1}_F(\omega) \\
&\quad \times \int_{\Omega_{\Lambda\setminus\Lambda'}} \exp[-\beta E^\phi(\omega \cup \omega')] \nu_{z\sigma^\tau}(d\omega') \nu_{z\sigma^\tau}(d\omega).
\end{aligned}
$$

Because $k \in L^1(\Omega_\Lambda, \nu_{z\sigma^\tau})$ (cf. Corollary 4.14), using Lemma 2.8 the inner integral can be rewritten as follows

$$
\int_{\Omega_{\Lambda\setminus\Lambda'}} \exp[-\beta E^\phi(\omega \cup \omega')] \nu_{z\sigma^\tau}(d\omega') = \int_{\Omega_{\Lambda\setminus\Lambda'}} (\exp^* k)(\omega \cup \omega') \nu_{z\sigma^\tau}(d\omega')
$$

$$= \exp\left(\int_{\Omega_{\Lambda\backslash\Lambda'}} k(\omega')\nu_{z\sigma\tau}(d\omega')\right) \exp^*\left(\mathbb{1}_{\Omega_{fin}\backslash\{\emptyset\}}(\cdot)\int_{\Omega_{\Lambda\backslash\Lambda'}} k(\omega'\cup\cdot)\nu_{z\sigma\tau}(d\omega')\right)(\omega).$$

∎

Proposition 5.2 *Let* $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ *be such that* $\Lambda' \subset \Lambda$.

1. *Let* $k_\Lambda^{\Lambda'}$ *be defined by*

$$k_\Lambda^{\Lambda'}(\omega) := \mathbb{1}_{\Omega_{fin}\backslash\{\emptyset\}}(\omega)\int_{\Omega_{\Lambda\backslash\Lambda'}} k(\omega\cup\omega')\nu_{z\sigma\tau}(d\omega'). \qquad (5.1)$$

Then for $\nu_{z\sigma\tau}$-*a.a.* $\omega \in \Omega_{fin}\backslash\{\emptyset\}$ *we have* $\lim_{\Lambda\nearrow X} k_\Lambda^{\Lambda'}(\omega) = k^{\Lambda'}(\omega)$, *where*

$$k^{\Lambda'}(\omega) = \mathbb{1}_{\Omega_{fin}\backslash\{\emptyset\}}(\omega)\int_{\Omega_{X\backslash\Lambda',fin}} k(\omega\cup\omega')\nu_{z\sigma\tau}(d\omega'). \qquad (5.2)$$

2. *We have also that* $\lim_{\Lambda\nearrow X} \tilde{Z}_\Lambda^{\Lambda'}(\emptyset) = \tilde{Z}^{\Lambda'}(\emptyset)$, *where*

$$\tilde{Z}^{\Lambda'}(\emptyset) = \exp\left(\int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} k^{\Lambda'}(\omega)\nu_{z\sigma\tau}(d\omega)\right) > 0. \qquad (5.3)$$

Proof. 1. We would like to estimate the following quantity

$$\left| k_\Lambda^{\Lambda'}(\omega) - \int_{\Omega_{X\backslash\Lambda',fin}} k(\omega\cup\omega')\nu_{z\sigma\tau}(d\omega')\right|.$$

According to the definition of $k_\Lambda^{\Lambda'}$ the above quantity is estimated by

$$\int_{\Omega_{X\backslash\Lambda',fin}\backslash\Omega_{\Lambda\backslash\Lambda'}} |k(\omega\cup\omega')|\nu_{z\sigma\tau}(d\omega'). \qquad (5.4)$$

Now let $\{\Lambda_n | n \in \mathbb{N}\}$ be a sequence of increasing volumes such that $\Lambda_n \nearrow X$. Then $\Omega_{\Lambda_n\backslash\Lambda'} \nearrow \Omega_{X\backslash\Lambda',fin}$. On the other hand (4.21) guarantees that there exists a $\nu_{z\sigma\tau}$-null set $N \in \mathcal{B}(\Omega_{fin})$ such that for all $\omega \in \Omega_{fin}\backslash(N\cup\{\emptyset\})$

$$\int_{\Omega_{fin}} |k(\omega\cup\omega')|\nu_{z\sigma\tau}(d\omega') < \infty.$$

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Therefore by Lebesgue's dominated convergence theorem it follows that (5.4) goes to zero for all fixed $\omega \in \Omega_{fin} \backslash (N \cup \{\emptyset\})$. Hence part 1 is proved.

To prove part 2 we note that

$$|k_\Lambda^{\Lambda'}(\omega)| \le \int_{\Omega_{fin}} |k(\omega \cup \omega')| \nu_{z\sigma\tau}(d\omega'),$$

and thus

$$\int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} |k_\Lambda^{\Lambda'}(\omega)| \nu_{z\sigma\tau}(d\omega) \le \int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} \int_{\Omega_{fin}} |k(\omega \cup \omega')| \nu_{z\sigma\tau}(d\omega') \nu_{z\sigma\tau}(d\omega) < \infty,$$

(5.5)

because of (4.18). This implies that

$$\lim_{\Lambda \nearrow X} \int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} k_\Lambda^{\Lambda'}(\omega) \nu_{z\sigma\tau}(d\omega) = \int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} \int_{\Omega_{X\backslash\Lambda',fin}} k(\omega \cup \omega') \nu_{z\sigma\tau}(d\omega') \nu_{z\sigma\tau}(d\omega),$$

and, of course, taking exponential, and having in mind the form of $k^{\Lambda'}$ in (5.2), the desired result (5.3) follows. ∎

Theorem 5.3 *For any z such that $|z| < \frac{1}{2e}(e^{2\beta B} C(\beta))^{-1}$, where $C(\beta)$ is given by integrability condition (3.9), the specification $\Pi_\Lambda^{\sigma^\tau, \phi}(\emptyset, d\omega)$ converges in the weak local sense to a measure μ, i.e., for any bounded $\mathcal{B}_{\Lambda'}(\Omega)$-measurable function F we have*

$$\int_\Omega F(\omega) \Pi_\Lambda^{\sigma^\tau, \phi}(\emptyset, d\omega) \to \frac{1}{\tilde{Z}^{\Lambda'}(\emptyset)} \int_{\Omega_{\Lambda'}} F(\omega)(\exp^* k^{\Lambda'})(\omega) \nu_{z\sigma\tau}(d\omega),$$

and thus

$$\mu^{\Lambda'}(d\omega) = \frac{1}{\tilde{Z}^{\Lambda'}(\emptyset)}(\exp^* k^{\Lambda'})(\omega) \nu_{z\sigma\tau}(d\omega). \qquad (5.6)$$

Proof. Let F be as stated above, then Lemma 5.1 implies

$$\int_\Omega F(\omega) \Pi_\Lambda^{\sigma^\tau, \phi}(\emptyset, d\omega) = \frac{1}{\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)} \int_{\Omega_{\Lambda'}} f(\omega) \sum_{n=0}^\infty \frac{1}{n!} \sum_{(\omega_1, \ldots, \omega_n) \in \mathfrak{P}^n(\omega)} \prod_{i=1}^n k_\Lambda^{\Lambda'}(\omega_i) \nu_{z\sigma\tau}(d\omega),$$

where $F = f \circ p_{\Lambda'}$. According to Proposition 5.2 we know already that $\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)$ converges to $\tilde{Z}^{\Lambda'}(\emptyset)$. In order to use the Lebesgue dominated convergence

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theorem one should estimate the integrand by a function which is integrable and independent of Λ. An appropriate bound is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)} \prod_{i=1}^{n} \int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega_i)|\nu_{z\sigma^\tau}(d\zeta)$$

$$= \exp^*\left(\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\cdot)|\nu_{z\sigma^\tau}(d\zeta)\right)(\omega).$$

Moreover, the integral of the bound is given by

$$\int_{\Omega_{\Lambda'}} \exp^*\left(\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\cdot)|\nu_{z\sigma^\tau}(d\zeta)\right)(\omega)\nu_{z\sigma^\tau}(d\omega)$$

$$= \exp\left(\int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} \int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega)|\nu_{z\sigma^\tau}(d\zeta)\nu_{z\sigma^\tau}(d\omega)\right) < \infty,$$

because of Corollary 2.7 and (4.18). Therefore we have the desired result

$$\lim_{\Lambda\nearrow X} \int_{\Omega_{\Lambda'}} F(\omega)\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset,d\omega) = \int_{\Omega_{\Lambda'}} F(\omega)\frac{1}{Z^{\Lambda'}(\emptyset)}(\exp^* k^{\Lambda'})(\omega)\nu_{z\sigma^\tau}(d\omega).$$

$\blacksquare$

The measure from Theorem 5.3 is not concentrated in all Ω, indeed we have the following:

Proposition 5.4 *Let A be a $\mathcal{B}(X\times S)$-measurable set such that $\sigma^\tau(A) = 0$, then the set of marked configurations not touching A, i.e.,*

$$\tilde{\Omega} = \{\omega = (\gamma,s)\in\Omega|(x,s_x)\in A^c, \forall x\in\gamma\},$$

has full μ-measure.

Proof: Let us prove that $\mu(\tilde{\Omega}^c) = 0$. To this end we write $\tilde{\Omega}^c$ as

$$\tilde{\Omega}^c = \{\omega = (\gamma,s)\in\Omega|(x,s_x)\in A, \text{ for some } x\in\gamma\}$$

$$= \bigcup_{n\in\mathbb{N}} p_{\Lambda_n}^{-1}(\{\omega = (\gamma,s)\in\Omega_{\Lambda_n}|(x,s_x)\in A, \text{ for some } x\in\gamma_{\Lambda_n}\}).$$

Therefore

$$\mu(\tilde{\Omega}^c) \leq \sum_{n=1}^{\infty} \mu^{\Lambda_n}(\{\omega = (\gamma, s) \in \Omega_{\Lambda_n} | (x, s_x) \in A, \text{ for some } x \in \gamma_{\Lambda_n}\}).$$

Since $\mu^{\Lambda_n} \ll \nu_{z\sigma\tau}$ (cf.(5.6)), it is enough to prove that

$$\nu_{z\sigma\tau}(\{\omega = (\gamma, s) \in \Omega | (x, s_x) \in A, \text{ for some } x \in \gamma_{\Lambda_n}\}) = 0.$$

According to the definition of $\nu_{z\sigma\tau}$ (cf. (2.8)) the left hand side of the above equality yields

$$\sum_{m=0}^{\infty} \frac{z^m}{m!} \sigma_m^{\tau}(\{(\hat{x}_1, \ldots, \hat{x}_m) \in (\Lambda_n \times S)^m / S_m | \hat{x}_i \in A \text{ for some } i\})$$

$$\leq \sum_{m=0}^{\infty} \frac{z^m m}{m!} (\sigma^{\tau}(\Lambda_n \times S))^{m-1} \sigma^{\tau}(A),$$

which is zero since $\sigma^{\tau}(A) = 0$. ∎

Remark 5.5 *Since the projections of tempered Gibbs measures at arbitrary temperature and fugacity are absolutely continuous with respect to the Lebesgue Poisson measure (cf. [Rue69] and [KK98]) the above considerations extends also to all Gibbs measures.*

5.2 Identification with Gibbs measures

If we additionally assume finite range of the potential, then it is a direct consequence that the limit measure from Theorem 5.3 verifies the DLR equations. Without this additional assumption a more detailed consideration is necessary, see. [KK98] and [Kun98].

Theorem 5.6 *For any finite range potential ϕ (c.f. 3.10) the measure μ from Theorem 5.3 fulfils the DLR equations.*

Proof. Let $\Lambda \in \mathcal{B}_c(X)$ be given. Then there exists a $\tilde{\Lambda} \in \mathcal{B}_c(X)$ such that $\Lambda \subset \tilde{\Lambda}$ and $\phi((x, s_x), (y, s_y)) = 0$ if $x \in \Lambda$ and $y \in \tilde{\Lambda}^c$. Whence the interaction energy is

$$W(\omega_\Lambda, \omega_{X \setminus \Lambda}) = \sum_{\hat{x} \in \omega_\Lambda} \sum_{\hat{y} \in \omega_{X \setminus \Lambda}} \phi(\hat{x}, \hat{y}) = \sum_{\hat{x} \in \omega_\Lambda} \sum_{\hat{y} \in \omega_{\tilde{\Lambda} \setminus \Lambda}} \phi(\hat{x}, \hat{y}) = W(\omega_\Lambda, \omega_{\tilde{\Lambda} \setminus \Lambda}),$$

and the sums are finite. Let F be a "locally" measurable set, i.e., there exists $\widehat{\Lambda} \in \mathcal{B}_c(X)$ with $F \in \mathcal{B}_{\widehat{\Lambda}}(\Omega)$, then

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) = \frac{\mathbb{1}_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi} < \infty\}}(\omega_{\tilde{\Lambda}\backslash\Lambda})}{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega_{\tilde{\Lambda}\backslash\Lambda})} \int_{\Omega_\Lambda} \mathbb{1}_F(\zeta \cup \omega_{X\backslash\Lambda}) e^{-\beta E_\Lambda^\phi(\zeta \cup \omega_{\tilde{\Lambda}\backslash\Lambda})} \nu_{z\sigma^\tau}(d\zeta),$$

which implies that $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, F)$ is a bounded $\mathcal{B}_{\tilde{\Lambda}\cup\widehat{\Lambda}}(\Omega)$-measurable function. Additionally, we have for all $\Lambda' \in \mathcal{B}_c(X)$ with $\Lambda \subset \Lambda'$

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) \Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, d\omega) = \Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, F),$$

(cf. Remark 3.2-(S4)). Since $\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, \cdot) \to \mu$ in the weak local sense also

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) \Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, d\omega) \to \int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) \mu(d\omega)$$

for $\Lambda' \nearrow X$. Moreover, $\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, F) \to \mu(F)$ which implies the DLR equations

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) \mu(d\omega) = \mu(F).$$

$\blacksquare$

5.3   Extension to standard Borel spaces

In this subsection we will present a natural generalization of our results. Except in Theorem 5.3 and 5.6 we use nothing else than the measurability structure of X and S and there we only apply the theorem of Kolmogorov for projective limits. Thus the construction works as well for X and S separable standard Borel spaces. To this end we recall the definition and properties of separable standard Borel space, see e.g., [Coh93], [Geo88] and [Par67].

Definition 5.7 *Let $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ be two measurable spaces.*

1. The spaces $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ are called isomorphic iff there exists a measurable bijective mapping $f : X \to X'$ such that its inverse f^{-1} is also measurable.

2. $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ are called σ-isomorphic iff there exists a bijective mapping $F : \mathfrak{F} \to \mathfrak{F}'$ between the σ-algebras which preserves the operations in a σ-algebra.

3. $(X, \mathfrak{F})$ is said to be countable generated iff there exists a denumerable class $\mathfrak{D} \subset \mathfrak{F}$ such that $\mathfrak{D}$ generates $\mathfrak{F}$.

4. $(X, \mathfrak{F})$ is said to be separable iff it is countably generated and for each $x \in X$ the set $\{x\} \in \mathfrak{F}$.

Definition 5.8 *Let $(X, \mathfrak{F})$ be a countably generated measurable space. Then $(X, \mathfrak{F})$ is called standard Borel space iff there exists a Polish space $(X', \mathfrak{F}')$ (i.e., metrizable, complete metric space which fulfills the second axiom of countability and the σ-algebra $\mathfrak{F}'$ coincides with the Borel σ-algebra) such that $(X, \mathfrak{F})$ and $(X', \mathcal{B}(X'))$ are σ-isomorphic.*

Example 5.9 *1. Every locally compact, σ-compact space is a standard Borel space.*

2. Polish spaces are standard Borel spaces.

We have the following proposition, cf. [Par67, Chap. V, Theorem 2.1].

Proposition 5.10 *1. If $(X, \mathfrak{F})$ is a countable generated measurable space, then there exists $E \subset \{0,1\}^{\mathbb{N}}$ such that $(X, \mathfrak{F})$ is σ-isomorphic to $(E, \mathcal{B}(E))$. Thus $(X, \mathfrak{F})$ is σ-isomorphic to a separable measurable space.*

2. Let $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ be separable measurable spaces. Then $(X, \mathfrak{F})$ is σ-isomorphic to $(X', \mathfrak{F}')$ iff they are isomorphic.

Finally we state some operations under which separable standard Borel space are closed, see e.g., [Par67] and [Coh93].

Theorem 5.11 *Let $(X_1, \mathfrak{F}_1), (X_2, \mathfrak{F}_2), \ldots$ be separable standard Borel spaces.*

1. Countable product, sums, and union are separable standard Borel spaces.

2. The projective limit is a separable standard Borel space.

3. *Any measurable subset of a separable standard Borel space is also a separable standard Borel space.*

We need also a version of Kolmogorov's theorem for separable standard Borel spaces.

Theorem 5.12 *(cf. [Par67, Chap. V Theorem 3.2]) Let $(X_n, \mathfrak{F}_n)$, $n \in \mathbb{N}$ be separable standard Borel spaces. Let $(X, \mathfrak{F})$ be the projective limit of the space $(X_n, \mathfrak{F}_n)$ relative to the maps $p_{n,m} : X_n \to X_m$, $m \le n$. If $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of probability measures such that μ_n is a measure on $(X_n, \mathfrak{F}_n)$ and $\mu_m = \mu_n \circ p_{n,m}^{-1}$ for $m \le n$; then there exists a unique measure μ on $(X, \mathfrak{F})$ such that $\mu_n = \mu \circ p_n^{-1}$ for all $n \in \mathbb{N}$ where p_n is the projection map from X on X_n.*

This theorem can be extended to an index set I which is a directed set with an order generating sequence, i.e., there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in I such that for every $\alpha \in I$ exists a $n \in \mathbb{N}$ with $\alpha < \alpha_n$.

Let us now apply this general framework to our marked configuration space Ω.

We assume, therefore, that $(X, \mathfrak{X})$, $(S, \mathfrak{S})$ are separable standard Borel spaces.

To use $\mathcal{B}_c(X)$ makes in this generality no sense, hence we have to introduce an abstract concept of "local" sets. Let $\mathfrak{I}_X$ be a subset of $\mathfrak{X}$ with the properties:

(I1) $\Lambda_1 \cup \Lambda_2 \in \mathfrak{I}_X$ for all $\Lambda_1, \Lambda_2 \in \mathfrak{I}_X$.

(I2) If $\Lambda \in \mathfrak{I}_X$ and $A \in \mathfrak{X}$ with $A \subset \Lambda$ then $A \in \mathfrak{I}_X$.

(I3) There exists a sequence $\{\Lambda_n, n \in \mathbb{N}\}$ from $\mathfrak{I}_X$ with $X = \bigcup_{n \in \mathbb{N}} \Lambda_n$ and such that if $\Lambda \in \mathfrak{I}_X$ then $\Lambda \subset \Lambda_n$ for some $n \in \mathbb{N}$.

Then we can construct the marked configuration space as in Subsection 2.1 replacing $\mathcal{B}_c(X)$ by $\mathfrak{I}_X$. Our aim is to show that $(\Omega, \mathcal{B}(\Omega)$ is a separable standard Borel space and thus by Theorem 5.12 the measure in Theorem 5.3 exists.

It follows from Theorem 5.11 that for any $\Lambda \in \mathfrak{I}_X$ and any $n \in \mathbb{N}$ the set $(\Lambda \times S)^n$ is a separable standard Borel space. Therefore, by the same argument $\widetilde{(\Lambda \times S)^n}/S_n$ is also a separable standard Borel space, see e.g., [Shi94].

Now taking into account the isomorphism (cf. 2.4) between $(\widetilde{\Lambda \times S})^n/S_n$ and $\Omega_\Lambda^{(n)}$ the same holds for $\Omega_\Lambda^{(n)}$, Hence Ω_Λ is also a separable standard Borel space as well as $\Omega_X^{(n)}$ by Theorem 5.11, (1).

Finally, the marked configuration space itself is a separable standard Borel space as the projective limit of the separable standard Borel spaces $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$, $\Lambda \in \mathfrak{I}_X$.

Furthermore, if on $(X, \mathfrak{X})$ is given a non-atomic measure σ with $\sigma(\Lambda) < \infty$ $\forall \Lambda \in \mathfrak{I}_X$ and a kernel $\tau : X \times \mathfrak{G} \to \mathbb{R}$ which fulfills (2.6), then the procedure from Subsection 2.2 can be done in an analogous way and as a result we obtain a probability measure $\pi_{z\sigma}^\tau$, $z > 0$ on $(\Omega, \mathcal{B}(\Omega))$. Specifications and marked Gibbs measures can also be defined analogously, see e.g., [Pre80]. All the contents of Sections 4 and 5 generalize straightforward and only in Theorem 5.3 we need the assumption that the spaces are standard Borel. In Theorem 5.6 we have to generalize the notion of finite range, actually we use in the proof only the following property: For any $\Lambda \in \mathfrak{I}_X$ exists a $\Lambda' \in \mathfrak{I}_X$ such that $\Lambda \subset \Lambda'$ and $\phi((x,s),(y,t)) = 0$ if $x \in \Lambda$ and $y \in X \backslash \Lambda'$.

5.4   Examples revisited

Here we will verify that our framework is sufficient to treat the examples stated in Subsection 3.2. Therefore the main task in this subsection is to verify the stability condition (S) (cf. (3.8)) and integrability condition (I) (cf. (3.9)) for each of the examples in Subsection 3.2. This enables us to apply Theorem 5.3 to the considered examples and therefore give us a limit measures corresponding to the specifications under consideration.

Proposition 5.13 *Let ϕ be a potential on $X \times S$ bounded below, then the following three conditions are equivalent*

(i) *the potential fulfills the integrability condition.*

(ii) *there exists a $\alpha > 0$ such that for*

$$A_\alpha := \{(x, s_x) \in X \times S \, | \, \phi((y, s_y), (x, s_x)) > \alpha\}$$

the following is fulfilled:

$$\operatorname*{ess\,sup}_{(y,s_y) \in X \times S} \sigma^\tau(A_\alpha) < +\infty$$

and

$$\operatorname*{ess\,sup}_{(y,s_y)\in X\times S}\int_{(X\times S)\backslash A_\alpha}|\phi((y,s_y),(x,s_x))|\,\tau(x,ds_x)\sigma(dx)<+\infty.$$

(iii) For every $\hat{y}\in X\times S$ there exists a $N_{\hat{y}}\in\mathcal{B}(X)\otimes\mathcal{B}(S)$ such that

$$\operatorname*{ess\,sup}_{\hat{y}\in X\times S}\sigma^\tau(N_{\hat{y}})<+\infty,$$

and

$$\operatorname*{ess\,sup}_{(y,s_y)\in X\times S}\int_{(X\times S)\backslash N_{\hat{y}}}|\phi((y,s_y),(x,s_x))|\,\tau(x,ds_x)\sigma(dx)<+\infty.$$

In particular, integrability condition (I) (cf. (3.9)) is independent of β.

Proof. Denote the lower bound for ϕ by B'. Using the fact that there exist constants $C_1,C_2>0$ such that

$$\begin{aligned}C_1\left(|x|\mathbb{1}_{[-B',\alpha)}(x)+\mathbb{1}_{(\alpha,\infty)}(x)\right)&\leq\ \mathbb{1}_{[-B',\infty)}(x)|e^{-x}-1|\\&\leq\ C_2\left(|x|\mathbb{1}_{[-B',\alpha)}(x)+\mathbb{1}_{(\alpha,\infty)}(x)\right),\end{aligned}$$

we see that (i) and (ii) are equivalent. Obviously, (ii) implies (iii) if we put $N_{\hat{y}}:=\{\hat{x}\in X\times S\,|\,\phi(\hat{y},\hat{x})\leq\alpha\}$. Conversely, using

$$\begin{aligned}\{\hat{x}\ \in\ X\times S\,|\,\phi(\hat{y},\hat{x})\leq\alpha\}\\=\ (\{\hat{x}\in X\times S\,|\,\phi(\hat{y},\hat{x})\leq\alpha\}\cap N_{\hat{y}})\sqcup(\{\hat{x}\in X\times S\,|\,\phi(\hat{y},\hat{x})\leq\alpha\}\backslash N_{\hat{y}}),\end{aligned}$$

we obtain that also (iii) implies (ii). ∎

Thus we have the following sufficient condition for the integrability condition.

Corollary 5.14 *Let X be a Riemannian manifold (d denotes the metric on X). Let us assume that there exists $R>0$ such that*

$$\operatorname*{ess\,sup}_{(y,s_y)\in X\times S}\int_{\{(x,s_x)\in X\times S\,|\,d((y,s_y),(x,s_x))\geq R\}}|\phi((y,s_y),(x,s_x))|\,\tau(x,ds_x)\sigma(dx)<+\infty.$$

Then the above conditions are fulfilled.

Proposition 5.15   *1. Let ϕ_1, ϕ_2 be two stable potentials then also $\phi_1 + \phi_2$ is stable.*

2. *Let ϕ_1, ϕ_2 be two potentials which fulfil the integrability condition then $\phi_1 + \phi_2$ also satisfies the integrability condition.*

3. *A potential bounded from below by a stable one is stable itself.*

Example 3.5 Φ is integrable on $\{|x| \geq R\}$ and because of monotonicity of J there exists a $C > 0$ such that

$$\int J(|x|)dx \leq C \sum_{q \in \mathbb{Z}^d} J\left(a|q|\right) < +\infty.$$

We can bound the potential ϕ below by

$$\phi((x, s_x), (y, s_y)) \geq \Phi(|x - y|) - K|J(|x - y|)|,$$

for $K := \sup_{s \in \text{supp}\,\tau} |s|^2$ and this potential is stable according to the Dobrushin-Fisher-Ruelle criterium (cf. Section 3.2.8 in [Rue69]). This potential fulfills also the integrability condition because we can bound it above by

$$|\phi((x, s_x), (y, s_y))| \leq |\Phi(|x - y|)| + K|J(|x - y|)|,$$

and this is integrable on $\{|x| \geq R\}$.

Example 3.6 The arguments are analogous to the above case.
Example 3.7 On the one hand in this model the potential is bounded below by

$$\phi((x, s_x), (y, s_y)) \geq \psi(|x - y|),$$

and thus stable. On the other hand its modulus is bounded above by

$$|\phi((x, s_x), (y, s_y))| \leq \varphi(|x - y|) + |\psi(|x - y|)|,$$

whence it fulfills the integrability condition, because the lower regularity of ψ implies that also ψ_- is integrable.

Example 3.8 The potential is stable since for all $\{(x_1, s_1), \ldots, (x_n, s_n)\}$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_0^\theta V(s_i(t) - s_j(t))dt \geq -nB \int_0^\theta dt = -nB\theta.$$

The potential fulfills the integrability condition because of the following arguments. Let $s_x \in \mathcal{L}^\theta(\mathbb{R}^d)$, $s_y \in \mathcal{L}^\theta(\mathbb{R}^d)$ and denote by $\tilde{s}_x := s_x - s_x(0)$ and $\tilde{s}_y := s_y - s_y(0)$ then

$$\int_{\mathbb{R}^d} \int_{\mathcal{L}^\theta(\mathbb{R}^d)} |\phi((x, s_x), (y, s_y))| \tau(x, ds_x) \sigma(dx)$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathcal{L}^\theta(\mathbb{R}^d)} \int_0^\theta |V(x + \tilde{s}_x(t) - s_y(t))| dt W_{0|0}(d\tilde{s}_x) dx$$

$$= \int_{\mathcal{L}^\theta(\mathbb{R}^d)} \int_0^\theta \int_{\mathbb{R}^d} |V(x + \tilde{s}_x(t) - s_y(t))| dx dt W_{0|0}(d\tilde{s}_x)$$

$$= \int_{\mathbb{R}^d} |V(x)| dx \int_{\mathcal{L}^\theta(\mathbb{R}^d)} \int_0^\theta dt W_{0|0}(d\tilde{s}_x) \leq \frac{1}{(2\pi\theta)^{d/2}} \theta \int_{\mathbb{R}^d} |V(x)| dx.$$

Thus according to Theorem 5.3 there exists a limiting measure μ on the marked configuration space $\Omega_{\mathbb{R}^d}(\mathcal{L}^\theta(\mathbb{R}^d))$, this are not configurations in loops but we can embed the loop space into $\mathbb{R}^d \times \mathcal{L}^\theta(\mathbb{R}^d)$ in the following way

$$I : \mathcal{L}^\theta(\mathbb{R}^d) \hookrightarrow \mathbb{R}^d \times \mathcal{L}^\theta(\mathbb{R}^d), \ s \mapsto (s(0), s),$$

and the image of this mapping

$$A := \{(x, s) \in \mathbb{R}^d \times \mathcal{L}^\theta(\mathbb{R}^d) | s(0) = x\},$$

is a measurable set of full measure, i.e.,

$$\int_{A^c} W_{x|x}(ds) dx = 0.$$

Whence according to Proposition 5.4 the set

$$\tilde{\Omega} := \{\{(x_1, s_{x_1}), (x_2, s_{x_2}), \ldots\} \in \Omega | s_{x_i}(0) = x_i \text{ for all } i\},$$

has full measure, i.e., $\mu(\tilde{\Omega}) = 1$ and thus we can define a measure on the loop space via the above embedding.

Acknowledgments

Appendix

A.1   Proof of Lemma 2.8

Lemma A.1 *The following results are valid.*

1. $\sigma^{\tau \otimes n}(\{(\hat{x}_1, \ldots, \hat{x}_n) \in (X \times S)^n | \exists i, j \; i \neq j \text{ with } x_i = x_j\}) = 0.$

2. $\sigma^{\tau \otimes n}((X \times S)^n \backslash \widetilde{(X \times S)}^n) = 0.$

3. *For all* $\omega \in \Omega_{fin}$ *the set* $A_\omega := \{\omega' \in \Omega_{fin} | \gamma_\omega \cap \gamma_{\omega'} \neq \emptyset\}$ *has zero* $\nu_{z\sigma^\tau}$-*measure.*

4. *The set* $A := \{(\omega, \omega') \in \Omega_{fin} \times \Omega_{fin} | \gamma_\omega \cap \gamma_{\omega'} \neq \emptyset\}$ *has zero* $\nu_{z\sigma^\tau} \otimes \nu_{z\sigma^\tau}$-*measure.*

Proof. 1. Because of the symmetry and the non-atomicity of σ we have

$$
\begin{aligned}
& \sigma^{\tau \otimes n}(\{(\hat{x}_1, \ldots, \hat{x}_n) \in (X \times S)^n | \exists i, j \; i \neq j \text{ with } x_i = x_j\}) \\
\leq \;& \binom{n}{2} \sigma^{\tau \otimes n}(\{(\hat{x}_1, \ldots, \hat{x}_n) \in (X \times S)^n | x_1 = x_2\}) \\
= \;& \binom{n}{2} \sigma^\tau(X \times S)^{n-2} \sigma^{\tau \otimes 2}(\{((x, s), (x, t)) | x \in X, \, s, t \in S\}) \\
= \;& 0
\end{aligned}
$$

2. Consequence of 1.

3. Let $\omega = \{y_1, \ldots, y_m\}$. According to (2.5) we can decompose the set A_ω as

$$
A_\omega = \bigsqcup_{n=0}^{\infty} \{\omega' \in \Omega^{(n)} | \gamma_\omega \cap \gamma_{\omega'} \neq \emptyset\},
$$

then the definition of $\nu_{z\sigma^\tau}$ applied to A yields

$$
\nu_{z\sigma^\tau}(A) = \sum_{n=0}^{\infty} \frac{z^n}{n!} (\sigma_n^\tau)(A_{\omega,n}),
$$

where $A_{\omega,n}$ is given by

$$
A_{\omega,n} := \{\{\hat{x}_1, \ldots, \hat{x}_n\} \in \Omega^{(n)} \mid \exists i, j \; i \neq j \text{ with } x_i = y_j\}.
$$

On the other hand we can estimate $\frac{1}{n!}(\sigma_n^\tau)(A_{\omega,n})$ by

$$nm\sigma^{\tau\otimes n}(\{(\hat{x}_1,\ldots,\hat{x}_n)\in(\widetilde{X\times S})^n\times(\widetilde{X\times S})^m|x_1=y_1\}),$$

then the definition of σ^τ and the non-atomicity of σ implies as above that this last expression is zero.

4. Consequence of 3. ∎

Lemma 2.8 Let $\psi\in\mathcal{A}$ and $\Lambda,\Lambda'\in\mathcal{B}_c(X)$ be given such that $\Lambda'\subset\Lambda$, suppose that $\psi\in L^1(\Omega_\Lambda,\nu_{z\sigma\tau})$. Then the following equality holds

$$\int_{\Omega_{\Lambda\setminus\Lambda'}}(\exp^*\psi)(\omega\cup\omega')\nu_{z\sigma\tau}(d\omega) \tag{A.7}$$

$$=\ \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}}\psi(\omega)\nu_{z\sigma\tau}(d\omega)\right)\exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}}\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot\cup\omega)\nu_{z\sigma\tau}(d\omega)\right)(\omega'),$$

for $\nu_{z\sigma\tau}$-a.e. $\omega\in\Omega_{\Lambda'}$.

Proof. First we clarify the existence of the integrals. It follows from Fubini's theorem that

$$\int_{\Omega_{\Lambda'}}\int_{\Omega_{\Lambda\setminus\Lambda'}}|(\exp^*\psi)(\omega\cup\omega')|\nu_{z\sigma\tau}(d\omega')\nu_{z\sigma\tau}(d\omega)$$

$$\leq\ \int_{\Omega_{\Lambda'}}\int_{\Omega_{\Lambda\setminus\Lambda'}}(\exp^*|\psi|)(\omega\cup\omega')\nu_{z\sigma\tau}(d\omega')\nu_{z\sigma\tau}(d\omega)$$

$$=\ \exp\left(\int_{\Omega_\Lambda}|\psi|(\omega)\nu_{z\sigma\tau}(d\omega)\right)<\infty,$$

thus for $\nu_{z\sigma\tau}$-a.e. $\omega\in\Omega_{\Lambda'}$ $(\exp^*|\psi|)(\cdot\cup\omega')$ and $(\exp^*\psi)(\cdot\cup\omega')$ belongs to $L^1(\Omega_{\Lambda\setminus\Lambda'},\nu_{z\sigma\tau})$. Hence the following manipulations are justified for $|\psi|$ and therefore also for the function ψ itself.

The left hand side of (A.7) for $\omega'\neq\emptyset$ is by definition equivalent to

$$\int_{\Omega_{\Lambda\setminus\Lambda'}}\sum_{n=1}^\infty\frac{1}{n!}\sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}_\emptyset^n(\omega\cup\omega')}\psi(\omega_1)\ldots\psi(\omega_n)\nu_{z\sigma\tau}(d\omega). \tag{A.8}$$

Without loss of generality we may assume $\gamma_\omega\cap\gamma_{\omega'}=\emptyset$ (cf. Lemma A.1). To each partition $(\omega_1,\ldots,\omega_n)\in\mathfrak{P}_\emptyset^n(\omega\cup\omega')$ we define in one to one form the

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following objects (we put together the ω_i's which have solely points from ω)

$$
\begin{aligned}
J &:= \{i \,|\, \omega_i \subset \omega\} \\
l &:= |J| \\
\eta_i &:= \omega_i, \ \forall i \in J \\
\xi_i &:= \omega_i \cap \omega, \ \forall i \notin J \\
\xi_i' &:= \omega_i \cap \omega', \ \forall i \notin J \\
\eta_0 &:= \omega \backslash (\sqcup_{i \in J} \omega_i),
\end{aligned}
$$

where $l \in \{0, \ldots, n-1\}$; $(\eta_0, \ldots, \eta_l) \in \mathfrak{P}_\emptyset^{l+1}(\omega)$; $(\xi_{l+1}, \ldots \xi_n) \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)$; $(\xi_{l+1}', \ldots \xi_n') \in \mathfrak{P}^{n-l}(\omega')$. This implies that (A.8) can be rewritten as

$$
\int_{\Omega_{\Lambda \backslash \Lambda'}} \psi(\omega \cup \omega') \nu_{z\sigma\tau}(d\omega) \tag{A.9}
$$

$$
+ \sum_{n=2}^{\infty} \sum_{l=0}^{n-1} \frac{1}{l!(n-l)!} \int_{\Omega_{\Lambda \backslash \Lambda'}} \sum_{(\eta_0, \ldots, \eta_l) \in \mathfrak{P}_\emptyset^{l+1}(\omega)} \prod_{i=1}^{l} \psi(\eta_i) \varphi_{n,l}(\eta_0) \nu_{z\sigma\tau}(d\omega),
$$

where

$$
\varphi_{n,l}(\eta_0) := \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \psi(\xi_i' \cup \xi_i).
$$

Then using Lemma 2.6 we obtain

$$
\sum_{n=2}^{\infty} \sum_{l=0}^{n-1} \frac{1}{l!(n-l)!} \left(\int_{\Omega_{\Lambda \backslash \Lambda'}} \psi(\omega) \nu_{z\sigma\tau}(d\omega) \right)^l \int_{\Omega_{\Lambda \backslash \Lambda'}} \varphi_{n,l}(\eta_0) \nu_{z\sigma\tau}(d\eta_0). \tag{A.10}
$$

First we look at the integral of $\varphi_{n,l}$,

$$
\int_{\Omega_{\Lambda \backslash \Lambda'}} \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \psi(\xi_i' \cup \xi_i) \nu_{z\sigma\tau}(d\eta_0)
$$

$$
= \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \int_{\Omega_{\Lambda \backslash \Lambda'}} \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \prod_{i=l+1}^{n} \psi(\xi_i' \cup \xi_i) \nu_{z\sigma\tau}(d\eta_0).
$$

Once more we apply Lemma 2.6 to the right hand side of the above equality to get

$$
\sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \int_{\Omega_{\Lambda \backslash \Lambda'}} \psi(\xi_i' \cup \xi_i) \nu_{z\sigma\tau}(d\xi_i). \tag{A.11}
$$

Hence interchanging the sums and putting together (A.10) and (A.11) we get

$$\sum_{l=0}^{\infty} \sum_{n=l+1}^{\infty} \frac{1}{l!} \left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega)\nu_{z\sigma\tau}(d\omega) \right)^l$$

$$\times \frac{1}{(n-l)!} \sum_{(\xi')_1^{n-l}\in\mathfrak{P}^{n-l}(\omega')} \prod_{i=1}^{n-l} \int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega\cup\xi_i')\nu_{z\sigma\tau}(d\omega)$$

$$= \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega)\nu_{z\sigma\tau}(d\omega) \right) \exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot\cup\omega)\nu_{z\sigma\tau}(d\omega) \right)(\omega').$$

∎

A.2 Proof of Proposition 4.10

Proposition 4.10 Let $\omega, \zeta \in \Omega_{fin}$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$. The solution of (4.13) for $\omega = \{\hat{x}_1, \ldots, \hat{x}_l\}$, $l \geq 1$ has the form

$$Q(\{\hat{x}_1, \ldots, \hat{x}_l\}, \zeta) = \sum_{(\omega_1, \ldots, \omega_l)\in\mathfrak{P}_\emptyset^l(\zeta)} Q(\{\hat{x}_1\}, \omega_1) \cdots Q(\{\hat{x}_l\}, \omega_l), \qquad (A.12)$$

where

$$Q(\{\hat{x}\}, \zeta) := (e^{2\beta B})^{|\zeta|+1} \sum_{T\in\mathfrak{T}(\{\hat{x}\}\cup\zeta)} \prod_{\{\hat{y},\hat{y}'\}\in T} |e^{-\beta\phi(\hat{y},\hat{y}')} - 1|, \qquad (A.13)$$

for $\zeta \neq \emptyset$ and $Q(\{\hat{x}\}, \emptyset) := e^{2\beta B}$. In the case $\omega = \emptyset$ we define $Q(\emptyset, \zeta)$ as in (4.12).

Proof. For $\omega = \emptyset$ the assertion follows by definition, hence we assume $\omega \neq \emptyset$. We prove the result by induction in $|\omega| + |\zeta|$.

For $|\omega| + |\zeta| = 1$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$ we have $\zeta = \emptyset$ and $\omega = \{\hat{x}\}$. On the one hand the r.h.s. of (4.13) yields

$$e^{2\beta B} \sum_{\hat{\omega}\subset\emptyset} Q_I(\hat{\omega}, \emptyset\setminus\hat{\omega})|k_{\hat{\omega}}(\hat{x})| = e^{2\beta B} Q_I(\emptyset, \emptyset) = e^{2\beta B},$$

on the other hand equation (A.13) gives

$$Q_I(\{\hat{x}\}, \emptyset) = e^{2\beta B} \sum_{T\in\mathfrak{T}(\{\hat{x}\})} \prod_{\{\hat{x},\hat{x}'\}\in\emptyset} |e^{-\beta\Phi(\hat{x},\hat{x}')} - 1| = e^{2\beta B}.$$

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Thus the initial induction step is verified.

Let us assume that the result is true for $|\omega|+|\zeta| = n-1$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$. Choose ω, ζ such that $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, $|\omega| + |\zeta| = n$, and denote $I(\omega) = x_0 \in \omega$. Using (A.12) for $n-1$ in the r.h.s. of (4.13) one obtains

$$e^{2\beta B} \sum_{\hat\omega \subset \zeta} |k_{\hat\omega}(\hat{x}_0)| \sum_{(\omega'_{\hat{x}})_{\hat{x} \in \omega \setminus \{\hat{x}_0\} \cup \hat\omega} \in \mathfrak{P}_\emptyset^{n'}(\zeta \setminus \hat\omega)} \prod_{\hat{x} \in \omega \setminus \{\hat{x}_0\}} Q(\{\hat{x}\}, \omega'_{\hat{x}}) \prod_{\hat{x} \in \hat\omega} Q(\{\hat{x}\}, \omega'_{\hat{x}}),$$
(A.14)

where $n' = |\omega| + |\hat\omega| - 1$. If $\hat\omega \neq \emptyset$, then define $\omega'_{\hat{x}_0} := \hat\omega \sqcup \bigsqcup_{\hat{x} \in \hat\omega} \omega'_{\hat{x}}$ and we make the following re-arrangement in one to one form

$$\emptyset \neq \hat\omega \subset \zeta, (\omega'_{\hat{x}})_{\hat{x} \in \omega \setminus \{\hat{x}_0\} \cup \hat\omega} \in \mathfrak{P}_\emptyset^{|\omega|+|\hat\omega|-1}(\zeta \setminus \hat\omega)$$

$$\updownarrow$$

$$((\omega'_{\hat{x}})_{\hat{x} \in \omega}, \hat\omega, (\omega'_{\hat{x}})_{\hat{x} \in \hat\omega}),$$

where $(\omega'_{\hat{x}})_{\hat{x} \in \omega} \in \mathfrak{P}_\emptyset^{|\omega|}(\zeta)$, $\omega'_{\hat{x}_0} \neq \emptyset$, $\emptyset \neq \hat\omega \subset \omega'_{\hat{x}_0}$, and $(\omega'_{\hat{x}})_{\hat{x} \in \hat\omega} \in \mathfrak{P}_\emptyset^{|\hat\omega|}(\omega'_{\hat{x}_0} \setminus \hat\omega)$. With this, the expression in (A.14) can be rewritten as

$$\sum_{\substack{(\omega'_{\hat{x}})_{\hat{x} \in \omega} \in \mathfrak{P}_\emptyset^{|\omega|}(\zeta) \\ \omega'_{\hat{x}_0} \neq \emptyset}} \prod_{\hat{x} \in \omega \setminus \{\hat{x}_0\}} Q(\{\hat{x}\}, \omega'_{\hat{x}}) \sum_{\emptyset \neq \hat\omega \subset \omega'_{\hat{x}_0}} e^{2\beta B} |k_{\hat\omega}(\hat{x}_0)| \quad \text{(A.15)}$$

$$\times \sum_{(\omega'_{\hat{x}})_{\hat{x} \in \hat\omega} \in \mathfrak{P}_\emptyset^{|\hat\omega|}(\omega'_{\hat{x}_0} \setminus \hat\omega)} \prod_{\hat{x} \in \hat\omega} Q(\{\hat{x}\}, \omega'_{\hat{x}}).$$

Next we use the explicit form for $Q(\{\hat{x}\}, \omega'_{\hat{x}})$ in (A.13) to write the term

$$e^{2\beta B} \sum_{\emptyset \neq \hat\omega \subset \omega'_{\hat{x}_0}} |k_{\hat\omega}(\hat{x}_0)| \sum_{(\omega'_{\hat{x}})_{\hat{x} \in \hat\omega} \in \mathfrak{P}_\emptyset^{|\hat\omega|}(\omega'_{\hat{x}_0} \setminus \hat\omega)} \prod_{\hat{x} \in \hat\omega} Q(\{\hat{x}\}, \omega'_{\hat{x}})$$

as

$$\sum_{\emptyset \neq \hat\omega \subset \omega'_{\hat{x}_0}} e^{2\beta B} \sum_{(\omega'_{\hat{x}})_{\hat{x} \in \hat\omega} \in \mathfrak{P}_\emptyset^{|\hat\omega|}(\omega'_{\hat{x}_0} \setminus \hat\omega)} \prod_{\hat{x} \in \hat\omega} (e^{2\beta B})^{|\omega'_{\hat{x}}|+1} \sum_{T_{\hat{x}} \in \mathfrak{T}(\{\hat{x}\} \cup \omega'_{\hat{x}})} |k_{T_{\hat{x}}}| \, |k_{\hat\omega}(\hat{x}_0)|$$

$$= \sum_{\emptyset \neq \hat\omega \subset \omega'_{\hat{x}_0}} (e^{2\beta B})^{1+|\omega'_{\hat{x}_0} \setminus \hat\omega| + |\hat\omega|} \sum_{(\omega'_{\hat{x}})_{\hat{x} \in \hat\omega} \in \mathfrak{P}_\emptyset^{|\hat\omega|}(\omega'_{\hat{x}_0} \setminus \hat\omega)} \prod_{\hat{x} \in \hat\omega} \sum_{T_{\hat{x}} \in \mathfrak{T}(\{\hat{x}\} \cup \omega'_{\hat{x}})} |k_{T_{\hat{x}}}| \, |k_{\hat\omega}(\hat{x}_0)|. \quad \text{(A.16)}$$

We again make a re-arrangement: for $\emptyset \neq \hat{\omega} \subset \omega'_{\hat{x}_0}$, $(\omega'_{\hat{x}})_{\hat{x} \in \hat{\omega}} \in \mathfrak{P}_{\emptyset}^{|\hat{\omega}|}(\omega'_{\hat{x}_0} \backslash \hat{\omega})$, and $(T_{\hat{x}})_{x \in \hat{\omega}} \in \times_{\hat{x} \in \hat{\omega}} \mathfrak{T}(\{\hat{x}\} \cup \omega'_{\hat{x}})$ we define

$$T := \bigsqcup_{\hat{x} \in \hat{\omega}} T_{\hat{x}} \sqcup \{(\hat{x}, \hat{x}_0) | \hat{x} \in \hat{\omega}\} \in \mathfrak{T}(\omega'_{\hat{x}_0} \cup \{\hat{x}_0\}),$$

and vice versa, given $T \in \mathfrak{T}(\omega'_{\hat{x}_0} \cup \{\hat{x}_0\})$ we define $\hat{\omega}$, $\omega'_{\hat{x}}$, and $(T_{\hat{x}})_{\hat{x} \in \hat{\omega}}$ by

$$\hat{\omega} := \{\hat{x} \in V(T) | (\hat{x}, \hat{x}_0) \in T\} \subset \omega'_{\hat{x}_0}$$

$$T_{\hat{x}_0} \oplus \bigoplus_{\hat{x} \in \hat{\omega}} T_{\hat{x}} := T \backslash \{(\hat{x}, \hat{x}_0) | \hat{x} \in \hat{\omega}\} \text{ with } \hat{x} \in V(T_{\hat{x}}) \text{ and } V(T_{\hat{x}_0}) = \hat{x}_0$$

$$\omega'_{\hat{x}} := V(T_{\hat{x}}) \backslash \{\hat{x}\}$$

Then (A.16) can be written (using A.13) as

$$(e^{2\beta B})^{1+|\omega'_{\hat{x}_0}|} \sum_{T \in \mathfrak{T}(\omega'_{\hat{x}_0} \cup \{\hat{x}_0\})} k_T = Q(\{\hat{x}_0\}, \omega'_{\hat{x}_0}).$$

Hence (A.15) now simplifies to

$$\sum_{\substack{(\omega'_{\hat{x}})_{\hat{x} \in \omega} \in \mathfrak{P}_{\emptyset}^{|\omega|}(\zeta) \\ \omega'_{\hat{x}_0} \neq \emptyset}} \prod_{\hat{x} \in \omega \backslash \{\hat{x}_0\}} Q(\{\hat{x}\}, \omega'_{\hat{x}}) Q(\{\hat{x}_0\}, \omega'_{\hat{x}_0}).$$

After an explicit calculation for the case $\hat{\omega} = \emptyset$ we see that the above expression is nothing but the required form for $Q(\omega, \zeta)$. ∎

A.3   Proof of Proposition 4.13

Lemma 4.12 For every $\hat{x} \in X \times S$, $Y \in \mathcal{B}(X)$, and $n \geq 1$ we have

$$\int_{(Y \times S)^n} Q(\{\hat{x}\}, \{\hat{y}\}_1^n) \sigma^\tau (d\hat{y})_1^n \qquad (A.17)$$

$$\leq e^{2\beta B(n+1)} C(\beta)^{n-1} (n+1)^{n-1} \int_{Y \times S} |e^{-\beta \phi(\hat{x}, \hat{y})} - 1| \sigma^\tau (d\hat{y}).$$

Proof. In the following we denote $\hat{y}_{n+1} := \hat{x}$. The equality (4.15) implies the following estimate for (A.17)

$$(e^{2\beta B})^{n+1} \sum_{T \in \mathfrak{T}([n+1])} \int_{(Y \times S)^n} \prod_{(i,j) \in T} |e^{-\beta \Phi(\hat{y}_i, \hat{y}_j)} - 1| \sigma^\tau (d\hat{y})_1^n. \qquad (A.18)$$

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We now estimate by induction in n the term

$$\int_{(Y \times S)^n} \prod_{(i,j) \in T} |e^{\beta \Phi(\hat{y}_i, \hat{y}_j)} - 1| \sigma^\tau(d\hat{y})_1^n. \tag{A.19}$$

For $n = 1$ all trees T are of the form $\{\{\hat{x}, \hat{y}_1\}\}$ and hence (A.19) is reduced to

$$\int_{(Y \times S)} |e^{\beta \Phi(\hat{x}, \hat{y}_1)} - 1| \sigma^\tau(d\hat{y}_1).$$

Let us assume that for $n = N - 1$ we have for all $T \in \mathfrak{T}([n+1])$

$$\int_{(Y \times S)^n} \prod_{(i,j) \in T} |e^{-\beta \Phi(\hat{y}_i, \hat{y}_j)} - 1| \sigma^\tau(d\hat{y})_1^n$$

$$\leq \ C(\beta)^{n-1} \int_{Y \times S} |e^{-\beta \phi(\hat{y}_{n+1}, \hat{y})} - 1| \sigma^\tau(d\hat{y}).$$

For the case $n = N$ we proceed as follows. Let $T \in \mathfrak{T}([n+1])$ be given. Choose $\hat{y}_{n+1}$ as a foot point of T. Then there exists a final pair $\{j_1, j_2\} \in T$ where $\hat{y}_{j_1}$ is the final vertex and $\hat{y}_{j_1} \neq \hat{y}_{n+1}$. This implies the following estimate

$$\int_{(Y \times S)^n} \prod_{(i,j) \in T} |e^{-\beta \phi(\hat{y}_i, \hat{y}_j)} - 1| \sigma^\tau(d\hat{y})_1^n$$

$$\leq \ \int_{(Y \times S)^{n-1}} \prod_{(i,j) \in T \setminus \{j_1, j_2\}} |e^{-\beta \phi(\hat{y}_i, \hat{y}_j)} - 1|$$

$$\times \int_{Y \times S} |e^{-\beta \phi(\hat{y}_{j_1}, \hat{y}_{j_2})} - 1| \sigma^\tau(d\hat{y}_{j_1}) \prod_{\substack{l=1 \\ l \neq j_1}}^n \sigma^\tau(d\hat{y}_l)$$

$$\leq \ C(\beta) \int_{(Y \times S)^{n-1}} \prod_{(i,j) \in T \setminus \{j_1, j_2\}} |e^{-\beta \phi(\hat{y}_i, \hat{y}_j)} - 1| \prod_{\substack{l=1 \\ l \neq j_1}}^n \sigma^\tau(d\hat{y}_l)$$

$$\leq \ C(\beta)^{n-1} \int_{Y \times S} |e^{-\beta \phi(\hat{y}_{n+1}, \hat{y})} - 1| \sigma^\tau(d\hat{y}).$$

where in the last inequality we used the induction step. Thus (A.18) yields

$$(e^{2\beta B})^{n+1} C(\beta)^{n-1} \int_{Y \times S} |e^{-\beta \phi(\hat{y}_{n+1}, \hat{y})} - 1| \sigma^\tau(d\hat{y}) \sum_{T \in \mathfrak{T}([n+1])} 1.$$

It follows from Proposition 2.4 that $|\mathfrak{T}([n+1])| = (n+1)^{n-1}$. ∎

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References

[AGL78]   M. Aizenman, S. Goldstein, and J. L. Lebowitz. Conditional equilibrium and the equivalence of microcanonical and grandcanonical ensembles in the thermodynamic limit. *Comm. Math. Phys.*, 62:279–302, 1978.

[AKR97a] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: The Gibbsian case. Preprint 97-091, SFB 343 Univ. Bielefeld, 1997. To appear in J. Funct. Anal.

[AKR97b] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Diffeomorphism groups and current algebras: configuration spaces analysis in quantum theory. Preprint 97-073, SFB 343 Univ. Bielefeld, 1997. To appear in Rev. Math. Phys.

[AKR98]   S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154:444–500, 1998.

[Coh93]   D. L. Cohn. *Measure Theory*. Birkhäuser, Boston, Basel, Stuttgart, 1993.

[Föl75]   H. Föllmer. Phase transition and Martin boundary. In *Séminaire de Probabilités IX*, pages 305–317. Springer Verlag, 1975.

[Geo88]   H.-O. Georgii. *Gibbs Measures and Phase Transitions*, volume XIV. de Gruyter, Berlin, 1988.

[GH96]   H.-O. Georgii and O. Häggström. Phase transition in continuum Potts models. *Comm. Math. Phys.*, 181(2):507–528, 1996.

[Gin71]   J. Ginibre. Some applications of functional integration in statistical mechanics. In C. De Witt and R. Stora, editors, *Statistical Mechanics and Quantum Field Theory*, pages 327–427, New York, London Paris, 1971. Gordon and Breach, Science Publishers, Inc.

[GZ93]   H.-O. Georgii and H. Zessin. Large deviations and the maximuum entropy principle for marked point random fields. *Probab. Theory Related Fields*, 96:117–204, 1993.

[Kin93]   J. F. C. Kingman. *Poisson Processes*. Oxford Studies in Probability·3. University of Bristol, 1993.

[KK98]   Yu. G. Kondratiev and T. Kuna. Correlation functional for marked Gibbs measures and Ruelle bound. In preparation, 1998.

[KLR+97] Yu. G. Kondratiev, E. W. Lytvynov, A. L. Rebenko, M. Röckner, and G. V. Shchepan'uk. Euclidean Gibbs states for quantum continuous systems with Boltzmann statistics via cluster expansion. *Methods of Functional Analysis and Topology*, 3(1):62–81, 1997.

[KSS97a]   Yu. G. Kondratiev, J. L. Silva, and L. Streit. Differential geometry on compound Poisson space. *Methods of Functional Analysis and Topology*, 3(4):–, 1997.

[KSS97b]   Yu. G. Kondratiev, J. L. Silva, and L. Streit. Generalized Appell systems. *Methods Functional Analysis and Topology*, 3(3):28–61, 1997.

[Kun98]   T. Kuna. Properties of marked Gibbs measures in high temperature regime. In preparation, 1998.

[MM91]   V. A. Malyshev and R. A. Minlos. *Gibbs Random Fields: Cluster Expansions*. Mathematics and its applications. Kluwer Academic Publishers, 1991.

[NV95]   D. Nualart and J. Vives. A duality formula on the Poisson space and some applications. In *Seminar on Stochastic Analysis, Random Fields and Applications*, volume 36, pages 205–213. Birkhäuser, Basel, 1995.

[Ore67]   O. Ore. *Theory of Graphs*, volume 38 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 3 edition, 1967.

[Par67]   K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Probability and mathematical statistics. Academic Press, 1967.

[Pen63]   O. Penrose. Convergence of fugacity expansions for fluids and lattice gases. *J. Math. Phys.*, 4(10):1312–1320, 1963.

[Pre76]   C. J. Preston. *Random Fields*, volume 534 of *Lectures Notes in Math*. Springer Verlag, 1976.

[Pre79]   J. C. Preston. Canonical and microcanonical Gibbs states. *Z. Wahrsch. verw. Gebiete*, 46:125–158, 1979.

[Pre80]   C. J. Preston. Specifications and their Gibbs states. Manuscript, unpublished, 1980.

[Röc98]   M. Röckner. Stochastic analysis on configuration spaces: Basic ideas and recent results. Technical Report 98-031, Univ. Bielefeld, SFB-343, 1998.

[Rue64]   D. Ruelle. Cluster property of the correlation functions of classical gases. *Rev. Modern Phys.*, 36:580–584, 1964.

[Rue69]   D. Ruelle. *Statitiscal Mechanics. Rigorous results*. Benjamin, New York, Amsterdam, 1969.

[Rue70]   D. Ruelle. Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.*, 18:127–159, 1970.

[RZ98]   S. Romano and V. A. Zagrebnov. Orientational ordering transition in a continuous-spin ferrofluid. Technical report, Uni. de la Méditerranée, 1998.

[Shi94]   H. Shimomura. Poisson measures on the configuration space and unitary representations of the group of diffeomorphism. *J. Math. Kyoto Univ.*, 34:599–614, 1994.