Scaling Limits for the Solution of Wick type Burgers Equation

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Abstract

Scaling limits for the solution of Burgers equation are investigated. It is shown that the limiting distribution for the solution of Wick type Burgers equation with Gaussian initial data is the same as that for the solution of Burgers equation with ordinary product calculated in [LPW96].

Contents

1 Introduction

The Burgers equation

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \cdot \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x), \quad t > 0,$$
$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d,\ 1 \leq k \leq d,\ \lambda, \nu > 0, \tag{1}$$

is the main object of study in this paper. The Burgers equation is a nonlinear diffusion equation and it is known that if the viscosity coefficient $\nu \to 0$, then the solutions of (1) converge to the (generalized) solutions of the Riemann equation

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \cdot \frac{\partial U_k}{\partial x_j}(t,x) = 0, \quad t > 0,$$
$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d, 1 \leq k \leq d, \tag{2}$$

see [DiP83a] and [DiP83b]. Equation (2) describes the hydro-dynamical flow of noninteracting particles moving along x with velocity U, see [Arn88]. The term $\nu \Delta U_k(t,x)$ in the Burgers equation is called dissipation term and it is softening the shock fronts of the Riemann equation (2). The initial velocity potential G is assumed to be random. This is natural if one keeps in mind that Burgers equation can be viewed as a special case of the Navier-Stokes equation

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \cdot \frac{\partial U_k}{\partial x_j}(t,x) = -\frac{\partial p}{\partial x_k}(t,x) + \nu \Delta U_k(t,x) + F_k(t,x)$$

describing turbulent fluid flow, where the pressure p and external force field F terms are neglected. In such a flow, the velocity field appears to be random even without random initial conditions and "this contrast is the source of much of what is interesting in turbulence theory", see [Cho75]. This statistical approach has been the established tool in the study of turbulence for a long time. The relationship between the Navier-Stokes equation and the Burgers equation is analyzed in e.g. [Kra68].

The Burgers turbulence is known to describe various physical phenomena, such as nonlinear acoustic shock waves and the formation of cell structures (sheets, filaments, nodes), and other types of irrotational flows.

In this paper we study the asymptotic behavior of the solution U of Wick type Burgers equation

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \diamond \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x), \quad t > 0,$$
$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d,\ 1 \le k \le d,$$

see e.g. [HLØ$^+$95], with random initial velocity potential G, as $t \to \infty$.

Bulinski and Molchanov [BM91], Albeverio et al. [AMS94], and Funaki et al. [FSW95] studied solutions of the Burgers equation when the initial condition was either a Gaussian random field or a shot-noise (or Gibbs-Cox) random fields with weak or strong dependence. They obtained Gaussian and non-Gaussian distributions as parabolic scaling limits ($U(t^2, ta)$, $t \to \infty$) of the random fields solving the Burgers equation. Leonenko et al. [LOR94], [LOP95] also obtained Gaussian and non-Gaussian distributions in the same context of parabolic scaling in the case when the initial condition is either a Gaussian random field or a chi-square field with long-range dependence. Analogous results under suitable non-Gaussian initial conditions with weak dependence can be found in Surgailis and Woyczynski [SW94], Hu and Woyczynski [HW94] and Leonenko and Deriev [LD94].

Other types of random problems for the Burger equation have also been considered recently in the literature. Sinai [Sin92], Molchanov and coauthors [MSW95], Avellaneda and Weinan [AW95], considered the statistics of shocks in the zero-viscosity limit and related problems of hyperbolic scaling limiting behavior ($U(t^2, t^2 a)$, $t \to \infty$). The zero-viscosity limit is of importance in physical applications. Newman [New81] for instance used the solution of Burgers equation in the zero-viscosity limit in order to investigate the Curie-Weiss model. He worked out the correspondence: inverse temperature $\sim$ time, field strength $\sim$ space, infinite volume $\sim$ zero-viscosity, phase transition $\sim$ shock wave, critical point $\sim$ shock formation, etc.

Albeverio et al. [AMS94] used parabolic scaling limits of Burgers equation solution in order to discuss the model for the existing large scale structure of the universe. Modern astrophysical data provide clear evidence that matter in the universe is highly non-uniformly distributed. The non-uniformity can be observed at very different scales. Principle elements of the large scale structure are clusters and super-clusters of Galaxies along with the voids between them. The scales in question are from about $1Mpc \approx 3 \times 10^{24} cm$ to a hundred or perhaps a couple of hundred Mpc, see [SZ89]. On the other hand the scales of the large scale structure are still much less than the horizon scale $\approx 10^4 Mpc$, therefore, for the description of motion that result in the large scale structure formation, one can safely use a Newtonian description of mechanics and gravitation, see [Pee80], [ZN83]. The reason for the large scale structure lies in the gravitational instability which shows up in different forms at different stages of the evolution. Gravitational instability means that small density fluctuations grow in an expanding universe under the action of gravity. The large scale structure is actively discussed in the modern literature, see e.g. the reviews [SDZ83], [SZ89], [ZMRS88], [ZMS83], and the numerous references therein. The corresponding model was developed by Zeldovich and his coauthors [ASZ82], [SDZ83], [SZ89], [Zel70], [ZMS83]. Utilizing the equations describing the evolution of density inhomogeneities in the hydro-dynamical approximation, see e.g. [Pee80], in [SZ89] the authors have shown that the evolution of the velocity field U of matter is described by the Burgers equation

$$\begin{aligned}
&\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{3} U_j(t,x) \cdot \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x), \\
&\text{rot} U(t,x) = 0, \quad t \geq 0,\, x \in \mathbb{R}^3, \\
&U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad 1 \leq k \leq 3,
\end{aligned} \tag{3}$$

while the density of matter ρ satisfies the continuity equation

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}(t,x) + \text{div}(\rho U)(t,x) = 0, \quad t \geq 0, \\
&\rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}^3.
\end{aligned} \tag{4}$$

The potential condition $\text{rot} U = 0$ reflects self-gravitation of the medium, and the viscosity parameter $\nu > 0$ its hydro-dynamical friction, i.e. the effect of numerous collisions of particles. The notation in (3) and (4) actually is a little simplified. The original equations in [SZ89] include a certain scaling of the velocity field and density by certain astrophysical quantities such as the Hubble expansion.

The parabolic scaling limit for the solution of Burgers equation is also discussed in [LPW96]. There the authors considered as initial velocity potential G a Gaussian random field with singular spectral density, that means, the spectral density is unbounded at zero. We consider the scaling limit for the solution of Wick type Burgers equation with the same initial condition as in [LPW96] which turns out to have the same distribution as that of Burgers equation with ordinary product.

The present paper is organized as follows. Wick type stochastic partial differential equations are formulated in the framework of white noise analysis or, more generally, Gaussian analysis. In Section 2 we introduce the concepts of Gaussian and white noise analysis as far as necessary in order to formulate the Wick type Burgers equation and calculate scaling limits of its solution. For a detailed exposition of Gaussian and white noise analysis we refer to the monographs [Hid80], [BK95], [HKPS93], [Oba94], [HØUZ96], and [Kuo96]. In Section 2.3, for example, we summarize some facts on Wick calculus in Gaussian analysis and in Lemma 2.11 we prove a continuity property of the Wick inverse. This Lemma is essential for the proof of the main result stated in Theorem 4.3. In Section 3 we present the existence and uniqueness theorem for the Wick type viscous Burgers equation with a stochastic source proved in [HLØ$^+$95]. An analysis of this theorem shows that the Wick type approach admits very singular noises. Then in Section 4 we calculate scaling limits for solutions of Wick type Burgers equation. The main result concerning the limiting behavior of the scaled solution of Wick type Burgers equation is proved in Section 4.1, Theorem 4.3, in Section 4.2 we present two corollaries of Theorem 4.3. In Corollary 4.6 we consider a slightly different scaling limit and in Corollary 4.8 we show that the addition of a compactly supported deterministic initial shape to the random initial data does not influence the scaling limit.

In Section 4.3, Theorem 4.9, we give a probabilistic interpretation of the main result proved in Theorem 4.3 and analyze the relationship of the scaling limit for the solution of Wick type Burgers equation with that of Burgers equation with ordinary product.

2 Gaussian analysis

2.1 Gaussian spaces

We start by considering a standard Gel'fand triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

$\mathcal{H}$ is a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$ and $\mathcal{N}$ is a separable nuclear space densely topologically embedded in $\mathcal{H}$. As is well-known, see e.g. [Pie72] and [Sch71], $\mathcal{N}$ is the projective limit of a family of Hilbert spaces $(\mathcal{H}_p)_{p\in\mathbb{N}}$, such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $\mathcal{H}_p \subset \mathcal{H}_{p_1}$ and $\mathcal{H}_p \subset \mathcal{H}_{p_2}$ and the embeddings are of Hilbert-Schmidt class. I.e., $\mathcal{N}$ is a countably Hilbert space in the sense of [GV68]. The dual space space $\mathcal{N}'$ is the inductive limit of the corresponding dual spaces $(\mathcal{H}_{-p})_{p\in\mathbb{N}}$. We denote by $\langle\cdot,\cdot\rangle$ the dual pairings between $\mathcal{H}_p$ and $\mathcal{H}_{-p}$ and between $\mathcal{N}$ and $\mathcal{N}'$ given by the extension of the inner product $(\cdot,\cdot)$ on $\mathcal{H}$. Furthermore, $|\cdot|_{\pm p}$ denote the norms on $\mathcal{H}_p$ and $\mathcal{H}_{-p}$ respectively and we preserve this notation for the norms on the complexifications $\mathcal{H}_{p,\mathbb{C}}$ and $\mathcal{H}_{-p,\mathbb{C}}$ and tensor powers of these spaces.

Additionally, we introduce the notion of symmetric tensor power of a nuclear space. The simplest way to do this is to start from usual symmetric tensor powers $\mathcal{H}_p^{\hat{\otimes}n}, n \in \mathbb{N}$, of Hilbert spaces. Using the definition

$$\mathcal{N}^{\hat{\otimes}n} := \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p^{\hat{\otimes}n}$$

one can prove, see e.g [Pie72] and [Sch71], that $\mathcal{N}^{\hat{\otimes}n}$ is a nuclear space which is called the n-th symmetric tensor power of $\mathcal{N}$. The dual space $\mathcal{N}'^{\hat{\otimes}n}$ can be written as

$$\mathcal{N}'^{\hat{\otimes}n} = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\hat{\otimes}n}.$$

All the results quoted above also hold for complex spaces.

In addition, we introduce the symmetric (or Boson) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ which is given by the completion of $\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ ($\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}0} := \mathbb{C}$) w.r.t. the Hilbertian norm

$$\| \vec{f} \|^2_{\Gamma(\mathcal{H})} := \sum_{n=0}^{\infty} n! |f^{(n)}|^2, \quad \vec{f} = (f^{(n)})_{n\in\mathbb{N}_0} \in \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}.$$

Example 2.1 *$\mathcal{N}$ can be chosen as the real Schwartz test function space $S(\mathbb{R})$ of rapidly decreasing smooth functions together with $\mathcal{H} = L^2(\mathbb{R},\mathbb{R})$, the space of real valued square integrable functions w.r.t. the Lebesgue measure. This particular choice is the usual one in white noise analysis, see e.g. [HKPS93].*

In order to introduce a probability measure on the vector space $\mathcal{N}'$ we consider the σ-algebra $\mathcal{C}_\sigma(\mathcal{N}')$ generated by cylinder sets. The canonical Gaussian measure μ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ is given by its characteristic function

$$\int_{\mathcal{N}'} \exp(i\langle x, \xi\rangle)\, d\mu(x) = \exp(-\tfrac{1}{2}|\xi|^2), \quad \xi \in \mathcal{N},$$

via Minlos' theorem, see e.g. [BK95], [Hid80] and [HKPS93].

The integral $\int_{\mathcal{N}'} f(x)\, d\mu(x)$ of a measurable function f defined on $\mathcal{N}'$ is called the expectation of f if f is integrable, i.e., the integral $\int_{\mathcal{N}'} |f(x)|\, d\mu(x)$ is finite. The space of functions integrable w.r.t. μ is denoted by $L^1(\mu) = L^1(\mathcal{N}', \mathcal{C}_{\sigma,\mu}(\mathcal{N}'), \mu)$ and the expectation of a function f w.r.t. μ is denoted by $\mathbb{E}_\mu(f)$.

The central space in our setup is the space of complex valued functions which are square integrable w.r.t. this measure $L^2(\mu) = L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$. An element of this space is the Wick exponential

$$\begin{aligned} :\exp(\langle x, \xi\rangle): \;&:=\; \frac{\exp(\langle x, \xi\rangle)}{\mathbb{E}_\mu(\exp(\langle \cdot, \xi\rangle))}, \quad x \in \mathcal{N}', \xi \in \mathcal{N}, \\ &=\; \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes n}:, \xi^{\otimes n}\rangle. \end{aligned} \tag{5}$$

The map $\mathcal{N}' \ni x \mapsto :x^{\otimes n}: \in \mathcal{N}'^{\hat{\otimes} n}$, $n \in \mathbb{N}$, is called the n-th Wick power of $x \in \mathcal{N}'$ ($\langle :x^{\otimes 0}:, \xi^{\otimes 0}\rangle := \xi^{\otimes 0} := 1$), see e.g. [BK95] or [HKPS93]. For any $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, $\varphi^{(0)} \in \mathbb{C}$, we define the smooth Wick monomial of order n corresponding to the kernel $\varphi^{(n)}$ as follows:

$$I(\varphi^{(n)})(x) \;:=\; \langle :x^{\otimes n}:, \varphi^{(n)}\rangle, \quad x \in \mathcal{N}', n \in \mathbb{N}_0.$$

Smooth Wick monomials of different order are orthogonal w.r.t. the standard inner product in $L^2(\mu)$. Furthermore, we can construct Wick monomials $I(f^{(n)})$ with kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ in the sense of measurable functions by using an approximation. More precisely, for any sequence $(\varphi_j^{(n)})_{j\in\mathbb{N}} \subset \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$

which converges to $f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ we have the convergence of $I(\varphi^{(n)})$ to $I(f^{(n)})$ in any $L^p(\mu), p \geq 1$, see e.g. [BK95]. We use $I(f^{(n)}) = \langle : x^{\otimes n} :, f^{(n)} \rangle$, as a formal notation for the monomial introduced above. For Wick monomials associated to the kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ and $h^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} m}$, $n, m \in \mathbb{N}_0$, we have the following orthogonality property:

$$\begin{aligned} \Big(I(f^{(n)}), I(h^{(m)}) \Big)_{L^2(\mu)} &= \int_{\mathcal{N}'} \overline{\langle : x^{\otimes n} :, f^{(n)} \rangle} \langle : x^{\otimes m} :, h^{(m)} \rangle \, d\mu(x) \\ &= \delta_{n,m} \, n! \, (\overline{f^{(n)}}, h^{(n)}) \end{aligned}$$

($\delta_{n,m}$ is the Kronecker delta).

Example 2.2 *(i) Let us consider the white noise triple*

$$S(\mathbb{R}) \subset L^2(\mathbb{R}, \mathbb{R}) \subset S'(\mathbb{R}),$$

see Example 2.1. Within this formalism a version of Wiener's Brownian motion is given by

$$B(t) = \langle \cdot, \mathbf{1}_{[0,t)} \rangle, \quad t > 0, \quad B(0) = 0,$$

where $\mathbf{1}_{[0,t)}$ is the indicator function of the interval $[0, t)$.
(ii) Next we look at d-parameter white noise, $d \in \mathbb{N}$. To do this we need the nuclear triple

$$S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d, \mathbb{R}) \subset S'(\mathbb{R}^d).$$

In this setting a version of the d-parameter Brownian sheet is realized by

$$\begin{aligned} &B(x_1, \ldots, x_d) = \langle \cdot, \mathbf{1}_{[x_1 \wedge 0, 0 \vee x_1]} \cdot \ldots \cdot \mathbf{1}_{[x_d \wedge 0, 0 \vee x_d]} \rangle, \\ &x = (x_1, \ldots, x_d) \in \mathbb{R}^d. \end{aligned}$$

(iii) It is also possible to consider vector valued white noise, see e.g. [SW93]. The starting point is the real separable Hilbert space $L^2_d(\mathbb{R}) := L^2(\mathbb{R}, \mathbb{R}^d), d \in \mathbb{N}$, of vector valued square-integrable functions. In this space we choose the densely embedded nuclear space $S_d(\mathbb{R})$ of vector valued Schwartz test functions. The topology on $S_d(\mathbb{R})$ may be represented by a system of Hilbertian norms

$$|f|_p^2 = \sum_{i=1}^{d} |f_i|_p^2, \quad f = (f_1, \ldots, f_d) \in S_d(\mathbb{R}), \ f_i \in S(\mathbb{R}), 1 \leq i \leq d, p \in \mathbb{N}_0.$$

For notational simplicity we identify $|\cdot|_0$ *with the norm on* $L^2_d(\mathbb{R})$. *Together with the dual space* $S'_d(\mathbb{R})$ *of vector valued tempered distributions we obtain the basic nuclear triple*

$$S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset S'_d(\mathbb{R}).$$

On $S'_d(\mathbb{R})$ *the canonical Gaussian measure* μ_d *is determined by the characteristic function*

$$C(f) = \exp\Big(-\frac{1}{2} \int_{\mathbb{R}} \sum_{i=1}^{d} f_i^2(t)\, dt \Big), \quad f \in S_d(\mathbb{R}).$$

Then we can introduce a d *dimensional Brownian motion by*

$$B(t)(\omega) := \langle \omega, \mathbf{1}_{[0,t)} \rangle := \Big(\langle \omega_i, \mathbf{1}_{[0,t)} \rangle \Big)_{1 \le i \le d}, \quad \omega = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}).$$

2.2 Generalized functions

For our considerations the space $L^2(\mu)$ is too small. A convenient way to solve this problem is to introduce spaces of test functions in $L^2(\mu)$ and to use their larger dual spaces. In Gaussian analysis there exist various triples of test and generalized functions with $L^2(\mu)$ as a central space, here we choose the spaces

$$(\mathcal{N})^1 \subset \mathcal{G}^1 \subset L^2(\mu) \subset \mathcal{G}^{-1} \subset (\mathcal{N})^{-1},$$

see [KLS96] and [GKS97]. In order to construct the spaces of test functions $(\mathcal{N})^1$ and $\mathcal{G}^1$ we define, for any given $p, q \in \mathbb{Z}$, the following Hilbertian norm for the smooth Wick polynomials $\varphi(x) = \sum_{n=0}^{N} \langle :x^{\otimes n}:, \varphi^{(n)} \rangle$, $x \in \mathcal{N}'$, $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$, $N \in \mathbb{N}_0$:

$$\| \varphi \|^2_{p,q,\pm 1} := \sum_{n=0}^{\infty} (n!)^{1\pm 1}\, 2^{nq} \Big| \varphi^{(n)} \Big|^2_p .$$

Then, for $p, q \in \mathbb{N}_0$, we define the Hilbert spaces $(\mathcal{H}_p)^1_q$ and G^1_q as the completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\| \cdot \|_{p,q,1}$ and $\| \cdot \|_{0,q,1}$, respectively. Or, equivalently,

$$\begin{aligned} (\mathcal{H}_p)^1_q &= \Big\{ f \in L^2(\mu) \Big| f(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle, \| f \|^2_{p,q,1} < \infty \Big\}, \\ G^1_q &= \Big\{ f \in L^2(\mu) \Big| f(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f^{(n)} \rangle, \| f \|^2_{0,q,1} < \infty \Big\}. \end{aligned}$$

Finally, the space of test functions $(\mathcal{N})^1$ and $\mathcal{G}^1$ are defined as the projective limit of the spaces $(\mathcal{H}_p)^1_q$ and G^1_q, respectively,

$$(\mathcal{N})^1 = \bigcap_{p,q\geq 0} (\mathcal{H}_p)^1_q, \quad \mathcal{G}^1 = \bigcap_{q\geq 0} G^1_q. \tag{6}$$

Let $(\mathcal{H}_{-p})^{-1}_{-q}$ and G^{-1}_{-q} be the dual w.r.t. $L^2(\mu)$ of $(\mathcal{H}_p)^1_q$ and G^1_q, respectively, and let $(\mathcal{N})^{-1}$ and $\mathcal{G}^{-1}$ be the dual of $(\mathcal{N})^1$ and $\mathcal{G}^1$, respectively. We know from general duality theory that

$$(\mathcal{N})^{-1} = \bigcup_{p,q\geq 0} (\mathcal{H}_{-p})^{-1}_{-q}, \quad \mathcal{G}^{-1} = \bigcup_{q\geq 0} G^{-1}_{-q}. \tag{7}$$

The bilinear dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $(\mathcal{N})^1$ and $(\mathcal{N})^{-1}$ is connected to the sesquilinear inner product on $L^2(\mu)$ by

$$\langle\!\langle f, \varphi \rangle\!\rangle = (\overline{f}, \varphi)_{L^2(\mu)}, \quad f \in L^2(\mu), \, \varphi \in (\mathcal{N})^1. \tag{8}$$

Since the constant function 1 is in $(\mathcal{N})^1$ we may extend the concept of expectation from integrable functions to distributions $\Phi \in (\mathcal{N})^{-1}$:

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle.$$

The chaos decomposition introduces the following natural decomposition of $\Phi \in (\mathcal{N})^{-1}$. Let $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$ be given. Then there exists a distribution $I(\Phi^{(n)})$ acting on test functions $\varphi \in (\mathcal{N})^1$ as

$$\left\langle\!\!\left\langle I(\Phi^{(n)}), \varphi \right\rangle\!\!\right\rangle = n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle.$$

We use $I(\Phi^{(n)}) = \langle : x^{\otimes n} :, \Phi^{(n)} \rangle$, as a formal notation for the distribution introduced above. Any $\Phi \in (\mathcal{N})^{-1}$ then has the unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \Phi^{(n)} \rangle,$$

where the sum converges in $(\mathcal{N})^{-1}$ and we have

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle \Phi^{(n)}, \varphi^{(n)} \rangle, \quad \varphi \in (\mathcal{N})^{-1},$$

see [KLS96].

From the definition we know that every distribution is of finite order, i.e., for every $\Phi \in \mathcal{G}^{-1}$ there exists $q \in \mathbb{N}_0$ such that $\Phi \in G^{-1}_{-q}$. Furthermore, it is not hard to see that G^{-1}_{-q} is a Hilbert space which can be described as follows

$$G^{-1}_{-q} = \left\{ \Phi \in (\mathcal{N})^{-1} \middle| \Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}, \| \Phi \|^2_{0,-q,-1} < \infty \right\}. \tag{9}$$

This description of G^{-1}_{-q} (and therefore also of $\mathcal{G}^{-1}$) shows that its elements have the property that the Wick monomials in their generalized chaos decomposition are square-integrable functions. This is the characteristic feature of so called regular generalized function which enables to endow spaces of them with a probabilistic structure, see [GKS97].

In $\mathcal{G}^1$ and $\mathcal{G}^{-1}$ we can define the operator $\Gamma(A)$ and the second quantization $d\Gamma(A)$ of a bounded operator A defined on $\mathcal{H}$. Let $A^{\otimes n}$, $n \in \mathbb{N}_0$, be the n-th tensor power of the operator A, see e.g. [RS75]. It is well-known that this operator can be extended to a bounded operator on $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ with norm $\| A \|^n_{op}$, where $\| A \|_{op}$ is the norm of the operator A. Then for $\Phi \in G^{\pm 1}_q$, $q \in \mathbb{Z}$, we define

$$\Gamma(A)\Phi := \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, A^{\otimes n} \Phi^{(n)} \rangle. \tag{10}$$

The image $\Gamma(A)\Phi$ is in $G^{\pm 1}_{q_1}$, $q_1 \in \mathbb{Z}$, if we choose $q_1 \leq q - 2\ln_2(\| A \|_{op})$. This follows from the estimate

$$\| \Gamma(A)\Phi \|^2_{0,q_1,\pm 1} = \sum_{n=0}^{\infty} (n!)^{1\pm 1} \, 2^{nq_1} \, \| A \|^{2n}_{op} \left| \Phi^{(n)} \right|^2 \leq \| \Phi \|^2_{0,q,\pm 1} \, .$$

Now under use of the representation of $\mathcal{G}^1$ and $\mathcal{G}^{-1}$ in (6) and (7), respectively, we can easily conclude that $\Gamma(A)$ is an well-defined operator in $\mathcal{G}^1$ and in $\mathcal{G}^{-1}$. The second quantization $d\Gamma(A)$ of the operator A we define to be the operator which acts on the kernel of the n-th chaos by the operator

$$A \otimes Id \otimes \ldots \otimes Id + Id \otimes A \otimes Id \otimes \ldots \otimes Id + \ldots + Id \otimes \ldots \otimes Id \otimes A$$

(Id denotes the identity operator on $\mathcal{H}$). In a similar way as for the operator $\Gamma(A)$ we can show that the second quantized operator $d\Gamma(A)$ is well-defined in $\mathcal{G}^1$ and in $\mathcal{G}^{-1}$. $N := d\Gamma(Id)$ for example is the number operator.

A useful tool in order to characterize $(\mathcal{N})^{-1}$ and $\mathcal{G}^{-1}$ is the S-transform. The S-transform of elements from $(\mathcal{N})^{-1}$ is defined as the dual paring with the Wick exponential, see (5). Since the Wick exponential is not an element of $(\mathcal{N})^1$ the S-transform of an element Φ from $(\mathcal{N})^{-1}$ is defined only locally, i.e.,

$$S\Phi(\eta) := \langle\!\langle \Phi, :\exp(\langle \cdot, \eta \rangle):\rangle\!\rangle, \quad \eta \in \mathcal{O} \subset \mathcal{N}_{\mathbb{C}},$$

where $\mathcal{O}$ is an open neighborhood of zero depending on $\Phi \in (\mathcal{N})^{-1}$. In the characterization theorem it is proved that the S-transform maps $(\mathcal{N})^{-1}$ isomorphically in to certain space of holomorphic functions, see [KLS96]. Of course, the S-transform is also defined locally for elements from $\mathcal{G}^{-1}$ and it gives an isomorphism between $\mathcal{G}^{-1}$ and an infinite dimensional version of th Hardy space, see [GKS97].

Example 2.3 *(i) Again we consider the white noise triple. Since the Dirac delta function at time $t \geq 0$ is a tempered distribution, i.e., $\delta_t \in S'(\mathbb{R})$, the generalized function*

$$\omega(t) := \langle \omega, \delta_t \rangle, \quad t \geq 0,$$

exists as an element of $(S(\mathbb{R}))^{-1}$. The generalized process $(\omega(t))_{t\geq 0}$ is the derivative of Brownian motion, see Example 2.2(i), in the weak sense, it is the Gaussian white noise process.
(ii) Within the formalism of $d+1$-parameter white noise we have a time-space white noise which can be realized as the derivative of the Brownian sheet, see Example 2.2(ii),

$$\omega(t, x_1, \ldots, x_d) = \frac{\partial^{d+1} B(t, x_1, \ldots, x_d)}{\partial t \partial x_1 \ldots \partial x_d}, \quad t, x_1, \ldots, x_d \geq 0,$$

in the weak sense. It has the chaos decomposition given by

$$\omega(t, x_1, \ldots, x_d) = \langle \omega, \delta_{t,x_1,\ldots,x_d} \rangle,$$

where $\delta_{t,x_1,\ldots,x_d} \in S'(\mathbb{R}^{d+1})$ is the Dirac delta function at $(t, x_1, \ldots, x_d)$. Thus, it is an element of $(S(\mathbb{R}^{d+1}))^{-1}$.

2.3 Wick calculus

Here we summarize some facts on Wick calculus in Gaussian analysis which originally has been derived in the space $(\mathcal{N})^{-1}$, see [KLS96]. In [GKU98] it has been shown that the Wick calculus can also be constructed as an invariant calculus in $\mathcal{G}^{-1}$. Since here we mainly work in the space $\mathcal{G}^{-1}$ we formulate all definitions and results in this space.

Definition 2.4 *Let $\Phi, \Psi \in \mathcal{G}^{-1}$. Then the Wick product is defined by*

$$\Phi \diamond \Psi := S^{-1}(S\Phi \cdot S\Psi) \in \mathcal{G}^{-1}.$$

This multiplication is clearly associative and for deterministic functions it coincides with the point-wise product.

Remark 2.5 *In terms of the chaos decomposition the Wick product may be described as follows: Let $\Phi, \Psi \in \mathcal{G}^{-1}$ correspond to the sequences of kernels $(\Phi^{(n)})_{n\in\mathbb{N}_0}$ and $(\Psi^{(n)})_{n\in\mathbb{N}_0}$, respectively. Then the Wick product $\Phi \diamond \Psi$ corresponds to $(\Xi^{(n)})_{n\in\mathbb{N}_0}$ where*

$$\Xi^{(n)} = \sum_{k+l=n} \Phi^{(k)} \hat{\otimes} \Psi^{(l)}.$$

By induction, we can define Wick powers

$$\Phi^{\diamond n} = S^{-1}((S\Phi)^n)$$

in $\mathcal{G}^{-1}$ and by taking finite linear combinations of them also Wick polynomials of finite order $\sum_{n=1}^{N} a_n \Phi^{\diamond n}$ can be defined in $\mathcal{G}^{-1}$.

In white noise analysis the Wick product is local in the following sense.

Proposition 2.6 *Let $\Phi_i \in \Gamma(\mathbf{1}_{A_i})\mathcal{G}^{-1}$, $1 \le i \le n$, $n \in \mathbb{N}$, then*

$$\Phi_1 \diamond \ldots \diamond \Phi_n \in \Gamma(\mathbf{1}_{\cup_{i=1}^n A_i})\mathcal{G}^{-1},$$

where A_i, $1 \le i \le n$, are Borel measurable subsets of $\mathbb{R}$ and $\mathbf{1}_A$ denotes the projection operator in $L^2(\mathbb{R})$ given by the product with the indicator function of the subset $A \subset \mathbb{R}$. The operator $\Gamma(\mathbf{1}_A)$ we defined in (10).

Proof: For $n = 1$ this is trivial. Let us assume that the statement of this proposition holds for $n-1$. We define $\Psi = \Phi_1 \diamond \ldots \diamond \Phi_{n-1} \in \Gamma(\mathbf{1}_{\cup_{i=1}^{n-1} A_i})\mathcal{G}^{-1}$ and obtain under use of Remark 2.5 for the m-th chaos

$$(\Phi_1 \diamond \ldots \diamond \Phi_n)^{(m)} = (\Psi \diamond \Phi_n)^{(m)} = \sum_{k+l=m} \Psi^{(k)} \hat{\otimes} \Phi_n^{(l)}, \quad m \in \mathbb{N}_0. \tag{11}$$

By assumption,

$$\Psi^{(k)} \in (\mathbf{1}_{\cup_{i=1}^{n-1} A_i} L^2(\mathbb{R}))^{\hat{\otimes} k}, \quad \Phi_n^{(l)} \in (\mathbf{1}_{A_n} L^2(\mathbb{R}))^{\hat{\otimes} l}, \quad k, l, \in \mathbb{N}_0. \tag{12}$$

From (11) together with (12) we can conclude that

$$(\Phi_1 \diamond \ldots \diamond \Phi_n)^{(m)} \in (\mathbf{1}_{\cup_{i=1}^{n} A_i} L^2(\mathbb{R}))^{\hat{\otimes} m}, \quad m \in \mathbb{N}_0.$$

Thus, the statement of Proposition 2.6 is true for n and Proposition 2.6 is proved by induction. ■

Not only Wick polynomials can be defined in $\mathcal{G}^{-1}$, it is even possible to define Wick analytic functions in $\mathcal{G}^{-1}$ under very general assumptions.

Theorem 2.7 *Let F be analytic in a neighborhood of the point $z_0 = \mathbb{E}_\mu(\Phi)$ in $\mathbb{C}$, $\Phi \in \mathcal{G}^{-1}$. Then $F^\diamond(\Phi)$ defined as $F^\diamond(\Phi) := S^{-1}(F(S\Phi))$ exists in $\mathcal{G}^{-1}$.*

Remark 2.8 *Let F be analytic at $z_0 = \mathbb{E}_\mu(\Phi)$, $\Phi \in \mathcal{G}^{-1}$, i.e., F has the power series representation $F(z) = \sum_n a_n(z - z_0)^n$, $z, a_n \in \mathbb{C}$. Then the Wick series $\sum_n a_n(\Phi - z_0)^{\diamond n}$ converges in $\mathcal{G}^{-1}$ and*

$$F^\diamond(\Phi) = \sum_{n=1}^{\infty} a_n(\Phi - z_0)^{\diamond n}.$$

In addition, we need the following two lemmas.

Lemma 2.9 *The Wick product is continuous on $\mathcal{G}^{-1}$, in particular the following estimate holds for $\Phi \in G^{-1}_{-q_1}$, $\Psi \in G^{-1}_{-q_2}$, $q = \max(q_1, q_2) + 1$:*

$$\| \Phi \diamond \Psi \|_{0,-q,-1} \leq C \| \Phi \|_{0,-q_1,-1} \| \Psi \|_{0,-q_2,-1} .$$

Lemma 2.10 *Let $\Phi \in \mathcal{G}^{-1}$ such that $\mathbb{E}_\mu(\Phi) = \Phi^{(0)} \neq 0$. Then $\Phi^{\diamond(-1)} \in \mathcal{G}^{-1}$. Moreover, if $\Phi \in G^{-1}_{-q_1}$ and*

$$q > q_1 + 3 + \log_2(1 \vee c), \quad c = |\Phi^{(0)}|^{-1} \| \Phi - \Phi^{(0)} \|_{0,-q_1,-1},$$

then the following estimate holds

$$\| \Phi^{\diamond(-1)} \|^2_{0,-q,-1} \leq 2|\Phi^{(0)}|^{-2}. \tag{13}$$

The next lemma is essential for proving the main result of this paper stated in Theorem 4.3.

Lemma 2.11 *The Wick inverse* $\Phi^{\diamond(-1)}, \Phi \in \mathcal{G}^{-1}$*, is continuous at the point* $\Phi_0 = 1 \in \mathcal{G}^{-1}$*. Moreover, if* $\Phi \in G^{-1}_{-q_1}, \mathbb{E}_\mu(\Phi) \neq -1$*, and*

$$q > q_1 + 1 + \log_2(1 \vee c), \quad c = |\Phi^{(0)} + 1|^{-1} \parallel \Phi - \Phi^{(0)} \parallel_{0,-q_1,-1},$$

then the following estimate holds

$$\begin{aligned} \parallel (\Phi+1)^{\diamond(-1)} - 1 \parallel_{0,-q,-1} \;\leq\; & |\Phi^{(0)}+1|^{-1}|\Phi^{(0)}| \\ + \; & \sqrt{2}|\Phi^{(0)}+1|^{-2} \parallel \Phi - \Phi^{(0)} \parallel_{0,-q_1,-1} . \end{aligned} \tag{14}$$

Proof: It is sufficient to prove (14) because this estimate implies continuity of the Wick inverse at the point $1 \in \mathcal{G}^{-1}$. We have

$$\begin{aligned} (\Phi+1)^{\diamond(-1)} - 1 &= -1 + (\Phi^{(0)}+1)^{-1} \sum_{n=0}^{\infty} (-1)^n (\Phi^{(0)}+1)^{-n} (\Phi - \Phi^{(0)})^{\diamond n} \\ &= -(\Phi^{(0)}+1)^{-1} \Big(\Phi^{(0)} + (\Phi - \Phi^{(0)}) \diamond (\Phi+1)^{\diamond(-1)} \Big) \end{aligned}$$

by using Remark 2.8 and Lemma 2.9. Now from Lemma 2.9 and 2.10 we can conclude

$$\begin{aligned} & \parallel (\Phi+1)^{\diamond(-1)} - 1 \parallel_{0,-q,-1} \\ \leq\; & |\Phi^{(0)}+1|^{-1} \Big(|\Phi^{(0)}| + \parallel \Phi - \Phi^{(0)} \parallel_{0,-q_1,-1} \parallel (\Phi+1)^{\diamond(-1)} \parallel_{0,-q_2,-1} \Big) \\ \leq\; & |\Phi^{(0)}+1|^{-1}|\Phi^{(0)}| + \sqrt{2}|\Phi^{(0)}+1|^{-2} \parallel \Phi - \Phi^{(0)} \parallel_{0,-q_1,-1}, \end{aligned}$$

where $q_2 = q - 1$. ■

2.4 Itô/Skorohod integration

One of the most striking features of the Wick product is its relation to Itô/Skorohod integration. Let $\omega(s)$, $s \in [0,t]$, be the Gaussian white noise process, see Example 2.3(i), and $B(s)$, $s \in [0,t]$, the Wiener process, then for any Skorohod integrable process $\Phi(s)$, $s \in [0,t]$, in $L^2(\mu)$ can be shown that

$$\int_0^t \Phi(s) \diamond \omega(s)\, ds = \int_0^t \Phi(s)\, \delta B(s). \tag{15}$$

The integral on the left hand side of (15) is a Pettis integral in $(S(\mathbb{R}))^{-1}$. In the case where the integrand is a Bochner integrable function with values in a Hilbert space $(\mathcal{H}_{-p})^{-1}_{-q}$, $p, q \in \mathbb{N}_0$, its Pettis integral as a $(\mathcal{N})^{-1}$-valued function coincides with its Bochner integral as a $(\mathcal{H}_{-p})^{-1}_{-q}$-valued function.

The integral on the right hand side of (15) is the Skorohod integral (which coincides with the Itô integral if Φ_s is adapted). See [LØU92], [HKPS93], and [Ben93] for a discussion of this relation. We can say that the functional integration in $(S(\mathbb{R}))^{-1}$ involving Wick product with $\omega(s)$, $s \in [0, t]$, generalizes the Skorohod/Itô integration.

3 The viscous Burgers equation with a stochastic source

In this section we recall the existence and uniqueness theorem for the Wick type viscous Burgers equation with a stochastic source derived in [HLØ+95]. For a general introduction to the theory of Wick type stochastic partial differential equations we refer to the monograph [HØUZ96]. The equations are formulated in the space $(\mathcal{N})^{-1}$ and the differentiation is defined in the weak sense. Let us discuss this in more details.

Let $\mathcal{O}$ be an open set in $\mathbb{R}^d, d \in \mathbb{N}$. Consider the mapping

$$\Phi(\cdot) : \mathcal{O} \to (\mathcal{N})^{-1}.$$

We say that Φ is weakly differentiable at $x_0 \in \mathcal{O}$ in direction e_k, $k = 1, \ldots, d$, if

$$\lim_{h \to 0} \frac{\Phi(x_0 + he_k) - \Phi(x_0)}{h}$$

exists as a weak limit in $(\mathcal{N})^{-1}$. Here we have set $(e_k)_{1 \le k \le d}$ as the canonical orthonormal basis of $\mathbb{R}^d$ and we denote

$$\frac{\partial \Phi}{\partial x_k}(x_0) = \lim_{h \to 0} \frac{\Phi(x_0 + he_k) - \Phi(x_0)}{h}.$$

If Φ has weak partial derivatives for any $x \in \mathcal{O}$ and in all directions e_k, $k = 1, \ldots, d$, such that for all $k = 1, \ldots, d$, $x \mapsto \partial\Phi/\partial x_k(x)$ is weakly continuous from $\mathcal{O}$ into $(\mathcal{N})^{-1}$, we say that Φ is weakly continuously differentiable in $\mathcal{O}$. We can repeat the preceding considerations for higher derivatives.

Using the Cole-Hopf transform the Burgers equation with ordinary product is transformed into the heat equation. The same concept also works for the Wick type equation. Hence, let us consider the Wick type stochastic heat equation with a stochastic potential

$$\begin{aligned}
&\frac{\partial Y}{\partial t}(t,x) = \frac{1}{2}\nu\Delta Y(t,x) + Y(t,x)\diamond V(t,x), \quad t>0,\, x\in\mathbb{R}^d,\\
&Y(0,x) = \exp^{\diamond}\Big(-\frac{G(x)}{2\nu}\Big), \quad x\in\mathbb{R}^d. \qquad (16)
\end{aligned}$$

Here $\nu > 0$, $d \in \mathbb{N}$, and the equation (16) is formulated in $(\mathcal{N})^{-1}$.

Let us suppose that V and G are weakly continuous $(\mathcal{N})^{-1}$-valued functions. Under this assumption it has been proved in [HLØ+95] that

$$\begin{aligned}
Y(t,x) \;=\; &\int_{S'_d(\mathbb{R})} \exp^{\diamond}\Big(\frac{G(\sqrt{2\nu}(B(t)(\hat\omega)+x))}{2\nu}\Big) \qquad (17)\\
&\diamond\; \exp^{\diamond}\Big(\int_0^t V(t-s,\sqrt{2\nu}(B(s)(\hat\omega)+x))\,ds\Big)\,d\mu_d(\hat\omega).
\end{aligned}$$

is the unique $(\mathcal{N})^{-1}$-solution of the stochastic heat equation (16).

Remark 3.1 *(i) Formula (17) is a consequence of the Feynman-Kac formula. The integrand*

$$\exp^{\diamond}\Big(\frac{G(\sqrt{2\nu}(B(t)(\hat\omega)+x))}{2\nu}\Big)\diamond\exp^{\diamond}\Big(\int_0^t V(t-s,\sqrt{2\nu}(B(s)(\hat\omega)+x))\,ds\Big),$$

where $\hat\omega \in S'_d(\mathbb{R})$, is a function from $S'_d(\mathbb{R})$ to $(\mathcal{N})^{-1}$ and the integral exists as a Pettis integral in $(\mathcal{N})^{-1}$.
(ii) A priori the initial condition $Y(0)$ and the stochastic potential V can be chosen as arbitrary weakly continuous functions having values in $(\mathcal{N})^{-1}$. In order to choose the initial condition $Y(0)$ independent of the stochastic potential V a natural choice for the underlying nuclear space is a product space $\mathcal{N}\times\mathcal{M}$ given by the product of two nuclear spaces $\mathcal{N}$ and $\mathcal{M}$.

Now we are prepared to state the existence and uniqueness theorem proved in [HLØ+95], Theorem 5.1.

Theorem 3.2 *Let Y be the $(\mathcal{N})^{-1}$-process given by (17) solving the stochastic heat equation (16), where V and G are $(\mathcal{N})^{-1}$-valued functions, weakly continuously differentiable in x, V weakly continuous w.r.t. t. Then the generalized stochastic processes*

$$U^k(t,x) = -\frac{2\nu}{\lambda}\frac{\partial \log^\diamond(Y)}{\partial x^k}(t,x), \quad 1 \le k \le d, \tag{18}$$

belong to $(\mathcal{N})^{-1}$ for $t \ge 0, x \in \mathbb{R}^d$, and $U = (U_1, \ldots, U_d)$ solves the stochastic Burgers equation

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \diamond \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x) + F_k(t,x), \quad t > 0,$$
$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d, 1 \le k \le d,$$

where

$$F_k(t,x) = -\frac{2\nu}{\lambda}\frac{\partial V}{\partial x_k}(t,x).$$

Here $\lambda, \nu > 0$ and $d \in \mathbb{N}$.

Example 3.3 *Theorem 3.2 admits forces given by very singular noises. For instance the potential*

$$V(t,x) = -\frac{\lambda}{2\nu}\int_0^x \langle \cdot, \delta_{t,y}\rangle \, dy \in (S(\mathbb{R}^2))^{-1}, \quad \delta_{t,x} \in S'(\mathbb{R}^2),$$

fulfills the requirements of Theorem 3.2. The corresponding force

$$F(t,x) = \langle \cdot, \delta_{t,x}\rangle \in (S(\mathbb{R}^2))^{-1}$$

is the time-space white noise discussed in Example 2.3(ii).

4 Scaling limits

Now we are prepared to calculate scaling limits for the solution of Wick type Burgers equation. Let us consider the Burgers equation with random initial data:

$$\frac{\partial U_k}{\partial t}(t,x) + \lambda \sum_{j=1}^{d} U_j(t,x) \diamond \frac{\partial U_k}{\partial x_j}(t,x) = \nu \Delta U_k(t,x), \quad t > 0,$$

$$U_k(0,x) = \frac{\partial G}{\partial x_k}(x), \quad x \in \mathbb{R}^d, 1 \leq k \leq d. \tag{19}$$

From Theorem 3.2 we know that its solution is given by

$$U_k(t,x) = I_k(t,x) \diamond J(t,x)^{\diamond(-1)}, \quad t > 0,\, x \in \mathbb{R}^d, \tag{20}$$

where

$$J(t,x) = Y(t,x) = \int_{\mathbb{R}^d} K(t,x-y) \exp^{\diamond}\Big(- \frac{G(y)}{2\nu}\Big)\, dy, \tag{21}$$

$$\begin{aligned} I_k(t,x) &= -\frac{2\nu}{\lambda}\frac{\partial Y}{\partial x_k}(t,x) \\ &= \int_{\mathbb{R}^d} \frac{x_k - y_k}{\lambda t} K(t,x-y) \exp^{\diamond}\Big(- \frac{G(y)}{2\nu}\Big)\, dy, \end{aligned} \tag{22}$$

and where

$$K(t,x) = (4\pi\nu t)^{-d/2} \exp\Big(- \frac{|x|^2}{4\nu t}\Big), \quad x \in \mathbb{R}^d,\, t > 0,$$

is the Gaussian heat kernel.

In order to formulate the conditions for the initial velocity potential G we need the following definition.

Definition 4.1 *A function L is said to be slowly varying at infinity if it is positive and measurable on $[a, \infty)$, for some $a > 0$, and if for each $p > 0$*

$$\lim_{s\to\infty} \frac{L(ps)}{L(s)} = 1.$$

Properties of slow varying functions are discussed in e.g. [Sen76].

Condition (A) The initial velocity potential G is a zero-mean, measurable, mean-square differentiable, homogeneous and isotropic complex Gaussian random field on $\mathbb{R}^d \times S'(\mathbb{R}^d)$. In addition, its variance $\mathbb{E}_\mu(\overline{G(x)}G(x)) = 1$, and its covariance is of the form

$$C(|x|) = \mathbb{E}_\mu(\overline{G(0)}G(x)) = \frac{L(|x|)}{|x|^\alpha}, \quad x \in \mathbb{R}^d,\, 0 < \alpha < d,$$

where the function L is slowly varying at infinity, bounded, and there exist $b, c > 0$ such that $L(t) \geq c$ for all $t \geq b$.

Condition (B) The initial velocity potential G has a spectral density $f(p) = \tilde{f}(|p|) \geq 0$, $p \in \mathbb{R}^d$, i.e.,

$$\begin{aligned} G(x,\omega) &= \langle \omega, \exp(i \cdot x) f^{1/2}(\cdot) \rangle, \quad x \in \mathbb{R}^d,\, \omega \in S'(\mathbb{R}^d) \\ &= \int_{\mathbb{R}^d} \exp(ipx) f^{1/2}(p)\, dB(p,\omega), \end{aligned}$$

where B is the d-parameter Brownian sheet, see Example 2.2(ii). The expression px, $p, x \in \mathbb{R}^d$, symbolizes the scalar product in $\mathbb{R}^d$. As a consequence of condition A together with the Tauberian theorem, see [LO91], the spectral density f has a singularity at zero. We assume that this singularity is integrable, the value of the integral of f is one, f is a bounded continuous function on $\mathbb{R}^d \backslash \{p \in \mathbb{R}^d \,|\, |p| < \epsilon\}$ for all $\epsilon > 0$, and there exist $p_0 > 0$ such that f is a decreasing function for $|p| \geq p_0$.

Before we can state the main theorem we have to introduce the vector valued Gaussian random field

$$X(t,a,\omega) = i\left(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\right)^{1/2} \langle \omega, t^{d/4} \exp(i\sqrt{t} \cdot a) g(\sqrt{t}\cdot) \rangle, \quad a \in \mathbb{R}^d, \tag{23}$$

$t > 0$, $\omega \in S'(\mathbb{R}^d)$, where

$$\begin{aligned} g(p) &= \frac{\exp(-\nu|p|^2)}{|p|^{(d-\alpha)/2}} \frac{p}{\lambda}, \quad p \in \mathbb{R}^d, \\ c_1(d,\alpha) &= 2^\alpha \Gamma\left(1 + \frac{\alpha}{2}\right)\Gamma\left(\frac{d}{2}\right) \Big/ \left(\frac{d-\alpha}{2}\right), \\ c_2(d) &= 2\pi^{d/2}/\Gamma(n/2), \end{aligned} \tag{24}$$

the latter constant is the area of the unit sphere in $\mathbb{R}^d$ (Γ is the well-known gamma function).

Remark 4.2 *Calculating the covariance of the Gaussian random vectors* $X(t,a_1)$ *and* $X(t,a_2)$, $a_1, a_2 \in \mathbb{R}^d$, *one easily finds that it is independent of* $t > 0$. *Hence, finite dimensional distributions of the random field* $X(t,a)$, $a \in \mathbb{R}^d$, *coincide with finite dimensional distributions of the random field* $X(a) :=$

$X(1,a)$, $a \in \mathbb{R}^d$, *for all* $t > 0$. *In particular, the random field*

$$\begin{aligned} X(a,\omega) &= i\Big(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\Big)^{1/2} \langle \omega, \exp(i\cdot a)g(\cdot)\rangle, \quad a \in \mathbb{R}^d, \\ &= i\Big(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\Big)^{1/2} \int_{\mathbb{R}^d} \exp(ipa)g(p)\, dB(p,\omega), \end{aligned}$$

$\omega \in S'(\mathbb{R}^d)$, *is a homogeneous Gaussian random field with mean zero.*

4.1 Main result

Theorem 4.3 *Let* $U(t,x)$, $t > 0$, $x \in \mathbb{R}^d$, $d \geq 1$, *be the solution of the initial value problem (19) with random initial data satisfying condition (A) and (B). Then for* α *as in condition (A) each component of the vector valued generalized random field*

$$Y(t,a) = \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U(t,\sqrt{t}a) - X(t,a)$$

converges uniformly in $a \in \mathbb{R}^d$ *to* $0 \in \mathcal{G}^{-1}$ *as* $t \to \infty$ *w.r.t. inductive limit topology. I.e., there exists a Hilbert space norm* $\|\cdot\|_{0,-q,-1}$, $q \in \mathbb{N}$, *such that*

$$\lim_{t\to\infty} \| Y_k(t,a) \|_{0,-q,-1} = 0$$

uniformly in $a \in \mathbb{R}^d$ *for all* $1 \leq k \leq d$.

Remark 4.4 *(i) As we already mentioned in the introduction there are several papers in which scaling limits for the solution of Burgers equation with ordinary product have been calculated. Under the use of condition (A) only in e.g. [LOR94] weak convergence of finite dimensional distributions of the field*

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U(t,\sqrt{t}a), \quad a \in \mathbb{R}^d, \tag{25}$$

as $t \to \infty$ *to a centered Gaussian random field has been proved (here the solution of Burgers equation with ordinary product is also denoted by* U*).*
(ii) Using condition (A) and (B) in [LPW96], Theorem 2.2, it has been proved that finite dimensional distributions of the field (25) converge weakly

as $t \to \infty$ to finite dimensional distributions of the Gaussian random field $X(a), a \in \mathbb{R}^d$, see Remark 4.2. A more detailed comparison of the scaling limit for the solution of the Wick type equation with that for the equation with ordinary product we give in Section 4.3.
(iii) Note that the solutions of the Wick type Burgers equation and the Burgers equation with ordinary product for finite times $t > 0$ are completely different random processes. Since the solution of the Wick type Burgers equation are generalized functions in the variable giving the randomness, they are in general not measurable functions and so we at first have to work out another kind of convergence. In Section 4.3 we derive consequences of the type of convergence we prove in this theorem, i.e., we interpret this convergence in terms of types of convergence which are well-known in probability theory.
(iv) Consider the initial value problem (19). The parameter λ is giving the nonlinearity of the Burgers equation. Notice that the limiting Gaussian random field X still depends on the parameter λ, see (23) and (24).

Proof of Theorem 4.3:
Step 1: Let us define

$$\begin{aligned} & D(t,a) := J(t, \sqrt{t}a) \\ = \ & (4\pi\nu t)^{-d/2} \int_{\mathbb{R}^d} \exp^\diamond \left(- \frac{|\sqrt{t}a - y|^2}{4\nu t} - \frac{G(y)}{2\nu} \right) dy \\ = \ & (4\pi\nu)^{-d/2} \int_{\mathbb{R}^d} \exp^\diamond \left(- \frac{|a-y|^2}{4\nu} - \frac{G(\sqrt{t}y)}{2\nu} \right) dy \\ = \ & \int_{\mathbb{R}^d} \exp^\diamond \left(- \frac{G(\sqrt{t}(a-y))}{2\nu} \right) d\gamma_{2\nu}(y), \quad t > 0, \, x \in \mathbb{R}^d, \end{aligned} \tag{26}$$

where $\gamma_{2\nu}$ is the centered Gaussian measure on $\mathbb{R}^d$ with variance 2ν. Our aim is to show that $D(t,a)$ converges to 1 in $L^2(\mu)$ as $t \to \infty$. Since the initial velocity potential G is a centered Gaussian random variable we have $\exp^\diamond(G) = :\exp(G):$. By the assumptions on the initial potential G the Wick exponential $:\exp(G(x)):$ as a function of $x \in \mathbb{R}^d$ is bounded in $L^2(\mu)$. Therefore, we can interpret the integral (26) as a Bochner integral in $L^2(\mu)$. Then under use of the chaos decomposition of the Wick exponential we obtain

$$\begin{aligned} & \| D(t,a) - 1 \|^2_{L^2(\mu)} \\ = \ & \sum_{n=1}^{\infty} \frac{1}{n!} (2\nu)^{-2n} \int_{\mathbb{R}^{2d}} C(\sqrt{t}|y-z|)^n \, d\gamma_{2\nu}(y) \, d\gamma_{2\nu}(z) \end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!}(2\nu)^{-2n} \int_{\mathbb{R}^{2d}} |C(\sqrt{t}|y-z|)| \, d\gamma_{2\nu}(y) \, d\gamma_{2\nu}(z)$$

$$\leq \exp\left(\frac{1}{4\nu^2}\right) \int_{\mathbb{R}^{2d}} |C(\sqrt{t}|y-z|)| \, d\gamma_{2\nu}(y) \, d\gamma_{2\nu}(z),$$

since the modulus of the covariance C is bounded by 1. Condition (A) implies that $\lim_{t\to\infty} C(t) = 0$ and therefore $\lim_{t\to\infty} |C(\sqrt{t}|y-z|)| = 0$ almost everywhere w.r.t. the measure $\gamma_{2\nu} \times \gamma_{2\nu}$. Thus, under use of Lebesgue's dominated convergence theorem we obtain

$$\lim_{t\to\infty} D(t,a) = 1$$

in $L^2(\mu)$. Obviously, this convergence is uniform in $a \in \mathbb{R}^d$.
Step 2: Next we examine the limit behavior of

$$\begin{aligned} N_k(t,a) &:= \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} I_k(t, \sqrt{t}a), \quad t > 0,\, x \in \mathbb{R}^d,\, 1 \leq k \leq d, \\ &= \frac{t^{\alpha/4}}{L^{1/2}(\sqrt{t})} \int_{\mathbb{R}^d} \frac{y_k}{\lambda} \exp^{\diamond}\left(- \frac{G(\sqrt{t}(a-y))}{2\nu}\right) d\gamma_{2\nu}(y). \end{aligned}$$

Here our goal is to prove that $N_k(t,a) - X_k(t,a)$ converges to zero in $L^2(\mu)$ as t tends to infinity. The square-norm of this difference splits into a sum of three parts:

$$\begin{aligned} &\| N_k(t,a) - X_k(t,a) \|^2_{L^2(\mu)} \\ =\; & \frac{t^{\alpha/2}}{L(\sqrt{t})} \left(\int_{\mathbb{R}^d} \frac{y_k}{\lambda} \, d\gamma_{2\nu}(y) \right)^2 + \| \tilde{N}_k(t,a) - X_k(t,a) \|^2_{L^2(\mu)} \qquad (27) \\ +\; & \sum_{n=2}^{\infty} \frac{1}{n!}(2\nu)^{-2n} \frac{t^{\alpha/2}}{L(\sqrt{t})} \int_{\mathbb{R}^{2d}} \frac{y_k z_k}{\lambda^2} C(\sqrt{t}|y-z|)^n \, d\gamma_{2\nu}(y) \, d\gamma_{2\nu}(z), \end{aligned}$$

where

$$\tilde{N}_k(t,a) = -\frac{t^{\alpha/4}}{2\nu L^{1/2}(\sqrt{t})} \int_{\mathbb{R}^d} \frac{y_k}{\lambda} G(\sqrt{t}(a-y)) \, d\gamma_{2\nu}(y).$$

The first part of the sum in (27) is identically to zero for all $t > 0$ because the integral is equal to zero. For the third part of the sum in (27) we have the following estimate

$$\begin{aligned}
& \sum_{n=2}^{\infty} \frac{1}{n!}(2\nu)^{-2n} \frac{t^{\alpha/2}}{L(\sqrt{t})} \int_{\mathbb{R}^{2d}} \frac{y_k z_k}{\lambda^2} C(\sqrt{t}|y-z|)^n \, d\gamma_{2\nu}(y)\, d\gamma_{2\nu}(z) \\
\leq \; & \sum_{n=2}^{\infty} \frac{1}{\lambda^2 n!}(2\nu)^{-2n} \int_{\mathbb{R}^{2d}} \left| \frac{L(\sqrt{t}(|z-y|))}{L(\sqrt{t})} \right| \frac{|y_k z_k|}{|z-y|^{\alpha}} \\
\cdot \; & |C(\sqrt{t}|y-z|)| \, d\gamma_{2\nu}(y)\, d\gamma_{2\nu}(z) \\
\leq \; & \exp\left(\frac{1}{4\nu^2}\right) \frac{1}{\lambda^2} \int_{\mathbb{R}^{2d}} \left| \frac{L(\sqrt{t}(|u|))}{L(\sqrt{t})} \right| \frac{|(u_k)^2 - (v_k)^2|}{8|u|^{\alpha}} \\
\cdot \; & |C(\sqrt{t}|u|)| \, d\gamma_{8\nu}(u)\, d\gamma_{8\nu}(v), \qquad (28)
\end{aligned}$$

by substituting the variables y and z by $u = (y-z)$ and $v = (y+z)$, respectively. Since L is slowly varying at infinity and $\lim_{t\to\infty} C(t) = 0$ the integrand in (28) converges to zero almost everywhere w.r.t. the measure $\gamma_{8\nu} \times \gamma_{8\nu}$. Condition (A) implies that

$$\left| \frac{L(\sqrt{t}(|u|))}{L(\sqrt{t})} \right| |C(\sqrt{t}|u|)|$$

is bounded if we choose t large enough and since $\alpha < d$

$$\frac{|(u_k)^2 - (v_k)^2|}{8|u|^{\alpha}}$$

is integrable w.r.t. the measure $\gamma_{8\nu} \times \gamma_{8\nu}$. Therefore, again by Lebesgue's dominated convergence theorem

$$\lim_{t\to\infty} \sum_{n=2}^{\infty} \frac{1}{n!}(2\nu)^{-2n} \frac{t^{\alpha/2}}{L(\sqrt{t})} \int_{\mathbb{R}^{2d}} \frac{y_k z_k}{\lambda^2} C(\sqrt{t}|y-z|)^n \, d\gamma_{2\nu}(y)\, d\gamma_{2\nu}(z) = 0.$$

Now let us consider the second part of the sum in (27). We have

$$\begin{aligned}
\tilde{N}_k(t,a,\omega) &= -\frac{t^{\alpha/4}}{2\nu L^{1/2}(\sqrt{t})} \int_{\mathbb{R}^d} \frac{y_k}{\lambda} \langle \omega, \exp(i\cdot\sqrt{t}(a-y)) f^{1/2}(\cdot)\rangle \, d\gamma_{2\nu}(y) \qquad (29) \\
&= -\frac{t^{\alpha/4}}{2\nu L^{1/2}(\sqrt{t})} \left\langle \omega, \int_{\mathbb{R}^d} \frac{y_k}{\lambda} \exp(i\cdot\sqrt{t}(a-y)) \, d\gamma_{2\nu}(y) f^{1/2}(\cdot) \right\rangle, \qquad (30)
\end{aligned}$$

$\omega \in S'(\mathbb{R}^d)$, as elements in $L^2(\mu)$ because the integral in (29) is a Bochner integral with values in $L^2(\mu)$. The kernel of the Gaussian random variable in (30) we can transform to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{y_k}{\lambda} \exp(i\sqrt{t}p(a-y))\, d\gamma_{2\nu}(y) \\
= \;& \frac{i}{\lambda\sqrt{t}} \exp(i\sqrt{t}pa) \frac{\partial}{\partial p_k} \int_{\mathbb{R}^d} \exp(-i\sqrt{t}py)\, d\gamma_{2\nu}(y) \\
= \;& \frac{i}{\lambda\sqrt{t}} \exp(i\sqrt{t}pa) \frac{\partial}{\partial p_k} \exp(-\nu t|p|^2) \\
= \;& -i2\nu\sqrt{t} \exp(i\sqrt{t}pa) \exp(-\nu t|p|^2) \frac{p_k}{\lambda}
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
& \| \tilde{N}_k(t,a) - X_k(t,a) \|^2_{L^2(\mu)} \\
= \;& \frac{\alpha}{c_1(d,\alpha)c_2(d)} \int_{\mathbb{R}^d} \Bigg| \frac{t^{1/2+\alpha/4}}{L(\sqrt{t})^{1/2}} \sqrt{\frac{c_1(d,\alpha)c_2(d)}{\alpha}} \\
\cdot \;& \exp(i\sqrt{t}pa) \exp(-\nu t|p|^2) f^{1/2}(p) \frac{p_k}{\lambda} - t^{d/4} \exp(i\sqrt{t}pa) g(\sqrt{t}p) \Bigg|^2 dp \\
= \;& \frac{\alpha}{c_1(d,\alpha)c_2(d)} \int_{\mathbb{R}^d} h(p) \left| \left(f\left(\frac{p}{\sqrt{t}}\right) \left|\frac{p}{\sqrt{t}}\right|^{(d-\alpha)} \frac{c_1(d,\alpha)c_2(d)}{\alpha L(\sqrt{t})} \right)^{1/2} - 1 \right|^2 dp
\end{aligned} \tag{31}$$

where

$$h(p) = \frac{{p_k}^2}{\lambda^2 |p|^{d-\alpha}} \exp(i2pa - 2\nu|p|^2) \in L^1(\mathbb{R}^d).$$

Condition (A) implies via the Tauberian theorem, see [LO91], the following asymptotics of the spectral density:

$$f(p) = \tilde{f}(|p|) \sim \frac{\alpha}{c_1(d,\alpha)c_2(d)} L\left(\frac{1}{|p|}\right) |p|^{\alpha-d}, \quad 0 < \alpha < d,\ p \in \mathbb{R}^d \backslash \{0\},$$

as $|p| \to 0$, so that the integrand in (31) converges to zero as $t \to \infty$ for all $p \in \mathbb{R}^d$. From the properties of the functions f and L we can conclude that there exists $C > 0$ such that

$$\left| f(u)|u|^{d-\alpha} \frac{c_1(d,\alpha)c_2(d)}{\alpha L(\sqrt{t})} \right| \le C, \quad \forall u \in \mathbb{R}^d,\ t \ge b.$$

Thus there exists $K > 0$ so that

$$\left| \left(f\left(\frac{p}{\sqrt{t}}\right) \left|\frac{p}{\sqrt{t}}\right|^{(d-\alpha)} \frac{c_1(d,\alpha)c_2(d)}{\alpha L(\sqrt{t})} \right)^{1/2} - 1 \right|^2 \leq K, \quad \forall p \in \mathbb{R}^d,\ t \geq b.$$

Hence, by Lebesgue's dominated convergence theorem we have

$$\lim_{t\to\infty} \| \tilde{N}_k(t,a) - X_k(t,a) \|_{L^2(\mu)} = 0$$

and therefore also

$$\lim_{t\to\infty} \| N_k(t,a) - X_k(t,a) \|_{L^2(\mu)} = 0.$$

It is easy to see that this convergence is uniform in $a \in \mathbb{R}^d$.

Step 3: From step 1 and step 2 together with the fact that the square-norm of $X_k(t,a)$ is independent of $t > 0$ and $a \in \mathbb{R}^d$ we can conclude under use of the continuity of the Wick product, see Lemma 2.9, and the Wick inverse, see Lemma 2.11, that there exist $q \in \mathbb{N}$ and $t_0 \geq 0$ such that

$$Y_k(t,a) \in G_{-q}^{-1}$$

for all $t \geq t_0$ and

$$\lim_{t\to\infty} \| Y_k(t,a) \|_{0,-q,-1} = 0$$

uniformly in $a \in \mathbb{R}^d$ for all $1 \leq k \leq d$. ■

Remark 4.5 *The proof of Theorem 4.3 can easily be generalized to the case of Poisson analysis. More precisely, if we use a Poisson measure, see e.g. [IK88], instead of a Gaussian measure, substitute the orthogonal system given by Hermite polynomial by Charlier polynomials, and use the Wick calculus in Poisson analysis, see [KSS97], then we can prove an analogous result. The initial velocity potential then is a compensated Poisson field, just as the limiting random field.*

4.2 Corollaries

Often the slightly different scaling $U(\beta^2 t, \beta a)$ is considered, see e.g. [AMS94] and [SW94]. This scaling preserves the time variable and therefore the authors found a limiting random process. Also in the case of Wick type Burgers equation we can examine this scaling.

Corollary 4.6 *Let $U(t,x)$, $t>0$, $x\in\mathbb{R}^d$, $d\geq 1$, be the solution of the initial value problem (19) with random initial data satisfying condition (A) and (B). Then for α as in condition (A) each component of the generalized random field*

$$Z(\beta,t,a)=\frac{\beta^{1+\alpha/2}}{L^{1/2}(\beta)}U(\beta^2 t,\beta a)-t^{-(1/2+\alpha/4)}X(\beta^2 t,a/\sqrt{t}),\quad t>0,$$

converge uniformly in $a\in\mathbb{R}^d$ to $0\in\mathcal{G}^{-1}$ as $\beta\to\infty$ w.r.t. inductive limit topology. I.e., there exists a Hilbert space norm $\|\cdot\|_{0,-q,-1}$, $q\in\mathbb{N}$, such that

$$\lim_{\beta\to\infty}\| Z_k(\beta,t,a)\|_{0,-q,-1}=0$$

uniformly in $a\in\mathbb{R}^d$ for all $t>0$, $1\leq k\leq d$.

Remark 4.7 *For fixed $t>0$ finite dimensional distributions of the Gaussian random field $t^{-(1/2+\alpha/4)}X(\beta^2 t,a/\sqrt{t}), a\in\mathbb{R}^d$, coincide with finite dimensional distributions of the random field*

$$\begin{aligned}Z(t,a,\omega) &= t^{-(1/2+\alpha/4)}X(a/\sqrt{t},\omega),\quad \omega\in S'(\mathbb{R}^d),\\ &= it^{-(1/2+\alpha/4)}\left(\frac{\alpha}{c_1(d,\alpha)c_2(d)}\right)^{1/2}\langle\omega,\exp(i\cdot a/\sqrt{t})g(\cdot)\rangle,\end{aligned}$$

for all $\beta>0$, see Remark 4.2.

Proof of Corollary 4.6: Let us define $\hat{t}=\beta^2 t$ and $\hat{a}=a/\sqrt{t}$. Then we can rewrite $Z(\beta,t,a)$ as

$$t^{-(1/2+\alpha/4)}\left(\frac{L^{1/2}(\sqrt{\hat{t}})}{L^{1/2}(\sqrt{\hat{t}}/\sqrt{t})}\frac{\hat{t}^{1/2+\alpha/4}}{L^{1/2}(\sqrt{\hat{t}})}U(\hat{t},\sqrt{\hat{t}}\hat{a})-X(\hat{t},\hat{a})\right).$$

Taking the limit $\hat{t}\to\infty$ is equivalent to taking the limit $\beta\to\infty$. Thus the corollary is an immediate consequence of Theorem 4.3 and the properties of the slow varying function L. ■

Let us add a deterministic part to the initial data. Then we have to consider the following equation:

$$\begin{aligned}&\frac{\partial U_k}{\partial t}(t,x)+\lambda\sum_{j=1}^d U_j(t,x)\diamond\frac{\partial U_k}{\partial x_j}(t,x)=\nu\Delta U_k(t,x),\quad t>0,\\ &U_k(0,x)=\frac{\partial G}{\partial x_k}(x)+\frac{\partial H}{\partial x_k}(x),\quad x\in\mathbb{R}^d, 1\leq k\leq d.\end{aligned}\qquad(32)$$

In the next corollary we show that if the initial shape given by H is compactly supported then the scaling limit is independent of it.
Condition (C) The deterministic function $H(x), x \in \mathbb{R}^d$, is continuously differentiable with compact support.

Corollary 4.8 *Let $U(t,x)$, $t > 0$, $x \in \mathbb{R}^d$, $d \geq 1$, be the solution of the initial value problem (32). We assume that the random part of the initial data satisfies condition (A) and (B) and that the deterministic part fulfills condition (C). Then for α as in condition (A) each component of the generalized random field*

$$W(t,a) = \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U(t, \sqrt{t}a) - X(t,a)$$

converge uniformly in $a \in \mathbb{R}^d$ to $0 \in \mathcal{G}^{-1}$ as $t \to \infty$ w.r.t. inductive limit topology. I.e., there exists a Hilbert space norm $\| \cdot \|_{0,-q,-1}$, $q \in \mathbb{N}$, such that

$$\lim_{t \to \infty} \| W_k(t,a) \|_{0,-q,-1} = 0$$

uniformly in $a \in \mathbb{R}^d$ for all $1 \leq k \leq d$.

Proof: The solution of (32) can be written as

$$U_k(t,x) = I_k(t,x) \diamond J(t,x)^{\diamond(-1)}, \quad t > 0,\ x \in \mathbb{R}^d,$$

where in this case

$$\begin{aligned} J(t,x) &= \int_{\mathbb{R}^d} K(t, x-y) \exp^\diamond \Big(- \frac{G(y) + H(y)}{2\nu} \Big)\, dy, \\ I_k(t,x) &= \int_{\mathbb{R}^d} \frac{x_k - y_k}{\lambda t} K(t, x-y) \exp^\diamond \Big(- \frac{G(y) + H(y)}{2\nu} \Big)\, dy, \end{aligned}$$

see Theorem 3.2. Since H is a deterministic function we have

$$\exp^\diamond \Big(- \frac{G(y) + H(y)}{2\nu} \Big) = \exp^\diamond \Big(- \frac{G(y)}{2\nu} \Big) \exp \Big(- \frac{H(y)}{2\nu} \Big)$$

Furthermore, $\exp(-H(y)/(2\nu))$ is a bounded function and therefore the proof of this corollary mainly is an easy generalization of the proof of Theorem 4.3. There is only one part which we should discuss in more detail. Consider the first part of the sum in (27) translated in the situation with additional deterministic initial data. It is given by

$$\frac{t^{\alpha/2}}{L(\sqrt{t})}\bigg(\int_{\mathbb{R}^d}\frac{y_k}{\lambda}\exp\Big(-\frac{H(\sqrt{t}(a-y))}{2\nu}\Big)\,d\gamma_{2\nu}(y)\bigg)^2.$$

For $H\equiv 0$ it is obviously zero for all $t>0$. Let $r>0$ be so large that the support of H is contained in the ball of radius r in $\mathbb{R}^d$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} y_k\exp\Big(-\frac{H(\sqrt{t}(a-y))}{2\nu}\Big)\,d\gamma_{2\nu}(y)\\
=&\int_{\{y\in\mathbb{R}^d\,|\,|a-y|\le r/\sqrt{t}\}} y_k\exp\Big(-\frac{H(\sqrt{t}(a-y))}{2\nu}\Big)\,d\gamma_{2\nu}(y)\\
+&\int_{\{y\in\mathbb{R}^d\,|\,|a-\tilde{y}|\le r/\sqrt{t}\}} y_k\exp\Big(-\frac{H(\sqrt{t}(a-y))}{2\nu}\Big)\,d\gamma_{2\nu}(y)\\
\le&\int_{\{y\in\mathbb{R}^d\,|\,|a-y|\le r/\sqrt{t}\}} C_1\,dy=C_2t^{-d/2},\quad \forall t\ge 1,
\end{aligned}\tag{33}$$

for suitable positive constants C_1 and C_2 and where $\tilde{y}=(y_1,\ldots,-y_k,\ldots,y_d)$. Estimate (33) gives us

$$\lim_{t\to\infty}\frac{t^{\alpha/2}}{L(\sqrt{t})}\bigg(\int_{\mathbb{R}^d}\frac{y_k}{\lambda}\exp\Big(-\frac{H(\sqrt{t}(a-y))}{2\nu}\Big)\,d\gamma_{2\nu}(y)\bigg)^2=0.$$

■

4.3 Probabilistic interpretation and related result

Again, we consider the solution $U(t,x)$, $t>0$, $x\in\mathbb{R}^d$, $d\ge 1$, of the initial value problem (19) as in Theorem 4.3. In step 3 of the proof of Theorem 4.3 we have proved that there exist $q\in\mathbb{N}$ and $t_0>0$ such that $Y_k(t,a)\in G_{-q}^{-1}$ and hence also $U_k(t,\sqrt{t}a)\in G_{-q}^{-1}$ for all $a\in\mathbb{R}^d$, $t\ge t_0$, $1\le k\le d$. Therefore, the Wick monomial of n-th order $U^{(n)}$ in its generalized chaos decomposition, see (15), is a square-integrable function.

Theorem 4.9 *Let $U(t,x)$, $t>0$, $x\in\mathbb{R}^d$, $d\ge 1$, be as in Theorem 4.3.*
(i) Each component of the scaled chaos of order n

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}U^{(n)}(t,\sqrt{t}a)$$

converges in mean square to zero, uniformly in $a \in \mathbb{R}^d$, as t tends to infinity for all $n \in \mathbb{N}_0$, $n \neq 1$.
(ii) Finite dimensional distributions of the vector valued Gaussian random field

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U^{(1)}(t, \sqrt{t}a), \quad a \in \mathbb{R}^d,$$

converge weakly to finite dimensional distribution of the random field $X(a)$, $a \in \mathbb{R}^d$, as t tends to infinity.
(iii) Each component of the renormalized random process

$$R(N)Y(t,a) = \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} R(N)U(t, \sqrt{t}a) - X(t,a), \quad t \geq t_0,$$

where

$$R(n) = (n!)^{-1/2}\, 2^{(n-1)q/2},\ n \in \mathbb{N},\ R(0) = 2^{-q/2},$$

and $N = d\Gamma(Id)$ is the number operator, see Section 2.2, is a process in $L^2(\mu)$. Moreover, it converges in mean square to zero, uniformly in $a \in \mathbb{R}^d$, as t tends to infinity.

Proof: (i): Since every chaos converges separately, this is an immediate consequence of Theorem 4.3.
(ii): Utilizing the rules for the Wick calculus, see Section 2.3, we find

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} U_k^{(1)}(t, \sqrt{t}a) = \tilde{N}_k(t,a), \quad t > 0,\ a \in \mathbb{R}^d,$$

see (30). One easily checks that for all $t > 0$ finite dimensional distributions of the Gaussian random field $\tilde{N}(t,a)$, $a \in \mathbb{R}^d$, $1 \leq k \leq d$, coincide with finite dimensional distributions of the random field

$$\hat{N}(t,a,\omega) = i\frac{t^{(\alpha-d)/4}}{L^{1/2}(\sqrt{t})} \langle \omega, K(t,a,\cdot)\rangle, \quad \omega \in S'(\mathbb{R}^d),$$

where

$$K(t,a,p) = \exp(ipa)\exp(-\nu t|p|^2) f^{1/2}(p/\sqrt{t}) \frac{p}{\lambda}, \quad p \in \mathbb{R}^d.$$

Now in the same way as we proved mean square convergence of $(\tilde{N}(t,a) - X_k(t,a))$ to zero we can prove mean square convergence of $\hat{N}(t,a)$ to $X_k(a)$ as t tends to infinity. Finally, applying the Cramér Wold device we can conclude statement (ii).
(iii): This follows by Theorem 4.3 together with the equality

$$\| R(N)Y(t,a) \|_{L^s(\mu)} = 2^{q/2} \| Y(t,a) \|_{0,-q,-1} .$$

■

Remark 4.10 *Theorem 4.9 shows that the scaling limit for the solution of Wick type Burgers equation is almost the same, including the type of convergence, as that of the Burgers equation with ordinary product. In [LPW96] it has been shown that finite dimensional distributions of the scaled solution of Burgers equation with ordinary product weakly converge to the random field $X(a)$, $a \in \mathbb{R}^d$, when t tends to infinity, see Remark 4.4(ii). In Theorem 4.9 (ii) we proved that finite dimensional distributions of the first chaos of the scaled solution of the Wick type Burgers equation weakly converge to the same random field. Since the solution of the Wick type Burgers equation is a generalized random field and therefore in general not measurable in the usual sense we can not prove convergence in distribution for the entire solution. Nevertheless, the limiting behavior of the scaled solution of Wick type Burgers equation is very close to that of the scaled solution of Burgers equation with ordinary product, because all other chaos, see Theorem 4.9 (i), in mean square converge to zero.*

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