THE COMPLEX SCALED FEYNMAN-KAC FORMULA FOR SINGULAR INITIAL DISTRIBUTIONS

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Abstract. We study complex scaling in the Feynman-Kac formula, using the fact that for a Dirac delta initial distribution complex scaling is well defined in White Noise Analysis.

1. Introduction

In many branches of theoretical physics, e.g. quantum field theory and polymer physics integrals over ”paths” are of particular interest. The initial idea of averages over paths has a mathematical meaning only for (real) measures as in the heat equation. In this case one can present the solution by a path integral, based on the Wiener measure. This is stated by the famous Feynman–Kac formula

$$E\left(\exp\left(\int_{t_0}^{t} V(x_0 + B_s)\, ds\right) f(x_0 + B_t)\right),$$

for suitable $f, V : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and $0 \leq t_0 \leq t \leq T < \infty$, see e.g. [26]. Here $(B_t)_{t\in[t_0,T]}$ denotes a Brownian motion starting at time t_0 in zero. Furthermore for nice potentials the heat kernel K_V is given by

$$\begin{aligned} K_V(x,t;x_0,t_0) = {}& \frac{1}{\sqrt{2\pi(t-t_0)}} \exp\left(-\frac{1}{2(t-t_0)}(x_0-x)^2\right) \\ & \times E\left(\exp\left(\int_{t_0}^{t} V\left(x_0 - \frac{s-t_0}{t-t_0}(x_0-x) + B_s - \frac{s-t_0}{t-t_0}B_t\right) ds\right)\right), \end{aligned} \tag{1.1}$$

for $0 < t_0 < t < T$, $x, x_0 \in \mathbb{R}$, see e.g. [15]. There have been a lot of attempts to write down solutions of complex scaled heat equations (like e.g. the Schrödinger equation) as a (path) integral in a mathematically rigorous way. The methods used in this context (e.g. analytic continuation, limits of finite dimensional approximations and Fourier transform) are always more involved and less direct than in the Euclidean – i.e. Feynman–Kac – case. Instead of giving a long list of publications with different approaches to path integrals, we refer to [1] and the large number of references therein.

We choose a white noise approach to construct a complex scaled heat kernel as the expectation of a generalized functions for a new class of potentials. White noise is a framework which offers various generalizations of concepts known from finite-dimensional analysis, like differential operators and Fourier transform. Detailed information concerning these methods can be found in the monographs [16], [3], [17], [22] and the articles [25], [20], [27]. The idea to realize Feynman path integrals within the white noise framework was first mentioned in the work of Hida and Streit [18]. The concept of integral has a natural extension in the dual pairing of generalized and test function and the construction of the Feynman integrand as a generalized function follows closely Feynman's fundamental idea of a sum over histories. The integrand is an element of a suitable space of distributions chosen so as to accommodate various classes of admissible interaction potentials. For this construction in case of the Schrödinger equation see e.g. [6], [19], [17], [10], [21], [27], [7]. The present article proposes a complementary strategy inspired by [8], see also [4] and [14]: we construct a complex scaled Feynman–Kac kernel with white noise methods for suitable potentials V by giving a meaning to

$$K\left(x,t \mid x_0,t_0\right) = \frac{1}{\sqrt{2\pi(t-t_0)z^2}} \exp\left(-\frac{1}{2(t-t_0)z^2}(x_0-x)^2\right)$$

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$$\times E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0+\frac{s-t_0}{t-t_0}(x-x_0)+zB_s-\frac{s-t_0}{t-t_0}B_t\right)\right)ds\right), \quad (1.2)$$

which is a scaled version of (1.1). Perturbation theory with its limitations on admissible potentials plays no role in this construction.

It is done by inserting Donsker's delta in order to fix the final point $x \in \mathbb{R}$, and taking a generalized expectation, i.e.,

$$K_V(x,t;x_0,t_0) = E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x+zB_s)ds\right)\sigma_z\delta(B_t-(x-x_0)))\right), \quad (1.3)$$

whenever the expression in the expectation is a generalized function of white noise, e.g. a Hida distribution. The use of White Noise Analysis is essential here: it allows us to transcend the L^1 limitations inherent in the work of Doss.

The fundamental concept in proving this is a Wick product representation of the integrand

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x+zB_s)ds\right)\sigma_z\delta(B_t-(x-x_0))$$
$$= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0+\frac{s-t_0}{t-t_0}(x-x_0)+z\left(B_s-\frac{s-t_0}{t-t_0}B_t\right)\right)ds\right)\diamond\sigma_z\delta\left(B_t-(x-x_0)\right),$$

whose generalized expectation coincides with (1.2).

After a short introduction to White Noise Analysis (see Section 2) we prove a general Wick relation between products of Donsker's deltas of Brownian motion and Brownian bridge in Section 3. In Section 4 we give a meaning to the integrand of 1.3 as a Hida distribution (see Theorem 4.8).

2. White Noise Analysis

2.1. **Hida distributions.** The white noise measure μ on Schwartz distribution space $S'(\mathbb{R})$ arises from the characteristic function

$$C(f) := \exp\left(-\tfrac{1}{2}(f,f)\right), \quad f \in S(\mathbb{R}),$$

via Minlos' theorem, see e.g. [3, 16, 17]:

$$C(f) = \int_{S'} \exp\left(i\langle\omega, f\rangle\right) d\mu(\omega).$$

Considering the Gel'fand triple

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}),$$

the dual pairing $\langle\cdot,\cdot\rangle$ of $S'(\mathbb{R})$ and $S(\mathbb{R})$ extends the usual scalar product $(\cdot,\cdot)$ of $L^2(\mathbb{R}) := L^2(\mathbb{R},\mathbb{R};dx)$. We define the space of complex valued square integrable functions over $S'(\mathbb{R})$ by

$$L^2(\mu) := L^2(S'(\mathbb{R}),\mathbb{C};\mu).$$

In the sense of an $L^2(\mu)$-limit we may extend $\langle\omega, f\rangle$ to $f = \mathbf{1}_A$, where $\mathbf{1}_A$ denotes the indicator function of $A \subset \mathbb{R}$. Informally

$$\langle\omega, \mathbf{1}_{[t_0,t)}\rangle = \int_{t_0}^{t} \omega(s)\,ds, \quad 0 < t_0 < t < \infty,$$

$$B_{t_0,t}(\omega) := \langle\omega, \mathbf{1}_{[t_0,t)}\rangle \quad (2.1)$$

is a representation of Wiener's Brownian motion starting in zero at time t_0. Likewise, for $0 < t_0 < t < \infty$ we can define a version of a Brownian bridge starting in zero at time t_0 and ending in zero at time t via

$$B^{0\to 0}_{t_0,t,s}(\omega) := \langle\omega, \mathbf{1}_{[t_0,s)}\rangle - \frac{s-t_0}{t-t_0}\langle\omega, \mathbf{1}_{[t_0,t)}\rangle, \quad t_0 \le s \le t. \quad (2.2)$$

Moreover, each $F \in L^2(\mu)$ possesses a chaos decomposition

$$F(\omega) = \sum_{n=0}^{\infty} I\left(F^{(n)}\right) = \sum_{n=0}^{\infty}\left\langle :\omega^{\otimes n}:, F^{(n)}\right\rangle, \quad F^{(n)} \in \widehat{L^2(\mathbb{R}^n)}_{\mathbb{C}},$$

where $:\omega^{\otimes n}:$ denotes the n-th Wick power of $\omega \in S'(\mathbb{R})$, see e.g. [17], and $F^{(n)}$ is called the kernel of its n-th chaos. Here $\widehat{L^2(\mathbb{R}^n)}_{\mathbb{C}}$ denotes the complexification of the space of symmetric elements from

$L^2(\mathbb{R}^n)$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing in the corresponding spaces extended to their complexification in a bilinear way. One then constructs a Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)'$$

of Hida test functions and distributions, see e.g. [17]. We introduce the S-transform of $\Phi \in (S)'$ by

$$(S\Phi)(f) := \langle\langle \Phi, : \exp(\langle \cdot, f \rangle) : \rangle\rangle, \quad f \in S(\mathbb{R}),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the bilinear dual pairing between $(S)'$ and (S) and $: \exp(\langle \cdot, f \rangle) : := \exp(\langle \cdot, f \rangle) \exp(-1/2(f, f)) \in (S)$. If $\Phi \in L^2(\mu)$ then

$$(S\Phi)(f) = \int_{S'(\mathbb{R})} \Phi(\omega + f) \, d\mu(\omega). \tag{2.3}$$

For all $\Phi \in (S)'$ one has a generalized chaos decomposition, i.e.,

$$\Phi = \sum_{n=0}^{\infty} I\left(\Phi^{(n)}\right), \quad \Phi^{(n)} \in \widehat{S'(\mathbb{R}^n)}_{\mathbb{C}}, \quad n \in \mathbb{N}_0$$

in the sense of Hida distributions. I.e. the S-transform of Φ is given by

$$S(\Phi)(f) = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, f^{\otimes n} \right\rangle, \quad f \in S(\mathbb{R}).$$

Since $\mathbf{1} \in (S)$, the expectation extends to Hida distributions Φ by

$$E(\Phi) := \langle\langle \Phi, \mathbf{1} \rangle\rangle.$$

Definition 2.1. *A function $F : S(\mathbb{R}) \to \mathbb{C}$ is called U-functional if:*

(i) *F is "ray-analytic", i.e., for all $f, g \in S(\mathbb{R})$ the mapping*

$$\mathbb{R} \ni y \mapsto F(f + yg) \in \mathbb{C}$$

extends to $y \in \mathbb{C}$ as an entire function.

(ii) *F is uniformly bounded of exponential order 2, i.e., there exist some constants $0 < K, D < \infty$ and a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that for all $z \in \mathbb{C}$, $f \in S(\mathbb{R})$*

$$|F(zf)| \leq K \exp(D|z|^2 \|f\|^2).$$

Theorem 2.2. *The following statements are equivalent:*

(i) *$F : S(\mathbb{R}) \to \mathbb{C}$ is a U-functional.*

(ii) *F is the S-transform of a unique Hida distribution $\Phi \in (S)'$.*

Theorem 2.3. *Let $(F_n)_{n\in\mathbb{N}}$ denote a sequence of U-functionals with the following properties:*

(i) *For all $f \in S(\mathbb{R})$, $(F_n(f))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

(ii) *There exists some constants $0 < P, Q < \infty$ and a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that*

$$\left|F_n(zf)\right| \leq P \exp(Q|z|^2 \, \|f\|^2),$$

for all $n \in \mathbb{N}$, $f \in S(\mathbb{R})$ and $z \in \mathbb{C}$. Here F_n denotes the analytic extension.

Then there exists a unique $\Phi \in (S)'$ such that $(S^{-1}F_n)_{n\in\mathbb{N}}$ converge strongly to Φ.

Theorem 2.4. *Let $(\Omega, \mathcal{F}, m)$ denote a measure space and $\lambda \mapsto \Phi(\lambda)$ a mapping from Ω to $(S)'$. Let $F(\lambda)$ denote the S-transform of $\Phi(\lambda)$ and fulfill the following conditions:*

(i) *$\lambda \mapsto F(\lambda, f)$ is a measurable function for all $f \in S(\mathbb{R})$.*

(ii) *There exists a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that for all $f \in S(\mathbb{R})$, $\lambda \in \Omega$ and $z \in \mathbb{C}$*

$$\left|F(\lambda, zf)\right| \leq P(\lambda) \exp((Q(\lambda)|z|^2 \|f\|^2),$$

for some $Q \in L^\infty(\Omega, m)$ and $P \in L^1(\Omega, m)$.

Then Φ is Bochner integrable in a Hilbert subspace of $(S)'$ and

$$\int_\Omega \Phi(\lambda)\, dm(\lambda) \in (S)'.$$

Moreover

$$S\left(\int_\Omega \Phi(\lambda)\, dm(\lambda)\right)(f) = \int_\Omega S(\Phi(\lambda))(f)\, dm(\lambda), \quad f \in S(\mathbb{R}).$$

For proofs and more see e.g. [20].

Example 2.5. *In order to 'pin' Brownian motion at a point $a \in \mathbb{R}$ we want to consider the informal composition of the Dirac delta distribution with Brownian motion: $\delta(B(t) - a)$. This can be given a precise meaning as a Hida distribution*

$$\delta(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - a) := \int_{\mathbb{R}} \exp\left(i\lambda(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - a)\right)\, d\lambda, \quad 0 \le t_0 < t < \infty,$$

in the sense of Theorem 2.4. Its S-transform is given by

$$S\left(\delta(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - a)\right)(f) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\int_{t_0}^t f(s)\, ds - a\right)^2\right).$$

Since (S) is an algebra the pointwise product of $\varphi, \psi \in (S)$ is a well-defined element of (S). In particular, the product of two monomials $I\left(\varphi^{(n)}\right)$, $\varphi^{(n)} \in \widehat{S(\mathbb{R}^n)}_{\mathbb{C}}$ and $I\left(\psi^{(m)}\right)$, $\psi^{(m)} \in \widehat{S(\mathbb{R}^m)}_{\mathbb{C}}$, $n, m \in \mathbb{N}$, has the chaos decomposition

$$\left\langle :\omega^{\otimes n}:, \varphi^{(n)}\right\rangle \left\langle :\omega^{\otimes m}:, \psi^{(m)}\right\rangle = \sum_{k=0}^{n\wedge m} k!\binom{n}{k}\binom{m}{k}\left\langle :\omega^{\otimes(n+m-2k)}:, \varphi^{(n)}\hat{\otimes}_k\psi^{(m)}\right\rangle, \tag{2.4}$$

see [17, Eq. (2.51)]. Having in mind the generalized chaos decomposition of elements from $(S)'$ by extending (2.4) we can define the informal product of $\Phi, \Psi \in (S)'$ with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$ and $\Psi^{(m)}$, $m \in \mathbb{N}$, respectively, by

$$\Phi\Psi := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n\wedge m} k!\binom{n}{k}\binom{m}{k}\left\langle :\omega^{\otimes(n+m-2k)}:, \Phi^{(n)}\hat{\otimes}_k\Psi^{(m)}\right\rangle, \tag{2.5}$$

whenever the sum on the right hand side is a Hida distribution.

Definition 2.6. *Let $\Phi, \Psi \in (S)'$. Then we define the Wick product $\Psi \diamond \Phi$ by*

$$\Psi \diamond \Phi := S^{-1}(S\Phi\, S\Psi) \in (S)'.$$

The Wick product is a well-defined object since the set of U-functionals is an algebra. Thus by the Characterization Theorem 2.2, there exists an element $\Psi \diamond \Phi \in (\mathcal{S})'$ such that $S(\Psi \diamond \Phi) = S(\Psi)\, S(\Phi)$.

2.2. **Regular distributions and independence.** We consider the Gel'fand triple of regular test and generalized functions

$$(S) \subset \mathcal{G} \subset L^2(\mathbb{R}) \subset \mathcal{G}' \subset (S)',$$

which was introduced in [24] and later generalized and characterized via the Bargmann–Segal transform in [11], see also [12]. Note that the characteristic property of regular test and generalized functions is that the Wick monomials in their (generalized) chaos decomposition are square integrable functions. Here we are interested in a definition of independence for elements of $\mathcal{G}'$. Hence we omit a repetition of its construction and characterization and refer to [24] and [11], respectively. Recall that a priori it is not clear how to define independence of elements from $\mathcal{G}'$, since they are not pointwisely defined.

To define independence we recall some results from [5], see also [2] and [9]. Let I be an interval in $\mathbb{R}$, and denote by $\mathcal{F}_I$ the σ-algebra generated by the random variables $\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - \langle \cdot, \mathbf{1}_{[t_0,s)}\rangle$, $t_0, s, t \in I$, $t_0 \le s \le t$.

Definition 2.7. *We call $\Phi \in \mathcal{G}'$ $\mathcal{F}_I$-measurable, if for all $f \in S(\mathbb{R})$,*

$$S(\Phi)(f) = S(\Phi)(\mathbf{1}_I f).$$

Remark 2.8. (i) *Recall that $F \in L^2(\mu)$ is $\mathcal{F}_I$-measurable iff $SF(f) = SF(\mathbf{1}_I f)$ for all $f \in S(\mathbb{R})$.*
(ii) *By characterization one knows the S-transform of an element in $\mathcal{G}'$ has a continuous extension from $S(\mathbb{R})$ to $L^2(\mathbb{R})$, hence for $\Phi \in \mathcal{G}'$ we have that $S(\Phi)(\mathbf{1}_I f)$ is well-defined for all intervals $I \subset \mathbb{R}$.*

Definition 2.9. *Two generalized random variables $\Phi,\ \Psi \in \mathcal{G}'$ are called independent if there exist intervals $I,\ J \subset \mathbb{R}$ whose intersection has Lebesgue measure zero, and such that Φ is $\mathcal{F}_I$-measurable, and Ψ is $\mathcal{F}_J$-measurable.*

Lemma 2.10. *The pointwise product of random variables as in* (2.5) *has a well-defined extension to pairs $\Psi,\ \Phi$ of independent generalized random variables in $\mathcal{G}'$ so that $\Psi \cdot \Phi \in \mathcal{G}'$. Moreover, the formula*

$$\Psi \cdot \Phi = \Psi \diamond \Phi,$$

holds. As a consquence

$$E(\Psi \cdot \Phi) = E(\Psi)\ E(\Phi).$$

Following (2.5) *the formal product of $\Psi\Phi$ of elements $\Psi,\ \Phi \in \mathcal{G}$ has square integrable Wick monomials in its chaos decomposition. Hence the pointwise product extends to elements from $\mathcal{G}'$ whenever the formal series of the corresponding chaos decompositions converge in $\mathcal{G}'$.*

2.3. **Scaling operator.** In this section we follow [17] and [28, Sec. 4.5], and define the scaling operator on the space of test function (S). Later we show in which case an extension to generalized functions make sense. This is an important tool for giving a mathematical rigorous definition of Feynman integrands.

Definition 2.11. *Let φ be the continuous version of an element of (S). Then for $z \in \mathbb{C}$ we define the scaling operator σ_z by*

$$(\sigma_z\varphi)(\omega) = \varphi(z\omega), \quad \omega \in S'(\mathbb{R}).$$

It is easy to calculate the chaos expansion

$$\sigma_z\varphi = \sum_{n=0}^{\infty} \left\langle :\omega^{\otimes n}:, \varphi_z^{(n)} \right\rangle, \tag{2.6}$$

with kernels

$$\varphi_z^{(n)} = z^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{z^2-1}{2}\right)^k \mathrm{tr}^k \varphi^{(n+2k)}. \tag{2.7}$$

Here $\mathrm{tr}^k\varphi^{(n+2k)}$ is defined by

$$\mathrm{tr}^k\varphi^{(n+2k)} := \left\langle \mathrm{Tr}^{\otimes k}, \varphi^{(n+2k)} \right\rangle \in \widehat{L^2(\mathbb{R}^n)}_{\mathbb{C}},$$

where $\mathrm{Tr} \in \mathrm{S}'(\mathbb{R}^2)$ (trace kernel) is defined by $\langle \mathrm{Tr}, \mathrm{f} \otimes \mathrm{g} \rangle = (\mathrm{f}, \mathrm{g})$, $f, g \in S(\mathbb{R})$.

Theorem 2.12. *The scaling operator has the following properties:*
(i) *For $z \in \mathbb{C}$ the mapping $\varphi \mapsto \sigma_z\varphi$ is continuous from (S) into itself.*
(ii) *For $\varphi, \psi \in (S)$ and $z \in \mathbb{C}$ the equation*

$$\sigma_z(\varphi\psi) = (\sigma_z\varphi)(\sigma_z\psi),$$

holds.

Remark 2.13. (i) *We extend the scaling operator to elements from $(S)'$, whenever* (2.7) *makes sense and* (2.6) *converges in $(S)'$.*
(ii) *Following the definition of the pointwise product in* (2.5) *for $\Phi, \Psi \in (S)'$ one obviously has*

$$\sigma_z(\Phi\Psi) = (\sigma_z\Phi)(\sigma_z\Psi),$$

whenever all operations are well-defined.

3. Donsker's delta of Brownian motion and Brownian bridge

3.1. **Complex scaling of Donsker's delta.** We consider the S-transform of Donsker's delta $\delta(\langle\cdot,h\rangle-a)$, $a\in\mathbb{R}$, and $h\in L^2(\mathbb{R})$:

$$F_{h,a}(f):=S(\delta(\langle\cdot,h\rangle-a))(f)=\frac{1}{\sqrt{2\pi\,(h,h)}}\exp\left(\frac{1}{2\,(h,h)}((f,h)-a)^2\right),$$

for all $f\in S(\mathbb{R})$. Clearly, $F_{h,a}(f)$ is analytic in the parameter a. Thus it is possible to extend it to complex a and the resulting expression is still a U-functional. Analogously,$F_{h,a}(f)$ has an analytic continuation to $h\in L^2_{\mathbb{C}}(\mathbb{R})$ (the complexification of $L^2(\mathbb{R})$), similar to Definition 2.1. One only has to be careful with the square root, hence we exclude $h\in L^2_{\mathbb{C}}(\mathbb{R})$ for which (h,h) is a negative, real number (note that due to the bilinear extension of $(\cdot,\cdot)$ to $L^2_{\mathbb{C}}(\mathbb{R})$, negative values are possible). In this case $F_{h,a}(f)$ is again a well-defined U-functional.

Theorem 3.1. *Let $a\in\mathbb{C}$ and $h\in L^2_{\mathbb{C}}(\mathbb{R})$ with $(h,h)\notin(-\infty,0]$. Then*

$$\delta(\langle\cdot,h\rangle-a)\in\mathcal{G}'.$$

Lemma 3.2. *Donsker's delta is homogeneous of degree -1 in $z\in S_\alpha:=\left\{z\in\mathbb{C}\mid \arg z\in(-\frac{\pi}{4}+\alpha,\frac{\pi}{4}+\alpha)\right\}$, i.e.,*

$$\sigma_z\delta(\langle\cdot,h\rangle-a)=\frac{1}{z}\delta\left(\langle\cdot,h\rangle-\frac{a}{z}\right).$$

See [23] for the proof and more details.

3.2. **Products of Donsker's deltas.**

Theorem 3.3. *One can define n-fold products of Donsker's deltas by*

$$\Phi:=\prod_{j=1}^{n}\sigma_z\delta(\langle\cdot,f_j\rangle-a_j):=\frac{1}{(2\pi)^n}\prod_{j=1}^{n}\int_\gamma\exp\left(i\lambda_j(z\langle\cdot,f_j\rangle-a_j)\right)d\lambda_j,$$

* *in the sense of Bochner integration, see Theorem 2.4. Here $\gamma=\left\{e^{-i\alpha}x\mid x\in\mathbb{R}\right\}$, $z\in\mathbb{C}$ such that $|\arg(z)|<\pi/4+\alpha$, $|\alpha|<\pi/4$, and h_j linear independent elements of $L^2(\mathbb{R})$ and $a_j\in\mathbb{C}$, $j=1,\dots,n$. Then Φ is an element of $(S)'$ and for $f\in S(\mathbb{R})$ its S-transform is given by*

$$S\Phi(f)=\frac{1}{\sqrt{(2\pi z^2)^n\det M}}\exp\left(-\frac{1}{2}\left((h,f)-\frac{1}{z}a\right)M^{-1}\left((h,f)-\frac{1}{z}a\right)^T\right),$$

where M denotes the Gram matrix to $h_1,\dots,h_n$ defined by $M:=(h_k,h_l)_{k,l=1,\dots,n}$, $(h,f):=((h_1,f),\dots,(h_n,f))$ and $a:=(a_1,\dots,a_n)$.

Again for the proof see [23]. In the following we only consider products for $z\in\overline{S_0}=\left\{z\in\mathbb{C}\mid \left|\arg z\right|\le\frac{\pi}{4}\right\}$. Of course, for different choices of $z\in\overline{S_0}$ one has to choose an appropriate α.

Lemma 3.4. *Let $0<t<\infty$, $z\in\overline{S_0}$, $n\in\mathbb{N}$, $x_k\in\mathbb{C}$, and $t_k:=t\frac{k}{n}$, $1\le k\le n$. Then the Gram matrix, defined as in Theorem 3.3, corresponding to Brownian motion at discrete times t_k, $1\le k\le n$, is given by $M_{1,n}=(\mathbf{1}_k,\mathbf{1}_l)_{k,l=1,\dots,n}=\frac{t}{n}\min(k,l)_{k,l=1,\dots,n}$, see (2.1). So for*

$$\Phi:=\prod_{k=1}^{n}\sigma_z\delta(\langle\cdot,\mathbf{1}_{[0,t_k)}\rangle-x_k)\in(S)'\tag{3.1}$$

one has that

$$S(\Phi)(f)=\frac{1}{\sqrt{(2\pi z^2\frac{t}{n})^n}}\times\exp\left(-\frac{n}{2t}\left(\left((\mathbf{1}_{[0,t_1)},f)-\frac{x_1}{z}\right)^2+\sum_{k=1}^{n-1}\left((\mathbf{1}_{[0,t_k)},f)-\frac{x_k}{z}-\left((\mathbf{1}_{[0,t_{k+1})},f)-\frac{x_{k+1}}{z}\right)\right)^2\right)\right),\tag{3.2}$$

for all $f\in S(\mathbb{R})$.

Proof. By induction one can show that for all $n \in \mathbb{N}$ the inverse of $M_{1,n}$ is given by

$$M_{1,n}^{-1} = \frac{n}{t} \begin{pmatrix} 2 & -1 & 0 & . & . & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & -1 & 2 & -1 & 0 \\ . & . & . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 1 \end{pmatrix}$$

By Gauss algorithm (here add the $k+1$-th row to the k-th row $k = n-1, \dots, 1$) one can show that $\det M_{1,n}^{-1} = \left(\frac{n}{t}\right)^n$, for all $n \in \mathbb{N}$. Hence Theorem 3.3 yields (3.2). □

Lemma 3.5. *Let $0 < t < \infty$, $z \in \overline{S_0}$, $n \in \mathbb{N}$, $y_k \in \mathbb{C}$. Then we define $t_k := t\frac{k}{n}$ for $1 \le k \le n$ and $h_k := \mathbf{1}_{[0,t_k)} - \frac{k}{n}\mathbf{1}_{[0,t)}$ for $1 \le k \le n-1$. The Gram matrix corresponding to Brownian bridge at discrete times t_k, $1 \le k \le n-1$, is given by $M_{2,n-1} = (h_k, h_l)_{k,l=1,\dots,n} = \left(\left(\mathbf{1}_{[0,t_k)} - \frac{k}{n}\mathbf{1}_{[0,t)}, \mathbf{1}_{[0,t_l)} - \frac{l}{n}\mathbf{1}_{[0,t)}\right)\right)_{k,l=1,\dots,n} = \frac{t}{n}\left(\min(k,l) - \frac{kl}{n}\right)_{k,l=1,\dots,n}$, see (2.2). Hence for*

$$\Psi := \prod_{k=1}^{n-1} \sigma_z \delta(\langle \cdot, h_k \rangle - y_k) \in (S)' \tag{3.3}$$

one has that

$$S(\Psi)(f) = \frac{1}{\sqrt{\frac{1}{n}(2\pi z^2 \frac{t}{n})^{n-1}}} \exp\left(-\frac{n}{2t}\left(\left((h_1, f) - \frac{y_1}{z}\right)^2 + \sum_{k=1}^{n-2}\left((h_k, f) - \frac{y_k}{z} - \left((h_{k+1}, f) - \frac{y_{k+1}}{z}\right)\right)^2 + \left((h_{n-1}, f) - \frac{y_{n-1}}{z}\right)^2\right)\right), \tag{3.4}$$

for all $f \in S(\mathbb{R})$.

Proof. One can easy calculate that

$$M_{2,n-1} = M_{2,n-1} M_{1,n-1}^{-1} M_{1,n-1} = \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & 1 & 0 \\ -\frac{1}{n} & -\frac{2}{n} & . & . & . -\frac{n-2}{n} & 1 - \frac{n-1}{n} \end{pmatrix} M_{1,n-1},$$

therefore

$$M_{2,n-1}^{-1} = M_{1,n-1}^{-1} \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & 1 & 0 \\ -\frac{\frac{1}{n}}{1-\frac{n-1}{n}} & -\frac{\frac{2}{n}}{1-\frac{n-1}{n}} & . & . & . -\frac{\frac{n-2}{n}}{1-\frac{n-1}{n}} & \frac{1}{1-\frac{n-1}{n}} \end{pmatrix}$$

$$= \frac{n}{t} \begin{pmatrix} 2 & -1 & 0 & . & . & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & -1 & 2 & -1 & 0 \\ . & . & . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 2 \end{pmatrix}.$$

As one can see in the proof of Lemma 3.4, the determinant of $M^{-1}_{2,n-1}$ is given by $\det M^{-1}_{2,n-1} = n \det M^{-1}_{1,n-1} = n\left(\frac{n}{t}\right)^{n-1}$. Now again by Theorem 3.3 we obtain (3.4). □

In the following Theorem we consider a relation between a product of Donsker's deltas for Brownian motion and a product of Donsker's deltas for Brownian bridge.

Theorem 3.6. *Let* $0 < t < \infty$, $z \in \overline{S_0}$, $n \in \mathbb{N}$ *and* $x_k \in \mathbb{R}$, $k = 1, \ldots, n$. *We define* $t_k := t\frac{k}{n}$, $k = 1, \ldots, n$, $y_k := x_k - \frac{k}{n}x_n$, $k = 1, \ldots, n-1$, *and* $h_k := \mathbf{1}_{[0,t_k)} - \frac{k}{n}\mathbf{1}_{[0,t)}$, $1 \le k \le n-1$. *Then for* Φ *and* Ψ *as in* (3.1) *and* (3.3)*, respectively, one has that*

$$S(\Phi)(f) = S(\Psi)(f)\frac{1}{z}S(\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - x_n/z))(f) = S(\Psi)(f)\frac{1}{\sqrt{2\pi z t}}\exp\left(\frac{1}{2t}(f, \mathbf{1}_{[0,t)}) - x_n/z\right)^2,$$

for $f \in S(\mathbb{R})$. *I.e.,*

$$\Phi = \Psi \diamond \sigma_z \delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - x).$$

Proof. From Lemma 3.4 we know that

$$S(\Phi)(f) = \frac{1}{\sqrt{(2\pi z^2 \frac{t}{n})^n}} \exp\left(-\frac{n}{2t}\left(\left((\mathbf{1}_{[0,t_1)}, f) - \frac{x_1}{z}\right)^2 + \sum_{k=1}^{n-1}\left((\mathbf{1}_{[0,t_k)}, f) - \frac{x_{k+1}}{z} - \left((\mathbf{1}_{[0,t_{k+1})}, f) - \frac{x_{k+1}}{z}\right)\right)^2\right)\right),$$

for all $f \in S(\mathbb{R})$. One can show that for $a_1, \ldots, a_{n-1}$, $n \in \mathbb{N}$, and $b \in \mathbb{C}$ one has that

$$\left(a_1 - \frac{1}{n}b\right)^2 + \sum_{k=1}^{n-2}\left(a_k + \frac{1}{n}b - a_{k+1}\right)^2 + \left(a_{n-1} - \frac{n-1}{n}b\right)^2 = a_1^2 + \sum_{k=1}^{n-2}(a_k - a_{k+1})^2 + (a_{n-1} - b)^2 - \frac{1}{n}b^2.$$

Using this formula and Lemma 3.5 one gets that

$$\begin{aligned} S(\Phi)(f) &= \frac{1}{\sqrt{(2\pi z^2 \frac{t}{n})^n}} \\ &\times \exp\left(-\frac{t}{2n}\left(\left((h_1, f) - \frac{y_1}{z}\right)^2 + \sum_{k=1}^{n-2}\left((h_k, f) - \frac{y_k}{z} - \left((h_{k+1}, f) - \frac{y_{k+1}}{z}\right)\right)^2 + \left((h_{n-1}, f) - \frac{y_{n-1}}{z}\right)^2\right)\right) \\ &= \frac{\sqrt{\frac{1}{n}(2\pi\frac{t}{n})^{n-1}}}{\sqrt{(2\pi\frac{t}{n})^n}} \exp\left(-\frac{1}{2t}\left((f, \mathbf{1}_{[0,t)}) - x_n/z\right)\right) S(\Psi)(f), \end{aligned}$$

for all $f \in S(\mathbb{R})$. Hence

$$S(\Phi)(f) = \frac{1}{\sqrt{2\pi z^2 t}} \exp\left(-\frac{1}{2t}\left((f, \mathbf{1}_{[0,t)}) - x_n/z\right)\right) S(\Psi)(f), \quad f \in S(\mathbb{R}).$$

□

Corollary 3.7. *The Wick product in Theorem 3.6 is a pointwise product, due to independence.*

Proof. By Lemma 2.10 one has only to show that the regular generalized functions Ψ and the scaled Donsker's delta $\sigma_z\delta\left(\langle\cdot,\mathbf{1}_{[0,t)}\rangle-(x-x_0)\right)$ are independent. This is true since for all $f\in S(\mathbb{R})$ on the one side

$$S\sigma_z\delta\left(\langle\cdot,\mathbf{1}_{[0,t)}\rangle-(x-x_0)\right)(f)=S\sigma_z\delta\left(\langle\cdot,\mathbf{1}_{[0,t)}\rangle-(x-x_0)\right)(\mathbf{1}_{[0,t)}f)$$

and on the other side by (2.3)

$$S(\Psi)(f)=S(\Psi)(f-\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)})=S(\Psi)\left(\mathbf{1}_{[0,t)^c}(f-\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)})\right)=S(\Psi)(\mathbf{1}_{[0,t)^c}f).$$

Here we used that for $k\in\{1,\dots,n\}$ we have that

$$\begin{aligned}(h_k,f-\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)})&=(h_k,f)-(h_k,\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)})\\&=(h_k,f)-(\mathbf{1}_{[0,t_k)}\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)})-\left(\frac{k}{n}\mathbf{1}_{[0,t)},\langle f,\mathbf{1}_{[0,t)}\rangle\mathbf{1}_{[0,t)}\right)=(h_k,f). \quad (3.5)\end{aligned}$$

□

3.3. **Finitely based functions in terms of products of Donsker's delta.** Following [28, Sec. 4.6.3], we can use Theorem 3.3 to construct an integral representation for finitely based functions. Let $h_j\in L^2(\mathbb{R})$, $1\le j\le n$ be linear independent, such that $M=\left(\left(h_k,h_j\right)_{L^2}\right)_{k,j=1,\dots,n}$ is positive definite, and $G\in L^p(\nu_M)$ for some $p>1$. Here ν_M denotes the measure on $\mathbb{R}^n$ having density $\exp\left(-\frac{1}{2}\sum_{k,j=1}^n x_kM^{-1}_{k,j}x_j\right)$ w.r.t. the Lebesgue measure on $\mathbb{R}^n$. In this case in terms of the image measure one can easily show that

$$G\left(\langle\cdot,h_1\rangle,\dots,\langle\cdot,h_n\rangle\right)\in L^p(\mu).$$

Lemma 3.8. *For $G\in L^p(\nu_M)$ the following relation holds*

$$G\left(\langle\cdot,h_1\rangle,\dots,\langle\cdot,h_n\rangle\right)=\int_{\mathbb{R}^n}G(x)\prod_{j=1}^n\delta\left(\langle\cdot,h_j\rangle-x_j\right)d^nx$$

where the integral in $(S)'$ is in the sense of Bochner as in Theorem 2.4.

4. Complex scaled Feynman–Kac kernel

In the Euclidean case, a solution to the heat equation is obtained by the famous Feynman–Kac formula

$$E\left(\exp\left(\int_{t_0}^t V(x_0+B_s)\,ds\right)f(x_0+B_t)\right),$$

for suitable $f,\ V:\mathbb{R}\to\mathbb{R}$, $x_0\in\mathbb{R}$ and $0\le t_0\le t\le T<\infty$, see e.g. [26]. Here $(B_t)_{t\in[t_0,T]}$ denotes a Brownian motion starting at time t_0 in zero. Moreover for $t\in(t_0,T]$ and $x,x_0\in\mathbb{R}$ the heat kernel $K_V:\mathbb{R}\times\mathbb{R}\times(0,T]\to\mathbb{R}$ is given by

$$\begin{aligned}K_V(x,t;x_0,t_0)=\frac{1}{\sqrt{2\pi(t-t_0)}}&\exp\left(-\frac{1}{2(t-t_0)}(x_0-x)^2\right)\\&E\left(\exp\left(\int_{t_0}^t V\left(x_0-\frac{s-t_0}{t-t_0}(x_0-x)+B_s-\frac{s-t_0}{t-t_0}B_t\right)ds\right)\right), \quad (4.1)\end{aligned}$$

for nice potentials, see e.g. [15]. Observe that in the integral we are dealing with a Brownian bridge (starting at time t_0 in x_0 and ending at time t in x). In White Noise Analysis one will construct the integral kernel by inserting Donsker's delta in order to fix the final point $x\in\mathbb{R}$, and taking the generalized expectation, i.e.,

$$K_V(x,t;x_0,t_0)=E\left(\exp\left(\int_{t_0}^t V(x_0+\langle\cdot,\mathbf{1}_{[t_0,s)}\rangle)ds\right)\delta(\langle\cdot,\mathbf{1}_{[t_0,t)}\rangle-(x-x_0))\right),$$

here the integrand is supposed to be e.g. a Hida distribution. We are not only interested in a mathematical meaning of the informal expression inside the expectation on the right hand side, we also want to answer the question for which potentials we can generalize this kernel to a complex scaled situation. One idea of complex scaling goes back to [8], see also [4] and [14]. In formulas this means for suitable potentials V, $z \in \mathbb{C}$, we are interested in the product

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x + z\langle\cdot, \mathbf{1}_{[t_0,s)}\rangle)ds\right)\sigma_z\delta(\langle\cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0)), \tag{4.2}$$

as a generalized function of white noise, with the goal of constructing solutions for

$$\left(\frac{\partial}{\partial t} - z^2\frac{1}{2}\Delta - \frac{1}{z^2}V(x)\right)K(x, t \mid x_0, t_0) = 0, \tag{4.3}$$

with $x, x_0 \in \mathbb{R}, 0 \leq t_0 < t < T$.

We show that

$$\begin{aligned}
&\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x + z\langle\cdot, \mathbf{1}_{[t_0,s)}\rangle)ds\right)\sigma_z\delta(\langle\cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0))\\
&= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{s - t_0}{t - t_0}(x - x_0) + z\left(\langle\cdot, \mathbf{1}_{[t_0,s)}\rangle - \frac{s - t_0}{t - t_0}\langle\cdot, \mathbf{1}_{[0,t)}\rangle\right)\right)ds\right) \diamond \sigma_z\delta\left(\langle\cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0)\right).
\end{aligned} \tag{4.4}$$

Taking its generalized expectation we obtain

$$\begin{aligned}
K(x, t \mid x_0, t_0) = &\frac{1}{\sqrt{2\pi(t - t_0)z^2}}\exp\left(-\frac{1}{2(t - t_0)z^2}(x_0 - x)^2\right)\\
&\times E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{s - t_0}{t - t_0}(x - x_0) + z\left(\langle\cdot, \mathbf{1}_{[t_0,s)}\rangle - \frac{s - t_0}{t - t_0}\langle\cdot, \mathbf{1}_{[t_0,t)}\rangle\right)\right)ds\right)\right),
\end{aligned}$$

which is a scaled version of (4.1). Of course this is only possible if $\sigma_z\delta(\langle\cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0))$ is a Hida distribution. Hence we only consider $z \in \overline{S_0}$. Furthermore the potential V should be defined on a certain subset of the complex plane such that $V(x + zy)$ is well-defined for $x \in \mathbb{R}$, $y \in \mathbb{R}$.

In Section 4.1 under appropriate assumptions on V we derive (4.4). First step in doing this is an approximation via finitely based functions, see Proposition 4.5. Then we show convergence of this *approximation* in the space of Hida distributions, see Theorem 4.8.

4.1. **Approximation by finitely based functions.** For $O \subset \mathbb{R}$, such that $\mathbb{R} \setminus O$ is a set of Lebesgue measure zero, and $z \in \mathbb{C} \setminus \{0\}$ the set $\mathcal{D}_z \subset \mathbb{C}$ is defined by

$$\mathcal{D}_z := \left\{x + zy \;\middle|\; x \in O \text{ and } y \in \mathbb{R}\right\}. \tag{4.5}$$

Assumption 4.1. *Let $z \in \overline{S_0}$ and $0 < T < \infty$. We assume that the potential $V : \mathcal{D}_z \to \mathbb{C}$ is analytic and that there exists a constant $0 < A < \infty$, a locally bounded function $B : O \to \mathbb{R}$ and some $0 < \varepsilon < \frac{1}{4T^2}$ such that for all $x_0 \in O$ and $y \in \mathbb{R}$ one has that*

$$\left|\exp\left(\frac{1}{z^2}V(y)\right)\right| \leq A\exp\left(\varepsilon y^2\right) \quad \text{and} \quad \left|\exp\left(\frac{1}{z^2}V\left(x_0 + zy\right)\right)\right| \leq B(x_0)\exp\left(\varepsilon y^2\right).$$

To illustrate this class we point out in particular the following admissible potentials.

Example 4.2. *Let $z = 1$ and $p : \mathbb{R} \to \mathbb{R}$ a polynomial. Then the potential*

$$\begin{aligned}
V : \mathbb{R} &\to \mathbb{R}\\
x &\mapsto -\exp(p(x))
\end{aligned}$$

fulfills Assumption 4.1.

Example 4.3. *Here for $n \in \mathbb{N}_0$ and $z = \sqrt{i}$, we have admissible potentials such as*

$$\begin{aligned}
V : \mathbb{C} &\to \mathbb{C}\\
x &\mapsto (-1)^{n+1}a_{4n+2}x^{4n+2} + \sum_{j=1}^{4n+1} a_j x^j,
\end{aligned}$$

for $a_0, \ldots, a_{4n+1} \in \mathbb{C}$ and $a_{4n+2} > 0$. It is easy to check that such potentials fulfill Assumption 4.1.

Example 4.4. *Consider $O = \mathbb{R} \setminus \{b\}$, $b \in \mathbb{R}$ and $z = \sqrt{i}$. Then the potentials*

(i)

$$V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$$
$$x \mapsto \frac{a}{|x-b|^n},$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$,

(ii)

$$V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$$
$$x \mapsto \frac{a}{(x-b)^n},$$

for $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $n \in \mathbb{N}$, fulfill Assumption 4.1. This can be obtained by using the natural representation of V, as in (i), as an analytic function

$$V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$$
$$x \mapsto \exp\Big(\log(a) - \frac{n}{2}\log\big((x-b)^2\big)\Big),$$

which can be estimated by

$$\Big|V\big(x+\sqrt{i}y\big)\Big| = |a|\,\Big|\exp\Big(-\frac{n}{2}\log\big((x-b+\sqrt{i}y)^2\big)\Big| \leq |a|\,\exp\Big(-\frac{n}{2}\log\Big(\frac{(x-b)^2}{2}\Big)\Big),$$

for all $x \in O$ and $y \in \mathbb{R}$. For the proof of this formula see, [8].

Similar examples for potentials are considered in [8] and [13]. Typically these potentials are singular in the sense of not obeying a Kato bound which would ensure convergent perturbation expansions.

In the remaining part of this section we assume that $z \in \overline{S_0}$ and $V : \mathcal{D}_z \to \mathbb{C}$ such that Assumption 4.1 holds. W.l.o.g. we consider the case $t_0 = 0$. Then for $n \in \mathbb{N}$, we define the decomposition of the time interval $[0,t]$, given by $t_k := t\frac{k}{n}$, $k = 1,\ldots,n$. Then by Assumption 4.1 the Riemann approximation,

$$\phi_n := \exp\Big(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + z\langle\cdot,\mathbf{1}_{[0,t_k)}\rangle\Big)\Big), \tag{4.6}$$

is a well-defined $L^2(\mu)$-function.

Proposition 4.5. *The product of a complex scaled Donsker's delta with the Riemann approximation defined as in* (4.6) *can be defined as an operation in $(S)'$. Hence*

$$\Phi_n = \exp\Big(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + z\langle\cdot,\mathbf{1}_{[0,t_k)}\rangle\Big)\Big)\sigma_z\delta(\langle\cdot,\mathbf{1}_{[0,t)}\rangle - (x-x_0)), \tag{4.7}$$

is a Hida distribution for all $x \in \mathbb{R}$.

Proof. We define

$$G : \mathbb{R}^{n-1} \to \mathbb{C}$$
$$y = (y_1,\ldots,y_{n-1}) \mapsto \exp\Big(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + y_k\Big)\Big). \tag{4.8}$$

Then by Assumption 4.1 $y \mapsto G(y)$ and $y \mapsto G(zy)$ are in $L^2_{\mathbb{C}}\big(\mathbb{R}^n, \nu_{M^{-1}_{1,n-1}}\big)$, where $M_{1,n-1}$ is defined as in the proof of Lemma 3.4. Therefore, by Lemma 3.8 the function ϕ_n, defined as in (4.6), can be represented as

$$\phi_n = \int_{\mathbb{R}^{n-1}} G(zy)\prod_{k=1}^{n-1}\delta\left(\langle\cdot,\mathbf{1}_{[0,t_k)}\rangle - y_k\right)\,d^{n-1}y,$$

for all $n \in \mathbb{N}$. From Lemma 3.2 we know that $\sigma_z\delta$ is homogeneous of degree -1, so we get that for all $y \in \mathbb{R}^{n-1}$ the function

$$\Phi_{n,y} := \prod_{k=1}^{n-1} \delta\left(\langle \cdot, \mathbf{1}_{[0,t_k)}\rangle - y_k\right) \sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0)) = \frac{1}{z}\prod_{k=1}^{n-1} \delta\left(\langle \cdot, \mathbf{1}_{[0,t_k)}\rangle - y_k\right) \delta\left(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - \frac{(x-x_0)}{z}\right),$$

is a Hida distribution for all $n \in \mathbb{N}$, by Theorem 3.3. Moreover as shown in Lemma 3.4 its S-transform evaluated at $f \in S(\mathbb{R})$ is given by

$$S\left(\Phi_{n,y}\right)(f) = \frac{1}{\sqrt{i(2\pi\frac{t}{n})^n}} \exp\left(-\frac{n}{2t}\left(((\mathbf{1}_{[0,t_1)}, f) - y_1)^2 + \sum_{k=1}^{n-1}((\mathbf{1}_{[0,t_k)}, f) - y_k - ((\mathbf{1}_{[0,t_{k+1})}, f) - y_{k+1}))^2\right)\right),$$

where $y_n := \frac{x-x_0}{z}$. Since $y \mapsto G(zy) \in L^2_{\mathbb{C}}\left(\mathbb{R}^{n-1}, \nu_{M^{-1}_{1,n-1}}\right)$, $M_{1,n-1}$ defined as in the proof of Lemma 3.4, the function $\mathbb{R}^{n-1} \ni y = (y_1, \dots, y_{n-1}) \mapsto G(zy)\, S\Phi_{n,y}(f)$, $f \in S(\mathbb{R})$, fulfills the assumptions of Theorem 2.4. Hence

$$\Phi_n := \int_{\mathbb{R}^{n-1}} G(zy)\, \Phi_{n,y}\, dy$$

is a Hida distribution and its S-transform is given by

$$S(\Phi_n)(f) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Phi_{n,y}(f)\, dy,$$

for all $f \in S(\mathbb{R})$. □

Theorem 4.6. *Let $h_k := \mathbf{1}_{[0,t_k)} - \frac{k}{n}\mathbf{1}_{[0,t)}$, $1 \le k \le n-1$. Moreover let $x \in O$ such that $x_0 + \frac{s}{t}(x-x_0)$ and Φ_n, $n \in \mathbb{N}$, be as in Proposition 4.5. Then for*

$$\Psi_n := \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\left(x_0 + \frac{k}{n}(x-x_0) + z\langle \cdot, h_k\rangle\right)\right) \in L^2(\mu), \qquad n \in \mathbb{N},$$

we have that

$$\Phi_n = \Psi_n \diamond \sigma_z\delta\left(\langle \omega, \mathbf{1}_{[0,t)}\rangle - (x-x_0)\right) \in (S)'.$$

Proof. As shown in the proof of Proposition 4.5 Φ_n is a Hida distribution for all $n \in \mathbb{N}$. Its S-transform at $f \in S(\mathbb{R})$ is given by

$$S(\Phi_n)(f) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Phi_{n,y}(f)\, dy,$$

where G and $\Phi_{n,y}$, $y \in \mathbb{R}^{n-1}$, are defined as in the proof of Proposition 4.5. By Theorem 3.6 one gets that

$$S(\Phi_n)(f) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(f)\, dy\; S\left(\sigma_z\delta\left(\langle \omega, \mathbf{1}_{[0,t)}\rangle - (x-x_0)\right)\right)(f),$$

where $\Psi_{n,y} := \prod_{k=1}^{n-1}\delta\left(\langle \cdot, h_k\rangle - y_k - \frac{k}{n}\frac{(x-x_0)}{z}\right)$. Hence it is left to show that

$$S(\Psi_n)(f) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(f)\, dy, \tag{4.9}$$

which can be done by the following integral transformations:

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(f)\, dy &= \int_{\mathbb{R}^{n-1}} G(zy)\, S\left(\prod_{k=1}^{n-1}\delta\left(\langle \cdot, h_k\rangle - \left(y_k - \frac{k}{n}\frac{(x-x_0)}{z}\right)\right)\right)(f)\, dy \\ &= \frac{1}{z}\int_{\gamma_{1,n-1}} G(u)\, S\left(\prod_{k=1}^{n-1}\delta\left(\langle \cdot, h_k\rangle - \frac{u_k - \frac{k}{n}(x-x_0)}{z}\right)\right)(f)\, du \\ &= \frac{1}{z}\int_{\gamma_{2,n-1}} G(u + w(x-x_0))\, S\left(\prod_{k=1}^{n-1}\delta\left(\langle \cdot, h_k\rangle - \frac{u_k}{z}\right)\right)(f)\, du,\end{aligned}$$

where $\gamma_{1,n-1} := \left\{u \in \mathbb{C}^{n-1} \middle| u := zy,\ y \in \mathbb{R}^{n-1}\right\}$, $\gamma_{2,n-1} := \left\{u \in \mathbb{C}^{n-1} \middle| u_k := z(y_k - \frac{k}{n}(x-x_0)),\ y \in \mathbb{R}^{n-1}\right\}$ and $w \in \mathbb{R}^{n-1}$ is given by $w_k = \frac{k}{n}$, $k = 1, \dots, n-1$. Again by an integral transformation

$$\int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(f)\, dy = \int_{\gamma_{3,n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta(\langle \cdot, h_k\rangle - y_k)\right)(f)\, dy$$

$$= \int_{\mathbb{R}^{n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta(\langle \cdot, h_k\rangle - y_k)\right)(f)\, dy,$$

where $\gamma_{3,n-1} := \left\{y \in \mathbb{R}^{n-1} \middle| y_k := x_k - \frac{k}{n}(x-x_0),\ y \in \mathbb{R}^{n-1}\right\}$. Since $x_0 + \frac{k}{n}(x-x_0) \in O$ it follows by Assumption 4.1 that

$$y \mapsto \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=0}^{n-1} V\left(zy + \frac{k}{n}(x-x_0)\right)\right) \in L^2\left(\mathbb{R}^{n-1}, \nu_{M^{-1}_{2,n-1}}\right),$$

where $M_{2,n-1}$ is defined as in the proof of Lemma 3.5. Hence again by Lemma 3.8 we get that

$$S(\Psi_n)(f) = \int_{\mathbb{R}^{n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta(\langle \cdot, h_k\rangle - y_k)\right)(f)\, dy,$$

and therefore (4.9) is true. □

Proposition 4.7. *Let ϕ_n and Ψ_n, $n \in \mathbb{N}$, be defined as in* (4.6) *and Theorem 4.6, respectively. Then $\phi_n, \Psi_n \in L^2(\mu)$ for all $n \in \mathbb{N}$. Moreover the sequences $(\Psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ converge in $L^2(\mu)$ to*

$$\phi := \exp\left(\frac{1}{z^2}\int_0^t V(x_0 + z\langle \cdot, \mathbf{1}_{[0,s)}\rangle)\, ds\right),$$

and

$$\Psi := \exp\left(\frac{1}{z^2}\int_0^t V\left(x_0 - \frac{s}{t}(x-x_0) + z\langle \cdot, h_s\rangle\right) ds\right),$$

respectively, where $h_s := \mathbf{1}_{[0,s)} - \frac{s}{t}\mathbf{1}_{[0,t)}$, $0 \le s \le t$.

Using Assumption 4.1 the proof follows directly by Lebesgue dominated convergence.

Theorem 4.8. *Let $0 \le t \le T < \infty$, $t_k := t\frac{k}{n}$ for $1 \le k \le n$, $n \in \mathbb{N}$, $x, x_0 \in O$ such that $x_0 + \frac{s}{t}(x-x_0) \in O$, for all $s \in [0,t]$, and Φ_n as in Proposition 4.5. Then the sequence of Hida distributions $(\Phi_n)_{n\in\mathbb{N}}$ converges in $(S)'$, and it is natural to identify the limit object with* (4.2)*, i.e.,*

$$\Phi := \exp\left(\frac{1}{z^2}\int_0^t V\big(x_0 + z\langle \cdot, \mathbf{1}_{[0,s)}\rangle\big) ds\right)\sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0)) := \lim_{n\to\infty}\Phi_n. \tag{4.10}$$

Moreover equation (4.4) *holds, which implies that the S-transform of Φ is given by*

$$S\Phi(f) = S\left(\sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0))\right)(f) S\Psi(f),$$

for all $f \in S(\mathbb{R})$.

Proof. The proof follows directly by continuity of the Wick product and the proofs of Theorem 4.6 and Proposition 4.7. □

Corollary 4.9. *The Wick product in* (4.4) *and Theorem 4.8 is due to independence a pointwise product.*

Proof. Theorem 3.1 and Proposition 4.7 implies that $\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0))$ and Ψ (defined as in Proposition 4.7) are regular generalized functions. Hence by Lemma 2.10 one only has to show that the $L^2(\mu)$-function Ψ and the scaled Donsker's delta $\sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0))$ are independent. This is true since for all $f \in S(\mathbb{R})$ on the one side

$$S\sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0))(f) = S\sigma_z\delta(\langle \cdot, \mathbf{1}_{[0,t)}\rangle - (x-x_0))(\mathbf{1}_{[0,t)}f)$$

and on the other side

$$S(\Psi)(f) = S(\Psi)(f - \langle f, \mathbf{1}_{[0,t)}\rangle \mathbf{1}_{[0,t)}) = S(\Psi)\left(\mathbf{1}_{[0,t)^c}(f - \langle f, \mathbf{1}_{[0,t)}\rangle \mathbf{1}_{[0,t)})\right) = S(\Psi)(\mathbf{1}_{[0,t)^c}f). \tag{4.11}$$

Here the first equality of (4.11) can be shown by using (2.3) close to (3.5). □

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