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Completeness and decidability results for a logic of contrary-to-duty conditionals

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Abstract

This article has two parts. In Part I, we briefly outline the analysis of 'contrary-to-duty' obligation sentences presented in our 2002 handbook chapter 'Deontic logic and contrary-to-duties', with a focus on the intuitions that motivated the basic formal-logical moves we made. We also explain that the present account of the theory differs in two significant respects from the earlier version, one terminological, the other concerning the way the constituent modalities interconnect. Part II is the principal contribution of this article, in which we show that it is possible to define a complete and decidable axiomatization for the Carmo and Jones logic, a problem that was still open. The axiomatization includes two new inference rules; we illustrate their use in proofs, and show that on the basis of this axiomatization we can recover all the axioms and rules considered in 'Deontic logic and contrary-to-duties', and used there in the analysis of contrary-to-duty conditional scenarios.

Keywords: deontic logic, contrary-to-duty conditionals (CTDs), completeness and decidability results.

1 Part I

Part I of this article rehearses in outline the principal considerations that motivate the analysis of 'contrary-to-duty' obligation sentences (CTDs) presented in [1]. In the course of this outline, we also indicate that there are two significant differences between the [1] version of the theory and the one presented here. The first difference is terminological: what we earlier called 'ideal obligations' will now be referred to as 'primary obligations'. The second difference pertains to the way in which primary and actual obligations are conceptually connected: we here adopt a logical principle that was discussed in [1], but to which we did not there commit.

The 'dog and warning sign' example (D&WS), due to [9, 10] provides a suitable point of departure. (The example is a variant of the Chisholm set [3], which initiated the discussion of CTDs in deontic logic.)

(D&WS)

(a) There ought to be no dog.
(b) If there is no dog, there ought to be no warning sign.
(c) If there is a dog, there ought to be a warning sign.
(d) There is a dog.

Any attempt to give an appropriate formal-logical analysis of sets of this kind must be able to give an answer to the questions: what is the actual obligation, in the circumstances described, of the agent

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to whom the norms apply ? Is his actual obligation to put up a warning sign ? Or is it, rather, to get rid of the dog ? The key to the theory developed in [1] is that the appropriate answer to these questions turns on the *status* assigned to the fact described in line (d), in the following sense: if the fact that there is a dog is not a fixed, unavoidable feature of the situation, then the actual obligation on the agent concerned is that expressed by line (a). However, if—for one reason or another—the presence of the dog is a necessary, fixed, unavoidable feature of the situation, so that the practical possibility of complying with the obligation expressed by line (a) has to all intents and purposes been eliminated, then the actual obligation applying in the circumstances is to put up a warning sign.

There are various different sorts of reasons why the facts of a given situation might be deemed to be fixed. In our view, those deontic logicians who proposed temporal solutions to CTD problems focussed on *one* common type of reason for fixity; for instance, if the books ought to be returned by date due, then there is no way one can meet *that* obligation after the due-date has passed: the practical possibility of satisfying *that* obligation has been eliminated. (Depending on the content of the library's regulations, the actual obligation that then applies, in the circumstances, might be, for instance, to pay a fine.)

We believe that one useful way of viewing the [1] theory is to see it as a kind of generalization of the insight contained in the temporal approach to CTDs. The need to generalize arises because temporal reasons, although very common, are not the *only* reasons why facts become fixed. An obvious example here is that the deed of killing, once performed, cannot be undone—not for reasons of *temporal* necessity, but rather *causal* necessity. But less obvious, perhaps, are cases in which fixity arises as a result of agents' decisions; consider, again, the dog example: it may be practically impossible in the situation to remove the dog because, for instance, the dog owner stubbornly refuses to do so, and everyone else in the vicinity has decided to keep well away from the animal, knowing it to be savage. The presence of the dog is a fixed fact because nobody is willing to get rid of it. From the practical perspective, the main point is that the possibility of satisfaction of the requirement that there be no dog has been effectively eliminated. Of course, *that* obligation has been violated, and as a consequence the dog owner may be liable to sanction. But in the circumstances the actual obligation, in virtue of line (c) of (D&WS), is to erect a warning sign.

Two lines from the example of the 'considerate assassin' (CA), also due to [10], allow us to develop these points further:

(CA)

(a) You should not kill Mr X.
(b) …..............
(c) But if you kill Mr X, you should offer him a cigarette.

The question is: when does the assassin have an actual obligation to offer Mr X a cigarette ? Not after killing him, obviously ! A plausible interpretation of the scenario is that the assassin's actual obligation to offer a cigarette comes into force when he firmly decides that he is going to kill Mr X, i.e. when it becomes a fixed, settled fact that Mr X will be killed.

Note that the examples considered indicate that two distinct notions of necessity—and their associated notions of possibility—will need to be taken into account. The dead cannot be offered the opportunity to smoke a cigarette, just as a book cannot be returned by date due if the deadline has passed. But, in contrast, the dog owner may have both the ability and opportunity to remove the dog, just as the assassin may have both the ability and the opportunity to refrain from killing Mr X. It was a fundamental feature of the [1] approach that these two species of necessity each had their respective roles to play in determining which of two types of normative conclusion could validly be drawn from a given CTD scenario. We have already referred to one of these two types of normative conclusion: the *actual* obligations. For the other type, we used in [1] the label 'ideal obligations', following the terminology employed in [8]. We now believe the term 'primary obligations' to be more suitable, for reasons that should become clearer in due course. We next proceed to outline how the two concepts of necessity, and the two notions of obligation, hang together in the [1] theory of CTDs.

Formulae of the form $\Box_a A$, where A is a place-holder for a sentence, denoted by a box $\Box$ with an arrow $\rightarrow$ inside in [1], represent what is actually fixed, or unavoidable, given—among other factors—what the agents concerned have decided to do or not to do. So, in (D&WS), if the relevant agents have decided that the dog is not going to be removed, then the sentence $\Box_a$dog is true of the situation, where 'dog' abbreviates the sentence 'there is a dog'. The dual possibility notion, $\Diamond_a A$, is defined by $\neg\Box_a\neg A$. Which *actual* obligations arise in the dog scenario will depend, in particular, on whether or not $\Diamond_a\neg$dog is true. We use $O_a A$ to represent 'it is actually obligatory that A'.

Given that the dog owner's will is firm, and he has stubbornly resolved that the dog stays, it is not actually possible that there is no dog. However, we nevertheless wish to say that it is *potentially* possible that there is no dog, in the sense that the dog owner does have the ability and opportunity to get rid of the dog and so, if he had decided differently, the dog might have been removed. We represent this second notion of possibility by $\Diamond_p A$, and its dual necessity notion (i.e. $\neg\Diamond_p\neg A$) by $\Box_p A$, denoted by a box $\Box$ in [1]. In the dog scenario, which primary obligations, represented by $O_p A$, are derivable will depend, in particular, on whether or not $\Diamond_p\neg$dog is true. (We note that O_p was denoted by O_i in [1].)

The discussion in [1] covers a range of CTD scenarios, each of which contained two types of component: sentences expressing obligation norms (e.g. (D&WS), (a), (b) and (c)) and sentences expressing facts (e.g. (D&WS), (d)). We represented the former component by means of a dyadic, conditional obligation operator O(A/B), where A and B are place-holders for sentences. For instance, (D&WS) (a)–(c) took the form:

(a) O(¬dog/T)
(b) O(¬sign/¬dog)
(c) O(sign/dog)

where 'dog' stands for the sentence 'there is a dog', 'sign' for the sentence 'there is a warning sign' and 'T' represents the tautologous condition.

Consider first the dyadic conditional obligation operator. How do we wish to interpret a sentence of the kind 'if there is a dog then there shall be a warning sign'? On our view, this sentence is to be understood as saying that in any context in which the presence of a dog is a fixed, unalterable fact, it is obligatory to have a warning sign, if this is possible. We think of a *context* as a set of worlds—the set of relevant worlds for the situation at hand. So the above sentence is to be understood as saying that, for any context in which there is a dog (i.e. for any context in which there is a dog in each world of that context), if it is possible to have a warning sign then it is obligatory to have a warning sign.[1] And in order to capture this idea we introduce in our models a function ob:$\wp(W)\rightarrow\wp(\wp(W))$ (to be presented in Part II) which picks out, for each context, the propositions which represent that which is obligatory in that context. That is, $||B||\in ob(X)$ (where $||B||$ denotes the truth set of B in the model in question) if and only if the proposition expressed by B represents something obligatory in context

[1]We are here using the term 'obligatory' in a *weak sense*; in a *strict sense*, for a sentence B to be obligatory in a context X we would also claim that there must exist at least one world in X where B is false (i.e. we would insist, for the strict sense, that obligations must be violable). Our actual and primary obligations will be considered in this *strict sense*.

X. Accordingly, we say that a sentence O(B/A) is true in a model if and only if, in any context X where A is true and B is possible (i.e. in any context having A true in each of its worlds and B true in at least one of its worlds), it is obligatory that B.[2]

On the basis of this operator we could now derive the obligations that were applicable in each context. The question is what are the types of contexts that we need to be able to talk about in our formal language, taking into account the obligations we want to derive. Our answer is that we want to be able to talk about the context of what is actually considered open to the agent (formed by the worlds we call the actual versions of the current world) and about the context of what was potentially possible/open to the agent (formed by the worlds we call the potential versions of the current world). The propositions that are obligatory in the former context we call *actual* obligations and we reserve the term *primary* obligations for the propostions that are obligatory in the latter context. And we use the two above-mentioned necessity operators (and their duals) to express in our formal language what is true or not true in these contexts.

So, depending on the modal status of line (d), above, as expressed in terms of our two pairs of notions of necessity and possibility, the logical theory licensed the inference of particular conclusions concerning primary and actual obligations.[3]

Primary obligations, we may say, represent what should have been done in a given situation. Accordingly, we express *violation* of obligation in terms of primary obligation. We may now outline, in summary, our analysis of the (D&WS) scenario: if it is a fixed fact that (actually necessary that) there is a dog (i.e. if $\Box_a$dog is true), but it is *actually* possible that a sign may be erected and *potentially* possible that there is no dog, then the [1] logic licenses the derivation of the actual obligation to erect a warning sign ($O_a$sign) and the primary obligation that there is no dog ($O_p\neg$dog). There is no *actual* obligation that there be no dog precisely because we are supposing that removal of the dog is not an *actual* possibility. But a violation has nevertheless occurred, as expressed by the conjunction $O_p\neg\text{dog} \wedge \text{dog}$.[4]

Those, in brief outline, are the basic principles of our approach. For a detailed discussion of their application to a broad spectrum of CTD scenarios, including those in which more than two levels of obligations are involved, we refer the reader to [1], especially pp. 298–314.

What remains to be mentioned is the second issue with respect to which the present article differs from [1]. At p. 319 of the latter we raised the question of how the relationship between actual obligation and (what we now call) primary obligation is to be characterized. Suppose it to be the case that there is a primary obligation that A, and that it is still actually possible to fulfill that obligation and still actually possible to violate it. That is, suppose:

$$O_pA \wedge \Diamond_a A \wedge \Diamond_a \neg A$$

Should it not then follow that it is actually obligatory that A ? In [1] we discussed this issue, but left it unresolved. We now decide in favour of an affirmative answer to the question; the formal

[2]We also add the further requirement that the conjunction of A and B is not contradictory, in order to avoid some 'absurd' vacuous conditional obligations, and to secure the result that if O(B/A) is true then $||B|| \in ob(||A||)$.

[3]One of the reviewers suggested that there is a tendency in the literature to reserve the term 'primary obligations' to refer to unconditional obligations contained in the initial set of premises that describe a given scenario. So we should make it explicit that our use here of the term 'primary obligation' departs from that tendency, since for us primary obligations are among the obligations that may be derived from a particular scenario. In [1] we used the term 'ideal obligation' instead; we now prefer the term 'primary obligation' to designate those derived obligations that concern what *first and foremost should have been done* in a given situation; clearly, the circumstances of a given situation may be such that there is a difference between what *should have been done* and what *is actually required to be done*.

[4]On the distinction between violation and mere non-fulfillment, see [1], pp. 318–319.

ramifications of that decision are described in Part II. But we conclude this section by noting that the inclusion of

$$O_p A \wedge \Diamond_a A \wedge \Diamond_a \neg A \rightarrow O_a A$$

as a valid sentence of the logic provides, we think, good reason for not calling 'actual' obligations 'secondary' obligations.

2 Part II

Part II is the technical part of the article, in which we show that is possible to define a complete and decidable axiomatization for the logic proposed in [1], a problem that was still open.[5] In Section 1, we describe the formal language and in Section 2 we describe the semantics. In Section 3, we describe the proposed axiomatization and show that it is sound and consistent. The given axiomatization includes two inference rules of a new kind, and in Section 4 we illustrate the use of these inference rules in proofs and show that within this axiomatization we can recover (prove) all the axioms and rules considered in [1], and used there in the analysis of the CTD scenarios. Section 5 is devoted to proving that the proposed axiomatization is complete and satisfies the finite model property. We omit many of the proofs, or make only a sketch of them. For the details, the reader is referred to the Appendix A.

2.1 Section 1. Formal language

The alphabet of our formal language consists of:

- An infinite countable set of propositional symbols
- $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- (,) (parentheses)
- $\Box$ (in all worlds)
- $\Box_p$ (in all *potential* versions of the current world)
- $\Box_a$ (in all *actual* versions of the current world)
- O(/) (dyadic deontic operator)
- O_p (monadic deontic operator for *primary* obligation)
- O_a (monadic deontic operator for *actual* obligation)

Instead of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, we could consider as primitive only $\neg$ and $\wedge$, or any other adequate set of propositional connectives. The 'in all worlds' necessity operator $\Box$ was not considered in [1].[6] With it we can define an axiomatization that expresses better the underlying semantic intuitions,

[5]As mentioned in [1] (p. 296, footnote 21), at that time we already had some unpublished completeness results. They concerned the $\Box$+O(/) fragment of our logic (i.e. without considering $\Box_a$, $\Box_p$, O_a and O_p). Specifically, we proved that schemas 1–8 of Section 3 provide a complete and decidable axiomatization for that $\Box$+O(/) fragment (regarding models of the form M=<W,ob,V>). However, the definition of a complete and decidable axiomatization for the whole logic was an open issue. The adaptation of Cresswell's mini-canonical model (using the terminology of one of the reviewers) is here non-trivial and much more complex than for the $\Box$+O(/) fragment (in particular, for this fragment the formula φ used for building the 'canonical model', in Section 5, could be simply the initial consistent formula ψ). We also want to mention that, recently (after the writing of the original version of this article), we discovered that [6] provided a preferential semantics for our system, and proved an axiomatization of a modification of it to be complete with respect to ranked structures.

[6]In fact, as we have already mentioned, in [1] the box $\Box$ was used, but to represent the in all *potential* versions operator $\Box_p$, and the box with an arrow $\rightarrow$ inside was used to represent $\Box_a$. And O_p was denoted by O_i.

allowing us to clarify what has to do with all worlds, what has to do with the potential versions of the current world and what has to do with the actual versions of the current world. The consideration of $\Box$ also facilitates the construction of a complete axiomatization.

The *rules for construction of well-formed sentences (formulae)* are as usual. In writing formulas, we may omit parentheses assuming the following precedence between the operators: 1st) the unary operators; 2nd) $\wedge$; 3rd) $\vee$; 4th) $\rightarrow$ and $\leftrightarrow$. We will use (the first Latin capital letters) A, A_1, ..., B, B_1, ... to generically refer to sentences (formulas), $\Gamma, \Gamma_1, \ldots, \Delta, \ldots$ to denote sets of sentences and q, q_1, ... to refer to atomic sentences (propositional symbols).

The duals of $\Box$, $\Box_p$ and $\Box_a$ are denoted, respectively, by $\Diamond$, $\Diamond_p$ and $\Diamond_a$, and defined as usual:

$$\Diamond A =_{df} \neg\Box\neg A, \Diamond_p A =_{df} \neg\Box_p\neg A \text{ and } \Diamond_a A =_{df} \neg\Box_a\neg A$$

We consider $T =_{df} q\rightarrow q$, for some propositional symbol q, and $\bot =_{df} \neg T$. Sequences of conjunctions and disjunctions are defined as usual (and $A_1\wedge\ldots\wedge A_n =_{df} T$, if $n = 0$, and $A_1\vee\ldots\vee A_n =_{df} \bot$, if $n = 0$). We consider also conjunctions and disjunctions of finite sets of formulae, defined as expected (for $n\geq 0$): $\wedge\{A_1, \ldots, A_n\} =_{df} A_1\wedge\ldots\wedge A_n$ and $\vee\{A_1, \ldots, A_n\} =_{df} A_1\vee\ldots\vee A_n$. In the metalanguage, we also use the usual quantifiers.

2.2 *Section 2. Semantics*

Our *models* are structures M=<W,av, pv, ob, V>, where:

(1) W is a non-empty set.
(2) V is a function assigning a truth set to each atomic sentence (i.e. $V(q) \subseteq W$).
(3) ‘av’ is a function (where $\wp(W)$ denotes the power set of W)
 $av : W \rightarrow \wp(W)$
 such that (where w denotes an arbitrary element of W):

 (3a) $av(w) \neq \emptyset$

(4) $pv : W \rightarrow \wp(W)$ is such that:

 (4a) $av(w) \subseteq pv(w)$
 (4b) $w \in pv(w)$

(5) and $ob : \wp(W) \rightarrow \wp(\wp(W))$ is such that (where X,Y, Z designate arbitrary subsets of W)[7]:

 (5a) $\emptyset \notin ob(X)$
 (5b) if $Y\cap X = Z \cap X$, then ($Y \in ob(X)$ iff $Z \in ob(X)$)
 (5c*) Let $\beta \subseteq ob(X)$ and $\beta \neq \emptyset$, i.e. let β be a non-empty set of elements of ob(X).
 If $(\cap\beta) \cap X \neq \emptyset$ (where $\cap\beta = \{w\in W: \forall_{Z\in\beta}\ w\in Z\}$)
 then $(\cap\beta) \in ob(X)$
 (5d) if $Y\subseteq X$ and $Y\in ob(X)$ and $X\subseteq Z$, then $((Z\text{-}X) \cup Y) \in ob(Z)$
 (5e) if $Y\subseteq X$ and $Z\in ob(X)$ and $Y\cap Z \neq \emptyset$, then $Z\in ob(Y)$

[7]As one of the reviewers has pointed out, the fact that *ob(X)* is not closed under supersets is relevant to the Ross paradox. See [1], section 6.1.

Given a model M=<W, … >, the elements of W are designated by *worlds* and (as above) in what follows we will use w, v, … to denote arbitrary worlds and X,Y, Z to denote arbitrary sets of worlds. Intuitively: av(w) denotes the set of actual versions of the world w; pv(w) denotes the set of potential versions of the world w; and ob(X) denotes the set of propositions which are obligatory in context X.

We write $M\models_w A$ to denote that formula A is true in the world w of a model M=<W, av, pv, ob, V>, and we define $||A||^M = \{w\in W: M\models_w A\}$. In order to simplify the presentation, whenever the model M is obvious from the context, we write ||A|| instead of $||A||^M$.

Truth in a world w *in a model* M=<W, av, pv, ob, V> is characterized as follows:

$M\models_w p$	iff	$w\in V(p)$
…		(the usual truth conditions for the connectives ¬, ∧, ∨, → and ↔)
$M\models_w \Box A$	iff	$\|\|A\|\| = W$
$M\models_w \Box_a A$	iff	$av(w)\subseteq \|\|A\|\|$
$M\models_w \Box_p A$	iff	$pv(w)\subseteq \|\|A\|\|$
$M\models_w O(B/A)$	iff	$\|\|A\|\|\cap\|\|B\|\|\neq\emptyset$ and $(\forall X)$(if $X\subseteq\|\|A\|\|$ and $X\cap\|\|B\|\|\neq\emptyset$, then $\|\|B\|\|\in ob(X)$)
$M\models_w O_a A$	iff	$\|\|A\|\|\in ob(av(w))$ and $av(w)\cap\|\|\neg A\|\|\neq\emptyset$
$M\models_w O_p A$	iff	$\|\|A\|\|\in ob(pv(w))$ and $pv(w)\cap\|\|\neg A\|\|\neq\emptyset$

A sentence *A* is said to be *true in a model* M=<W, av, pv, ob, V>, written $M\models A$, iff $||A||^M = W$; and *A* is said to be *valid*, written $\models A$, iff $M\models A$ in all models M.

OBSERVATION II-2-1

(1) An *alternative* to the proposed definition of $M\models_w O(B/A)$ would be to define the dyadic obligation operator in the following *strict sense*[8]:

$M\models_w O(B/A)$ iff $||A||\cap||B||\neq\emptyset$ and $||A||\cap||\neg B||\neq\emptyset$ and $(\forall X)$(if $X\subseteq||A||$ and $X\cap||B||\neq\emptyset$ and $X\cap||\neg B||\neq\emptyset$, then $||B||\in ob(X)$)

In that case we would require that:

if $Y\in ob(X)$, then $X\cap(W-Y)\neq\emptyset$

and the truth in a world w of O_aA (respectively O_pA) would be defined simply as follows:

$M\models_w O_a$ iff $||A||\in ob(av(w))$ (resp. $M\models_w O_pA$ iff $||A||\in ob(pv(w))$)

With both approaches we get exactly the same semantics for O_aA and O_pA.

(2) Condition (5a) is obvious and means that for us a contradiction cannot be obligatory.

(3) Condition (5b) means that if, from the point of view of a context X, two propositions Y and Z are indistinguishable, then one of them is obligatory (in that context X) iff the other is.

Since $Y\cap X = (Y\cap X)\cap X$, from (5b) the condition (5b′) follows:

(5b′) $Y\in ob(X)$ iff $Y\cap X\in ob(X)$

And, taking into account (5a), we can deduce that our models also satisfy the following condition:

(5ab) if $Y\in ob(X)$, then $Y\cap X\neq\emptyset$

(a condition which in turn implies (5a))

(4) If W is finite, then (5c*) (which means (5c) generalized) is equivalent to the following condition[9]:

(5c) if $Y, Z\in ob(X)$ and $Y\cap Z\cap X\neq\emptyset$, then $Y\cap Z\in ob(X)$

[8] In the *strict sense*, any obligation should be possible to fulfill and to violate.

[9] Referred to as (5c−) in [1], p. 323. (In [1] condition (5c) is 'if $Y,Z\in ob(X)$, then $Y\cap Z\in ob(X)$').

which means that the conjunction of obligatory propositions (in some context X) is also obligatory, unless we are in the presence of contradictory obligations (within that context).

(5) Condition (5d) states that if a subset Y of X is an obligatory proposition in a context X, then in a bigger context Z it is obligatory to be either in Y or else in that part of Z which is not in X.
The first lemma (below), which can be used to simplify some proofs, states a ‘generalization’ of condition (5d) (implied by (5d), assuming, as we do, condition (5b)).

(6) Condition (5e) states that if Z is an obligatory proposition in a context X, then Z is also obligatory in any subcontext Y of Z where it is possible to fulfill Z.
Condition (5e) was discussed in [1], where its adoption was left open. Here we assume it.

(7) It is obvious that $M|=_w O(B/A)$ implies $||B|| \in ob(||A||)$. As stated below (result II-2-2), the inverse is also true, assuming condition (5e). Thus, besides some other good consequences, the adoption of (5e) allows us to simplify the definition of $M |=_w O(B/A)$.

LEMMA II-2-1
Assuming condition (5b), the following condition below is equivalent to (5d):
(5bd) if $Y \in ob(X)$ and $X \subseteq Z$, then $((Z\text{-}X) \cup Y) \in ob(Z)$

RESULT II-2-1
Let M_1 and M_2 be two models that differ at most in the valuation of the propositional symbols, i.e. $M_1 = \langle W, av, pv, ob, V_1 \rangle$ and $M_2 = \langle W, av, pv, ob, V_2 \rangle$.
If $V_1(q) = V_2(q)$, for any propositional symbol q occurring in A, then
$\forall_{w \in W}$ $(M_1|=_w A$ iff $M_2|=_w A)$

PROOF. Standard.

RESULT II-2-2 (redefinition of $M |=_w O(B/A)$)
Adopting, as we are doing, condition (5e) (besides condition (5ab)), then:
$M |=_w O(B/A)$ iff $||B|| \in ob(||A||)$

LEMMA II-2-2
With condition (5c*) (plus conditions (5a), (5b) and (5d)), we obtain:
ob-∪*) Let β be a non-empty set of subsets of W, and let $H = \cup\beta$ (= $\{w \in W: \exists_{Z \in \beta}\ w \in Z\}$).
If $\forall_{Z \in \beta}$ $X \in ob(Z)$, then $X \in ob(H)$

PROOF (sketch).
Let β be a non-empty set of subsets of W, $H = \cup\beta$ and suppose that $\forall_{Z \in \beta}$ $X \in ob(Z)$.
Let $\beta' = \{(H\text{-}Z) \cup X: Z \in \beta\}$.We have that $\cap\beta' = (\cap\{H\text{-}Z: Z \in \beta\}) \cup X = \emptyset \cup X = X$
We can also prove that: β' is non-empty, $\beta' \subseteq ob(H)$ and $(\cap\beta') \cap H \neq \emptyset$.
Thus, by condition (5c*),$(\cap\beta') \in ob(H)$, i.e. $X \in ob(H)$ (as we wish to prove).

2.3 *Section 3. Axiomatization*

2.3.1 Axioms and inference rules

In what follows we introduce the axioms and rules for the modal operators. For some of the axioms and theorems, we introduce special labels—if there is no standard label—in order to facilitate reference to them.

We assume as axioms all tautologies and we assume the Modus Ponens (MP) inference rule. And we will use PC (from Propositional Calculus) as reference for the use of any tautology or any tautological rule (both the primitive Modus Ponens and the other derivable tautological inference rules).

Characterization of $\Box$:

(1) $\Box$ is a normal modal operator of type S5 (*using Chellas classification*, see [2])

Characterization of O:		Reference label
(2)	$O(B / A) \rightarrow \Diamond(B \wedge A)$	$(O\rightarrow\Diamond)$
(3)[10]	$\Diamond(A \wedge B \wedge C) \wedge O(B / A) \wedge O(C / A) \rightarrow O(B \wedge C / A)$	(O-C)
(4)	the principle of strengthening of the antecedent:	
	$\Box(A \rightarrow B) \wedge \Diamond(A \wedge C) \wedge O(C / B) \rightarrow O(C / A)$	(O-SA)
(5)	the 'RE—axiom' wrt the antecedent:	
	$\Box(A \leftrightarrow B) \rightarrow (O(C/A) \leftrightarrow O(C/B))$	(O-REA)
(6)	the 'contextual RE—axiom' wrt the consequent:	
	$\Box(C \rightarrow (A \leftrightarrow B)) \rightarrow (O(A / C) \leftrightarrow O(B / C))$	(O-CONT-REC)
(7)	$O(B / A) \rightarrow \Box O(B / A)$	$(O\rightarrow\Box O)$
(8)[11]	$O(B / A) \rightarrow O(A\rightarrow B / T)$	$(O\rightarrow O\rightarrow)$

Characterization of $\Box_p$:

(9) $\Box_p$ is a normal modal operator of type KT (*using Chellas classification*)

Characterization of $\Box_a$:

(10) $\Box_a$ is a normal modal operator of type KD (*using Chellas classification*)

Relationships between $\Box$, $\Box_p$ *and* $\Box_a$:

(11)	$\Box A \rightarrow \Box_p A$	$(\Box \rightarrow \Box_p)$
(12)	$\Box_p A \rightarrow \Box_a A$	$(\Box_p \rightarrow \Box_a)$

Relationships between O_a(*respectively:* O_p) *and* $\Box_a$ (*resp.:* $\Box_p$):

(13)	$\Box_a A \rightarrow (\neg O_a A \wedge \neg O_a \neg A)$	$(\neg O_a)$
	$\Box_p A \rightarrow (\neg O_p A \wedge \neg O_p \neg A)$	$(\neg O_p)$
(14)	$\Box_a(A \leftrightarrow B) \rightarrow (O_a A \leftrightarrow O_a B)$	$(\leftrightarrow O_a)$
	$\Box_p(A \leftrightarrow B) \rightarrow (O_p A \leftrightarrow O_p B)$	$(\leftrightarrow O_p)$

Relationships between O, O_a(*resp.:* O_p) *and* $\Box_a$ (*resp.:* $\Box_p$) - factual detachment axioms:

(15)	$O(B/A) \wedge\Box_a A \wedge \Diamond_a B \wedge\Diamond_a\neg B \rightarrow O_a B$	(O_a-FD)
	$O(B/A) \wedge\Box_p A \wedge\Diamond_p B \wedge\Diamond_p\neg B \rightarrow O_p B$	(O_p-FD)

Finally, in order to get a complete axiomatization for the whole logic, we introduce two primitive inference rules, of a new type, which we refer as *rules to consistently add O(/) formulas*. This name comes from the use given to these rules in the completeness proof. Briefly, these rules are used there to construct the maximal consistent sets, that will be our 'canonical worlds', in such a way that

[10] The weaker version of schema C for the consequent of O(/), which we identify simply by (O-C) (referred to as (O-C−) in [1], p. 323).

[11] One of the reviewers has drawn our attention to [6], the content of which was not familiar to us at the time of the original writing of this article. While there are matters here for further consideration, we agree with the reviewer that the adoption of schema 8 is the reason—or at least a principal reason—why the incompleteness example presented in [6] is no longer valid. There is some discussion of schema 8 in [1], at p. 321.

whenever a relevant O_aA belongs to one of those worlds, we can also add to that world a formula $\Box_a q \wedge O(A/q)$ for an appropriate propositional symbol q (and similarly, if O_pA belongs, we can add $\Box_p q \wedge O(A/q)$ for an appropriate q).

(16) *Rules to consistently add O(/) formulas*:

$(O_a - \Box_a O)$-rule: If the propositional symbol q does not occur in any of the formulas $B_1, \ldots, B_n$, A
and $|{-}B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Box(O_aA \rightarrow \Box_a q \wedge O(A/q))$
then $|{-}B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Diamond O_aA$ (i.e. $|{-}B_1 \wedge \ldots \wedge B_n \rightarrow \Box\neg O_aA$)

$(O_p - \Box_p O)$-rule: If the propositional symbol q does not occur in any of the formulae $B_1, \ldots, B_n$, A
and $|{-}B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Box(O_pA \rightarrow \Box_p q \wedge O(A/q))$
then $|{-}B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Diamond O_pA$

2.3.2 Soundness and consistency

LEMMA II-3-1

From conditions (5b) and (5d) (of our models), it follows that:

if $M \models_w O(B/A)$ and Z is such that $Z \cap ||A|| \cap ||B|| \neq \emptyset$, then $||A \rightarrow B|| \in ob(Z)$

RESULT II-3-1

(a) The $(O_a - \Box_a O)$-rule preserves validity, i.e.
If the propositional symbol q does not occur in any of the formulae $B_1, \ldots, B_n$, A
and $\models B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Box(O_aA \rightarrow \Box_a q \wedge O(A/q))$
then $\models B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Diamond O_aA$

(b) The $(O_p - \Box_p O)$-rule preserves validity

PROOF OF (a) (the proof of b) is similar). Suppose that the propositional symbol q does not occur in any of the formulae $B_1, \ldots, B_n$, A and that

(1) $\models B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Box(O_aA \rightarrow \Box_a q \wedge O(A/q))$

and suppose, by reductio ad absurdum, that $\not\models B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Diamond O_aA$.

Then, there exists some model $M = \langle W, av, pv, ob, V\rangle$ and some world $w \in W$ such that
(2) $M \models_w B_1 \wedge \ldots \wedge B_n$
and $M \models_w \Diamond O_aA$, a condition that implies that there exists a world $v \in W$ such that
(3) $M \models_v O_aA$.
Let $X = \{y \in W: M \models_y O_aA\} = ||O_aA||^M$.
Let M_1 be the model $M_1 = \langle W, av, pv, ob, V_1\rangle$ with V_1 defined as follows:
if $q_1 \neq q$ then $V_1(q_1) = V(q_1)$, and $V_1(q) = \cup_{x \in X}\, av(x)$
Since q does not occur in any of the formulae $B_1, \ldots, B_n$, A, by result II-2-1, we have that
(4) $||B_1||^M = ||B_1||^{M1}, \ldots, ||B_n||^M = ||B_n||^{M1}$, $||A||^M = ||A||^{M1}$ and $X = ||O_aA||^M = ||O_aA||^{M1}$.
And, from (2) and (4) it follows that (5) $M_1 \models_w B_1 \wedge \ldots \wedge B_n$.
Let $\beta = \{av(y): y \in X\}$. We have that:

(i) $\beta \neq \emptyset$, since $X \neq \emptyset$ (by (3), $v \in X$).
(ii) $\cup\beta = V1(q) = ||q||^{M1}$.
(iii) For every $Z \in \beta$, $||A||^{M1} \in ob(Z)$. (As a matter of fact, if $Z \in \beta$, then there exists $y \in X$ such that $Z = av(y)$, and $y \in X = ||O_aA||^{M1}$ implies $M_1 \models_y O_aA$, which implies that $||A||^{M1} \in ob(av(y))$.)

Thus, from condition ob-∪*) (see lemma II-2-2), we conclude that (6) $||A||^{M1} \in ob(||q||^{M1})$.

Now, let $z \in W$ be any world such that $M_1 |=_z O_a A$ (i.e. $z \in X$).
Since $z \in X$, we have that $av(z) \subseteq \cup_{x \in X} av(x) = V_1(q) = ||q||^{M1}$. Thus (7) $M_1 |=_z \Box_a q$
And from (6) (by result II-2-2) it follows that (8) $M_1 |=_z O(A/q)$
Thus, from (7) and (8) (since z was an arbitrary world such that $M_1 |=_z O_a A$)
(9) $M_1 |=_w \Box(O_a A \rightarrow \Box_a q \wedge O(A/q))$
But then, from (5) and (9), we conclude that
$M_1 |=_w B_1 \wedge \ldots \wedge B_n \wedge \Box(O_a A \rightarrow \Box_a q \wedge O(A/q))$
and so
$M_1 |\neq_w B_1 \wedge \ldots \wedge B_n \rightarrow \neg\Box(O_a A \rightarrow \Box_a q \wedge O(A/q))$
which contradicts (1).

Result II-3-2
The previous axiomatization is *sound* (i.e. all theorems are valid).

Observation II-3-1
The complicated part of the soundness proof was established in result II-3-1, and proved above.

With respect to the rest of the soundness proof, which we have omitted above (and included in the Appendix A), we make next only some brief comments.

The validity of many of the axioms follows (more or less directly) from the relevant truth conditions.

The semantic condition (5c*) is needed only to prove result II-3-1. For the rest of the soundness proof, it suffices to consider condition (5c) (which is used to prove the validity of schema 3 (O-C)).

Lemma II-3-1 is useful to prove the validity of schema 8 (O→O→).

In the soundness proof, the semantic condition (5e) is needed to prove both the result II-3-1 and the validity of schema 8 (O→O→).

Result II-3-3
The proposed axiomatization is consistent, i.e.[12] $\perp$ is not a theorem.

Proof. By result II-3-2, it suffices to prove that there is (at least) a model for our logic.

Let M=<W, av, pv, ob, V> be defined as follows:

(1) W is any non empty set;
(2) V is any total function from the set of propositional symbols into $\wp(W)$;
(3) $av : W \rightarrow \wp(W)$ is defined as follows: $\forall_{w \in W}\ av(w) = W$;
(4) $pv : W \rightarrow \wp(W)$ is defined as follows: $\forall_{w \in W}\ pv(w) = W$; and
(5) $ob : \wp(W) \rightarrow \wp(\wp(W))$ is defined as follows: $\forall_{X \subseteq W}\ ob(X) = \emptyset$

It is easy to see that M, so defined, satisfies all conditions of the models of our logic.

2.4 Section 4. Some theorems and derived (proof) rules

In this section we show that we can recover (prove) all the axioms and rules considered in [1], and used there in the analysis of the CTD scenarios, and we illustrate the use of the new inference rules in proofs.

The axiomatization proposed here differs from the one considered in [1] in three principal ways. First, we have included here, in our formal language, the 'in all worlds' necessity operator, which

[12]Which is equivalent to stating that the set of theorems is consistent in the following sense: 'for any $n(\geq 1)$ theorems $A_1, \ldots, A_n$, $\neg(A_1 \wedge \ldots \wedge A_n)$ is not a theorem'.

has permitted a 'reformulation' of some of the axioms and rules in [1]. The list of these axioms and rules can be seen in result II-4-1 below, where it is stated that they can still be deduced.

Second, we have assumed here semantic condition (5e), whose adoption was left open in [1]. Without (5e), schema (O→O→) is not valid and should be replaced, in our axioms, by the schemas

$O(B/A) \wedge \Diamond_a(A \wedge B) \wedge \Diamond_a(A \wedge \neg B) \rightarrow O_a(A \rightarrow B)$ $(O \rightarrow O_a \rightarrow)$

$O(B/A) \wedge \Diamond_p(A \wedge B) \wedge \Diamond_p(A \wedge \neg B) \rightarrow O_p(A \rightarrow B)$ $(O \rightarrow O_p \rightarrow)$

(schemas that can be deduced from (O→O→): see observation II-4-2 below).

Finally, we have extended semantic condition (5c) (referred to as (5c–) in [1, p. 323]) to condition (5c*), an extension that seems uncontroversial and that, together with the adoption of condition (5e), has allowed us to 'validate' the two new primitive rules, which are the main innovation of this axiomatization as compared with the one in [1]. These rules are essential to our completeness proof, as will be clear in the next section. In this section, we illustrate the use of these rules in proofs, showing, in particular, that they allow us to deduce, as theorems, the following schemas

$\Diamond_a(A \wedge B) \wedge O_aA \wedge O_aB \rightarrow O_a(A \wedge B)$ $(O_a\text{-}C)$

$\Diamond_p(A \wedge B) \wedge O_pA \wedge O_pB \rightarrow O_p(A \wedge B)$ $(O_p\text{-}C)$

which are valid assuming condition (5c), and were assumed as axioms in [1], as well as

$O_pA \wedge \Diamond_a \neg A \wedge \Diamond_a A \rightarrow O_aA$ $(O_p \rightarrow O_a)$

The latter is an important schema, relating primary and actual obligations; it is valid assuming (5e) (and would be assumed as an axiom, if we were to adopt (5e) and if we did not adopt the new primitive rules).

LEMMA II-4-1

Let C[A/B] denote a formula that we can obtain by replacing in formula C one or more occurrences of formula A by formula B. Then:

$|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow C[A/B])$ (REQ) – theorem (theorem of replacement of equivalents)

PROOF. The proof is standard and uses (besides PC) the fact that □ is an S5-operator and axioms (O-REA), (O-CONT-REC), $(\Box \rightarrow \Box_p)$, $(\Box_p \rightarrow \Box_a)$, $(\leftrightarrow O_a)$ and $(\leftrightarrow O_p)$.

OBSERVATION II-4-1

(a) By □-necessitation and PC, from the (REQ)–theorem, above, it follows trivially the following rule of replacement of equivalents (where the notation C[A/B] means the same as in the previous lemma)

If $|\text{-} A \leftrightarrow B$ then $|\text{-} C \leftrightarrow C[A/B]$ (REQ) – rule

(b) Similarly, by □-necessitation and PC, from axiom (O-SA) it follows the rule

If $|\text{-} A \rightarrow B$ then $|-\Diamond(A \wedge C) \wedge O(C/B) \rightarrow O(C/A)$

(c) From axiom (O→ □O) and the normality of □ (i.e. the fact that □ is a normal modal operator), it follows $|-\Diamond O(B/A) \rightarrow \Diamond\Box O(B/A)$. But, since of □ is of type S5, $|-\Diamond\Box O(B/A) \rightarrow \Box O(B/A)$. Thus, the following theorem follows:

$|-\Diamond O(B/A) \rightarrow \Box O(B/A)$ label: $(\Diamond O \rightarrow \Box O)$

And from this theorem, the normality of □ and PC, the following theorem also follows:

$|-\Diamond\neg O(B/A) \rightarrow \Box\neg O(B/A)$ label: $(\Diamond\neg O \rightarrow \Box\neg O)$

As discussed in [1], p. 294, axiom (O→ □O) and these theorems reflect the fact that norms which comprise the deontic component of a CTD scenario are themselves taken to be fixed. This is a reasonable assumption to make, given that our concern is *not* with the dynamics of normative systems, but with the determination of which primary and actual obligations may be derived from a *fixed* set of norms, given the facts of the case.

RESULT II-4-1
(a) The following schemas (which are axioms in [1]) can be deduced as theorems:

- $\neg O(\bot/A)$
- $\Diamond_p(A\wedge B\wedge C) \wedge O(B/A) \wedge O(C/A) \rightarrow O(B\wedge C/A)$
- $O(B/A) \rightarrow O(B/A\wedge B)$
- $\Diamond_p O(B/A) \rightarrow \Box_p O(B/A)$
- $\Diamond_p(A\wedge B\wedge C) \wedge O(C/B) \rightarrow O(C/A\wedge B)$

(b) The following rules (which are primitive in [1]) can be derived:

- If $|\!-A \leftrightarrow B$ then $|\!- O(C/A) \leftrightarrow O(C/B)$
- If $|\!-C \rightarrow (A \leftrightarrow B)$ then $|\!- O(A/C) \leftrightarrow O(B/C)$

OBSERVATION II-4-2
That $(O\rightarrow O_a\rightarrow)$ can be deduced from $(O\rightarrow O\rightarrow)$ (and other axioms), can be seen as follows:

(1) $|\!- O(B/A) \wedge\Diamond_a(A \wedge B) \wedge\Diamond_a(A \wedge \neg B) \rightarrow O(B/A) \wedge\Diamond_a(A \rightarrow B) \wedge\Diamond_a\neg(A \rightarrow B)$
using PC and known properties of normal operators
(2) $|\!- O(B / A) \rightarrow O(A\rightarrow B / T)$ — $(O\rightarrow O\rightarrow)$
(3) $|\!- O(A\rightarrow B / T) \wedge\Box_a T \wedge \Diamond_a(A \rightarrow B) \wedge\Diamond_a\neg(A \rightarrow B)$
$\rightarrow O_a(A \rightarrow B)$ — $(O_a\text{-FD})$
(4) $|\!-\Box_a T$ — $\Box_a$ is a normal operator
(5) $|\!- O(A\rightarrow B / T) \wedge\Diamond_a(A \rightarrow B) \wedge\Diamond_a\neg(A \rightarrow B) \rightarrow O_a(A \rightarrow B)$ — from (3) and (4) by PC
(6) $|\!- O(B/A) \wedge\Diamond_a(A \wedge B) \wedge\Diamond_a(A \wedge \neg B) \rightarrow O_a(A \rightarrow B)$ — from (1), (2) and (5) by PC

Analogously we can derive $(O\rightarrow O_p\rightarrow)$ from $(O\rightarrow O\rightarrow)$ and $(O_p\text{-FD})$.

LEMMA II-4-2

(a) With the $(O_a-\Box_a O)$-rule, we can deduce the following derived proof rule:
label: $(\rightarrow\neg O_a)$-rule
If the propositional symbol q does not occur in any of the formulae $B_1, \dots, B_n, A$
and $|\!-B_1\wedge \dots \wedge B_n\wedge O_a A \rightarrow \neg(\Box_a q \wedge O(A/q))$
then $|\!-B_1\wedge \dots \wedge B_n \rightarrow \neg O_a A$
(b) With the $(O_p-\Box_p O)$-rule, we can deduce the following derived proof rule:
label: $(\rightarrow\neg O_p)$-rule
If the propositional symbol q does not occur in any of the formulae $B_1, \dots, B_n, A$
and $|\!-B_1\wedge \dots \wedge B_n\wedge O_p A \rightarrow \neg(\Box_p q \wedge O(A/q))$
then $|\!-B_1\wedge \dots \wedge B_n \rightarrow \neg O_p A$

PROOF OF (a) (the proof of (b) is similar). Suppose that the propositional symbol q does not occur in any of the formulae $B_1, \dots, B_n$, A and that
$|\!- B_1\wedge \dots \wedge B_n\wedge O_a A \rightarrow \neg(\Box_a q \wedge O(A/q))$
Then $|\!-B_1\wedge \dots \wedge B_n \rightarrow \neg O_a A$, as detailed below:

(1) $|-B_1\wedge \ldots \wedge B_n\wedge O_aA \rightarrow \neg(\Box_a q \wedge O(A/q))$ assumption
(2) $|-B_1\wedge \ldots \wedge B_n\wedge O_aA \rightarrow (O_aA \wedge \neg(\Box_a q \wedge O(A/q)))$ from (1) by PC
(3) $|-B_1\wedge \ldots \wedge B_n\wedge O_aA \rightarrow \neg(O_aA \rightarrow \Box_a q \wedge O(A/q))$ from (2) by PC
(4) $|-\neg(O_aA \rightarrow \Box_a q \wedge O(A/q)) \rightarrow \neg\Box(O_aA \rightarrow \Box_a q \wedge O(A/q))$ by the properties of $\Box$
(5) $|-B_1\wedge \ldots \wedge B_n\wedge O_aA \rightarrow \neg\Box(O_aA \rightarrow \Box_a q \wedge O(A/q))$ from (3) and (4) by PC
(6) $|-B_1\wedge \ldots \wedge B_n\wedge O_aA \rightarrow \neg\Diamond O_aA$ from (5) by the $(O_a-\Box_aO)$-rule
(considering the following $n+1$ B's in the $(O_a-\Box_aO)$-rule: $B_1, \ldots, B_n$ and $B_{n+1}= O_aA$)
(7) $|-\neg\Diamond O_aA \rightarrow \neg O_aA$ by the properties of $\Box$
(8) $|-B_1\wedge \ldots \wedge B_n \rightarrow \neg O_aA$ from (6) and (7) by PC

RESULT II-4-2

(a) With the $(O_p-\Box_pO)$-rule we can derive (the following theorem schema)
$|- O_pA \wedge\Diamond_a\neg A \wedge\Diamond_aA \rightarrow O_aA$ $(O_p \rightarrow O_a)$
(b) With the $(O_a-\Box_aO)$-rule we can derive
$|-\Diamond_a(A \wedge B) \wedge O_aA \wedge O_aB \rightarrow O_a(A \wedge B)$ $(O_a\text{-C})$
(c) With the $(O_p-\Box_pO)$-rule we can derive
$|-\Diamond_p(A \wedge B) \wedge O_pA \wedge O_pB \rightarrow O_p(A \wedge B)$ $(O_p\text{-C})$

PROOF. (a) Let A be any formula and suppose that q is any propositional symbol that does not occur in A.
(1) $|-O(A/q) \wedge\Box_a q \wedge\Diamond_aA \wedge\Diamond_a\neg A \rightarrow O_aA$ $(O_a\text{-FD})$
(2) $|-\Diamond_aA \wedge\Diamond_a\neg A \rightarrow (\neg O_aA \rightarrow \neg(\Box_a q \wedge O(A/q)))$ from (1) by PC
(3) $|-\Box_p q \wedge O(A/q) \rightarrow \Box_a q \wedge O(A/q)$ $(\Box_p \rightarrow \Box_a)$ and PC
(4) $|-\Diamond_aA \wedge\Diamond_a\neg A \rightarrow (\neg O_aA \rightarrow \neg(\Box_p q \wedge O(A/q)))$ from (2) and (3) by PC
(5) $|-\Diamond_aA \wedge\Diamond_a\neg A \wedge\neg O_aA \wedge O_pA \rightarrow \neg(\Box_p q \wedge O(A/q))$ from(4) by PC
(6) $|-\Diamond_aA \wedge\Diamond_a\neg A \wedge\neg O_aA \rightarrow \neg O_pA$ from (5) by the $(\rightarrow \neg O_p)$-rule
(see previous lemma II-4-2, considering $n = 3$ and $B_1 =\Diamond_aA$, $B_2 =\Diamond_a\neg A$ and $B_3 =\neg O_aA$)
(7) $|\text{-}\ O_pA \wedge\Diamond_a\neg A \wedge \Diamond_aA \rightarrow O_aA$ from (6) by PC

(b) Let A and B be any two formulas and suppose that q and r are two distinct propositional symbols that do not occur in A or B.

- We will start by proving that
  $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O(B/q) \wedge\Box_a q \rightarrow O_a(A\wedge B)$

The detailed proof is as follows:

(1) $|-r\wedge q \rightarrow q$ PC
(2) $|-\Diamond(r\wedge q\wedge B) \wedge O(B/q) \rightarrow O(B/r\wedge q)$ from (1) using the rule in observation II-4-1-(b)
(3) $|-\Diamond_a(A\wedge B) \wedge\Box_a q \wedge\Box_a r \rightarrow\Diamond_a(r\wedge q\wedge A\wedge B)$ using properties of normal operators and PC
(4) $|-\Diamond_a(r\wedge q\wedge A\wedge B) \rightarrow \Diamond(r\wedge q\wedge A\wedge B)$ from $(\Box \rightarrow \Box_p)$, $(\Box_p \rightarrow \Box_a)$ and PC
(5) $|-\Diamond(r\wedge q\wedge A\wedge B) \rightarrow \Diamond(r\wedge q\wedge B)$ using properties of normal operators and PC
(6) $|-\Diamond_a(A\wedge B) \wedge\Box_a q \wedge\Box_a r \wedge O(B/q) \rightarrow O(B/r\wedge q)$ from (3), (4), (5) and (2) by PC
analogously we get
(7) $|-\Diamond_a(A\wedge B) \wedge\Box_a q \wedge\Box_a r \wedge O(A/r) \rightarrow O(A/r\wedge q)$
(8) $|-\Diamond(r\wedge q\wedge A\wedge B) \wedge O(A/r\wedge q) \wedge O(B/r\wedge q) \rightarrow O(A\wedge B/r\wedge q)$ (O-C)
(9) $|-\Box_a q \wedge\Box_a r \rightarrow \Box_a(r\wedge q)$ using properties of normal operators
(10) $|-O(A\wedge B/r\wedge q) \wedge\Box_a(r\wedge q) \wedge\Diamond_a(A\wedge B) \wedge\Diamond_a\neg(A\wedge B) \rightarrow O_a(A\wedge B)$ $(O_a\text{-FD})$
(11) $|-O_aA\rightarrow \Diamond_a\neg A$ from $(\neg O_a)$ and PC using the properies of $\Box_a$ (is a normal operator)
(12) $|-\Diamond_a\neg A\rightarrow \Diamond_a\neg(A\wedge B)$ using PC and the properies of $\Box_a$ (is a normal operator)

(13) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O(B/q) \wedge\Box_a q \wedge\Box_a r \wedge O(A/r) \rightarrow O_a(A\wedge B)$
from (11), (12), (9), (7), (6), (3), (4), (8) and (10) by PC
(14) $|-\Diamond_a(A\wedge B) \wedge O(B/q) \wedge \Box_a q \wedge\neg O_a(A\wedge B) \wedge O_aA \rightarrow \neg(\Box_a r \wedge O(A/r))$ from 13) by PC
(15) $|-\Diamond_a(A\wedge B) \wedge O(B/q) \wedge \Box_a q \wedge\neg O_a(A\wedge B) \rightarrow \neg O_aA$ from 14) by the $(\rightarrow \neg O_a)$-rule
(see lemma II-4-2: note that r does not appear in any formula in the antecedent of (14))
(16) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O(B/q) \wedge\Box_a q \rightarrow O_a(A\wedge B)$ from (15) by PC

• And using this last theorem we can prove that $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O_aB \rightarrow O_a(A\wedge B)$.
The detailed proof is as follows:

(1) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O(B/q) \wedge\Box_a q \rightarrow O_a(A\wedge B)$ just proved
(2) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge \neg O_a(A\wedge B) \rightarrow \neg(\Box_a q \wedge O(B/q))$ from (1) by PC
(3) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge \neg O_a(A\wedge B) \wedge O_aB \rightarrow \neg(\Box_a q \wedge O(B/q))$ from (2) by PC
(4) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge \neg O_a(A\wedge B) \rightarrow \neg O_aB$ from (3) by the $(\rightarrow \neg O_a)$-rule
(5) $|-\Diamond_a(A\wedge B) \wedge O_aA \wedge O_aB \rightarrow O_a(A\wedge B)$ from 4) by PC

(c) The proof of (c) is similar to the proof of (b).

OBSERVATION II-4-3
As can be seen by the examples analysed in [1], the theorems that play the dominant role in determining what may, or may not, be derived from a chosen representation of a CTD scenario, are:

- the 'factual detachment' axioms (O_a-FD) and (O_p-FD);
- axioms $(\neg O_a)$ and $(\neg O_p)$;
- axioms $(\leftrightarrow O_a)$ and $(\leftrightarrow O_p)$;
- the theorems $(O\rightarrow O_a \rightarrow)$ and $(O\rightarrow O_p \rightarrow)$;
- the theorems
  $|-\Box_aA\rightarrow (O_aB \rightarrow O_a(A\wedge B))$ label: $(O_a \rightarrow O_a\wedge)$
  $|-\Box_pA\rightarrow (O_pB \rightarrow O_p(A\wedge B))$ label: $(O_p \rightarrow O_p\wedge)$
  (Note that $(O_a \rightarrow O_a\wedge)$ follows from $(\leftrightarrow O_a)$ and PC, since $|-\Box_aA\rightarrow \Box_a(B\leftrightarrow A\wedge B)$.
  And analogously for O_p.)
- and the 'deontic detachment' theorems
  $|- O_aA \wedge O(B/A) \wedge\Diamond_a(A\wedge B) \rightarrow O_a(A\wedge B)$ label: (O_a-DD)
  $|- O_pA \wedge O(B/A) \wedge\Diamond_p(A\wedge B) \rightarrow O_p(A\wedge B)$ label: (O_p-DD)
  (In Appendix A we prove that (Oa-DD) can be deduced from axiom $(\leftrightarrow O_a)$ and theorems (O_a-C) and $(O\rightarrow O_a \rightarrow)$, using the normality of $\Box_a$, and PC. And analogously for Op.)

Besides these theorems, extensive use is also made of the T-normality of $\Box_p$, the D-normality of $\Box_a$, and of axiom $(\Box_p \rightarrow \Box_a)$ (and of the theorem $|-\Diamond_aA\rightarrow \Diamond_pA$ which follows from it).

2.5 *Section 5. Completeness (and decidability)*

In this section, we will show that any consistent formula has a finite model where it is satisfiable. In this way, we show that our axiomatization is complete and has the finite model property, and so our logic is also decidable.

We start by stating and fixing some general notations and conventions that will be used in the rest of this article, and we state some preliminary results related with maximal consistent sets. We will use the standard results for normal modal operators, without giving proofs. The notions of consistency and maximal consistency are assumed herein to be known and defined as in e.g. the book [7].

2.5.1 General assumptions and notations

We will reserve ψ to denote the formula (assumed to be consistent) whose satisfiability we wish to show (i.e. we want to demonstrate that it is true in some world of some model).

From this initial (fixed, although arbitrary) consistent formula ψ, we will build another formula, to be denoted by φ, that is also consistent (assuming that ψ is consistent), and whose truth in a world (of a model) implies the truth of ψ in the same world. And we build a finite model where φ is true in some world. The way we build φ from ψ will be described later.

We assume a fixed enumeration (without repetitions) of all the formulae, and when we say that B_1, B_2, … , B_n constitutes an enumeration of a set of formulae Γ, we assume that B_1, B_2, … , B_n is the enumeration (without repetitions) of the formulae in Γ obtained according to the fixed enumeration of all the formulae.

Subf(A) denotes the set of all the subformulae of A, and it is defined as usual. (Thus, A is a subformula of A). And $\mathrm{Subf}(\Gamma)=\cup_{A\in\Gamma}\mathrm{Subf}(A)$.

The boolean closure of a set of formulae Γ, will be denoted by bc(Γ) and it is defined, inductively, as expected, i.e. bc(Γ) is the smallest set of formulae such that:

(i) $\Gamma \subseteq \mathrm{bc}(\Gamma)$
(ii) if $A \in \mathrm{bc}(\Gamma)$ then $\neg A \in \mathrm{bc}(\Gamma)$
(iii) if $A, B \in \mathrm{bc}(\Gamma)$ then $A\wedge B$, $A\vee B$, $A\rightarrow B$, $A\leftrightarrow B \in \mathrm{bc}(\Gamma)$

If the set Γ is closed under subformulae, then bc(Γ) is also closed under subformulae. (However, bc(Γ) is not finite, even if Γ is finite.)

We use Γ^* to denote the Lindenbaum extension of Γ. (We write A^* instead of $\{A\}^*$). Γ^* can be defined (e.g. as in [7]) as follows: assuming that B_1, B_2, … is the enumeration of all formulae, then Γ^* is the union of all the Γ_n's ($n \geqslant 0$), with:

(i) $\Gamma_0=\Gamma$;
(ii) $\Gamma_{n+1}=\Gamma_n\cup\{B_{n+1}\}$, if $\Gamma_n\cup\{B_{n+1}\}$ is consistent; otherwise, $\Gamma_{n+1}=\Gamma_n\cup\{\neg B_{n+1}\}$.

Since we are assuming a fixed enumeration of the formulae, Γ^* is uniquely determined.

As is well known, if Γ is consistent, Γ^* is maximal consistent.

NOTATION II-5-1

(i) We use Ω to denote set of all the subformulae of φ, i.e. $\Omega = \mathrm{Subf}(\varphi)$.
(ii) We use $a_1,\ldots, a_n$ to denote the subformulae of φ, i.e. more precisely, we assume that $a_1,\ldots, a_n$ (where $n = \#\Omega > 0$) constitutes an enumeration of Ω.
(iii) We write 'Γ is a mc' or, simply, 'Γ mc' to mean that Γ is a maximal consistent set of sentences.

Informally, we say that a formula A is a Boolean combination of subformulae of φ, if $A\in\mathrm{bc}(\Omega)$. We will assume that T is an abbreviation of $q\rightarrow q$, for some propositional symbol q occurring in the formula φ (and, as already mentioned, $\perp$ is an abbreviation of $\neg T$). Thus, T and $\perp$ are Boolean combinations of subformulae of φ (i.e. $T,\perp\in\mathrm{bc}(\Omega)$).

2.5.2 The notion of descriptor

We now define the notion of descriptor according to $\Omega = \mathrm{Subf}(\varphi)$,

DEFINITION II-5-1

The set DESCRIPTORS, of all *descriptors* according to Ω = Subf(φ), is defined as follows[13]:

DESCRIPTORS = {$A_1 \wedge \ldots \wedge A_n$: for $i = 1, \ldots, n$, A_i is either a_i or $\neg a_i$}, where n=#Ω

NOTATION II-5-2

(1) By a descriptor we understand a descriptor according to Ω = Subf(φ), and we use d, d_1, … to generically refer to a descriptor.
(2) For a descriptor d = $A_1 \wedge \ldots \wedge A_n$ and a formula B, although d is not a set, we may write 'B$\in$d' to mean that there is an $1 \leq i \leq n$ such that B is A_i (i.e. B$\in$d means that B is a conjunct of d).
(3) For Γ mc, d(Γ) denotes the unique element of $\Gamma \cap$ DESCRIPTORS.
(From the properties of the mc sets, it is trivial to see that d(Γ) is well defined.)

We may see d(Γ) as the descriptor of Γ, according to Ω, in the following sense:

If we define the following equivalence relation on the set of all the maximal consistent sets Θ:
'$\Gamma_1 \sim_\Omega \Gamma_2$ iff $\forall_{A\in\Omega}$ (A$\in \Gamma_1$ iff A$\in \Gamma_2$)'
then any set in each equivalent class $\Theta/\approx$ will have the same descriptor.

Using the terminology of [7, p. 137], for Γ a mc set, d(Γ) is the *characteristic* Ω*-formula* for [Γ] (the equivalence class of Γ). (In [7] these formulae are defined from a semantic point of view, but the idea is similar.)

Given a descriptor d, there exists a mc set Γ such that d(Γ) = d iff the descriptor d is consistent.

LEMMA II-5-1

Let Γ_1 and Γ_2 be two mc sets such that d(Γ_1) = d(Γ_2).

If A$\in$bc(Ω) (i.e. A is any Boolean combination of subformulae of φ), then
A$\in \Gamma_1$ iff A$\in \Gamma_2$

PROOF. The proof is by induction on the structure of the Boolean combination of subformulae of φ, and follows from the properties of the mc sets.

LEMMA II-5-2

If Γ is mc and $\Diamond$A$\in \Gamma$ (where A is any formula), there is (at least) one descriptor d such that $\Diamond$(d$\wedge$A)$\in \Gamma$

OUTLINE OF THE PROOF. The desired descriptor d = $A_1 \wedge \ldots \wedge A_n$, can be inductively built as follows ($i = 1, \ldots, n$, for n=#Ω):

if $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge a_i \wedge A) \in \Gamma$, then $A_i = a_i$; otherwise, $A_i = \neg a_i$.

LEMMA II-5-3

Let Γ be mc, d a descriptor and A a Boolean combination of subformulae of φ, i.e. A$\in$bc(Ω). Then
if $\Diamond$(d$\wedge$A)$\in \Gamma$ then $\Box$(d$\rightarrow$A)$\in \Gamma$

PROOF. The proof is by induction on the structure of the Boolean combination of subformulae of φ, and uses the properties of the mc sets and the fact that $\Box$ is a normal modal operator.

[13] In general, some descriptors according to Ω = Subf(φ) are inconsistent, and some descriptors contain redundant conjuncts. (For instance, if some formula $\neg$B is a subformula of φ, then B is also a subformula of φ, and so there exist descriptors containing both B and $\neg$B as conjuncts, descriptors containing both B and $\neg\neg$B as conjuncts, descriptors containing $\neg$B twice as a conjunct, and descriptors containing both $\neg$B and $\neg\neg$B as conjuncts.)

2.5.3 The set of worlds W(φ)

Later we will describe how we build our formula φ from our initial consistent formula ψ, Until then, in what follows it is only assumed that φ is a consistent formula.

DEFINITION II-5-2

(a) $\Box^{-1}\Gamma = \{A: \Box A \in \Gamma\}$.
(b) $s(A) = (\Box^{-1}\varphi^* \cup \{A\})^*$ (for any formula A).
(c) $W(\varphi) = \{s(d): d \in$ DESCRIPTORS and $\Diamond d \in \varphi^*\}$.
In what follows we write simply W, instead of W(φ), assuming φ implicit.

RESULT II-5-1

(a) Each $w \in W$ is mc.
(b) W is finite.

PROOF.

(a) Standard (using the fact that $\Box$ is a normal operator).
(b) Obvious, since the set DESCRIPTORS is finite.

NOTATION II-5-3
$|A| = \{w \in W: A \in w\}$.

2.5.4 Some results that depend only on the properties of the operator $\Box$ (and of the mc sets)

LEMMA II-5-4
(Where A and B can be any formulae:)

(a) If $|A| \neq \emptyset$, then $\Diamond A \in \varphi^*$ (i.e. if $A \in w$, for $w \in W$, then $\Diamond A \in \varphi^*$)
(b) If $\Box(A \rightarrow B) \in \varphi^*$, then $|A| \subseteq |B|$
(c) If $\Box(A \leftrightarrow B) \in \varphi^*$, then $|A| = |B|$
(d) If $\vdash A \rightarrow B$, then $|A| \subseteq |B|$
(e) If $\vdash A \leftrightarrow B$, then $|A| = |B|$
(f) $(\forall w \in W)$ ($\Box A \in w$ iff $\Box A \in \varphi^*$)

LEMMA II-5-5
If $A,B \in bc(\Omega)$ (i.e. A and B are any Boolean combinations of subformulae of φ), then:

(a) If $\Diamond A \in \varphi^*$, then $|A| \neq \emptyset$ (i.e. if $\Diamond A \in \varphi^*$ then there exists $w \in W$ such that $A \in w$)
(b) If $|A| \subseteq |B|$, then $\Box(A \rightarrow B) \in \varphi^*$
(c) If $|A| = |B|$, then $\Box(A \leftrightarrow B) \in \varphi^*$

COROLLARY II-5-1
If $A \in bc(\Omega)$, then: $\Box(A \leftrightarrow \bot) \in \varphi^*$ iff $|A| = \emptyset$

PROOF. Follows from lemmas II-5-4 and II-5-5, since $\bot$ is a Boolean combination of subformulae of φ.

RESULT II-5-2
There exists (at least) a $w \in W$ such that $\varphi \in w$.

PROOF. Since $|-\varphi \rightarrow \Diamond\varphi$, and $\varphi \in \varphi^*$, we conclude that $\Diamond\varphi \in \varphi^*$. And the result follows from Lemma II-5-5-(a).

We have defined d(Γ), for Γ a mc set. Since each world w is a mc set, this allow us to talk about d(w). Now we extend the notation to (finite) sets of worlds.

NOTATION II-5-4
For X⊆W (which implies that X is finite), we define
$d(X) = \vee\{d(w): w \in X\}$ ($=\vee_{w \in X} d(w)$)
Note that d(X) is a Boolean combination of subformulae of φ, i.e. $d(X) \in bc(\Omega)$.

LEMMA II-5-6

(a) (Where A and B can be any formulae:)

(i) $|A \vee B| = |A| \cup |B|$
(ii) $|A \wedge B| = |A| \cap |B|$
(iii) $|\neg A| = W - |A|$

(b) (i) $|d(w)| = \{w\}$ (for each $w \in W$)

(ii) $|d(X \cup Y)| = |d(X) \vee d(Y)|$ (for $X, Y \subseteq W$)
(iii) $|d(X)| = X$ (for each $X \subseteq W$)
(iv) $|d(X \cap Y)| = |d(X) \wedge d(Y)|$ (for $X, Y \subseteq W$)
(v) $|d(W-X)| = |\neg d(X)|$ (for $X \subseteq W$)

COROLLARY II-5-2 (corollary of lemmas II-5-6, II-5-5 and II-5-4)
Let $B \in bc(\Omega)$ (i.e. B is a Boolean combination of subformulae of φ).

(i) $\Box(d(|B|) \leftrightarrow B) \in \varphi^*$
(ii) $\Box(d(|B|) \leftrightarrow B) \in w$, for any $w \in W$

2.5.5 Some results that depend also on the properties of the operator O(/)

LEMMA II-5-7
(Where A and B can be any formulae:)
$(\forall w \in W)$ ($O(A/B) \in w$ iff $O(A/B) \in \varphi^*$)

PROOF. Consider any world w (i.e. $w \in W$).
Suppose $O(A/B) \in \varphi^*$. Since $|-O(A/B) \rightarrow \Box O(A/B)$ (is an axiom), we have that $\Box O(A/B) \in \varphi^*$. But then $O(A/B) \in w$.
Suppose now that $O(A/B) \notin \varphi^*$. Since $|-\neg O(A/B) \rightarrow \Diamond\neg O(A/B)$ and $|-\Diamond\neg O(A/B) \rightarrow \Box\neg O(A/B)$ (see observation II-4-1-c)), we have that $\Box\neg O(A/B) \in \varphi^*$. But then $\neg O(A/B) \in w$ and so $O(A/B) \notin w$.

LEMMA II-5-8
Let $B \in bc(\Omega)$ (i.e. B is a Boolean combination of subformulae of φ) and let A be any formula.

(a) (i) $O(A / d(|B|)) \in \varphi^*$ iff $O(A / B) \in \varphi^*$

(ii) $O(A / d(|B|)) \in w$ iff $O(A / B) \in w$, for any $w \in W$

(b) (i) $O(d(|B|) / A) \in \varphi^*$ iff $O(B / A) \in \varphi^*$

(ii) $O(d(|B|) / A) \in w$ iff $O(B / A) \in w$, for any $w \in W$

PROOF.

(a-i) Use corollary II-5-2 and axiom (O-REA).
(b-i) Use corollary II-5-2 and e.g. the (REQ) – theorem schema stated in lemma II-4-1.

(a-ii) and (b-ii) follow from, respectively, (a-i) and (b-i), by lemma II-5-7.

2.5.6 The construction of the relevant formula φ from the initial consistent formula ψ

Let ψ denote a (any) consistent sentence (which will be assumed fixed from now on).

- We are now going to build a sequence of sets of formulae $\Delta_0, \Delta_1, \ldots$as follows:
  Let $q_{p_1}, \ldots, q_{a_1}, \ldots$be a sequence of *distinct* propositional symbols not occurring in the formula ψ. (The number of q_{p_j}'s that are needed is equal to or less than the number of subformulae of ψ of the form O_pA; likewise the number of q_{a_j}'s that are needed is equal to or less than the number of subformulae of ψ of the form O_aA.)

  (a) $\Delta_0 = \{\psi\}$
  (b) Let $O_pA_1, O_pA_2, \ldots, O_pA_k$ (k$\geqslant$0) be the enumeration of the subformulae of ψ of the form O_pA.
  For j = 1,…,k:
  if $\Delta_{j-1} \cup \{\Diamond O_pA_j\}$ is inconsistent, define $\Delta_j = \Delta_{j-1} \cup \{\neg\Diamond O_pA_j\}$;
  otherwise, define $\Delta_j = \Delta_{j-1} \cup \{\Diamond O_pA_j\} \cup \{\Box(O_pA_j \rightarrow \Box_p q_{p_j} \wedge O(A_j/q_{p_j}))\}$
  (c)[14] Let $O_aA_1, O_aA_2, \ldots, O_aA_r$ (r$\geqslant$0) be the enumeration of the subformulae of ψ of the form O_aA.
  For j = 1,…,r:
  if $\Delta_{k+j-1} \cup \{\Diamond O_aA_j\}$ is inconsistent, define $\Delta_{k+j} = \Delta_{k+j-1} \cup \{\neg\Diamond O_aA_j\}$;
  otherwise, define $\Delta_{k+j} = \Delta_{k+j\ 1} \cup \{\Diamond O_aA_j\} \cup \{\Box(O_aA_j \rightarrow \Box_a q_{a_j} \wedge O(A_j/q_{a_j}))\}$
- Let $\Delta = \Delta_{k+r}$. Note that Δ is finite.
- Let $\varphi = \wedge\Delta$.

The rest of the notations are as before. In particular, Ω=Subf(φ) and bc(Ω) is the boolean closure of Ω. Note that Ω is finite and contains all the subformulae of the initial formula ψ (since ψ belongs to Δ).

OBSERVATION II-5-1

(1) Since $q_{p_1}, \ldots, q_{p_k}, \ldots, q_{a_1}, \ldots, q_{ar}$ do not occur in ψ, it follows that:

- if O_pA is a subformula of ψ, then $q_{p_1}, \ldots, q_{p_k}, \ldots, q_{a_1}, \ldots, q_{ar}$ do not occur in any of the formulae A, O_pA, $\Diamond O_pA$ and $\neg\Diamond O_pA$;
- and, if O_aA is a subformula of ψ, then $q_{p_1}, \ldots, q_{p_k}, \ldots, q_{a_1}, \ldots, q_{ar}$ do not occur in any of the formulae A, O_aA, $\Diamond O_aA$ and $\neg\Diamond O_aA$.

Thus, since the propositional symbols $q_{p_1}, \ldots, q_{p_k}, \ldots, q_{a_1}, \ldots, q_{ar}$ are all distinct, it is easy to see that, in the construction above:

- each q_{p_j} (with j = 1,…,k) does not occur in any of the formulae in Δ_{j-1};

[14]There is no specific reason to work with the primary obligations before the actual obligations. (We could have done it the other way around).

- and each q_{a_j} (with $j = 1,\ldots,r$) does not occur in any of the formulae in Δ_{k+j-1}.

(2) It is also easy to see that if a formula of the form O_pA belongs to Ω=Subf(φ) (or to bc(Ω), since a formula of the form O_pA belongs to bc(Ω) iff it belongs to Ω), then O_pA belongs to Subf(ψ). Analogously, if a formula of the form O_aA belongs to Ω (or to bc(Ω)), then it belongs to Subf(ψ).

RESULT II-5-3

(a) If (i) Γ is a consistent set of formulae
(ii) $\Diamond O_aA \in \Gamma$ (i.e. $\neg\Box\neg O_aA \in \Gamma$)
(iii) the propositional symbol q does not occur in any of the formulae in Γ
then $\Gamma \cup \{ \Box(O_aA \rightarrow \Box_a q \wedge O(A/q)) \}$ is consistent
(b) If (i) Γ is a consistent set of formulae
(ii) $\Diamond O_pA \in \Gamma$ (i.e. $\neg\Box\neg O_pA \in \Gamma$)
(iii) the propositional symbol q does not occur in any of the formulae in Γ
then $\Gamma \cup \{ \Box(O_pA \rightarrow \Box_p q \wedge O(A/q)) \}$ is consistent

PROOF OF (a) (the proof of (b) is similar). Suppose that the conditions (i), (ii) and (iii) are verified and suppose, by reductio ad absurdum, that $\Gamma \cup \{ \Box(O_aA \rightarrow \Box_a q \wedge O(A/q)) \}$ is inconsistent.

Then there exist $n(\geqslant 0)$ formulas $B_1,\ldots,B_n \in \Gamma$ such that $|-\neg(B_1\wedge\ldots\wedge B_n\wedge\Box(O_aA \rightarrow \Box_a q \wedge O(A/q)))$.

But then, by PC, $|\text{-}\ B_1\wedge\ldots\wedge B_n \rightarrow \neg\Box(O_aA\rightarrow \Box_a q\wedge O(A/q))$, which implies, by the $(O_a - \Box_a O)$-rule (note that q also does not occur in A, since $\Diamond O_aA \in \Gamma$ and q does not occur in any of the formulae in Γ), that $|\text{-}\ B_1\wedge\ldots\wedge B_n \rightarrow \neg\Diamond O_aA$, i.e., by PC, $|-\neg(B_1\wedge\ldots\wedge B_n\wedge\Diamond O_aA)$.

But this contradicts the consistency of Γ (since $B_1,\ldots,B_n, \Diamond O_aA \in \Gamma$).

RESULT II-5-4

(a) The set Δ is consistent.
(b) The formula φ is consistent.

PROOF.

(a) We prove below (by simple induction) that each set of formulae $\Delta_0, \Delta_1,\ldots,\Delta_{k+r}(=\Delta)$ is consistent.

(i) By hypothesis, $\Delta_0 = \{\psi\}$ is consistent.
(ii) Consider any $j = 1,\ldots,k$, and suppose that Δ_{j-1} is consistent. Then
Either $\Delta_{j-1}\cup\{\Diamond O_pA_j\}$ is inconsistent, and (as is known) $\Delta_j = \Delta_{j-1}\cup\{\neg\Diamond O_pA_j\}$ is consistent;
or $\Delta_{j-1}\cup\{\Diamond O_pA_j\}$ is consistent, and the consistency of
$\Delta_j = \Delta_{j-1}\cup \{\Diamond O_pA_j\} \cup \{\Box(O_pA_j \rightarrow \Box_p q_{p_j} \wedge O(A_j/ q_{p_j}))\}$
follows from result II-5-3, since q_{p_j} does not occur in $\Delta_{j-1}\cup\{\Diamond O_pA_j\}$ (see observation II-5-1).
(iii) Analogously we can show that each $\Delta_{k+1},\ldots,\Delta_{k+r}$ is consistent.

(b) Since Δ is finite, (a) and (b) are equivalent. (Recall that we are following the notion of consistency in [7]: see page 17 there.)

2.5.7 Definition of the model M(φ)

DEFINITION II-5-3
Let M(φ)=<W(φ), av, pv, ob, V> where:

- W(φ) is as in Definition II-5-2;
- av : W $\rightarrow \wp$(W), is defined as follows:
  v $\in$ av(w) iff $\forall_{A\in bc(\Omega)}$(if $\Box_a$A$\in$w then A$\in$v)
- pv : W $\rightarrow \wp$(W), is defined as follows:
  v $\in$ pv(w) iff $\forall_{A\in bc(\Omega)}$(if $\Box_p$A$\in$w then A$\in$v)
- ob : $\wp$(W) $\rightarrow$ $\wp$($\wp$(W)), is defined as follows: X $\in$ ob(Y) iff O(d(X) / d(Y)) $\in \varphi^*$
- w $\in$ V(p) iff p $\in$ w

OBSERVATION II-5-2
In M(φ)=<W(φ), av, pv, ob, V> above, in the definition of av and pv we cannot replace the set bc(Ω) by the smaller set Ω (otherwise we cannot make one of the steps in the proof of result II-5-6 below), neither can we replace bc(Ω) by the bigger set of all formulas (otherwise we are not able to do the proof of (a) and (b) of next lemma).

NOTATION II-5-5
In what follows, we assume φ implicit, and write (simply):

- W instead of W(φ)
- M instead of M(φ)
- ||A|| instead of $||A||^{M(\varphi)}$ (= {w$\in$W(φ): M(φ)$|=_w$A})

LEMMA II-5-9

(a) Let w$\in$W, B$\in$bc(Ω) and Γ = {C: $\Box_a$C$\in$w} $\cup$ {B}
If Γ is consistent, then there exists v$\in$W such that v$\in$av(w) and B$\in$v
(b) Let w$\in$W, B$\in$bc(Ω) and Γ = {C: $\Box_p$C$\in$w} $\cup$ {B}
If Γ is consistent, then there exists v$\in$W such that v$\in$pv(w) and B$\in$v
(c) Let w$\in$W and X=av(w). Then $\Box_a$d(X)$\in$w
(d) Let w$\in$W and X=pv(w). Then $\Box_p$d(X)$\in$w
(e) Let w$\in$W and B$\in$bc(Ω) and suppose that $\neg\Box_a$B$\in$w. Then there exists v$\in$W such that v$\in$av(w) and $\neg$B$\in$v
(f) Let w$\in$W and B$\in$bc(Ω) and suppose that $\neg\Box_p$B$\in$w. Then there exists v$\in$W such that v$\in$pv(w) and $\neg$B$\in$v
(g) If B$\in$bc(Ω) and ||B||=|B|, then ||$\Box_a$B|| = |$\Box_a$B| (even if $\Box_a$B$\notin$bc(Ω))
(h) If B$\in$bc(Ω) and ||B||=|B|, then ||$\Box_p$B|| = |$\Box_p$B| (even if $\Box_p$B$\notin$bc(Ω))

RESULT II-5-5
M (defined as in Definition II-5-3) satisfies all the conditions of our models.

OUTLINE OF THE PROOF.

- That W$\neq\emptyset$ is a particular consequence of result II-5-2.
- (proof of) condition (3a): use the D-axiom for $\Box_a$ and lemma II-5-9-e).
- condition (4a): use axiom ($\Box_p \rightarrow \Box_a$).
- condition (4b): use the T-axiom for $\Box_p$.
- condition (5a): use $|-\neg$O($\perp$/d(X)) (see result II-4-1-a).
- condition (5b): use lemmas II-5-6 and II-5-5 and axiom (O-CONT-REC).

- condition (5c*): since W is finite, M satisfies condition (5c*) iff M satisfies condition (5c). To prove (5c), use lemmas II-5-6, II-5-4-(a), II-5-5-(c) and theorem (REQ) (lemma II-4-1).
- condition (5d): assuming condition (5b), by lemma II-2-1, (5d) is equivalent to the condition
  (5bd) if $Y \in ob(X)$ and $X \subseteq Z$, then $((Z\text{-}X) \cup Y) \in ob(Z)$
  And, assuming (5b), since $((Z\text{-}X) \cup Y) \cap Z = ((W\text{-}X) \cup Y) \cap Z$, condition (5bd) is equivalent to
  if $Y \in ob(X)$ and $X \subseteq Z$, then $((W\text{-}X) \cup Y) \in ob(Z))$
  Thus, since we have proved that M satisfies condition (5b), we only need to prove that M satisfies the previous simpler condition. For that, use lemmas II-5-6-(a), II-5-6-(b), II-5-5-(b), II-5-5-(c) and II-5-4-(e), axioms $(O\rightarrow \Diamond)$, $(O\rightarrow O\rightarrow)$ and (O-SA), and theorem (REQ) (lemma II-4-1).
- condition (5e): use lemmas II-5-6, II-5-5-(b) and II-5-4-(a) and axiom (O-SA).

RESULT II-5-6[15]
$\forall_{A\in\Omega}\forall_{w\in W}$ (M $|=_w$ A iff $A \in w$)

PROOF.
Let *(A) denote '$\forall_{w\in W}$(M $|=_w$A iff $A\in w$)', i.e. $||A|| = |A|$
o(A) denote 'if $A\in \Omega$(=Subf(φ)) then *(A)'
We prove that $\forall_A$ o(A) by induction on the structure of A.

Base:

(i) A is an atomic sentence, i.e. A is a propositional symbol q. Thesis: o(A). Proof: standard.

Induction step:

(ii) $A=\neg B$ and o(B). Thesis: o(A). Proof: standard.
(iii) $A=B\wedge C$ (or $A=B\vee C$, or $A=B\rightarrow C$, or $A=B\leftrightarrow C$) and o(B) and o(C). Thesis: o(A). Proof: standard.
(iv) $A=\Box B$ and o(B). Thesis: o(A). Outline of the proof:
Use lemmas II-5-4-(a) and II-5-5-(f) (and the definition of W).
(v) A= O(B/C) and o(B1) and o(B2). Thesis: o(A). Outline of the proof:
Use lemmas II-5-7 and II-5-8 and result II-2-2 (and the definition of ob).
(vi) $A=\Box_a B$ and o(B). Thesis: o(A). Outline of the proof: use lemma II-5-9-(g).
(vii) $A=\Box_p B$ and o(B). Thesis: o(A). Outline of the proof: use lemma II-5-9-(h).
(viii) $A = O_a B$ and o(B). Thesis: o(A). Proof:
Suppose $O_aB\in$Subf(φ) which implies that $B\in$Subf(φ), and let w be any world.

- Suppose $O_aB\in w$. We want to prove that M $|=_w$ O_aB, i.e. $||B|| \in ob(av(w))$ and $av(w)\cap||\neg B|| \neq \emptyset$.
  By o(B) (since $B\in$Subf(φ)), $||B|| = |B|$. Thus we need to prove that
  (*) $|B| \in ob(av(w))$ and (**) $av(w)\cap|\neg B| \neq \emptyset$.
  From axiom $(\neg O_a)$, we derive that $|\text{-}\ O_aB\rightarrow \neg\Box_a B$. Thus $\neg\Box_a B\in w$ and, by lemma II-5-9-(e), there exists $v\in W$ such that $v\in av(w)$ and $\neg B\in v$. Thus, we have (**) $av(w)\cap|\neg B| \neq \emptyset$.
  Let us now prove (*) $|B| \in ob(X)$, with $X = av(w)$.
  We have that $O_aB\in$Subf(φ) implies that $O_aB\in$Subf(ψ).
  Suppose, then, that O_aB is the formula number j in the enumeration $O_aA_1, O_aA_2,\ldots,O_aA_r$ of the subformulae of ψ of the form O_aC.

[15]It is easy to prove that from this result it also follows that $\forall_{A\in bc(\Omega)}\forall_{w\in W}$ (M$|=_w$A iff $A\in w$). But this result II-5-6 is enough for our purposes.

If $\Delta_{k+j-1}\cup\{\Diamond O_aA_j\}$ is inconsistent, then $\Delta_{k+j}=\Delta_{k+j-1}\cup\{\neg\Diamond O_aA_j\}$ and so $\neg\Diamond O_aA_j\in\Delta\subseteq\varphi^*$. But then $\Box\neg O_aA_j=\Box\neg O_aB\in\varphi^*$, and, by the definition of W, $\neg O_aB$ would belong to all words. Thus O_aB would not belong to w, contradicting our assumption that $O_aB\in w$.

Thus $\Delta_{k+j-1}\cup\{\Diamond O_aA_j\}$ is consistent, and

$$\Delta_{k+j}=\Delta_{k+j-1}\cup\{\Diamond O_aA_j\}\cup\{\Box(O_aA_j\to\Box_a q_{a_j}\wedge O(A_j/q_{a_j}))\}$$

But then $\Box(O_aA_j\to\Box_a q_{a_j}\wedge O(A_j/q_{a_j}))\in\Delta$, and so also, successively:

$\Box(O_aA_j\to\Box_a q_{a_j}\wedge O(A_j/q_{a_j}))\in\varphi^*$

$O_aA_j\to\Box_a q_{a_j}\wedge O(A_j/q_{a_j})\in w$ (by the definition of W)

$\Box_a q_{a_j}\wedge O(A_j/q_{a_j})\in w$ (since $O_aA_j=O_aB\in w$)

$\Box_a q_{a_j}\in w$ and $O(B/q_{a_j})\in w$ (since $B=A_j$)

And, since q_{aj} is a subformula of φ, from $\Box_a q_{a_j}\in w$, it follows that $q_{a_j}\in v$ for every $v\in av(w)$. Thus $X=av(w)\subseteq|q_{a_j}|$. But $|d(X)|=X$ (lemma II-5-6-(b)-(iii)) and d(X) and q_{a_j} are Boolean combinations of subformulae of φ. Thus, by lemma II-5-5-(b), $\Box(d(X)\to q_{a_j})\in\varphi^*$, and so (by lemma II-5-4-(f)) $\Box(d(X)\to q_{a_j})\in w$.

On the other hand, from axiom $(\neg O_a)$, it follows that $\vdash\Box_a\neg B\to\neg O_a\neg\neg B$. Thus, since $O_aB\in w$, we have $\neg\Box_a\neg B=\Diamond_aB\in w$. And, by lemma II-5-9-(c), $\Box_a d(X)\in w$. So $\Diamond_a(d(X)\wedge B)\in w$, which implies that $\Diamond(d(X)\wedge B)\in w$.

And (axiom (O-SA)) $\vdash\Box(d(X)\to q_{a_j})\wedge\Diamond(d(X)\wedge B)\wedge O(B/q_{a_j})\to O(B/d(X))$.

Thus, from $\Box(d(X)\to q_{a_j})\in w$, $\Diamond(d(X)\wedge B)\in w$ and $O(B/q_{a_j})\in w$, it follows that $O(B/d(X))\in w$. And, by lemma II-5-7, $O(B/d(X))\in\varphi^*$, and by lemma II-5-8 (since $B\in bc(\Omega)$) $O(d(|B|/d(X))\in\varphi^*$. And, finally, (*) $|B|\in ob(X)$ follows, by the definition of ob.

- Suppose now that $M\models_w O_aB$. We want to prove that $O_aB\in w$.
  If $M\models_w O_aB$, then $||B||\in ob(av(w))$ and $av(w)\cap||\neg B||\neq\emptyset$. Let X=av(w).
  By o(B), $||B||=|B|$. Thus $|B|\in ob(X)$ and: (by the definition of ob) $O(d(|B|)/d(X))\in\varphi^*$; by lemma II-5-8 (since $B\in bc(\Omega)$), $O(B/d(X))\in\varphi^*$; and, by lemma II-5-7, $O(B/d(X))\in w$.
  By lemma II-5-9-(c), $\Box_a d(X)\in w$.
  On the other hand, since $av(w)\cap||\neg B||\neq\emptyset$. there exists $v\in av(w)$ such that $M\not\models_v B$. Thus $M\not\models_w\Box_aB$, and, by lemma II-5-9-(g) (since $B\in bc(\Omega)$ and $||B||=|B|$), $\Box_aB\notin w$, and so $\Diamond_a\neg B\in w$.
  Suppose now that $\Diamond_aB\notin w$, i.e. $\Box_a\neg B\in w$. Then (since[16] $\neg B\in bc(\Omega)$) $\neg B\in v$ for every $v\in av(w)$. So $X=av(w)\subseteq|\neg B|$. But (lemma II-5-6-(b)-(iii)) $|d(X)|=X$ and d(X) and $\neg B$ are Boolean combinations of subformulae of φ. Thus, by lemma II-5-5-(b), $\Box(d(X)\to\neg B)\in\varphi^*$. And, by axiom $(O\to\Diamond)$, from $O(B/d(X))\in\varphi^*$ it follows that $\Diamond(B\wedge d(X))\in\varphi^*$. But then $\neg\Diamond(B\wedge d(X))\notin\varphi^*$, and so $\Box(d(X)\to\neg B)\notin\varphi^*$, and a contradiction results. Thus $\Diamond_aB\in w$.
  But, from $O(B/d(X))\in w$, $\Box_a d(X)\in w$, $\Diamond_a\neg B\in w$, $\Diamond_aB\in w$ and axiom (O_a-FD), it follows that $O_aB\in w$ (as we wish to prove).

(ix) $A=O_pB$ and o(B). Thesis: o(A). Proof: similar to case (viii).

COROLLARY II-5-3

If ψ is consistent, there is a finite model M=<W,av,pv,ob,V> and a world $w\in W$ such that $M\models_w\psi$.

PROOF. By results II-5-5 and II-5-6, there is a finite model M=<W,av,pv,ob,V> such that

$\forall_{A\in\Omega}\forall_{v\in W}(M\models_v A$ iff $A\in v)$

[16] It is this step that does not allow us to replace the set $bc(\Omega)$ by the smaller set Ω in the definition of 'av'. As a matter of fact, $B\in\Omega=Subf(\varphi)$ does not imply that $\neg B\in\Omega$.

By result II-5-2, there exists (at least) a w∈W such that $\varphi \in$w. And, since w is a mc set and $|-\varphi \rightarrow \psi$ (recall that $\varphi = \wedge\Delta$ and that ψ belongs to Δ), we conclude that $\psi \in$w.

But $\psi \in \Omega$=Subf(φ). Thus $M|=_w \psi$ (as we wish to prove).

3 Conclusion

This article has supplemented the work reported in [1] by (a) resolving an issue there left open on the relationship between actual and primary obligations, and (b) providing a complete and decidable axiomatization of the logic. It thereby consolidates our earlier work, establishing a firmer foundation for our formal analysis of CTD scenarios. A next step would be to offer systematic comparisons between our theory and the recent works of Christian Straβer [17] [12] and by Dov Gabbay and Karl Schlechta [4–6, 11] on closely related issues in deontic logic, including ranked preferential structures.

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[17] In particular, it would be interesting to explore the possibility (mentioned in [12]) of complementing our framework, turning it into a form of adaptive logic, in order to be able to deal with exceptions and defeasibility, but keeping our treatment of contrary-to-duties.

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Appendix A

In this appendix, we use ⇒ meaning 'implies', ⇐ meaning 'is implied by' and ⇔ meaning 'iff'.

LEMMA II-2-1
Assuming condition (5b), the following condition is equivalent to (5d):
(5bd) if Y∈ob(X) and X⊆Z, then ((Z-X) ∪ Y) ∈ ob(Z)

PROOF. (5bd) ⇒ (5d): Obvious.
(5d) ⇒ (5bd):

Suppose Y∈ob(X) and X⊆Z.
We have Y∩X⊆X.
Since Y∈ob(X), from (5b'), we also have that Y∩X ∈ ob(X).
Thus, from (5d), ((Z-X) ∪ (Y∩X)) ∈ ob(Z).
But, (Z-X) ∪ Y = (Z-X) ∪ ((Y∩X)∪(Y-X)) = ((Z-X)∪(Y-X)) ∪ (Y∩X) = ((Z∪Y)-X)) ∪ (Y∩X)
Thus, ((Z-X) ∪ Y) ∩ Z = (((Z∪Y)-X)) ∪ (Y∩X)) ∩ Z = (((Z∪Y)-X) ∩ Z) ∪ ((Y∩X) ∩ Z)
= ((Z-X) ∩ Z) ∪ ((Y∩X) ∩ Z) = ((Z-X) ∪ (Y∩X)) ∩ Z
That is, ((Z-X) ∪ (Y∩X)) ∩ Z = ((Z-X) ∪ Y) ∩ Z. But then, since ((Z-X) ∪ (Y∩X)) ∈ ob(Z), from (5b), it follows that ((Z-X) ∪ Y) ∈ ob(Z) (as we wish to prove).

RESULT II-2-1
Let M_1 and M_2 be two models that differ at most in the valuation of the propositional symbols, i.e.
$M_1 = \langle W, av, pv, ob, V_1 \rangle$ and $M_2 = \langle W, av, pv, ob, V_2 \rangle$.
If $V_1(q)=V_2(q)$, for any propositional symbol q occurring in A, then $\forall_{w\in W}$ ($M_1|=_w A$ iff $M_2|=_w A$)

PROOF (standard)
Let *(A) denote '$\forall_{w\in W}$ ($M_1|=_w A$ iff $M_2|=_w A$)', i.e. $||A||^{M1} = ||A||^{M2}$
o(A) denote '$V_1(q) = V_2(q)$, for any propositional symbol q occurring in A'
We want to prove that $\forall_a$ (o(A)⇒*(A)). We will prove this by induction on the structure of A.

BASE

(i) A is an atomic sentence, i.e. A is a propositional symbol q. Thesis: o(A)⇒*(A). Proof:
Suppose o(A) is the case. Then (where w is any world, i.e. any member of W):
$M_1|=_w q$ iff (by definition of $M_1|=_w$) $w\in V_1(q)$ iff ($V_1(q)=V_2(q)$) $w\in V_2(q)$ iff $M_2|=_w q$

Induction step:

(ii) A=¬B and o(B)⇒*(B). Thesis: o(A)⇒*(A). Proof:
Suppose o(A) is the case, which implies that o(B) is the case, and let w be any world. Then:
$M_1|=_w \neg B$ iff $M_1|\neq_w B$ iff (by *(B), since o(B) is the case) $M_2|\neq_w B$ iff $M_2|=_w \neg B$

(iii) $A = B \wedge C$ and $o(B) \Rightarrow *(B)$ and $o(C) \Rightarrow *(C)$. Thesis: $o(A) \Rightarrow *(A)$. Proof:
Suppose o(A) is the case, which implies that o(B) and o(C) are the case, and let w be any world. Then:
$M_1 |=_w B \wedge C$ iff $M_1 |=_w B$ and $M_1 |=_w C$
iff (by *(B) and *(C) since o(B) and o(C) are the case) $M_2 |=_w B$ and $M_2 |=_w C$
iff $M_2 |=_w B \wedge C$

(iv) $A = B \vee C$ and $o(B) \Rightarrow *(B)$ and $o(C) \Rightarrow *(C)$. Thesis: $o(A) \Rightarrow *(A)$. Proof: Similar to case (iii).

(v) $A = B \rightarrow C$ and $o(B) \Rightarrow *(B)$ and $o(C) \Rightarrow *(C)$. Thesis: $o(A) \Rightarrow *(A)$. Proof: Similar to case (iii).

(vi) $A = B \leftrightarrow C$ and $o(B) \Rightarrow *(B)$ and $o(C) \Rightarrow *(C)$. Thesis: $o(A) \Rightarrow *(A)$. Proof: Similar to case (iii).

(vii) $A = \Box B$ and $o(B) \Rightarrow *(B)$. Thesis: $o(A) \Rightarrow *(A)$. Proof:
Suppose o(A) is the case, which implies that o(B) is the case, and let w be any world. Then:
$M_1 |=_w \Box B$ iff $||B||^{M1} = W$ iff (by *(B), since o(B) is the case) $||B||^{M2} = W$ iff $M_2 |=_w \Box B$

(viii) $A = \Box_a B$ and $o(B) \Rightarrow *(B)$. Thesis: $o(A) \Rightarrow *(A)$. Proof:
Suppose o(A) is the case, which implies that o(B) is the case, and let w be any world. Then:
$M_1 |=_w \Box_a B$ iff $av(w) \subseteq ||B||^{M1}$ iff (by *(B), since o(B) is the case) $av(w) \subseteq ||B||^{M2}$
iff $M_2 |=_w \Box_a B$

(ix) $A = \Box_p B$ and $o(B) \Rightarrow *(B)$. Thesis: $o(A) \Rightarrow *(A)$. Proof: Similar to case (viii).

(x) $A = O(B/C)$ and $o(B) \Rightarrow *(B)$ and $o(C) \Rightarrow *(C)$. Thesis: $o(A) \Rightarrow *(A)$. Proof:
Suppose o(A) is the case, which implies that o(B) and o(C) are the case, and let w be any world. Then:
$M_1 |=_w O(B/C)$ iff
$||C||^{M1} \cap ||B||^{M1} \neq \emptyset$ and $(\forall X)$(if $X \subseteq ||C||^{M1}$ and $X \cap ||B||^{M1} \neq \emptyset$, then $||B||^{M1} \in ob(X)$)
iff (by *(B) and *(C) since o(B) and o(C) are the case)
$||C||^{M2} \cap ||B||^{M2} \neq \emptyset$ and $(\forall X)$(if $X \subseteq ||C||^{M2}$ and $X \cap ||B||^{M2} \neq \emptyset$, then $||B||^{M2} \in ob(X)$)
iff $M_2 |=_w O(B/C)$

(xi) $A = O_a B$ and $o(B) \Rightarrow *(B)$. Thesis: $o(A) \Rightarrow *(A)$. Proof:
Suppose o(A) is the case, which implies that o(B) is the case, and let w be any world. Then:
$M_1 |=_w O_a B$ iff $||B||^{M1} \in ob(av(w))$ and $av(w) \cap (W - ||B||^{M1}) \neq \emptyset$
iff (by *(B), since o(B) is the case) $||B||^{M2} \in ob(av(w))$ and $av(w) \cap (W - ||B||^{M2}) \neq \emptyset$
iff $M_2 |=_w O_a B$

(xii) $A = O_p B$ and $o(B) \Rightarrow *(B)$. Thesis: $o(A) \Rightarrow *(A)$. Proof: Similar to case (xi).

RESULT II-2-2 (redefinition of $M |=_w O(B/A)$)
Adopting condition (5e) (besides condition (5ab)), then:
$M |=_w O(B/A)$ iff $||B|| \in ob(||A||)$

PROOF.

$\Rightarrow$: Direction $\Rightarrow$ follows directly from the definition of $M |=_w O(B/A)$ (since $||A|| \subseteq ||A||$).

$\Leftarrow$: Suppose now that $||B|| \in ob(||A||)$. We want to prove that $M |=_w O(B/A)$, i.e. that
$||A|| \cap ||B|| \neq \emptyset$ and $(\forall X)$(if $X \subseteq ||A||$ and $X \cap ||B|| \neq \emptyset$, then $||B|| \in ob(X)$)
That $||B|| \in ob(||A||)$ implies $||A|| \cap ||B|| \neq \emptyset$ follows from condition (5ab).
Let $X \subseteq ||A||$ and $X \cap ||B|| \neq \emptyset$. Then, since $||B|| \in ob(||A||)$, by condition (5e), it follows that $||B|| \in ob(X)$ (as we wish to prove).

LEMMA II-2-2
With condition (5c*) (plus conditions (5a), (5b) and (5d)), we obtain:
ob-∪*) Let β be a non-empty set of subsets of W, and let $H = \cup\beta$ ($= \{w \in W: \exists_{Z \in \beta}\, w \in Z\}$).
If $\forall_{Z \in \beta}\, X \in ob(Z)$, then $X \in ob(H)$

PROOF. Let β be a non-empty set of subsets of W, H = $\cup\beta$ and suppose that $\forall_{Z\in\beta}$ X∈ob(Z).
Let β'= {(H-Z)∪X: Z∈β}.

- (i) Since β is non-empty, β' is non-empty,
- (ii) Let Z∈β. By condition (5bd), since X∈ob(Z) and Z⊆H, we conclude that ((H-Z)∪X)∈ob(H) Thus β' ⊆ ob(H)
- (iii) We have that ∩β' = ∩{(H-Z)∪X: Z∈β} = (∩{H-Z: Z∈β}) ∪ X =[18] ∅ ∪ X = X
- (iv) On the other hand, since β is non-empty, there exists some Y∈β, and by condition (5ab), since X∈ob(Y), we have that X∩Y≠∅. Thus X∩H≠∅. That is, (∩β')∩H≠∅.

Thus, from (i)-(ii) and (iv), by condition (5c*), it follows that (∩β') ∈ ob(H), i.e. (by iii) X ∈ ob(H).

LEMMA II-3-1
From conditions (5b) and (5d) , it follows that:
if M $|=_w$O(B/A) and Z is such that Z ∩ ||A||∩ ||B||≠∅, then ||A→B|| ∈ ob(Z)

PROOF. Suppose that M $|=_w$ O(B/A) and that Z is such that Z ∩||A||∩||B||≠∅, and let X = Z ∩||A||.
Then X ⊆ ||A|| and X ∩ ||B||≠∅. Thus, from M $|=_w$ O(B/A), it follows that ||B|| ∈ ob(X).
Thus, from condition (5b′) (see observation II-2-1), X∩||B||∈ob(X), i.e. Z∩||A||∩||B||∈ob(Z∩||A||).
And, since Z∩||A||∩||B||⊆ Z∩||A||, Z∩||A||∩||B||∈ob(Z∩||A||) and Z∩||A||⊆ Z, from condition (5d), it follows that (Z-(Z∩||A||))∪(Z∩||A||∩||B||)∈ob(Z).
But (Z-(Z∩||A||))∪(Z∩||A||∩||B||)=(Z-||A||)∪(Z∩||A||∩||B||)=((W-||A||)∩Z)∪(||A||∩||B||∩Z)=
((W-||A||)∪(||A||∩||B||))∩Z = (((W-||A||)∪||A||)∩((W-||A||)∪||B||))∩Z
= ((W-||A||)∪||B||)∩ Z
Thus ((W-||A||)∪||B||)∩Z ∈ob(Z).
And, from condition (5b′), this implies that (W-||A||))∪||B||∈ob(Z), i.e. ||A→B|| ∈ ob(Z).

RESULT II-3-2
The previous axiomatization is *sound* (i.e. all theorems are valid).

PROOF.

1. The proof that □ is a normal modal operator of type S5 is standard (from its truth condition).
2. That |= O(B / A) → ◇(B ∧ A) follows from the relevant truth conditions. More detailed: Consider a (any) model M and a (any) world w of M, and suppose that M $|=_w$ O(B/A). Then (by the definition of M $|=_w$ O(B/A)), ||A||∩||B|| ≠∅. Thus there exists v∈W such that M$|=_v$ B∧A, and so M $|=_w$ ◇(B ∧ A).
3. That |=◇(A ∧ B ∧ C) ∧ O(B / A) ∧ O(C / A) → O(B ∧ C / A) follows from the relevant truth condition and semantic condition (5c) (implied by (5c*)). More detailed (and without using result II-2-2, i.e. true even if we do not assume condition (5e)):
Suppose M $|=_w$ ◇(A∧B∧C) ∧ O(B/A) ∧ O(C/A) (for M any model and w any world of M).
Since M $|=_w$ ◇(A ∧ B ∧ C) then ||B ∧ C ||∩ ||A||≠ ∅.
Let X ⊆ ||A|| and X ∩ ||B∧C||≠∅. Then X ⊆ ||A|| and X ∩ ||B||≠∅, and, from M $|=_w$O(B/A), it follows that ||B||∈ob(X). Analogously, we get that ||C||∈ob(X). And, since X∩||B||∩||C||= X∩||B∧C||≠∅, by (semantic) condition (5c), we get that ||B||∩||C||=||B∧C||∈ob(X).
Thus M $|=_w$ O(B∧C/A).

[18]Suppose, by *reductio ad absurdum*, that there exists x ∈∩{(H-Z): Z∈β}. Since β≠∅, there exists some Y∈β. Thus, we have that x∈(H-Y), which, in turn, implies that x∈H. But then, since H=∪β, there exists some V∈β, such that x∈V. But, since x∈∩{H-Z): Z∈β} and V∈β, we conclude that x∈(H-V). Thus x∈V and x∉V (a contradiction).

(4) That $|=\Box(A \rightarrow B) \wedge \Diamond(A \wedge C) \wedge O(C / B) \rightarrow O(C / A)$ follows from the relevant truth condition. More detailed (and without using result II-2-2):
Suppose that $M |=_w \Box(A \rightarrow B) \wedge \Diamond(A \wedge C) \wedge O(C / B)$.
Since $M |=_w \Diamond(A \wedge C)$, we have that $||A|| \cap ||C|| \neq \emptyset$.
Let $X \subseteq ||A||$ and $X \cap ||C|| \neq \emptyset$. Then (since $M |=_w \Box(A \rightarrow B)$) $X \subseteq ||B||$ and $X \cap ||C|| \neq \emptyset$, and, from $M |=_w O(C/B)$, it follows that $||C|| \in ob(X)$.
Thus $M |=_w O(C/A)$.
(5) That $|=\Box(A \leftrightarrow B) \rightarrow (O(C/A) \leftrightarrow O(C/B))$ follows (trivially) from the relevant truth condition.
(6) That $|=\Box(C \rightarrow (A \leftrightarrow B)) \rightarrow (O(A/C) \leftrightarrow O(B/C))$ follows from the relevant truth condition, using condition (5b). More detailed (and without using result II-2-2):
Suppose that $M |=_w \Box(C \rightarrow (A \leftrightarrow B))$.
Then $||C|| \subseteq ||A \leftrightarrow B||$, and so (*) $||A|| \cap ||C|| = ||B|| \cap ||C||$.
We want to prove that $M |=_w O(A/C) \rightarrow O(B/C)$ (the proof that $M |=_w O(B/C) \rightarrow O(A/C)$ is analogous).
Suppose (**) $M |=_w O(A/C)$.
Then, in particular, $||A|| \cap ||C|| \neq \emptyset$. Thus, by (*), $||B|| \cap ||C|| \neq \emptyset$.
Let $X \subseteq ||C||$ and $X \cap ||B|| \neq \emptyset$. Then, by (*), $X \cap ||A|| \neq \emptyset$. So, by (**), $||A|| \in ob(X)$).
But, by (*) (since $X \subseteq ||C||$), $||B|| \cap X = ||A|| \cap X$. Thus, by condition (5b), from $||A|| \in ob(X)$, it follows that $||B|| \in ob(X)$).
Thus $M |=_w O(B / C)$ (as we wish to prove).
(7) That $|= O(B/A) \rightarrow \Box O(B/A)$ follows (trivially) from the relevant truth conditions.
(8) That $|= O(B / A) \rightarrow O(A \rightarrow B / T)$ follows from condition (5e) and condition (5ab), using result II-2-2 and lemma II-3-1 (plus the relevant truth conditions). More detailed:
Suppose that $M |=_w O(B/A)$. Then $||B|| \in ob(||A||)$ and, by condition (5ab), $||A|| \cap ||B|| \neq \emptyset$.
Let $Z=||T||=W$. Then $M |=_w O(B/A)$ and $Z \cap ||A|| \cap ||B|| \neq \emptyset$. Thus, by lemma II-3-1, $||A \rightarrow B|| \in ob(Z)$, i.e. $||A \rightarrow B|| \in ob(||T||)$. And, by result II-2-2, $M |=_w O(A \rightarrow B/T)$ (as we wish to prove).
(9) The proof that $\Box_p$ is a normal modal operator is standard (from its truth condition). The validity of the T-schema follows from condition (4b).
(10) The proof that $\Box_a$ is a normal modal operator is standard (from its truth condition). The validity of the D-schema follows from condition (3a)
(11) That $|=\Box A \rightarrow \Box_p A$ follows simply from the relevant truth conditions.
(12) That $|=\Box_p A \rightarrow \Box_a A$ follows from condition (4a) (plus the relevant truth conditions).
(13) That $|=\Box_a A \rightarrow (\neg O_a A \wedge \neg O_a \neg A)$ (respectivelly $|= \Box_p A \rightarrow (\neg O_p A \wedge \neg O_p \neg A)$) follows from condition (5ab). More detailed:
Suppose that $M |=_w \Box_a A$.
Then $av(w) \subseteq ||A||$. Thus $av(w) \cap ||\neg A|| = \emptyset$, and so $M |\neq_w O_a A$.
And, by condition (5ab), $av(w) \cap ||\neg A|| = \emptyset$ implies that $||\neg A|| \notin ob(av(w))$. Thus $M |\neq_w O_a \neg A$.
(14) That $|=\Box_a(A \leftrightarrow B) \rightarrow (O_a A \leftrightarrow O_a B)$ (respectivelly $|= \Box_p(A \leftrightarrow B) \rightarrow O_p A \leftrightarrow O_p B)$) follows from condition (5b). More detailed:
Suppose that $M |=_w \Box_a(A \leftrightarrow B)$. Then (*) $av(w) \subseteq ||A \leftrightarrow B||$.
We want to prove that $M |=_w O_a A \rightarrow O_a B$ (the proof that $M |=_w O_a B \rightarrow O_a A$ is analogous).
Suppose $M |=_w O_a A$, i.e. $||A|| \in ob(av(w))$ and $av(w) \cap ||\neg A|| \neq \emptyset$.
By (*), $||\neg A|| \cap av(w) = ||\neg B|| \cap av(w)$. Thus $av(w) \cap ||\neg B|| \neq \emptyset$.
Also by (*), $||A|| \cap av(w) = ||B|| \cap av(w)$. Thus, by condition (5b), $||B|| \in ob(av(w))$.
Thus $M |=_w O_a B$ (as we wish to prove).

(15) The validity of (O_a-FD) and (O_p-FD) follows simply from the relevant truth conditions.

And, taking into account result II-3-1, we can conclude that every theorem is valid (as we wish).

LEMMA II-4-1
Let C[A/B] denote a formula that we can obtain by replacing in formula C one or more occurrences of formula A by formula B. Then:

$|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow C[A/B])$ (REQ) – (theorem of replacement of equivalents)

PROOF. Let A and B by any two formulas (fixed from now on, in this proof) and let o(D,C,A/B,n) denote that formula D can be obtained by replacing, in formula C, $n(\geqslant 0)$ occurrences of formula A by formula B.

We will prove the desired result in three steps, by proving:

(a) $\forall_{C,D}$ (if o(D,C,A/B,0), then $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$)
(b) $\forall_{C,D}$ (if o(D,C,A/B,1), then $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$)
(c) $\forall_{n\geqslant 1}\forall_{C,D}$ (if o(D,C,A/B,n), then $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$)

PROOF OF (a) (obvious). Let C and D be any two formulae, and suppose that o(D,C,A/B,0) is the case. Then D=C and:

(1) $|- C \leftrightarrow C$ PC
(2) $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow C)$ from (1) by PC

PROOF OF (b) We will proof, by induction on the structure of formula C, that

$\forall_C$*(C)

where: *(C) means $\forall_D$(if o(D,C,A/B,1), then $|- \Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$)

Base:

(i) C is an atomic sentence, i.e. C is a propositional symbol q, and let D be any formula such that o(D,C,A/B,1) is the case.
In such case we must have that (q=)C=A and D=B, and, since $\Box$ is an S5-operator (and so, satisfies the T-schema), $|-\Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$

Induction step:

(ii) $C=\neg C_1$ and $*(C_1)$. Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then two cases are possible:
Case 1: C=A and D=B, and that $|-\Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in (i).
Case 2: $D=\neg D_1$ and o(D_1,C_1,A/B,1) is the case. Then
(1) $|-\Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by $*(C_1)$ (since o(D_1,C_1,A/B,1) is the case)
(2) $|-\Box(A \leftrightarrow B) \rightarrow (\neg C_1 \leftrightarrow \neg D_1)$ from (1) by PC

(iii) $C = C_1 \wedge C_2$ and $*(C_1)$ and $*(C_2)$. Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then three cases are possible:
Case 1: C=A and D=B, and that $|-\Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in (i).
Case 2: $D=D_1 \wedge D_2$ and o(D_1,C_1,A/B,1) and o(D_2,C_2,A/B,0) are the case. Then

(1) $|-\Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by $*(C_1)$ (since o(D_1,C_1,A/B,1) is the case)
(2) $|-\Box(A \leftrightarrow B) \rightarrow (C_2 \leftrightarrow D_2)$ by (a)

(3) $|- (C_1 \leftrightarrow D_1) \wedge (C_2 \leftrightarrow D_2) \rightarrow (C_1 \wedge C_2 \leftrightarrow D_1 \wedge D_2)$ PC
(4) $|- \Box(A \leftrightarrow B) \rightarrow (C_1 \wedge C_2 \leftrightarrow D_1 \wedge D_2)$ from (1), (2) and (3) by PC

Case 3: $D=D_1 \wedge D2$ and $o(D_1,C_1,A/B,0)$ and $o(D_2,C_2,A/B,1)$ are the case. Then the desired theorem follows as in case 2.

(iv) $C = C_1 \vee C_2$ and $*(C_1)$ and $*(C_2)$. Thesis: *(C). Proof:
Similar to case (iii).

(v) $C = C_1 \rightarrow C_2$ and $*(C_1)$ and $*(C_2)$. Thesis: *(C). Proof:
Similar to case (iii).

(vi) $C = C_1 \leftrightarrow C_2$ and $*(C_1)$ and $*(C_2)$. Thesis: *(C). Proof:
Similar to case (iii).

(vii) $C=\Box C_1$ and $*(C_1)$. Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then two cases are possible:
Case 1: C=A and D=B, and that $|- \Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in (i).
Case 2: $D=\Box D_1$ and $o(D_1,C_1,A/B,1)$ is the case. Then

(1) $|- \Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by $*(C_1)$ (since $o(D_1,C_1,A/B,1)$ is the case)
(2) $|- \Box\Box(A \leftrightarrow B) \rightarrow \Box(C_1 \leftrightarrow D_1)$ from (1) since $\Box$ is a normal (modal) operator
(3) $|- \Box(C_1 \leftrightarrow D_1) \rightarrow (\Box C_1 \leftrightarrow \Box D_1)$ using known properties of the normal operators
(4) $|- \Box(A \leftrightarrow B) \rightarrow \Box\Box(A \leftrightarrow B)$ $\Box$ is a S5 operator
(5) $|- \Box(A \leftrightarrow B) \rightarrow (\Box C_1 \leftrightarrow \Box D_1)$ from (4), (2) and (3) by PC

(viii) $C=\Box_a C_1$ and $*(C_1)$. Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then two cases are possible:
Case 1: C=A and D=B, and that $|- \Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in i).
Case 2: $D=\Box_a D_1$ and $o(D_1,C_1,A/B,1)$ is the case. Then

(1) $|- \Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by *(C1) (since $o(D_1,C_1,A/B,1)$ is the case)
(2) $|- \Box\Box(A \leftrightarrow B) \rightarrow \Box(C_1 \leftrightarrow D_1)$ from (1) since $\Box$ is a normal (modal) operator
(3) $|- \Box(C_1 \leftrightarrow D_1) \rightarrow \Box_a(C_1 \leftrightarrow D_1)$ by axioms $(\Box \rightarrow \Box_p)$ and $(\Box_p \rightarrow \Box_a)$ and PC
(4) $|- \Box_a(C_1 \leftrightarrow D_1) \rightarrow (\Box_a C_1 \leftrightarrow \Box_a D_1)$ using known properties of the normal operators
(5) $|- \Box(A \leftrightarrow B) \rightarrow \Box\Box(A \leftrightarrow B)$ $\Box$ is a S5 operator
(6) $|- \Box(A \leftrightarrow B) \rightarrow (\Box_a C_1 \leftrightarrow \Box_a D_1)$ from (5), (2), (3) and (4) by PC

(ix) $C=\Box_p C_1$ and $*(C_1)$. Thesis: *(C). Proof:
Similar to case (viii).

(x) $C = O(C_1/C_2)$ and $*(C_1)$ and $*(C_2)$. Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then three cases are possible:
Case 1: C=A and D=B, and that $|- \Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in (i).
Case 2: $D=O(D_1/D_2)$ and $o(D_1,C_1,A/B,1)$ and $o(D_2,C_2,A/B,0)$ are the case. Then

(1) $|- \Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by $*(C_1)$ (since $o(D_1,C_1,A/B,1)$ is the case)
(2) $|- \Box(A \leftrightarrow B) \rightarrow (C_2 \leftrightarrow D_2)$ by (a)
(3) $|- \Box\Box(A \leftrightarrow B) \rightarrow \Box(C_1 \leftrightarrow D_1)$ from (1) since $\Box$ is a normal (modal) operator
(4) $|- \Box\Box(A \leftrightarrow B) \rightarrow \Box(C_2 \leftrightarrow D_2)$ from (2) since $\Box$ is a normal (modal) operator
(5) $|- \Box(A \leftrightarrow B) \rightarrow \Box\Box(A \leftrightarrow B)$ $\Box$ is a S5 operator
(6) $|- \Box(A \leftrightarrow B) \rightarrow \Box(C_1 \leftrightarrow D_1)$ from (5) and (3) by PC

(7) $|-\Box(A \leftrightarrow B) \rightarrow \Box(C_2 \leftrightarrow D_2)$ from (5) and (4) by PC
(8) $|-\Box(A \leftrightarrow B) \rightarrow (O(C_1/C_2) \leftrightarrow O(C_1/D_2))$ from (7) and (O-REA) by PC
(9) $|-\Box(C_1 \leftrightarrow D_1) \rightarrow \Box(D_2 \rightarrow (C_1 \leftrightarrow D_1))$ using PC and known properties of normal operators
(10) $|-\Box(A \leftrightarrow B) \rightarrow (O(C_1/D_2) \leftrightarrow O(D_1/D_2))$ from (6), (9) and (O-CONT-REC) by PC
(11) $|-\Box(A \leftrightarrow B) \rightarrow (O(C_1/C_2) \leftrightarrow O(D_1/D_2))$ from (8) and (10) by PC

Case 3: D=O(D_1/D_2) and o(D_1,C_1,A/B,0) and o(D_2,C_2,A/B,1) are the case. Then the desired theorem follows as in case 2.

(xi) C=O_aC_1 and *(C_1). Thesis: *(C). Proof:
Let D be any formula such that o(D,C,A/B,1) is the case.
Then two cases are possible:
Case 1: C = A and D = B, and that $|-\Box(A \leftrightarrow B) \rightarrow (A \leftrightarrow B)$ follows as in (i).
Case 2: D=O_aD_1 and o(D_1,C_1,A/B,1) is the case. Then

(1) $|-\Box(A \leftrightarrow B) \rightarrow (C_1 \leftrightarrow D_1)$ by *(C1) (since o(D_1,C_1,A/B,1) is the case)
(2) $|-\Box\Box(A \leftrightarrow B) \rightarrow \Box(C_1 \leftrightarrow D_1)$ from (1) since $\Box$ is a normal (modal) operator
(3) $|-\Box(C_1 \leftrightarrow D_1) \rightarrow \Box_a(C_1 \leftrightarrow D_1)$ by axioms ($\Box \rightarrow \Box_p$) and ($\Box_p \rightarrow \Box_a$) and PC
(4) $|-\Box(A \leftrightarrow B) \rightarrow \Box\Box(A \leftrightarrow B)$ $\Box$ is a S5 operator
(5) $|-\Box(A \leftrightarrow B) \rightarrow \Box_a(C_1 \leftrightarrow D_1)$ from (4), (2) and (3) by PC
(6) $|-\Box_a(C_1 \leftrightarrow D_1) \rightarrow (O_aC_1 \leftrightarrow O_aD_1)$ ($\leftrightarrow O_a$)
(7) $|-\Box(A \leftrightarrow B) \rightarrow (O_aC_1 \leftrightarrow O_aD_1)$ from (5) and (6) by PC

(xii) A=O_pC_1 and *(C_1). Thesis: *(C). Proof:
Similar to case (xi).

PROOF OF (c). We will prove, by (simple) induction on $n \geqslant 1$, that
$\forall_{n\geqslant 1}$**(n)

where: **(n) means $\forall_{C,D}$(if o(D,C,A/B,n), then $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$)

Base:

n= 1. We have that **(1), by (b).

Induction step:

Let $n \geqslant 1$ and assume that **(n) is the case. We want to prove that **(n+1) is also the case.
Let C and D be any two formulae such that o(D,C,A/B,n+1) is the case.
Then there exists a formula E such that o(E,C,A/B,n) and o(D,E,A/B,1) are the case. But then:

(1) $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow E)$ by **(n) (since o(E,C,A/B,n) is the case)
(2) $|-\Box(A \leftrightarrow B) \rightarrow (E \leftrightarrow D)$ by the base case (i.e. b)) (since o(D,E,A/B,1) are the case)
(3) $|-\Box(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)$ from 1) and 2) by PC

RESULT II-4-1

(a) The following schemas (that are axioms in [1]) can be deduced as theorems:

(i) $\neg O(\bot/A)$
(ii) $\Diamond_p(A \wedge B \wedge C) \wedge O(B/A) \wedge O(C/A) \rightarrow O(B \wedge C/A)$
(iii) $O(B/A) \rightarrow O(B/A \wedge B)$
(iv) $\Diamond_p O(B/A) \rightarrow \Box_p O(B/A)$
(v) $\Diamond_p(A \wedge B \wedge C) \wedge O(C/B) \rightarrow O(C/A \wedge B)$

(b) The following rules (that are primitive in [1]) can be derived:

(i) If $\vdash A \leftrightarrow B$ then $\vdash O(C/A) \leftrightarrow O(C/B)$
(ii) If $\vdash C \rightarrow (A \leftrightarrow B)$ then $\vdash O(A/C) \leftrightarrow O(B/C)$

PROOF

(a-i) Follows from axiom ($O\rightarrow\Diamond$) and the normality of $\Box$ (i.e. the fact that $\Box$ is a normal modal operator).
(a-ii) Follows from (O-C) and ($\Box \rightarrow \Box_p$).
(a-iii) We have the following derivation:

(1) $\vdash \Box(A\wedge B\rightarrow A) \wedge \Diamond(A\wedge B\wedge B) \wedge O(B/A) \rightarrow O(B/A\wedge B)$ (O-SA)
(2) $\vdash O(B/A) \rightarrow \Diamond(B\wedge A)$ ($O\rightarrow\Diamond$)
(3) $\vdash \Box(A\wedge B\rightarrow A)$ PC and $\Box$-necessitation
(4) $\vdash \Diamond(B\wedge A) \rightarrow \Diamond(A\wedge B\wedge B)$ by PC and known properties of normal modal operators
(5) $\vdash O(B/A) \rightarrow O(B/A\wedge B)$ from (2), (4), (3) and (1) by PC

(a-iv) Follows from ($\Diamond O\rightarrow \Box O$) and ($\Box \rightarrow \Box_p$).
(a-v) We have the following derivation:

(1) $\vdash \Box(A\wedge B\rightarrow B) \wedge \Diamond(A\wedge B\wedge C) \wedge O(C/B) \rightarrow O(C/A\wedge B)$ (O-SA)
(2) $\vdash \Box(A\wedge B\rightarrow B)$ PC and $\Box$-necessitation
(3) $\vdash \Diamond(A \wedge B \wedge C) \wedge O(C / B) \rightarrow O(C / A \wedge B)$ from (2) and (1) by PC
(4) $\vdash \Diamond_p(A\wedge B\wedge C) \rightarrow \Diamond(A\wedge B\wedge C)$ from ($\Box \rightarrow \Box_p$) (and PC)
(5) $\vdash \Diamond_p(A\wedge B\wedge C) \wedge O(C / B) \rightarrow O(C / A \wedge B)$ from (4) and (3) by PC

(b-i) Since $\Box$ verifies the necessitation proof rule, it follows trivially from (O-REA) and PC.
(b-ii) Since $\Box$ verifies the necessitation proof rule, it follows trivially from (O-CONT-REC) and PC.

OBSERVATION II-4-3
We have the 'deontic detachment' theorems
$\vdash O_aA \wedge O(B/A) \wedge\Diamond_a(A\wedge B) \rightarrow O_a(A\wedge B)$ label: (O_a-DD)
$\vdash O_pA \wedge O(B/A) \wedge\Diamond_p(A\wedge B) \rightarrow O_p(A\wedge B)$ label: (O_p-DD)

PROOF.
(O_a-DD) follows from axioms ($\leftrightarrow O_a$) and theorems (O_a-C) and ($O\rightarrow O_a\rightarrow$). In detail:

(1) $\vdash \Diamond_a(A\wedge(A\rightarrow B)) \wedge O_aA \wedge O_a(A\rightarrow B) \rightarrow O_a(A\wedge(A\rightarrow B))$ (O_a-C)
(2) $\vdash \Diamond_a(A\wedge B) \rightarrow \Diamond_a(A\wedge(A\rightarrow B))$ using PC and known properties of normal operators
(3) $\vdash \Diamond_a(A \wedge B) \wedge O_aA \wedge O_a(A\rightarrow B) \rightarrow O_a(A\wedge(A\rightarrow B))$ from (1) and (2) by PC
(4) $\vdash \Box_a(A\wedge B \leftrightarrow A\wedge(A\rightarrow B))$ from PC and $\Box_a$-necessitation
(5) $\vdash \Box_a(A\wedge B \leftrightarrow A\wedge(A\rightarrow B)) \rightarrow (O_a(A\wedge B) \leftrightarrow O_a(A\wedge(A\rightarrow B))$ ($\leftrightarrow O_a$)
(6) $\vdash O_a(A\wedge B) \leftrightarrow O_a(A\wedge(A\rightarrow B))$ from (4) and (5) by PC
(7) $\vdash \Diamond_a(A\wedge B) \wedge O_aA \wedge O_a(A\rightarrow B) \rightarrow O_a(A\wedge B)$ from (3) and (6) by PC
(8) $\vdash O(B/A) \wedge\Diamond_a(A\wedge B) \wedge \Diamond_a(A\wedge\neg B) \rightarrow O_a(A\rightarrow B)$ ($O\rightarrow O_a\rightarrow$)
(9) $\vdash \Diamond_a(A\wedge\neg B) \rightarrow (O_aA \wedge O(B/A) \wedge\Diamond_a(A\wedge B) \rightarrow O_a(A\wedge B))$ from (8) and (7) by PC
(10) $\vdash \neg\Diamond_a(A\wedge\neg B) \rightarrow \Box_a(A\leftrightarrow A\wedge B)$ using PC and known properties of normal operators
(11) $\vdash \Box_a(A\leftrightarrow A\wedge B) \rightarrow (O_aA \leftrightarrow O_a(A\wedge B))$ ($\leftrightarrow O_a$)
(12) $\vdash \neg\Diamond_a(A\wedge\neg B) \rightarrow (O_aA \wedge O(B/A) \wedge\Diamond_a(A\wedge B) \rightarrow O_a(A\wedge B))$ from (10) and (11) by PC
(13) $\vdash O_aA \wedge O(B/A) \wedge\Diamond_a(A\wedge B) \rightarrow O_a(A\wedge B)$ from (9) and (12) by PC

The proof of (O_p-DD) is analogous.

LEMMA II-5-1

Let Γ_1 and Γ_2 be two mc sets such that $d(\Gamma_1) = d(\Gamma_2)$.

If A∈bc(Ω) (i.e. A is any Boolean combination of subformulae of φ), then
A∈ Γ_1 iff A∈ Γ_2

PROOF. Let Γ_1 and Γ_2 be two mc sets such that $d(\Gamma_1) = d(\Gamma_2)$.

We will proof, by induction on the structure of the Boolean combination of subformulae of φ, that $\forall_{A \in bc(\Omega)}$o(A) where o(A) means that 'A∈ Γ_1 iff A∈ Γ_2'.

Base:

(i) A ∈ Ω=Subf(φ). Thesis: o(A). Proof:
Suppose A ∈ Ω=Subf(φ).
If A ∈ Ω, then, for any descriptor d, either A∈d or (A∉d and) ¬A∈d.
And for any mc set Γ, by the definition of d(Γ), we have that d(Γ) ∈ Γ.
Thus (by the properties of the mc sets):
A∈ Γ_1 iff A∈d(Γ_1) iff A∈d(Γ_2) iff A∈ Γ_2

Induction step:

(ii) *A*=¬*B* and o(*B*). Thesis: o(*A*). Proof (using standard properties of the mc sets):
¬B∈ Γ_1 iff B∉ Γ_1 iff (by o(*B*)) B∉ Γ_2 iff ¬B∈ Γ_2

(iii-vi) Similarly, using standard properties of the mc sets, we can prove that o(B) and o(C) imply o(B∧C), o(B∨C), o(B→C) and o(B↔C).

LEMMA II-5-2

If Γ is mc and ◇A∈ Γ (where A is any formula), there is (at least) one descriptor d such that ◇(d∧A)∈ Γ

OUTLINE OF THE PROOF

The desired descriptor d = $A_1 \wedge \ldots \wedge A_n$, can be inductively built as follows (i = 1,…,*n*):

if $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge a_i \wedge A) \in \Gamma$, then $A_i = a_i$; otherwise, $A_i = \neg a_i$.

(That is, we go through the list $a_1,\ldots,a_n$ of all the subformulae of φ. We start by checking if $\Diamond(a_1 \wedge A) \in$ Γ. If it is, we take a_1 as the first conjunct A_1 of d; if not, we make A1 be $\neg a_1$. And so on.)

The proof that 'if Γ is mc and ◇A∈ Γ, then ◇(d∧A)∈ Γ' is then as follows:

Assume, by induction hypothesis, that (*) $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge A) \in \Gamma$.
If $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge a_i \wedge A) \in \Gamma$, we are done.
Suppose that $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge a_i \wedge A) \notin \Gamma$.
Since Γ is mc and □ is a normal operator, then $\neg\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge a_i \wedge A) \in \Gamma$ and so, also
(**) $\Box(A_1 \wedge \ldots \wedge A_{i-1} \wedge A \rightarrow \neg a_i) \in \Gamma$.
And, from (*) and (**), it follows that $\Diamond(A_1 \wedge \ldots \wedge A_{i-1} \wedge \neg a_i \wedge A) \in \Gamma$ (as we wish to prove).

LEMMA II-5-3

Let Γ be mc, d a descriptor and A∈bc(Ω) (i.e. A is any Boolean combination of subformulae of φ). Then if '◇(d∧A)∈ Γ then □(d→A)∈ Γ' (property designated by o(A) in the proof below).

PROOF. Let Γ be any mc set and d any descriptor. We will proof, by induction on the structure of the Boolean combination of subformulae of φ, that $\forall_{A \in bc(\Omega)}$o(A).

Base:

(i) A ∈ Ω=Subf(φ). Thesis: o(A). Proof:
Suppose A ∈ Ω=Subf(φ) and suppose that ◇(d∧A)∈ Γ.

If A $\in\Omega$, then either A$\in$d or (A$\notin$d and) $\neg$A$\in$d
If A$\in$d, then $|-$ d$\rightarrow$A. Thus, by $\Box$-necessitation, $|-\Box$(d$\rightarrow$A), and so (since Γ is mc) $\Box$(d$\rightarrow$A)$\in\Gamma$.
If A$\notin$d, then $|-$ d$\rightarrow\neg$A. Thus, analogously, $\Box$(d$\rightarrow\neg$A)$\in\Gamma$, which contradicts the hypothesis that $\Diamond$(d$\wedge$A)$\in\Gamma$. Thus, (if $\Diamond$(d$\wedge$A)$\in\Gamma$ then) we cannot have the case where A$\notin$d.

Induction step:

(ii) *A*=$\neg$*B* and o(*B*). Thesis: o(*A*). Proof:
Suppose that $\Diamond$(d$\wedge\neg$B)$\in\Gamma$. Then (by the properties of the mc sets and the normality of $\Box$) $\Box$(d$\rightarrow$B)$\notin\Gamma$, and so (by o(*B*)) $\Diamond$(d$\wedge$B)$\notin\Gamma$. But then $\Box$(d$\rightarrow\neg$B)$\in\Gamma$.

(iii) A=B$\wedge$C and o(B) and o(C). Thesis: o(A).
Suppose that $\Diamond$(d$\wedge$B$\wedge$C)$\in\Gamma$. Then (by the properties of the mc sets and the normality of $\Box$) $\Diamond$(d$\wedge$B)$\in\Gamma$ and $\Diamond$(d$\wedge$C)$\in\Gamma$, and so (by o(B) and o(C)) $\Box$(d$\rightarrow$B)$\in\Gamma$ and $\Box$(d$\rightarrow$C)$\in\Gamma$. But then (by the properties of the mc sets and the normality of $\Box$), $\Box$(d$\rightarrow$ B$\wedge$C)$\in\Gamma$.

(iv) A=B$\vee$C and o(B) and o(C). Thesis: o(A).
We can prove this directly, or using (ii) and (iii) (since $|-$ B$\vee$C $\leftrightarrow\neg(\neg$B$\wedge\neg$C)). Directly:
Suppose that $\Diamond$(d$\wedge$(B$\vee$C))$\in\Gamma$. Then $\Diamond$(d$\wedge$B)$\vee\Diamond$(d$\wedge$C)$\in\Gamma$, and so $\Diamond$(d$\wedge$B)$\in\Gamma$ or $\Diamond$(d$\wedge$C)$\in\Gamma$. If $\Diamond$(d$\wedge$B)$\in\Gamma$, then (by o(B)) $\Box$(d$\rightarrow$B)$\in\Gamma$, and so $\Box$(d$\rightarrow$B$\vee$C)$\in\Gamma$. If $\Diamond$(d$\wedge$C)$\in\Gamma$, analogously (using now o(C)) we get that $\Box$(d$\rightarrow$B$\vee$C)$\in\Gamma$.

(v) A=B$\rightarrow$C and o(B) and o(C). Thesis: o(A).
Suppose that $\Diamond$(d$\wedge$(B$\rightarrow$C))$\in\Gamma$. Then (since $|-$ (B$\rightarrow$C) $\leftrightarrow\neg$B$\vee$C)), $\Diamond$(d$\wedge$($\neg$B$\vee$C))$\in\Gamma$. By (ii), we have o($\neg$B). Thus, by (iv), $\Box$(d$\rightarrow\neg$B$\vee$C)$\in\Gamma$, and so $\Box$(d$\rightarrow$(B$\rightarrow$C))$\in\Gamma$.

(vi) A=B$\leftrightarrow$C and o(B) and o(C). Thesis: o(A).
Suppose that $\Diamond$(d$\wedge$(B$\leftrightarrow$C))$\in\Gamma$.
Then (since $|-$ (B$\leftrightarrow$C)$\leftrightarrow$(B$\rightarrow$C)$\wedge$(C$\rightarrow$B)), $\Diamond$(d$\wedge$(B$\rightarrow$C)$\wedge$(C$\rightarrow$B))$\in\Gamma$. By (v), we have o(B$\rightarrow$C) and o(C$\rightarrow$B). Thus, by (iii), $\Box$(d$\rightarrow$(B$\rightarrow$C)$\wedge$(C$\rightarrow$B))$\in\Gamma$, and so $\Box$(d$\rightarrow$(B$\leftrightarrow$C))$\in\Gamma$.

RESULT II-5-1

(a) Each w$\in$W is mc.
(b) W is finite.

PROOF.

(a) Standard (using the fact that $\Box$ is a normal operator):
Since $\Diamond$d$\in\varphi^*$, and φ^* is consistent (because φ^* is mc), we have that $\Box^{-1}\varphi^*\cup\{$d$\}$ is consistent (see e.g. lemma 2.3a, page 22, of [7]). Thus s(d) = $(\Box^{-1}\varphi^*\cup\{d\})^*$ is mc.
(b) Obvious, since the set DESCRIPTORS is finite.

LEMMA II-5-4
(Where A and B can be any formulae:)

(a) If |A|$\neq\emptyset$, then $\Diamond$A$\in\varphi^*$
(i.e. if A$\in$w, for w$\in$W, then $\Diamond$A$\in\varphi^*$).
(b) If $\Box$(A$\rightarrow$B)$\in\varphi^*$, then |A| $\subseteq$ |B|.
(c) If $\Box$(A$\leftrightarrow$B)$\in\varphi^*$, then |A|$=$|B|.
(d) If $|-$ A$\rightarrow$B, then |A|$\subseteq$ |B|.
(e) If $|-$ A$\leftrightarrow$B, then |A| = |B|.
(f) ($\forall$w$\in$W) ($\Box$A $\in$ w iff $\Box$A $\in\varphi^*$)

PROOF.

(a) If $\Diamond A \notin \varphi^*$, then $\Box\neg A \in \varphi^*$. Thus (by the definition of W) $\neg A \in w$ for all $w \in W$, and |A| would be empty.
(b) If it is not the case that $|A| \subseteq |B|$, then there exists $w \in W$ such that $A \in w$ and $B \notin w$. Thus $A \wedge \neg B \in w$, which implies that $\neg(A \to B) \in w$. But then we cannot have $\Box(A \to B) \in \varphi^*$, since that would imply (by the definition of W) that $(A \to B) \in v$ for all $v \in W$.
(c) Follows from (b) (since, by the normality of $\Box$ and the properties of the mc sets, $\Box(A \leftrightarrow B) \in \varphi^*$ implies that both $\Box(A \to B)$ and $\Box(B \to A)$ belong to φ^*).
(d) If $|\!- A \to B$, then (by $\Box$-necessitation) $|\!- \Box(A \to B)$, which implies (since φ^* is a mc set) that $\Box(A \to B) \in \varphi^*$. But then $|A| \subseteq |B|$ follows from (b).
(e) Follows from (d).
(f) Consider any world w (i.e. $w \in W$).
Since $|\!- \Box A \to \Box\Box A$, we have that $\Box A \in \varphi^*$ implies that $\Box\Box A \in \varphi^*$, which, in turn, implies that $\Box A \in w$.
Suppose now that $\Box A \notin \varphi^*$. Then $\neg\Box A \in \varphi^*$ and so $\Diamond\neg A \in \varphi^*$. But $|\!- \Diamond\neg A \to \Box\Diamond\neg A$. Thus $\Box\Diamond\neg A \in \varphi^*$, and so $\Diamond\neg A \in w$. But then $\Box A \notin w$.

LEMMA II-5-5
If $A,B \in bc(\Omega)$ (i.e. A and B are any Boolean combinations of subformulae of φ), then:

(a) If $\Diamond A \in \varphi^*$, then $|A| \neq \emptyset$" (i.e. if $\Diamond A \in \varphi^*$ then there exists $w \in W$ such that $A \in w$).
(b) If $|A| \subseteq |B|$, then $\Box(A \to B) \in \varphi^*$.
(c) If $|A| = |B|$, then $\Box(A \leftrightarrow B) \in \varphi^*$.

PROOF.

(a) By lemma II-5-2, if $\Diamond A \in \varphi^*$, there is (at least) one descriptor d such that $\Diamond(d \wedge A) \in \varphi^*$. And, by lemma II-5-3 (since A is a Boolean combination of subformulae of φ), this implies that $\Box(d \to A) \in \varphi^*$. And (by the normality of $\Box$) $\Diamond(d \wedge A) \in \varphi^*$ implies that $\Diamond d \in \varphi^*$. Thus, by the definition of W, there exists w such that $\{d, d \to A\} \subseteq w$. And, since w is mc (result II-5-1), $A \in w$.
(b) If $\Box(A \to B) \notin \varphi^*$, then $\Diamond(A \wedge \neg B) \in \varphi^*$. Thus, by (a), $|A \wedge \neg B| \neq \emptyset$, i.e. there exists w such that $A \wedge \neg B \in w$. But this implies that $A \in w$ and $B \notin w$, and so it is not the case that $|A| \subseteq |B|$.
(c) Follows from (b) (using the normality of $\Box$ and the properties of the mc sets).

LEMMA II-5-6

(a) (Where A and B can be any formulae:)

(i) $|A \vee B| = |A| \cup |B|$
(ii) $|A \wedge B| = |A| \cap |B|$
(iii) $|\neg A| = W - |A|$

(b) (i) $|d(w)| = \{w\}$ (for each $w \in W$)

(ii) $|d(X \cup Y)| = |d(X) \vee d(Y)|$ (for $X,Y \subseteq W$)
(iii) $|d(X)| = X$ (for each $X \subseteq W$)
(iv) $|d(X \cap Y)| = |d(X) \wedge d(Y)|$ (for $X,Y \subseteq W$)
(v) $|d(W-X)| = |\neg d(X)|$ (for $X \subseteq W$)

PROOF.

(a) Follows from the properties of the mc sets.

(b) (i) Suppose w∈W. That | d(w) | = {w}, follows from the definition of d(Γ), for Γ a mc set, and the definition of W.

As a matter of fact:

By the definition of |A|, we have that: | d(w) | = {v∈W: d(w)∈v}.

By the definition of d(Γ), for Γ mc, we have that: d(w)∈w. Thus, w∈|d(w)|.

Suppose now that v∈W is such that v∈|d(w)|, i.e. d(w)∈v.

Then, by the definition of W, we have that there exists d∈DESCRIPTORS and ◇d∈φ* and w=s(d)= ($\Box^{-1}\varphi^* \cup$ {d})* (since w∈W) and there exists d_1 ∈DESCRIPTORS and ◇d_1 ∈φ* and v=s(d_1) =($\Box^{-1}\varphi^* \cup$ {d_1})* (since v∈W). And so d∈w and d_1 ∈v.

But, for Γ mc, d(Γ) denotes the unique element of Γ∩ DESCRIPTORS. Thus d(w)∈w and d∈w implies that d(w)=d, and d(w)∈v and d_1 ∈v implies that d(w)=d_1. That is, d(w)=d=d_1.

But then, w=s(d) and v=s(d_1) implies that w=v.

(ii) Let X,Y⊆W. That |d(X∪Y)| = |d(X)∨d(Y)| follows from the properties of the mc sets.

As a matter of fact:

|d(X∪Y)| ={w∈W: d(X∪Y)∈w} ={w∈W: $\vee_{v\in X\cup Y}$d(v)∈w}

Now, $\vee_{v\in X\cup Y}$d(v) is equivalent to ($\vee_{v\in X}$d(v))∨($\vee_{v\in Y}$d(v)) (i.e. the equivalence between them is a theorem). Thus, since w is a mc set:

($\vee_{v\in X\cup Y}$d(v))∈w iff ($\vee_{v\in X}$d(v))∨($\vee_{v\in Y}$d(v))∈w iff (d(X)∨d(Y))∈w iff w∈|d(X)∨d(Y)| .

Thus:

|d(X∪Y)| = |d(X)∨d(Y)|

(iii) Suppose X⊆W. That | d(X) | = X follows from (a-i) and previous (i) and (ii).

As a matter of fact:

|d(X)| = |d($\cup_{v\in X}${v})| = (by (b-ii)) |$\vee_{v\in X}$d({v})| = |$\vee_{v\in X}$d(v)| = (by (a-i)) $\cup_{v\in X}$|d(v)|
= (by (b-i)) $\cup_{v\in X}${v} = X

(iv) Let X,Y⊆W. That |d(X∩Y)| = |d(X)∧d(Y)| follows from (a-ii) and (b-iii).

As a matter of fact:

By (b-iii) (since X∩Y⊆W), |d(X∩Y)| = X∩Y = |d(X)|∩|d(Y)| = (by (a-ii)) |d(X)∧d(Y)|

(v) Let X⊆W. That | d(W-X) | = |¬d(X) | follows from (a-iii) and (b-iii).

As a matter of fact:

By (b-iii) (since (W-X)⊆W), |d(W-X)| = W-X = W-|d(X)| = (by (a-iii)) |¬d(X)|

Corollary II-5-2 (corollary of lemmas II-5-6, II-5-5 and II-5-4)

Let B∈bc(Ω) (i.e. B is a Boolean combination of subformulae of φ).

(i) □(d(|B|)↔B) ∈φ*
(ii) □(d(|B|)↔B) ∈ w, for any w∈W

Proof. Let B∈bc(Ω).

(i) By lemma II-5-6-(b)-(iii)), |d(|B|)| = |B|. On the other hand, by the definition of d(X) (for X⊆W), d(X) is always a Boolean combination of subformulae of φ. Thus d(|B|) is a Boolean combination of subformulae of φ. and, by hypothesis, B is also a Boolean combination of subformulae of φ. Thus, by lemma II-5-5-(c), □(d(|B|)↔B)∈φ*.
(ii) Follows from (i), by lemma II-5-4-(f).

Lemma II-5-8

Let B∈bc(Ω) (i.e. B is a Boolean combination of subformulae of φ) and let A be any formula.

(a) (i) O(A / d(|B|)) ∈φ* iff O(A / B) ∈ φ*
(ii) O(A / d(|B|)) ∈ w iff O(A / B) ∈ w, for any w∈W

(b) (i) $O(d(|B|) / A) \in \varphi^*$ iff $O(B / A) \in \varphi^*$
(ii) $O(d(|B|) / A) \in w$ iff $O(B / A) \in w$, for any $w \in W$

PROOF. Let $B \in bc(\Omega)$.

(a-i) By corollary II-5-2-(i), (since $B \in bc(\Omega)$) we have that $\Box(d(|B|) \leftrightarrow B) \in \varphi^*$. Thus, by the axiom (O-REA), $O(A/d(|B|)) \leftrightarrow O(A/B) \in \varphi^*$. And so $O(A/d(|B|)) \in \varphi^*$ iff $O(A/B) \in \varphi^*$.
(a-ii) Follows from (a-i), by lemma II-5-7.
(b-i) By corollary II-5-2-(i), (since $B \in bc(\Omega)$) we have that $\Box(d(|B|) \leftrightarrow B) \in \varphi^*$. Thus, by (REQ) – theorem (lemma II-4-1), $O(d(|B|)/A) \leftrightarrow O(B/A) \in \varphi^*$. And so $O(d(|B|)/A) \in \varphi^*$ iff $O(B/A) \in \varphi^*$.
(b-ii) Follows from (b-i), by lemma II-5-7.

LEMMA II-5-9

(a) Let $w \in W$, $B \in bc(\Omega)$ and $\Gamma = \{C: \Box_a C \in w\} \cup \{B\}$
If Γ is consistent, then there exists $v \in W$ such that $v \in av(w)$ and $B \in v$
(b) Let $w \in W$, $B \in bc(\Omega)$ and $\Gamma = \{C: \Box_p C \in w\} \cup \{B\}$
If Γ is consistent, then there exists $v \in W$ such that $v \in pv(w)$ and $B \in v$
(c) Let $w \in W$ and $X = av(w)$. Then $\Box_a d(X) \in w$
(d) Let $w \in W$ and $X = pv(w)$. Then $\Box_p d(X) \in w$
(e) Let $w \in W$ and $B \in bc(\Omega)$ and suppose that $\neg\Box_a B \in w$. Then there exists $v \in W$ such that $v \in av(w)$ and $\neg B \in v$
(f) Let $w \in W$ and $B \in bc(\Omega)$ and suppose that $\neg\Box_p B \in w$. Then there exists $v \in W$ such that $v \in pv(w)$ and $\neg B \in v$
(g) If $B \in bc(\Omega)$ and $||B|| = |B|$, then $||\Box_a B|| = |\Box_a B|$ (even if $\Box_a B \notin bc(\Omega)$)
(h) If $B \in bc(\Omega)$ and $||B|| = |B|$, then $||\Box_p B|| = |\Box_p B|$ (even if $\Box_p B \notin bc(\Omega)$)

PROOF.

(a) Let $w \in W$, $B \in bc(\Omega)$ and $\Gamma = \{C: \Box_a C \in w\} \cup \{B\}$, and suppose Γ is consistent, which implies that Γ^* is mc. Let $d = d(\Gamma)$.
If there exists $v \in W$ such that $d(v)=d$, then (by lemma II-5-1), for any $D \in bc(\Omega)$, $D \in v$ iff $D \in \Gamma^*$. Thus, $B \in v$ and (by the same lemma II-5-1),[19] for any $C \in bc(\Omega)$ such that $\Box_a C \in w$, we also have that $C \in v$ (since such C belongs to Γ). But then, $v \in av(w)$ and $B \in v$ (as we wish to prove).
Thus, we only need to prove that there exists $v \in W$ such that $d(v)=d$.
If such was not the case, it would mean that[20] $\Diamond d \notin \varphi^*$. But then $\Box\neg d \in \varphi^*$, which implies (by lemma II-5-4-(f)) that $\Box\neg d \in w$, which, in turn (since $|-\Box\neg d \rightarrow \Box_a \neg d$), implies that $\Box_a \neg d \in w$. But then $\neg d \in \Gamma$, which implies that $d \neq d(\Gamma^*)$, contradicting our hypothesis.
(b) The proof of (b) is similar to the proof of (a).
(c) Let $w \in W$ and $X = av(w)$ and suppose, to reach a contradiction, that $\Box_a d(X) \notin w$, which implies that $\neg\Box_a d(X) \in w$.

[19] It is this step that makes it necessary to consider $v \in av(w)$ iff $\forall_{A \in bc(\Omega)}$(if $\Box_a A \in w$ then $A \in v$), and not, simply, $v \in av(w)$ iff $\forall_A$(if $\Box_a A \in w$ then $A \in v$)).

[20] As a matter of fact, if $\Diamond d \in \varphi^*$ then $(\Box^{-1}\varphi^* \cup \{d\})$ is consistent and $(\Box^{-1}\varphi^* \cup \{d\})^*$ is a member of W (recall definition II-5-(2). But then, using v to denote such an element of W, we must have $d(v) = d$, since $d(v)$ denotes the unique element of $v \cap$DESCRIPTORS and $d \in v \cap$DESCRIPTORS.

Then the set $\Gamma = \{C: \Box_a C \in w\} \cup \{\neg d(X)\}$ is consistent (the proof is standard, since $\Box_a$ is a normal modal operator). Thus, since $\neg d(X) \in bc(\Omega)$, by (a), there exists $v \in W$ such that $v \in av(w)$ and $\neg d(X) \in v$.

But $d(X) = \vee_{w \in X} d(w)$ and $v \in av(w) = X$. Thus (by the properties of the mc sets) $\neg d(v) \in v$, and so $d(v) \notin v$, and we get a contradiction, since (by definition of d(v)) $d(v) \in v$.

(d) The proof of (d) is similar to the proof of (c).

(e) Let $w \in W$ and $B \in bc(\Omega)$ and suppose that $\neg \Box_a B \in w$.
Then the set $\Gamma = \{C: \Box_a C \in w\} \cup \{\neg B\}$ is consistent (the proof is standard, since $\Box_a$ is a normal modal operator). Thus, since $\neg B \in bc(\Omega)$, by (a), there exists $v \in W$ such that $v \in av(w)$ and $\neg B \in v$.

(f) The proof of (f) is similar to the proof of (e).

(g) Suppose $B \in bc(\Omega)$ and $||B|| = |B|$, and let w be any world.
We want to prove that $M \mid=_w \Box_a B$ iff $\Box_a B \in w$

- Suppose $\Box_a B \in w$. We want to prove that $M \mid=_w \Box_a B$.
Let v be any world belonging to av(w). We want to prove that $M \mid=_v B$.
By the definition of av(w) (since $B \in bc(\Omega)$), if $v \in av(w)$ and $\Box_a B \in w$, then $B \in v$, which implies (since $||B|| = |B|$) that $M \mid=_v B$ (as we wish to prove).
- Suppose now that $\Box_a B \notin w$, which implies that $\neg \Box_a B \in w$.
Then (since $B \in bc(\Omega)$), by (e) of this lemma, there exists $v \in W$ such that $v \in av(w)$ and $\neg B \in v$.
But then $B \notin v$, and (since $||B|| = |B|$) $M \mid\neq_v B$. Thus $M \mid\neq_w \Box_a B$ (as we wish to prove).

(h) The proof of (h) is similar to the proof of (g).

Result II-5-5

M (defined as in Definition II-5-3) satisfies all the conditions of our models.

Proof. That $W \neq \emptyset$ is a particular consequence of result II-5-2.

- Condition (3-a), i.e. $av(w) \neq \emptyset$ (for any $w \in W$). Proof:
Let $B \in \Omega = Subf(\varphi)$ (there exists one such B, since $Subf(\varphi) \neq \emptyset$).
Either (1) $\Box_a B \in w$ or (2) $\Box_a B \notin w$.
In case (1), by D-axiom, $\Diamond_a B \in w$, and by lemma II-5-9-(e), there exists $v \in W$ such that $v \in av(w)$ and $\neg\neg B \in v$. Thus $av(w) \neq \emptyset$.
In case (2), $\neg \Box_a B \in w$, and by the same lemma II-5-9-(e), there exists $v \in W$ such that $v \in av(w)$ and $\neg B \in v$. Thus $av(w) \neq \emptyset$.
- Condition (4-a), i.e. $av(w) \subseteq pv(w)$ (for any $w \in W$). Proof:
Let $v \in av(w)$ and let B be any Boolean combination of subformulae of φ. Suppose $\Box_p B \in w$. Then, since $\mid\!- \Box_p B \rightarrow \Box_a B$ (axiom $(\Box_p \rightarrow \Box_a)$), we have that $\Box_a B \in w$. But, since $v \in av(w)$, this implies that $B \in v$. Thus $v \in pv(w)$ (as we wish to prove).
- Condition (4-b), i.e. $w \in pv(w)$ (for any $w \in W$). Proof:
Let B be any Boolean combination of subformulae of φ, and suppose $\Box_p B \in w$. Then, since (by the T-axiom) $\mid\!- \Box_p B \rightarrow B$, we have that $B \in w$. Thus $w \in pv(w)$ (as we wish to prove).
- It remains to prove that ob verifies semantic conditions (5-a), (5-b), (5-c*), (5-d) and (5-e).

(5-a) '$\emptyset \notin ob(X)$'. Proof:
Suppose, by *reductio ad absurdum*, that $\emptyset \in ob(X)$, i.e. $O(d(\emptyset)/d(X)) \in \varphi^*$. Note that $d(\emptyset) = \bot$.
But $\mid\!- \neg O(\bot/d(X))$ (see result II-4-1-(a)). Thus $O(\bot/d(X)) \notin \varphi^*$, and a contradiction obtains.

(5-b) 'if $Y \cap X = Z \cap X$, then ($Y \in ob(X)$ iff $Z \in ob(X)$)'. Proof:

Suppose that $Y\cap X=Z\cap X$ and that $Y\in ob(X)$ (the case $Z \in ob(X)$ is similar).
Then, by the definition of ob, $O(d(Y)/d(X)) \in \varphi^*$.
On the other hand (by lemma II-5-6), $|d(Y)\wedge d(X)| = |d(Y\cap X)| = |d(Z\cap X)| = |d(Z)\wedge d(X)|$.
Thus, by lemma II-5-5, $\Box(d(Y)\wedge d(X)\leftrightarrow d(Z)\wedge d(X))\in\varphi^*$, and so $\Box(d(X)\rightarrow(d(Y)\leftrightarrow d(Z)))\in\varphi^*$.
So, by the axiom (O-CONT-REC), $O(d(Z)/d(X))\in\varphi^*$, i.e. $Z\in ob(X)$.

(5-c*) Since W is finite, M satisfies condition 5-c*) iff M satisfies condition 5-c). Thus we proof that M satisfies this simpler condtion.

(5-c) 'if $Y, Z \in ob(X)$ and $Y\cap Z\cap X \neq \emptyset$, then $Y\cap Z \in ob(X)$'. Proof:
Suppose $Y, Z \in ob(X)$ and $Y\cap Z\cap X \neq \emptyset$.
Then, by the definition of ob, $O(d(Y)/d(X))\in\varphi^*$ and $O(d(Z)/d(X))\in\varphi^*$.
On the other hand, by lemma II-5-6, $Y\cap Z\cap X=|d(Y\cap Z\cap X)| = |d(Y)\wedge d(Z)\wedge d(X)|$. Thus $|d(Y)\wedge d(Z)\wedge d(X)| \neq \emptyset$, and so, by lemma II-5-4-(a), $\Diamond(d(Y)\wedge d(Z)\wedge d(X))\in\varphi^*$.
Thus, using the axiom (O-C), we conclude that $O(d(Y)\wedge d(Z)/d(X))\in\varphi^*$.
But, by lemma II-5-6, $|d(Y\cap Z)| = |d(Y)\wedge d(Z)|$.
Thus, by lemma II.5-5-(c), $\Box(d(Y\cap Z)\leftrightarrow d(Y)\wedge d(Z))\in\varphi^*$.
But then, by theorem (REQ) (lemma II-4-1), from $O(d(Y)\wedge d(Z)/d(X))\in\varphi^*$ it follows that $O(d(Y\cap Z)/d(X))\in\varphi^*$. i.e. $Y\cap Z \in ob(X)$.

(5-d) Assuming condition (5-b), by lemma II-2-1, (5-d) is equivalent to the condition
(5-bd) if $Y\in ob(X)$ and $X\subseteq Z$, then $((Z-X) \cup Y) \in ob(Z)$
And, assuming (5-b), since $((Z-X)\cup Y)\cap Z = ((W-X)\cup Y)\cap Z$, condition (5-bd) is equivalent to 'if $Y\in ob(X)$ and $X\subseteq Z$, then $((W-X) \cup Y) \in ob(Z)$'.
Thus, since we have proved that M satisfies condition (5-b), we only need to prove that M satisfies the previous simpler condition.
Suppose, then, that $Y\in ob(X)$ and $X\subseteq Z$.
Then, by the definition of ob, $O(d(Y)/d(X))\in\varphi^*$. And:
- since we have axiom ($O\rightarrow\Diamond$), this implies that $\Diamond(d(Y)\wedge d(X))\in\varphi^*$.
- since we have axiom ($O\rightarrow O\rightarrow$), this also implies that $O(d(X)\rightarrow d(Y)/T)\in\varphi^*$.
On the other hand, by lemma II-5-6-(b), $|d(Z)|=Z$ and $|d(X)|=X$. Thus $|d(X)| \subseteq |d(Z)|$. And by lemma II-5-5-(b), $\Box(d(X)\rightarrow d(Z))\in\varphi^*$. Thus $\Diamond(d(Y)\wedge d(Z))\in\varphi^*$, which in turn implies that $\Diamond(d(Z)\wedge(d(X)\rightarrow d(Y)))\in\varphi^*$.
By lemma II-5-5-(b) (since $|T|=W$), we also have $\Box(d(Z)\rightarrow T)\in\varphi^*$.
Thus we have $O(d(X)\rightarrow d(Y)/T)\in\varphi^*$, $\Diamond(d(Z)\wedge(d(X)\rightarrow d(Y)))\in\varphi^*$ and $\Box(d(Z)\rightarrow T)\in\varphi^*$.
Thus, by axiom (O-SA), we conclude that $O((d(X)\rightarrow d(Y))/d(Z))\in\varphi^*$.
But (by lemma II-5-6-(b)-(iii)) $|d(X)\rightarrow d(Y)| = |d(|d(X)\rightarrow d(Y)|)|$.
Thus, by lemma II-5-5-(c), $\Box((d(X)\rightarrow d(Y))\leftrightarrow d(|d(X)\rightarrow d(Y)|))\in\varphi^*$, and so, by theorem (REQ) (lemma II-4-1), also $O(d(|d(X)\rightarrow d(Y)|)/d(Z))\in\varphi^*$, i.e. $|d(X)\rightarrow d(Y)| \in ob(Z)$.
But $|d(X)\rightarrow d(Y)|$ = (by lemma II-5-4-(e)) $|\neg d(X) \vee d(Y)|$ = (by lemma II-5-6-(a)) $|\neg d(X)|\cup|d(Y)|$ = (by lemma II-5-6-(a)) $(W-|d(X)|)\cup|d(Y)|$ = (by lemma II-5-6-(b)) $(W-X) \cup Y$.
Thus $((W-X) \cup Y) \in ob(Z)$.

(5-e) 'if $Y\subseteq X$ and $Z\in ob(X)$ and $Y\cap Z \neq\emptyset$, then $Z\in ob(Y)$'. Proof:
Suppose $Y\subseteq X$ and $Z\in ob(X)$ and $Y\cap Z \neq \emptyset$. Then, by the definition of ob, $O(d(Z)/d(X))\in\varphi^*$.
By lemma II-5-6, $|d(Y)|=Y$ and $|d(X)|=X$. Thus $|d(Y)| \subseteq |d(X)|$. And, by lemma II-5-5-(b), this implies that $\Box(d(Y)\rightarrow d(X))\in\varphi^*$.
And (similarly to what we have done before), by lemma II-5-6, $Y\cap Z = |d(Y\cap Z)| = |d(Y)\wedge d(Z)|$. Thus $|d(Y)\wedge d(Z)| \neq \emptyset$ and so, by lemma II-5-4-(a), $\Diamond(d(Y)\wedge d(Z))\in\varphi^*$.

Thus, by axiom (O-SA), O(d(Z)/d(Y))∈φ*, i.e. Z∈ob(Y).

RESULT II-5-6

$\forall_{A\in\Omega}\forall_{w\in W}$ (M |=$_w$A iff A ∈ w)

PROOF.

Let *(A) denote '$\forall_{w\in W}$ (M |=$_w$A iff A ∈ w)', i.e. ||A|| = |A|

o(A) denote 'if A∈ Ω(=Subf(φ)) then *(A)'

We want to prove that $\forall_A$ o(A). We will prove this by induction on the structure of A.

Base:

(i) A is an atomic sentence, i.e. A is a propositional symbol q. Thesis: o(A). Proof: In this case we have *(A) even if A is not a subformula of φ. The proof is standard.

Induction step:

(ii) A=¬B and o(B). Thesis: o(A). Proof: standard.

(iii) A = B∧C (or A = B∨C, or A = B→C, or A= B↔C) and o(B) and o(C). Thesis: o(A). Proof: standard.

(iv) A=□B and o(B). Thesis: o(A). Proof:
Suppose □B∈Subf(φ), which implies that B∈Subf(φ), and let w be any world. Then:
– If □B∈w then (by lemma II-5-4-(f)) □B∈φ* and thus (by the definition of W) B belongs to any world in W, i.e. |B| = W. Thus, by o(B) (since B∈Subf(φ)), ||B||=W, i.e. (∀v∈W)M|=$_v$B, and so M|=$_w$ □B.
– If □B∉w then (by lemma II-5-4-(f)) □B∉φ*. But then ◇¬B∈φ*, and (by lemma II-5-5-(a)) |¬B| ≠ ∅, i.e. there exists v such that ¬B∈v. Thus B∉v and, by o(B) (since B∈Subf(φ)), M| ≠$_v$B, and so M | ≠$_w$ □B.

(v) A= O(B/C) and o(B) and o(C). Thesis: o(A). Proof:
Suppose O(B/C)∈Subf(φ), which implies that B,C∈Subf(φ), and let w be any world.
(v-a) • Suppose M |=$_w$O(B/C). Then we can prove that O(B/C)∈w as follows:
If M |=$_w$O(B/C), then (see observation II-2-1-(7)) ||B|| ∈ob(||C||). But, by o(B) and o(C) (since B,C∈Subf(φ)), ||B|| = |B| and ||C|| = |C|. Thus |B| ∈ob(|C|). But this means (by the definition of ob) that O(d(|B|)/d(|C|))∈φ*.
And, by lemma II-5-7 and lemma II-5-8 (applicable since B and C are Boolean combinations of subformulae of φ, once they are subformulae of φ), we can conclude that O(B/C)∈w.
(v-b) • Suppose now that O(B/C)∈w, which implies, by lemma II-5-7, that O(B/C)∈φ*. We want to prove that M |=$_w$O(B/C).
Suppose, by *reductio ad absurdum*, that M | ≠$_w$O(B/C). Then (by result II-2-2[21]) ||B|| ∉ob(||C||).
Then, by o(B) and o(C) (since B,C∈Subf(φ)), |B| ∉ob(|C|), i.e. by definition of ob: O(d(|B|)/d(|C|))∉φ*
But, since B and C are Boolean combinations of subformulae of φ, by lemma II-5-8, from O(B/C)∈φ*, we can conclude that O(d(|B|)/d(|C|))∈φ*, and a contradiction obtains.

(vi) A=□$_a$B and o(B). Thesis: o(A). Proof:
Suppose □$_a$B∈Subf(φ). Then B∈Subf(φ) (and so also B∈bc(Ω)).

[21]We have already proved that M satisfies all conditions of our models. Thus we can use result II-2-2. We note that we can prove (v-b), above, without relying on result II-2-2, although the proof becomes a little longer and more complicated.

Since B∈Subf(φ) and o(B) is the case, we have that $||B|| = |B|$.
Thus, from lemma II-5-9-(g), it follows that $||\Box_a B|| = |\Box_a B|$ (as we wish to prove).

(vii) A = $\Box_p$B and o(B). Thesis: o(A).
Proof: similar to case (vi).

(viii) A = O_aB and o(B). Thesis: o(A). Proof:
The proof of (viii) was included in the article.

(ix) A = O_pB and o(B). Thesis: o(A).
Proof: similar to case (viii).