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ROLES OF LOG-CONCAVITY, LOG-CONVEXITY, AND GROWTH ORDER IN WHITE NOISE ANALYSIS

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In this paper we will develop a systematic method to answer the questions (Q1) (Q2) (Q3) (Q4) (stated in Sec. 1) with complete generality. As a result, we can solve the difficulties (D1) (D2) (discussed in Sec. 1) without uncertainty. For these purposes we will introduce certain classes of growth functions u and apply the Legendre transform to obtain a sequence which leads to the weight sequence $\{\alpha(n)\}$ first studied by Cochran *et al.*[6] The notion of (nearly) equivalent functions, (nearly) equivalent sequences and dual Legendre functions will be defined in a very natural way. An application to the growth order of holomorphic functions on $\mathcal{E}_c$ will also be discussed.

1. Introduction

Let $\mathcal{E}$ be a real nuclear space with topology given by a sequence of inner product norms $\{|\cdot|_p\}_{p=0}^{\infty}$. Let $\mathcal{E}_p$ be the completion of $\mathcal{E}$ with respect to the norms $|\cdot|_p$. We will assume the following conditions:

(a) There exists a constant $0 < \rho < 1$ such that $|\cdot|_0 \le \rho|\cdot|_1 \le \cdots \le \rho^p|\cdot|_p \le \cdots$.
(b) For any $p \ge 0$, there exists some $q \ge p$ such that the inclusion mapping $i_{q,p} : \mathcal{E}_q \to \mathcal{E}_p$ is a Hilbert–Schmidt operator.

We denote the complexification of $\mathcal{E}$ by $\mathcal{E}_c$.

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The generalized and test functions in $[\mathcal{E}]_\alpha^*$ and $[\mathcal{E}]_\alpha$ (CKS-space for short) are characterized in terms of their S-transforms F under a very general setting by Refs. 6 and 2, respectively. There are two conditions on F. The first one is the analyticity. The second one is the growth condition. The exponential generating functions

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n, \qquad G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} r^n \tag{1.1}$$

are adopted as the growth functions for generalized functions[6] and test functions,[2] respectively. Recently, Asai *et al.* have shown in Ref. 4 that the conditions (A1) (A2) (B2) ($\tilde{\mathrm{B}}$2) (C2) ($\tilde{\mathrm{C}}$2) (see the Appendix) are minimal assumptions on $\alpha(n)$ not only for the construction of CKS-space, but also for white noise operator theory on it (see also Refs. 3, 12 and 19).

On the other hand, the recent paper[7] by Gannoun *et al.* is closely connected with Ref. 6 by Cochran *et al.* and the series of papers.[1–5] They study the spaces of holomorphic functions $\mathcal{G}_{\theta^*}$ on $\mathcal{E}_c$ and $\mathcal{F}_\theta$ on the dual space $\mathcal{E}_c'$ considering functions $\exp(\theta(r)^*)$ and $\exp(\theta(r))$ as growth functions where θ^* is the dual function of Young function θ. They remark briefly the relationship between $\mathcal{G}_{\theta^*}$ and $[\mathcal{E}]_\alpha^*$ by taking $\theta(r)^* = \log G_\alpha(r^2)$ and $\alpha(n) := n!\theta_n = \inf_{r>0} \frac{\exp(\theta(r)^*)}{r^n}$.

However, Refs. 6 and 7 do not provide a general method to solve the following delicate problems:

(D1) Technical difficulties mentioned after (D2) cannot be canceled out by systematic way. Moreover, Young function $\theta(r)$ cannot be obtained explicitly from $\theta(r)^* = \log G_\alpha(r^2)$ even for the important cases, $\alpha(n) = (n!)^\beta$ and Bell's numbers $b_k(n)$ given in Ref. 6.

(D2) It is not checked whether $n!\theta_n$ satisfies (A2) (B2) ($\tilde{\mathrm{B}}$2) (C2) ($\tilde{\mathrm{C}}$2). Since precise estimates are required in general, this is not obvious problem at all. Refer to Refs. 1 and 12 for the case of Bell's numbers.

Now we shall justify our claim (D1) as follows. In Ref. 6 the following growth condition is used for generalized functions:

- There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K G_\alpha(a|\xi|_p^2)^{1/2}, \qquad \xi \in \mathcal{E}_c\,. \tag{1.2}$$

On the other hand, in Ref. 2 the following growth condition is used for test functions:

- For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K G_{1/\alpha}(a|\xi|_{-p}^2)^{1/2}, \qquad \xi \in \mathcal{E}_c\,. \tag{1.3}$$

In the case of Kondratiev–Streit space, $G_\alpha(r)$ and $G_{1/\alpha}(r)$ are given by

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1-\beta}} r^n, \qquad G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1+\beta}} r^n\,. \tag{1.4}$$

These series cannot be summed up in closed forms unless $\beta = 0$ (the case of Hida–Kubo–Takenaka space). Fortunately, we have the estimates:

$$\exp[(1-\beta)r^{\frac{1}{1-\beta}}] \le G_\alpha(r) \le 2^\beta \exp[(1-\beta)2^{\frac{\beta}{1-\beta}} r^{\frac{1}{1-\beta}}]\,. \tag{1.5}$$

$$2^{-\beta}\exp[(1+\beta)2^{-\frac{\beta}{1+\beta}} r^{\frac{1}{1+\beta}}] \le G_{1/\alpha}(r) \le \exp[(1+\beta)r^{\frac{1}{1+\beta}}]\,. \tag{1.6}$$

Therefore we can substitute the growth functions G_α and $G_{1/\alpha}$ in Eqs. (1.2) and (1.3) by the following functions $\tilde{G}_\alpha$ and $\tilde{G}_{1/\alpha}$, respectively,

$$\tilde{G}_\alpha(r) = \exp[(1-\beta)r^{\frac{1}{1-\beta}}], \qquad \tilde{G}_{1/\alpha}(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}]\,. \tag{1.7}$$

These are the growth functions used in Refs. 9 and 10 (see Refs. 20 and 16 for $\beta = 0$, respectively).

Now, for the case of CKS-space, the growth functions G_α and $G_{1/\alpha}$ in Eqs. (1.2) and (1.3) are not practical to use since in general we cannot find closed forms for the sums of the infinite series. For example, for the case of $\alpha(n) = b_k(n)$, even though $G_\alpha(r) = \exp_k(r)$, the kth iterated exponential function, we simply do not have a closed form for the corresponding $G_{1/\alpha}$.

The main purpose of this paper is to consider the following questions:

(Q1) In Eqs. (1.2) and (1.3) can we replace the growth functions G_α and $G_{1/\alpha}$ by elementary functions U and u, respectively?

(Q2) How to find U and u from G_α and $G_{1/\alpha}$? In particular, how to find $\tilde{G}_\alpha(r)$ and $\tilde{G}_{1/\alpha}(r)$ in Eq. (1.7) from G_α and $G_{1/\alpha}$ in Eq. (1.4) and without appealing to Eqs. (1.5) and (1.6)? This is related to (D1).

(Q3) Are there any other general criteria to check (A2) (B2) ($\tilde{\text{B}}$2) (C2) ($\tilde{\text{C}}$2) for a given growth function U or u without technical estimates? This is connected with (D2).

(Q4) For (Q1)–(Q3) what kinds of conditions do we have to impose on functions U and u?

The Legendre transform is the key tool to solve the above four questions. The answers to (Q3) can be found in Theorems 3.4, 3.6 and 3.10. For (Q1), (Q2), Theorems 3.13, 4.6 and 4.8 will play fundamental roles. About (Q4), refer to Definition 2.1, Eqs. (3.3) and (4.9) for quick reference.

This paper is organized as follows. In Sec. 2, several kinds of log-convex functions will be prepared for the Legendre transform. In Sec. 3, we examine properties of the Legendre transform for various log-convex functions. In addition, we will introduce the notion of equivalent functions and sequences motivated from Theorem 3.13. In Sec. 4, we discuss the dual Legendre functions, nearly equivalent functions and sequences. In Sec. 5, an application to the growth order of holomorphic functions on $\mathcal{E}_c$ will be discussed under quite general assumptions. In paticular, Theorem 3.13 will be useful to prove the topological isomorphism between spaces $\mathcal{G}_u$ and $\mathcal{K}_u$. Further related works with the present paper can be found in Refs. 3 and 5.

2. Log-, (log, exp)-, and (log, x^k)-Convex Functions

In this section, we shall consider three kinds of convexity for later use. Before giving their definitions, let us start with the following, which stems from the proof of Theorem 4.3 in Ref. 6 and is connected with (Q1), (Q2). It explains our viewpoint.

Let $u(x) = \sum_{n=0}^{\infty} u_n x^n$ be an entire function with $u_n > 0$ and the sequence $\{u_n\}$ being log-concave (see (C2) (C3) in the Appendix), i.e.

$$u_n u_{n+2} \le u_{n+1}^2, \qquad \forall\, n \ge 0\,. \tag{2.1}$$

It is shown in the proof of Theorem 4.3 in Ref. 6 that

$$1 \le \frac{1}{u_n} \inf_{r>0} \frac{u(r)}{r^n} \le (n+1)e\,.$$

But $n+1 \le 2^n$. Hence

$$u_n \le \inf_{r>0} \frac{u(r)}{r^n} \le e\, 2^n u_n\,. \tag{2.2}$$

These inequalities imply that

$$u(x) \le \sum_{n=0}^{\infty} \left(\inf_{r>0} \frac{u(r)}{r^n} \right) x^n \le e\, u(2x)\,. \tag{2.3}$$

Thus if we can regard the last series as the function $G_{1/\alpha}$, then the function u can be used as the growth function for test functions. But to get a satisfactory answer, we need to describe u by function properties instead of expressing it as an infinite series. This leads to a general question:

(Q) What functions U and u can serve as growth functions in Eqs. (1.2) and (1.3) for generalized and test functions, respectively?

Now we are in a position to define the three kinds of log-convexity of functions as follows.

Definition 2.1. Let u be a positive continuous function on $[0, \infty)$.

(a) The function u is called *log-convex* if $\log u$ is convex on $[0, \infty)$.
(b) The function u is called *(log, exp)-convex* if $\log u(e^x)$ is convex on $\mathbb{R}$.
(c) The function u is called *(log, x^k)-convex* if $\log u(x^k)$ is convex on $[0, \infty)$. Here k is a positive real number.

The concept of (log, exp)-convexity will play an important role in this paper. The (log, exp)-convex functions are the appropriate functions to replace those given by infinite series $u(r) = \sum_{n=0}^{\infty} u_n r^n$ as mentioned above. In fact, it has been shown in Ref. 11 that if $u(r) = \sum_{n=0}^{\infty} u_n r^n$ is an entire function with $u_n \ge 0$ and $u(r) > 0$ for all $r \ge 0$, then the function u is a (log, exp)-convex function. This fact also follows from Eq. (2.4) below and the equality

$$u(r)u''(r) - u'(r)^2 + \frac{1}{r} u(r)u'(r) = u_0 u_1 \frac{1}{r} + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[(n+2)/2]} (n+2-2j)^2 u_j u_{n+2-j} \right) r^n\,.$$

However, a (log, exp)-convex function u may not be given by an entire function with positive coefficients in the series expansion. For instance, $u(r) = \exp[r^2 - r^3 + r^4]$ is such a function.

Example 2.2. Suppose u is a positive C^2-function on $[0,\infty)$. It is easy to check by direct calculations the following assertions:

(1) u is log-convex if and only if

$$u(r)u''(r) - u'(r)^2 \geq 0, \qquad \forall\, r > 0\,.$$

(2) u is (log, exp)-convex if and only if

$$u(r)u''(r) - u'(r)^2 + \frac{1}{r}u(r)u'(r) \geq 0, \qquad \forall\, r > 0\,. \tag{2.4}$$

(3) u is (log, x^k)-convex if and only if

$$u(r)u''(r) - u'(r)^2 + \frac{k-1}{kr}u(r)u'(r) \geq 0, \qquad \forall\, r > 0\,.$$

Observe from this example that if u is log-convex then it is convex. When u is an increasing function, we have the implications: (i) log-convex $\Longrightarrow$ (log, x^k)-convex for any $k \geq 1$, (ii) (log, x^k)-convex for some $k > 0 \Longrightarrow$ (log, exp)-convex. In fact, these implications are true in general.

Proposition 2.3. *Let u be a positive continuous function on $[0,\infty)$.*

(a) *If u is log-convex, then it is convex.*

When u is also increasing, we have the assertions:

(b) *If u is log-convex, then it is (log, x^k)-convex for any $k \geq 1$.*
(c) *If u is (log, x^k)-convex for some $k > 0$, then it is (log, exp)-convex.*

Proof. To prove assertion (a), let $r, s \geq 0$ and $\lambda \in [0,1]$. Since by assumption u is log-convex, we have

$$\log u(\lambda r + (1-\lambda)s) \leq \lambda \log u(r) + (1-\lambda)\log u(s)\,. \tag{2.5}$$

But e^x is a convex function. Hence

$$e^{\lambda \log u(r) + (1-\lambda)\log u(s)} \leq \lambda e^{\log u(r)} + (1-\lambda)e^{\log u(s)} = \lambda u(r) + (1-\lambda)u(s)\,. \tag{2.6}$$

Take the exponential in Eq. (2.5) and then use Eq. (2.6) to get

$$u(\lambda r + (1-\lambda)s) \leq \lambda u(r) + (1-\lambda)u(s)\,.$$

Hence u is convex and we have proved assertion (a). For assertion (b) we use the fact that x^k is convex for any $k \geq 1$ and the assumption that u is increasing to get

$$u((\lambda r + (1-\lambda)s)^k) \leq u(\lambda r^k + (1-\lambda)s^k)\,. \tag{2.7}$$

Suppose u is log-convex. Then

$$\log u(\lambda r^k + (1-\lambda)s^k) \le \lambda \log u(r^k) + (1-\lambda)\log u(s^k)\,. \tag{2.8}$$

Upon taking logarithm in Eq. (2.7) and then use Eq. (2.8) we obtain

$$\log u((\lambda r + (1-\lambda)s)^k) \le \lambda \log u(r^k) + (1-\lambda)\log u(s^k)\,.$$

This shows that $\log u(x^k)$ is convex, i.e. u is (log, x^k)-convex and so assertion (b) is proved. For the third assertion, note that e^x is convex and the function $u(x^k)$ is increasing. Hence we have

$$u((e^{\lambda r/k + (1-\lambda)s/k})^k) \le u((\lambda e^{r/k} + (1-\lambda)e^{s/k})^k)\,. \tag{2.9}$$

But by assumption u is (log, x^k)-convex. Hence

$$\log u((\lambda e^{r/k} + (1-\lambda)e^{s/k})^k) \le \lambda \log u((e^{r/k})^k) + (1-\lambda)\log u((e^{s/k})^k)\,. \tag{2.10}$$

Upon taking logarithm in Eq. (2.9) and then use Eq. (2.10) we obtain

$$\log u(e^{\lambda r + (1-\lambda)s}) \le \lambda \log u(e^r) + (1-\lambda)\log u(e^s)\,.$$

Hence the function $\log u(e^x)$ is convex, i.e. u is (log, exp)-convex and so assertion (c) is proved. □

Lemma 2.4. *Let f be a convex function on $\mathbb{R}$ such that $\lim_{x\to-\infty} f(x)$ exists. Then the function f is an increasing function.*

Proof. Let $x < x_1 < x_2$ with x_1 and x_2 being fixed. Then

$$x_1 = \frac{x_2 - x_1}{x_2 - x}x + \frac{x_1 - x}{x_2 - x}x_2\,.$$

Since the function f is convex,

$$f(x_1) \le \frac{x_2 - x_1}{x_2 - x}f(x) + \frac{x_1 - x}{x_2 - x}f(x_2)\,.$$

Letting $x \to -\infty$, we get $f(x_1) \le f(x_2)$ and so the function f is increasing. □

Lemma 2.5. *If a positive continuous function u on $[0,\infty)$ is (log, exp)-convex, then it is an increasing function.*

Proof. Let $f(x) = \log u(e^x)$. By Lemma 2.4, f is increasing. It follows that the function u is also increasing. □

We want to point out that $u(r)$ being defined at $r = 0$ is crucial for Lemma 2.5. For example, let $u(r) = \exp[(\log r)^2 - 2\log r]$. Obviously, the function $\log u(e^x)$ is convex on $\mathbb{R}$. But u is not an increasing function on $(0,\infty)$.

3. Legendre Transform

For the characterization theorems in the paper by Cochran *et al.*[6] the following condition is imposed

$$\limsup_{n\to\infty}\left(\frac{n!}{\alpha(n)}\inf_{r>0}\frac{u(r)}{r^n}\right)^{1/n}<\infty\,.$$

This condition leads to the consideration of the sequence

$$\inf_{r>0}\frac{u(r)}{r^n},\qquad n=0,1,2,\ldots. \tag{3.1}$$

Moreover, as we pointed out in Sec. 2 that if $u(x)=\sum_{n=0}^{\infty}u_nx^n$ is an entire function with $u_n>0$ and $\{u_n\}$ being log-concave, then Eq. (2.3) holds, i.e. the functions u is "equivalent" to the following function

$$\sum_{n=0}^{\infty}\left(\inf_{r>0}\frac{u(r)}{r^n}\right)x^n\,. \tag{3.2}$$

The above discussion raises three questions: (i) What function u can we define the sequence in Eq. (3.1)? (ii) What is the new function in Eq. (3.2)? (iii) Is this new function "equivalent" to the function u? In this section we will give answers to these questions.

Notation. Let $C_{+,\log}$ denote the set of all positive continuous functions u on $[0,\infty)$ satisfying the condition

$$\lim_{r\to\infty}\frac{\log u(r)}{\log r}=\infty\,. \tag{3.3}$$

Observe that the condition in Eq. (3.3) means that u grows faster than all polynomials. The set $C_{+,\log}$ includes all entire functions $u(r)=\sum_{n=0}^{\infty}u_nr^n$ with $u_n\geq 0$ for all n and $u_n>0$ for infinitely many n's. If u is a function in the set $C_{+,\log}$, then we can define the sequence in Eq. (3.1). In fact, we will define Eq. (3.1) as a function on $[0,\infty)$.

Definition 3.1. The *Legendre transform* ℓ_u of a function $u\in C_{+,\log}$ is defined to be the function

$$\ell_u(t)=\inf_{r>0}\frac{u(r)}{r^t},\qquad t\in[0,\infty)\,. \tag{3.4}$$

The next lemma is immediate from the definition of Legendre transform.

Lemma 3.2. (a) *For $u\in C_{+,\log}$ and $a>0$, let θ_au be the function $\theta_au(r)=u(ar)$. Then $\theta_au\in C_{+,\log}$ and $\ell_{\theta_au}(t)=a^t\ell_u(t)$ for all $t\geq 0$.*
(b) *Suppose $u,v\in C_{+,\log}$ and $u(r)\leq v(r)$ for all $r\geq 0$. Then $\ell_u(t)\leq\ell_v(t)$ for all $t\geq 0$.*

Definition 3.3. A positive function f on $[0, \infty)$ is called *log-concave* if $\log f$ is a concave function, or equivalently, for any $t_1, t_2 \geq 0$ and $0 \leq \lambda \leq 1$, we have

$$f(\lambda t_1 + (1-\lambda)t_2) \geq f(t_1)^\lambda \, f(t_2)^{1-\lambda} \,. \tag{3.5}$$

Put $t_1 = n, t_2 = n+2$ and $\lambda = 1/2$ in Eq. (3.5) to get

$$f(n)f(n+2) \leq f(n+1)^2, \qquad \forall\, n \geq 0 \,.$$

This shows that if a positive function f on $[0, \infty)$ is log-concave, then the sequence $\{f(n)\}$ is log-concave [see Eq. (2.1)].

Theorem 3.4. *The Legendre transform ℓ_u of a function $u \in C_{+,\log}$ is log-concave. (Hence the function $\ell_u(t)$ is continuous and the sequence $\{\ell_u(n)\}_{n=0}^\infty$ is log-concave.)*

Proof. For any $t_1, t_2 \geq 0$ and $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \ell_u(\lambda t_1 + (1-\lambda)t_2) &= \inf_{r>0} \frac{u(r)}{r^{\lambda t_1 + (1-\lambda)t_2}} = \inf_{r>0} \frac{u(r)^\lambda}{r^{\lambda t_1}} \frac{u(r)^{1-\lambda}}{r^{(1-\lambda)t_2}} \\ &\geq \left(\inf_{r>0} \frac{u(r)}{r^{t_1}} \right)^\lambda \left(\inf_{r>0} \frac{u(r)}{r^{t_2}} \right)^{1-\lambda} = \ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} \,. \end{aligned}$$

Hence by Eq. (3.5) the function ℓ_u is log-concave. □

Now, we consider those functions in $C_{+,\log}$ which are (log, exp)-convex. Let u be such a function. Then the left-hand derivative $u'_-(r)$ and the right-hand derivative $u'_+(r)$ exist. For convenience, define

$$\tau_-(r) = \frac{ru'_-(r)}{u(r)}, \qquad \tau_+(r) = \frac{ru'_+(r)}{u(r)} \,.$$

Both τ_- and τ_+ are increasing functions. Since u is increasing by Lemma 2.5, we have $0 \leq \tau_-(r) \leq \tau_+(r)$ for all $r \geq 0$. Moreover, the condition in Eq. (3.3) implies that $\tau_-(r) \to \infty$ as $r \to \infty$. Note that for any $r \geq 0$ and $t \in [\tau_-(r), \tau_+(r)]$,

$$\frac{u(s)}{s^t} \geq \frac{u(r)}{r^t}, \qquad \forall\, s > 0 \,.$$

Hence $\inf_{s>0} u(s)/s^t = u(r)/r^t$ and so

$$\ell_u(t) = \frac{u(r)}{r^t}, \qquad \text{for any } t \in [\tau_-(r), \tau_+(r)] \,.$$

In particular, let $\rho(t)$ be a solution of the equation $\tau_-(r) = t$, i.e. $\tau_-(\rho(t)) = t$. Then we have

$$\ell_u(t) = \frac{u(\rho(t))}{\rho(t)^t} \,.$$

We sum up the above discussion in the next lemma.

Lemma 3.5. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then*

$$\ell_u(t) = \frac{u(r)}{r^t}, \qquad \text{for any } t \in [\tau_-(r), \tau_+(r)], \tag{3.6}$$

where $\tau_-(r) = ru'_-(r)/u(r)$ and $\tau_+(r) = ru'_+(r)/u(r)$. In particular, let $\rho(t)$ be a solution of the equation $\tau_-(r) = t$, i.e. $\tau_-(\rho(t)) = t$. Then

$$\ell_u(t) = \frac{u(\rho(t))}{\rho(t)^t}. \tag{3.7}$$

Theorem 3.6. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then its Legendre transform $\ell_u(t)$ is decreasing for large t and $\lim_{t\to\infty} \ell_u(t)^{1/t} = 0$.*

Proof. Let $s \geq t$ be fixed. Use Lemma 3.5 to get

$$\ell_u(t) = \frac{u(\rho(t))}{\rho(t)^t} = \rho(t)^{s-t}\frac{u(\rho(t))}{\rho(t)^s}.$$

Then by the definition of the Legendre transform,

$$\ell_u(t) \geq \rho(t)^{s-t}\ell_u(s). \tag{3.8}$$

Recall that $\tau_-(r)$ increases to ∞ monotonically. Hence $\rho(t)$ also increases to ∞ monotonically. Choose t_0 such that $\rho(t) > 1$ for all $t \geq t_0$. Then it follows from Eq. (3.8) that

$$\ell_u(t) \geq \ell_u(s), \qquad \forall\, s \geq t \geq t_0.$$

Hence $\ell_u(t)$ is decreasing for large t. Moreover, from Eq. (3.8) we have

$$\ell_u(s) \leq \rho(t)^{t-s}\ell_u(t), \qquad \forall\, s \geq t \geq t_0.$$

Therefore,

$$\ell_u(s)^{1/s} \leq \rho(t)^{t/s-1}\ell_u(t)^{1/s}, \qquad \forall\, s \geq t \geq t_0.$$

Hold t fixed and let $s \to \infty$ to get

$$\limsup_{s\to\infty} \ell_u(s)^{1/s} \leq \rho(t)^{-1}, \qquad \forall\, t \geq t_0.$$

But $\rho(t) \to \infty$ as $t \to \infty$. Hence we can conclude that $\lim_{s\to\infty} \ell_u(s)^{1/s} = 0$. □

Lemma 3.7. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then*

$$u(r) = \sup_{t\geq 0} \ell_u(t) r^t, \qquad \forall\, r \geq 0.$$

Proof. From the definition of the Legendre transform in Eq. (3.4) we have

$$\ell_u(t) r^t \leq u(r), \qquad \forall\, r \geq 0. \tag{3.9}$$

On the other hand, for any fixed $r \geq 0$, we can choose $t \in [\tau_-(r), \tau_+(r)]$ in Lemma 3.5 to get

$$\ell_u(t) r^t = u(r). \tag{3.10}$$

Equations (3.9) and (3.10) imply that $u(r) = \sup_{t\geq 0} \ell_u(t) r^t$. □

The next lemma follows immediately from Lemmas 3.2(b) and 3.7.

Lemma 3.8. *Let $u, v \in C_{+,\log}$ be (log, exp)-convex. Then*

(a) *$u = v$ if and only if $\ell_u = \ell_v$.*
(b) *$\ell_u(t) \le \ell_v(t)$ for all $t \ge 0$ if and only if $u(r) \le v(r)$ for all $r \ge 0$.*

Now, we consider functions u in $C_{+,\log}$ which are (log, x^k)-convex.

Lemma 3.9. *Let $u \in C_{+,\log}$ and $k > 0$. Then u is (log, x^k)-convex if and only if $\ell_u(t)t^{kt}$ is log-convex.*

Proof. Let $t_1, t_2 \ge 0$ and $0 \le \lambda \le 1$. Then

$$\ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} = \inf_{r,s>0} \left(\frac{u(r)}{r^{t_1}}\right)^\lambda \left(\frac{u(s)}{s^{t_2}}\right)^{1-\lambda} = \inf_{x,y>0} \frac{u(x^k)^\lambda\, u(y^k)^{1-\lambda}}{x^{\lambda k t_1} y^{(1-\lambda)k t_2}}\,.$$

Suppose u is (log, x^k)-convex. Then

$$u((\lambda x + (1-\lambda)y)^k) \le u(x^k)^\lambda u(y^k)^{1-\lambda}\,.$$

Therefore,

$$\ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} \ge \inf_{x,y>0} \frac{u((\lambda x + (1-\lambda)y)^k)}{x^{\lambda k t_1} y^{(1-\lambda)k t_2}}\,.$$

Make a change of variables $z = \lambda x + (1-\lambda)y$ to get

$$\ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} \ge (1-\lambda)^{(1-\lambda)kt_2} \inf_{z>0} u(z^k) \inf_{0<\lambda x<z} \frac{1}{x^{\lambda k t_1}(z-\lambda x)^{(1-\lambda)kt_2}}\,. \qquad (3.11)$$

It is straightforward to check that for fixed $z > 0$,

$$\sup_{0<\lambda x<z} x^{\lambda r_1}(z-\lambda x)^{(1-\lambda)r_2} = \frac{r_1^{\lambda r_1}((1-\lambda)r_2)^{(1-\lambda)r_2} z^{\lambda r_1 + (1-\lambda)r_2}}{(\lambda r_1 + (1-\lambda)r_2)^{\lambda r_1 + (1-\lambda)r_2}}\,. \qquad (3.12)$$

Apply Eq. (3.12) with $r_1 = kt_1, r_2 = kt_2$ to Eq. (3.11) to obtain

$$\ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} \ge \left(\frac{(\lambda t_1 + (1-\lambda)t_2)^{\lambda t_1 + (1-\lambda)t_2}}{t_1^{\lambda t_1} t_2^{(1-\lambda)t_2}}\right)^k \inf_{z>0} \frac{u(z^k)}{z^{\lambda k t_1 + (1-\lambda)kt_2}}\,.$$

The last infimum is nothing but $\ell_u(\lambda t_1 + (1-\lambda)t_2)$. Hence we have proved that

$$\ell_u(t_1)^\lambda \ell_u(t_2)^{1-\lambda} \ge \left(\frac{(\lambda t_1 + (1-\lambda)t_2)^{\lambda t_1 + (1-\lambda)t_2}}{t_1^{\lambda t_1} t_2^{(1-\lambda)t_2}}\right)^k \ell_u(\lambda t_1 + (1-\lambda)t_2)\,.$$

This inequality shows that $\ell_u(t)t^{kt}$ is log-convex.

Conversely, suppose $\ell_u(t)t^{kt}$ is log-convex. We can carry out similar calculations as above backward to show that u is (log, x^k)-convex. □

Theorem 3.10. *Let $u \in C_{+,\log}$ be (log, x^k)-convex, $k > 0$. Then*

$$\ell_u(n)\ell_u(m) \le \ell_u(0)2^{k(n+m)}\ell_u(n+m), \qquad \forall\ n, m \ge 0\,. \qquad (3.13)$$

Remark. Let $u \in C_{+,\log}$. By Theorem 3.4 the sequence $\{\ell_u(n)\}_{n=0}^{\infty}$ is log-concave. Then we can apply Theorem 2(b) in Ref. 1 with $\alpha(n) = n!\ell_u(n)/\ell_u(0)$ to get

$$\ell_u(0)\ell_u(n+m) \le \ell_u(n)\ell_u(m), \qquad \forall\, n, m \ge 0\,. \tag{3.14}$$

Proof. By Lemma 3.9 the sequence $\{\ell_u(n)n^{kn}\}_{n=0}^{\infty}$ is log-convex (here $0^0 = 1$ by convention). Then apply Theorem 2(a) in Ref. 1 with $\alpha(n) = \ell_u(n)n^{kn}/\ell_u(0)$ to get

$$\ell_u(n)\ell_u(m) \le \ell_u(0)\ell_u(n+m)\left(\frac{(n+m)^{n+m}}{n^n m^m}\right)^k. \tag{3.15}$$

Let $A = (n+m)^{n+m}/(n^n m^m)$ and $x = n/(n+m)$. Then it is easily checked that

$$\frac{1}{n+m}\log A = -x\log x - (1-x)\log(1-x)\,. \tag{3.16}$$

But the maximum of the function $-x\log x - (1-x)\log(1-x)$ for $x \in (0,1)$ obviously occurs at $x = 1/2$ with a value of $\log 2$. Hence

$$\frac{(n+m)^{n+m}}{n^n m^m} = A \le 2^{n+m}\,. \tag{3.17}$$

Thus Eqs. (3.15) and (3.17) yield Eq. (3.13). □

3.1. *Inverse Legendre transform*

In view of Lemma 3.7, we can define the inverse Legendre transform as follows. Let f be a positive continuous function on $[0,\infty)$ such that $\lim_{t\to\infty} f(t)^{1/t} = 0$ or equivalently $\lim_{t\to\infty} t^{-1}\log f(t) = -\infty$. Then we define

$$\theta_f(r) = \sup_{t\ge 0} f(t)r^t, \qquad r \ge 0\,. \tag{3.18}$$

Suppose $u \in C_{+,\log}$ is (log, exp)-convex. By Theorems 3.4 and 3.6, θ_{ℓ_u} is defined. Moreover, by Lemma 3.7, we have

$$\theta_{\ell_u}(r) = u(r), \qquad \forall\, r \ge 0\,.$$

Hence $\theta_{\ell_u} = u$ for any (log, exp)-convex funtion u in $C_{+,\log}$.

On the other hand, let f be a positive continuous function on $[0,\infty)$ satisfying the conditions:

(a) $\lim_{t\to\infty} f(t)^{1/t} = 0$,
(b) f is decreasing for large t,
(c) f is log-concave.

We can carry out similar calculations as before to show that $\ell_{\theta_f} = f$. Therefore, θ is the inverse Legendre transform.

Now, we come to questions (ii) and (iii) related to Eq. (3.2) at the beginning of this section. Note that the coefficient of x^n in Eq. (3.2) is $\ell_u(n)$. Hence the new function that we mentioned in question (ii) is the series $\sum_{n=0}^{\infty} \ell_u(n)x^n$. Since we will often refer to this function we give it a name.

Definition 3.11. Let $u \in C_{+,\log}$ and $\lim_{n\to\infty} \ell_u(n)^{1/n} = 0$. The *L-function* of u is defined to be the function

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n, \qquad r \geq 0. \tag{3.19}$$

Note that $\mathcal{L}_u$ is an entire function. Let $u \in C_{+,\log}$ be (log, exp)-convex. Then (i) by Theorem 3.6 $\mathcal{L}_u$ is defined and (ii) by Theorem 3.4 the sequence $\{\ell_u(n)\}_{n=0}^{\infty}$ of coefficients in $\mathcal{L}_u$ is log-concave.

Lemma 3.12. *Let $u \in C_{+,\log}$ be (log, x^k)-convex, $k > 0$. Then*

$$r\mathcal{L}_u(r) \leq \frac{\ell_u(0)}{\ell_u(1)} \mathcal{L}_u(2^k r), \qquad \forall\, r \geq 0. \tag{3.20}$$

Proof. By Theorem 3.10 with $m = 1$ we have

$$\ell_u(n) \leq \frac{\ell_u(0)}{\ell_u(1)} 2^{k(n+1)} \ell_u(n+1).$$

Hence for any $r \geq 0$,

$$r\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^{n+1} \leq \frac{\ell_u(0)}{\ell_u(1)} \sum_{n=0}^{\infty} 2^{k(n+1)} \ell_u(n+1) r^{n+1} \leq \frac{\ell_u(0)}{\ell_u(1)} \mathcal{L}_u(2^k r). \qquad \square$$

Theorem 3.13. (a) *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then its L-function $\mathcal{L}_u$ is also (log, exp)-convex and for any constant $a > 1$,*

$$\mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \qquad \forall\, r \geq 0. \tag{3.21}$$

(b) *Let $u \in C_{+,\log}$ be increasing and (log, x^k)-convex, $k > 0$. Then there exists a constant C, independent of k, such that*

$$u(r) \leq C\mathcal{L}_u(2^k r), \qquad \forall\, r \geq 0. \tag{3.22}$$

Remarks. (a) From the proof below the constant C is given as follows. Note that if u is increasing and (log, x^k)-convex for some $k > 0$, then by Proposition 2.3 u is (log, exp)-convex. Hence by Theorem 3.6 its Legendre transform $\ell_u(t)$ is decreasing for large t. Let n_0 be a natural number such that $\ell_u(t)$ is decreasing for $t \geq n_0$. The constant C is given by

$$C = \max\left\{ \frac{u(1)}{\ell_u(0)}, \frac{\ell_u(0)}{\ell_u(1)}, \frac{u(1)}{\ell_u(n_0+1)} \right\}.$$

(b) If $u \in C_{+,\log}$ is increasing and (log, x^k)-convex for some $k > 0$, then we can combine the inequalities in Eqs. (3.21) and (3.22) together to get

$$\frac{1}{C} u(2^{-k} r) \leq \mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \qquad \forall\, r \geq 0. \tag{3.23}$$

Proof. To prove the inequality in Eq. (3.21), note that from Eq. (3.8) we have

$$\ell_u(s) \leq \rho(t)^{t-s}\ell_u(t), \qquad \forall\, s, t \geq 0\,,$$

where $\rho(t)$ is given in Lemma 3.5. Hence for any fixed $t \geq 0$,

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n \leq \sum_{n=0}^{\infty} \rho(t)^{t-n}\ell_u(t) r^n = \rho(t)^t \ell_u(t) \sum_{n=0}^{\infty} (r\rho(t)^{-1})^n\,.$$

Thus for $0 < r < \rho(t)$ we have

$$\mathcal{L}_u(r) \leq \rho(t)^t \ell_u(t)(1 - r\rho(t)^{-1})^{-1}\,.$$

Use this inequality to get

$$\begin{aligned}\ell_{\mathcal{L}_u}(t) &= \inf_{r>0} \frac{\mathcal{L}_u(r)}{r^t} \leq \inf_{0<r<\rho(t)} \frac{\mathcal{L}_u(r)}{r^t} \leq \inf_{0<r<\rho(t)} \frac{\rho(t)^t \ell_u(t)(1 - r\rho(t)^{-1})^{-1}}{r^t} \\ &= \rho(t)^t \ell_u(t) \inf_{0<r<\rho(t)} (r^t(1 - r\rho(t)^{-1}))^{-1}\,.\end{aligned}$$

But it is easily checked that

$$\sup_{0<r<\rho(t)} r^t(1 - r\rho(t)^{-1}) = \frac{t^t \rho(t)^t}{(t+1)^{t+1}}\,.$$

Therefore, for any $t > 0$, we have

$$\ell_{\mathcal{L}_u}(t) \leq \ell_u(t) \frac{(t+1)^{t+1}}{t^t}\,.$$

Now, for any constant $a > 1$, the inequality $(t+1)^{t+1}/t^t \leq (ea/\log a)a^t$ holds for all $t > 0$. Hence

$$\ell_{\mathcal{L}_u}(t) \leq \frac{ea}{\log a}\, a^t \ell_u(t), \qquad \forall\, t > 0\,.$$

By Lemma 3.2 $\ell_{\theta_a u}(t) = a^t \ell_u(t)$. Thus $\ell_{\mathcal{L}_u}(t) \leq (ea/\log a)\ell_{\theta_a u}(t)$ for all $t \geq 0$. Then apply Lemma 3.8(b) to conclude that

$$\mathcal{L}_u(r) \leq \frac{ea}{\log a}\, u(ar), \qquad \forall\, r \geq 0\,.$$

Now, we prove the inequality in Eq. (3.22). First suppose $0 \leq r \leq 1$. Since u and $\mathcal{L}_u$ are increasing functions and $\mathcal{L}_u(r) \geq \ell_u(0)$, we get

$$u(r) \leq u(1) \leq \frac{u(1)}{\ell_u(0)} \mathcal{L}_u(r) \leq \frac{u(1)}{\ell_u(0)} \mathcal{L}_u(2^k r), \qquad \forall\, r \in [0, 1]\,. \tag{3.24}$$

Before we consider $r > 1$, let us note that by Proposition 2.3 the function u, being increasing and (log, x^k)-convex, is also (log, exp)-convex. Hence by Theorem 3.6 $\ell_u(t)$ is decreasing for large t. Let n_0 be a natural number such that $\ell_u(t)$ is decreasing for $t \geq n_0$.

Let $r > 1$ be fixed. By Lemma 3.7, we have $u(r) = \sup_{t \geq 0} \ell_u(t) r^t$. Hence there exists $\tau = \tau(r) \geq 0$ such that

$$u(r) = \ell_u(\tau) r^\tau .$$

Let $j = j(r)$ be the integer such that $j \leq \tau < j + 1$.

Case 1: $j \geq n_0$. In this case we have $u(r) \leq \ell_u(j) r^{j+1}$ and so by Lemma 3.12

$$u(r) \leq r \ell_u(j) r^j \leq r \mathcal{L}_u(r) \leq \frac{\ell_u(0)}{\ell_u(1)} \mathcal{L}_u(2^k r) . \qquad (3.25)$$

Case 2: $j < n_0$. In this case, we use the fact that $u(1) = \sup_{t \geq 0} \ell_u(t)$ to get

$$u(r) \leq u(1) r^\tau \leq u(1) r^{n_0+1} \leq \frac{u(1)}{\ell_u(n_0+1)} \mathcal{L}_u(r) \leq \frac{u(1)}{\ell_u(n_0+1)} \mathcal{L}_u(2^k r) . \qquad (3.26)$$

Let $C = \max\{u(1)/\ell_u(0), \ell_u(0)/\ell_u(1), u(1)/\ell_u(n_0+1)\}$. We can put Eqs. (3.24)–(3.26) together to get Eq. (3.22). □

Now, observe that the inequalities in Eq. (3.23) are similar to those in Eq. (2.3). Thus the functions u and $\mathcal{L}_u$ are what we called "equivalent" in the beginning of this section. We now make this concept a formal definition.

Definition 3.14. Two positive functions u and v on $[0, \infty)$ are called *equivalent* if there exist positive constants c_1, c_2, a_1, a_2 such that

$$c_1 u(a_1 r) \leq v(r) \leq c_2 u(a_2 r), \qquad \forall\, r \in [0, \infty) .$$

Suppose $u \in C_{+,\log}$ is increasing and (log, x^2)-convex. Then by Theorem 3.13 the function u is equivalent to its L-function $\mathcal{L}_u$. Note that $\mathcal{L}_u$ is (log, exp)-convex and entire with positive coefficients. Moreover, Eq. (3.22) implies that $\mathcal{L}_u \in C_{+,\log}$. Hence we can state that each increasing (log, x^2)-convex function in $C_{+,\log}$ is equivalent to a (log, exp)-convex entire function with positive coefficients in $C_{+,\log}$.

Example 3.15. Consider the function $u(r) = \exp[(1+\beta) r^{1/(1+\beta)}], 0 \leq \beta < 1$. Obviously, $u \in C_{+,\log}$ is increasing and (log, x^2)-convex. Its Legendre transform is easily checked to be

$$\ell_u(n) = \begin{cases} \left(\dfrac{e}{n}\right)^{(1+\beta)n}, & \text{if } n \geq 1 ; \\ 1, & \text{if } n = 0 . \end{cases}$$

Hence the L-function of u is given by

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} (\frac{e}{n})^{(1+\beta)n} r^n , \qquad (3.27)$$

where $0^0 = 1$ by convention. We can use the Stirling formula (see p. 357 in Ref. 15) to get the inequalities

$$\frac{1}{n!} \leq \left(\frac{e}{n}\right)^n \leq \frac{e\, 2^{n/2}}{n!}, \qquad \forall\, n \geq 0 . \qquad (3.28)$$

It follows from Eqs. (3.27) and (3.28) that

$$G_{1/\alpha}(r) \leq \mathcal{L}_u(r) \leq e^{1+\beta} G_{1/\alpha}(2^{(1+\beta)/2} r), \qquad r \geq 0,$$

where $G_{1/\alpha}(r) = \sum_{n=0}^{\infty} (n!)^{-(1+\beta)} r^n$ as defined in Eq. (1.4). Thus $\mathcal{L}_u$ and $G_{1/\alpha}$ are equivalent. On the other hand, by Theorem 3.13, u is equivalent to $\mathcal{L}_u$. Hence we conclude that u and $G_{1/\alpha}$ are equivalent.

On the other hand, consider the function $v(r) = \exp[(1-\beta) r^{1/(1-\beta)}]$. By a similar argument as above we can show that v and the function G_α defined in Eq. (1.4) are equivalent. Note that the functions u and v are nothing but $\tilde{G}_{1/\alpha}$ and $\tilde{G}_\alpha$, respectively, in Eq. (1.7). Thus the equivalence of $\tilde{G}_{1/\alpha}$ and $\tilde{G}_\alpha$ to $G_{1/\alpha}$ and G_α, respectively, has been proved without using the inequalities in Eqs. (1.5) and (1.6) (cf. (Q2) in Sec. 1).

At the end of this section we define the equivalence of two sequences and state a simple fact which will be convenient for future reference.

Definition 3.16. Two sequences $\{a(n)\}$ and $\{b(n)\}$ of non-negative numbers are said to be *equivalent* if there exist positive constants K_1, K_2, c_1, c_2 such that

$$K_1 c_1^n a(n) \leq b(n) \leq K_2 c_2^n a(n), \qquad \forall\, n. \tag{3.29}$$

Let $f(r)$ and $g(r)$ be positive functions on $[0, \infty)$. We want to point out that the equivalence of functions f and g (in the sense of Definition 3.14) is quite different from the equivalence of sequences $\{f(n)\}$ and $\{g(n)\}$. Moreover, suppose $u(r) = \sum_{n=0}^{\infty} u_n r^n$ is an entire function with $u_n > 0$ and $\{u_n\}$ being log-concave. Then by Eq. (2.2) the sequences $\{u_n\}$ and $\{\ell_u(n)\}$ are equivalent.

Lemma 3.17. *Suppose $\{a(n)\}$ and $\{b(n)\}$ are equivalent sequences of non-negative numbers such that $a(n)^{1/n} \to 0$ or $b(n)^{1/n} \to 0$ as $n \to \infty$. Then the functions $A(r) = \sum_{n=0}^{\infty} a(n) r^n$ and $B(r) = \sum_{n=0}^{\infty} b(n) r^n$ defined on $[0, \infty)$ are equivalent.*

4. Dual Legendre Function

In this section we will develop a crucial machinery for the next section and the application to white noise analysis in the forthcoming paper.[3]

We will think of the exponential generating function $G_{1/\alpha}$ as $\mathcal{L}_u$ for some u. Equivalently, the sequence $\{\alpha(n)\}$ and the function u are related by the Legendre transform as follows:

$$\ell_u(n) = \frac{1}{n!\alpha(n)}. \tag{4.1}$$

In that case the exponential function G_α is given by

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n = \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)(n!)^2} r^n. \tag{4.2}$$

But by Eq. (3.28) the sequences $\{n!\}$ and $\{(n/e)^n\}$ are equivalent. Hence by Lemma 3.17 G_α is equivalent to the function defined by the series

$$\sum_{n=0}^{\infty} \frac{e^{2n}}{\ell_u(n) n^{2n}} r^n . \tag{4.3}$$

A good way to understand this new function is to regard it as $\mathcal{L}_v$ for some v, i.e. we need to find v such that

$$\ell_v(t) = \frac{e^{2t}}{\ell_u(t) t^{2t}}, \qquad t \geq 0 , \tag{4.4}$$

where $0^0 = 1$ by convention. The function v, defined as u^* in Definition 4.1 below, belongs to $C_{+,\log}$. Moreover, it is (log, exp)-convex by Proposition 2.3 and Lemma 4.5 below. Hence we can apply Lemma 3.7 to get

$$v(r) = \sup_{t \geq 0} \ell_v(t) r^t = \sup_{t>0} \frac{e^{2t} r^t}{\ell_u(t) t^{2t}} . \tag{4.5}$$

Then use the definition of the Legendre transform to show

$$v(r) = \sup_{t,s>0} \frac{e^{2t} r^t s^t}{u(s) t^{2t}} = \sup_{s>0} \frac{1}{u(s)} \sup_{t>0} \frac{(e^2 rs)^t}{t^{2t}} . \tag{4.6}$$

But it can be easily checked that for $a > 0$,

$$\sup_{t>0} \frac{a^t}{t^{2t}} = e^{2\sqrt{a}/e} . \tag{4.7}$$

Put Eq. (4.7) with $a = e^2 rs$ into Eq. (4.6) to conclude that

$$v(r) = \sup_{s>0} \frac{e^{2\sqrt{rs}}}{u(s)} . \tag{4.8}$$

This equation suggests a new transform and raises a question of finding u for which this new transform can be defined.

Notation. Let $C_{+,j}, j > 0$, denote the set of all positive continuous functions u on $[0, \infty)$ satisfying the condition

$$\lim_{r \to \infty} \frac{\log u(r)}{r^j} = \infty . \tag{4.9}$$

We will mostly be concerned with the set $C_{+,1/2}$ because the right-hand side of Eq. (4.8) exists for all $r \geq 0$ when $u \in C_{+,1/2}$. On the other hand, observe that $C_{+,j} \subset C_{+,\log}$ for all $j > 0$.

Definition 4.1. The *dual Legendre function* u^* of $u \in C_{+,1/2}$ is defined to be the function

$$u^*(r) = \sup_{s>0} \frac{e^{2\sqrt{rs}}}{u(s)}, \qquad r \geq 0 . \tag{4.10}$$

Remark. In Ref. 7 by Gannoun *et al.*, they adopted the relation:

$$\theta(r)^* := \sup_{s\geq 0}\{sr - \theta(s)\}, \quad r \geq 0\,.$$

Hence

$$\log u(s) = 2\theta(\sqrt{s}), \quad \log u(r)^* = 2\theta(\sqrt{r})^* \tag{4.11}$$

hold.

Example 4.2. For the function $u(r) = e^r$, we have $u^*(r) = e^r$. This is the case for the Hida–Kubo–Takenaka space.

Example 4.3. For the function $u(r) = \exp[(1+\beta)r^{1/(1+\beta)}]$, we can easily check that $u^*(r) = \exp[(1-\beta)r^{1/(1-\beta)}]$. This is the case for the Kondratiev–Streit space.

Example 4.4. Let $u(r) = \exp[e^x]$. To find $u^*(r)$ we need to find the maximum of the function $2\sqrt{rs} - e^s$. The critical point s_0 of this function satisfies the equation

$$\sqrt{r} = \sqrt{s}\, e^s\,.$$

Obviously, we have $\lim_{r\to\infty} s_0 = \infty$. Hence $s_0 \sim \log\sqrt{r}$ for large r and so

$$\sup_{s>0}(2\sqrt{rs} - e^s) = 2\sqrt{rs_0} - e^{s_0} = 2\sqrt{rs_0} - \frac{\sqrt{r}}{\sqrt{s_0}} \sim 2\sqrt{rs_0} \sim 2\sqrt{r\log\sqrt{r}}\,.$$

Thus although we cannot find the exact form of u^*, the function u^* is equivalent to the function $\exp\left[2\sqrt{r\log\sqrt{r}}\right]$. In general let

$$u(r) = \exp_k(r) = \exp(\exp(\cdots(\exp(r)))), \qquad k\text{th iteration}\,.$$

Its dual Legendre function u^* is equivalent to the function

$$\exp\left[2\sqrt{r\log_{k-1}\sqrt{r}}\right],$$

where $\log_j$ is defined by

$$\log_1(r) = \log(\max\{r, e\}), \quad \log_j(r) = \log_1(\log_{j-1}(r)), \quad j \geq 2\,.$$

This example is for the Gel'fand triple associated with the Bell numbers in the paper by Cochran *et al.*[6]

Lemma 4.5. *Let $u \in C_{+,1/2}$. Then its dual Legendre function u^* belongs to $C_{+,1/2}$ and is an increasing $(\log, x^2)$-convex function.*

Proof. From the definition of $u^*(r)$ we have $\log u^*(r) \geq 2\sqrt{rs} - \log u(s)$ for any $s > 0$. Hence

$$\frac{\log u^*(r)}{\sqrt{r}} \geq \frac{2\sqrt{rs} - \log u(s)}{\sqrt{r}} = 2\sqrt{s} - \frac{\log u(s)}{\sqrt{r}}\,.$$

This implies that

$$\liminf_{r\to\infty} \frac{\log u^*(r)}{\sqrt{r}} \geq 2\sqrt{s}, \qquad \forall\ s > 0\,.$$

Therefore, $\lim_{r\to\infty} \log u^*(r)/\sqrt{r} = \infty$, which shows that $u^* \in C_{+,1/2}$. To show that $u^*(r)$ is increasing, let $r_1 < r_2$. Note that there exists some $s_1 > 0$ such that $u^*(r_1) = e^{2\sqrt{r_1 s_1}}/u(s_1)$. Hence

$$u^*(r_1) = \frac{e^{2\sqrt{r_1 s_1}}}{u(s_1)} \le \frac{e^{2\sqrt{r_2 s_1}}}{u(s_1)} \le u^*(r_2)\,.$$

To show that $u^*(r)$ is (log, x^2)-convex, let $r_1, r_2 \ge 0$, and $0 \le \lambda \le 1$. Then

$$\begin{aligned} u^*((\lambda r_1 + (1-\lambda) r_2)^2) &= \sup_{s>0} \frac{e^{2(\lambda r_1 + (1-\lambda) r_2)\sqrt{s}}}{u(s)} \\ &\le \left(\sup_{s_1>0} \frac{e^{2r_1\sqrt{s_1}}}{u(s_1)}\right)^{\lambda} \left(\sup_{s_2>0} \frac{e^{2r_2\sqrt{s_2}}}{u(s_2)}\right)^{1-\lambda} \\ &= u^*(r_1^2)^{\lambda} u^*(r_2^2)^{1-\lambda}\,. \end{aligned}$$

Thus $u^*(r)$ is (log, x^2)-convex. □

Theorem 4.6. *Let $u \in C_{+,1/2}$ be (log, x^2)-convex. Then the Legendre transform of u^* is given by*

$$\ell_{u^*}(t) = \frac{e^{2t}}{\ell_u(t) t^{2t}}\,. \tag{4.12}$$

Proof. Note that $u^* \in C_{+,\log}$ since $u^* \in C_{+,1/2}$ by Lemma 4.5 and $C_{+,j} \subset C_{+,\log}$ for all $j > 0$. Hence the Legendre transform ℓ_{u^*} is defined. By assumption u is (log, x^2)-convex and so by Lemma 3.9 the function $\ell_u(t)t^{2t}$ is log-convex. Hence $(\ell_u(t)t^{2t})^{-1}$ is log-concave. Since e^{2t} is also log-concave, we see that the function

$$w(t) = \frac{e^{2t}}{\ell_u(t) t^{2t}}$$

is log-concave. Note that $\ell_u(t)^{1/t} t^2$ increases to ∞ as $t \to \infty$ since the function u is (log, x^2)-convex. Hence the inverse Legendre transform θ in Eq. (3.18) is defined at w by

$$\theta_w(r) = \sup_{t\ge 0} \frac{e^{2t} r^t}{\ell_u(t) t^{2t}}\,. \tag{4.13}$$

Moreover, $\ell_{\theta_w} = w$. On the other hand, from the motivation for the dual Legendre function in Eqs. (4.5), (4.8) and (4.10) we have

$$u^*(r) = \sup_{t\ge 0} \frac{e^{2t} r^t}{\ell_u(t) t^{2t}}\,. \tag{4.14}$$

It follows from Eqs. (4.13) and (4.14) that $\theta_w = u^*$. But we also have $\ell_{\theta_w} = w$. Hence $\ell_{u^*} = w$ and the theorem is proved. □

Remark. Let $u \in C_{+,1/2}$ be (log, x^2)-convex. Suppose u is increasing on the interval $[r_0, \infty)$. Then $(u^*)^*(r) = u(r)$ for all $r \ge r_0$. Observe that if u is an increasing (log,

x^2)-convex function in $C_{+,1/2}$, then we have $(u^*)^* = u$. Since we will not use this involution property elsewhere in this paper, we skip the proof.

As we mentioned at the beginning of this section the exponential generating function $G_{1/\alpha}$ is thought of as the L-function $\mathcal{L}_u$ for some function u. Then the corresponding exponential generating function G_α, expressed in terms of $\ell_u(n)$'s, is given by the second series in Eq. (4.2). We give this series a name for future reference.

Definition 4.7. Let $u \in C_{+,1/2}$ and suppose $\lim_{n\to\infty}(\ell_u(n)(n!)^2)^{-1/n} = 0$. The $L^\#$*-function* of u is defined to be the function

$$\mathcal{L}_u^\#(r) = \sum_{n=0}^\infty \frac{1}{\ell_u(n)(n!)^2}\, r^n, \qquad r \geq 0\,.$$

Note that $\mathcal{L}_u^\#(r)$ is an entire function. It follows from Theorem 4.6 and Eqs. (4.2) and (4.3) that $\mathcal{L}_u^\#(r)$ is defined for any (log, x^2)-convex function u in $C_{+,1/2}$.

Theorem 4.8. *Let $u \in C_{+,1/2}$ be (log, x^2)-convex. Then the functions $\mathcal{L}_{u^*}$ and $\mathcal{L}_u^\#$ are equivalent.*

Remark. Let $u \in C_{+,1/2}$. Then by Lemma 4.5 its dual Legendre transform u^* belongs to $C_{+,1/2}$ and is increasing and (log, x^2)-convex. Hence we can apply Theorem 3.13 to u^* to conclude that the functions u^* and $\mathcal{L}_{u^*}$ are equivalent. Therefore, under the assumption of the above theorem, the functions $u^*, \mathcal{L}_{u^*}, \mathcal{L}_u^\#$ are all equivalent.

Proof. Note that the function $\mathcal{L}_u^\#$ is the second series in Eq. (4.2). But from the discussion for Eqs. (4.3)–(4.5) with $v = u^*$, we see easily that $\mathcal{L}_u^\#$ is equivalent to the function $\sum_{n=0}^\infty \ell_{u^*}(n) r^n$, which is exactly the function $\mathcal{L}_{u^*}$. □

Below we list some facts concerning equivalent functions and sequences. These facts can be easily checked by using the previous results or the techniques in the proofs.

(1) If $u \in C_{+,\log}$ and v is equivalent to u, then $v \in C_{+,\log}$ and the sequences $\{\ell_u(n)\}$ and $\{\ell_v(n)\}$ are equivalent.
(2) If $u, v \in C_{+,\log}$, u is increasing and (log, x^2)-convex, and the sequences $\{\ell_u(n)\}$ and $\{\ell_v(n)\}$ are equivalent, then the functions u and v are equivalent.
(3) If $u \in C_{+,\log}$, u is increasing and (log, x^2)-convex, and u and v are equivalent, then the L-functions $\mathcal{L}_u$ and $\mathcal{L}_v$ are equivalent.
(4) If $u \in C_{+,1/2}$ and v is equivalent to u, then $v \in C_{+,1/2}$ and the functions u^* and v^* are equivalent.
(5) If $u \in C_{+,1/2}$, u is (log, x^2)-convex, and u and v are equivalent, then the functions $\mathcal{L}_u^\#$ and $\mathcal{L}_v^\#$ are equivalent.

Many properties of a function or sequence remain true for equivalent functions or sequences. For convenience, we make the following definition.

Definition 4.9. Let P be a property of functions or sequences. A function u is said to be *nearly* P if there exists a P function which is equivalent to u. A sequence $\{a(n)\}$ is said to be *nearly* P if there exists a P sequence which is equivalent to $\{a(n)\}$.

For example, a positive function u is nearly (log, exp)-convex if there exists a (log, exp)-convex function which is equivalent to u. A positive sequence $\{a(n)\}$ is nearly log-concave if there exists a log-concave sequence which is equivalent to $\{a(n)\}$.

Here we list some results concerning functions and sequences that are "nearly" something.

(6) Let $u, v \in C_{+,\log}$ be increasing and nearly (log, x^2)-convex. Then the functions u and v are equivalent if and only if the sequences $\{\ell_u(n)\}$ and $\{\ell_v(n)\}$ are equivalent.
(7) Let $u, v \in C_{+,1/2}$ be nearly (log, x^2)-convex. Then u and v are equivalent if and only if u^* and v^* are equivalent.
(8) Let $u(r) = \sum_{n=0}^{\infty} u_n r^n$ and $v(r) = \sum_{n=0}^{\infty} v_n r^n$ be entire functions with $u_n, v_n > 0$. Suppose $\{u_n\}$ and $\{v_n\}$ are nearly log-concave sequences. Then $\{u_n\}$ and $\{v_n\}$ are equivalent if and only if u and v are equivalent if and only if $\{\ell_u(n)\}$ and $\{\ell_v(n)\}$ are equivalent.
(9) If $u \in C_{+,\log}$, then the sequence $\{\ell_u(n)\}$ is log-concave. On the other hand, if $u \in C_{+,1/2}$ is (log, x^2)-convex, then the sequence $\{(\ell_u(n)(n!)^2)^{-1}\}$ is nearly log-concave.

We make two remarks about Item 9: (a) Let $\{b_k(n)\}$ be the Bell numbers of order k. It has been shown in Ref. 1 that $\{b_k(n)/n!\}$ is log-concave and $\{b_k(n)\}$ is log-convex. Note that $\{b_k(n)\}$ being log-convex implies that $\{(b_k(n)n!)^{-1}\}$ is log-concave.

(b) The near log-concavity of the sequence $\{(\ell_u(n)(n!)^2)^{-1}\}$ has been shown in Ref. 11 to be a necessary condition for the characterization theorem of generalized functions in the Gel'fand triple introduced by Cochran *et al.*[6]

5. Growth Order of Holomorphic Functions

Recall that the S-transform F of a generalized function is a function on the complexification $\mathcal{E}_c$ of $\mathcal{E}$. It is a holomorphic function on $\mathcal{E}_c$ in the sense that for any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$. Moreover, it satisfies the growth conditions in Eqs. (1.2) and (1.3) for generalized and test functions, respectively. In this section we will study the representation of holomorphic functions F on $\mathcal{E}_c$ satisfying the growth conditions in Eqs. (1.2) and (1.3) with G_α and $G_{1/\alpha}$

being replaced by certain functions. The characterization theorems will be given in our forthcoming paper.[3]

Lemma 5.1. *Let $u \in C_{+,\log}$. Suppose F is a holomorphic function on $\mathcal{E}_c$ and there exist constants $K, a, p \geq 0$ such that*

$$|F(\xi)| \leq K\, u(a|\xi|_{-p}^2)^{1/2}, \qquad \forall\, \xi \in \mathcal{E}_c\,. \tag{5.1}$$

Let $q \in [0, p]$ be an integer such that $i_{p,q}$ is a Hilbert–Schmidt operator. Then there exist functions $f_n \in \mathcal{E}_{q,c}^{\hat{\otimes} n}$ such that $F(\xi) = \sum_{n=0}^\infty \langle f_n, \xi^{\hat{\otimes} n}\rangle$ and

$$|f_n|_q^2 \leq K^2 (ae^2 \|i_{p,q}\|_{\mathrm{HS}}^2)^n \ell_u(n)\,, \tag{5.2}$$

where ℓ_u is the Legendre transform of u.

Proof. We follow the same argument as in the proof of Theorem 8.9 in Ref. 15. Since F is a holomorphic function on $\mathcal{E}_c$, it has the expansion

$$F(\xi) = \sum_{n=0}^\infty J_n(\xi, \xi, \ldots, \xi)\,,$$

where J_n is a symmetric n-linear functional on $\mathcal{E}_c \times \cdots \times \mathcal{E}_c$ given by

$$J_n(\xi_1, \ldots, \xi_n) = \frac{1}{n!} \frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} F(z_1\xi_1 + \cdots + z_n\xi_n)|_{z_1 = \cdots = z_n = 0}\,.$$

Apply the Cauchy formula to show that

$$J_n(\xi_1, \ldots, \xi_n) = \frac{1}{n!}\frac{1}{(2\pi i)^n} \int_{|z_1|=r_1} \cdots \int_{|z_n|=r_n} \frac{F(z_1\xi_1 + \cdots + z_n\xi_n)}{z_1^2 \cdots z_n^2}\, dz_1 \cdots dz_n\,.$$

Let $R > 0$. For nonzero ξ_j's, take $r_j = R/|\xi_j|_{-p}$, $1 \leq j \leq n$ and use the maximum modulus principle to derive that

$$|J_n(\xi_1, \ldots, \xi_n)| \leq \frac{1}{n!}\frac{1}{R^n} \left(\sup_{|\xi|_{-p} = nR} |F(\xi)| \right) |\xi_1|_{-p} \cdots |\xi_n|_{-p}\,.$$

Use the growth condition in Eq. (5.1) to get

$$|J_n(\xi_1, \ldots, \xi_n)| \leq K \frac{1}{n!} \frac{u(an^2R^2)^{1/2}}{R^n}\, |\xi_1|_{-p} \cdots |\xi_n|_{-p}\,.$$

This inequality holds for any $R > 0$ and $\xi_1, \ldots, \xi_n \in \mathcal{E}_p'$. Let $an^2R^2 = r$. Then

$$|J_n(\xi_1, \ldots, \xi_n)| \leq K \frac{a^{n/2} n^n}{n!} \left(\frac{u(r)}{r^n} \right)^{1/2} |\xi_1|_{-p} \cdots |\xi_n|_{-p}\,.$$

Now, take the infimum over $r > 0$ to obtain

$$|J_n(\xi_1, \ldots, \xi_n)| \leq K \frac{a^{n/2} n^n}{n!} \ell_u(n)^{1/2} |\xi_1|_{-p} \cdots |\xi_n|_{-p}\,.$$

Then use the same argument as in the proof of Theorem 8.9 in Ref. 15 to conclude that $J_n(\xi_1, \ldots, \xi_n) = \langle f_n, \xi_1 \otimes \cdots \xi_n \rangle$ with $f_n \in \mathcal{E}_{q,c}^{\hat{\otimes} n}$, $q \in [0, p]$, and

$$|f_n|_q^2 \le K^2 a^n \left(\frac{n^n}{n!}\right)^2 \ell_u(n) \|i_{p,q}\|_{\mathrm{HS}}^{2n} .$$

This inequality implies the one in Eq. (5.2) because $n^n \le n! e^n$. □

Lemma 5.2. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Suppose $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\otimes n} \rangle$ is a holomorphic function on $\mathcal{E}_c$ and there exist $K, a, p \ge 0$ such that*

$$|f_n|_p \le K a^n \ell_u(n)^{1/2}, \qquad \forall\, n \ge 0 .$$

Then for any $\xi \in \mathcal{E}_c$,

$$|F(\xi)| \le \sqrt{2}\, e K u(2ea^2 |\xi|_{-p}^2)^{1/2} . \tag{5.3}$$

Proof. By assumption we have $|\langle f_n, \xi^{\otimes n} \rangle| \le K a^n \ell_u(n)^{1/2} |\xi|_{-p}^n$. Hence

$$\begin{aligned} |F(\xi)| &\le \sum_{n=0}^{\infty} K a^n \ell_u(n)^{1/2} |\xi|_{-p}^n = K \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n \left(\ell_u(n)^{1/2} (\sqrt{2}\, a |\xi|_{-p})^n\right) \\ &\le K\sqrt{2} \left(\sum_{n=0}^{\infty} \ell_u(n) (2a^2 |\xi|_{-p}^2)^n\right)^{1/2} = K\sqrt{2} \mathcal{L}_u(2a^2 |\xi|_{-p}^2)^{1/2} , \end{aligned} \tag{5.4}$$

where $\mathcal{L}_u$ is the L-function of u. But by Theorem 3.13(a) with $a = e$ we have

$$\mathcal{L}_u(r) \le e^2 u(er), \qquad r \ge 0 . \tag{5.5}$$

The conclusion in Eq. (5.3) follows from Eqs. (5.4) and (5.5). □

Now, let $u \in C_{+,\log}$ be a fixed function. Suppose F is an entire function on $\mathcal{E}_c$ with the expansion $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\otimes n} \rangle$. Being motivated by the norm given in Ref. 6 (with $\ell_u(n)$ replacing $(n!\alpha(n))^{-1}$ as noted before) we define for each $p \ge 0$,

$$\|F\|_{u,p} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2\right)^{1/2} .$$

Let $\mathcal{K}_{u,p} = \{F; \|F\|_{u,p} < \infty\}$. Then $\mathcal{K}_{u,p}$ is a Hilbert space with norm $\|\cdot\|_{u,p}$.

On the other hand, being motivated by the work of Lee[17] and Sec. 15.2 in Ref. 15, we define $|||F|||_{u,p}$ for a holomorphic function F on $\mathcal{E}_c$ and for each $p \ge 0$ by

$$|||F|||_{u,p} = \sup_{\xi \in \mathcal{E}_c} |F(\xi)| u(|\xi|_{-p}^2)^{-1/2} .$$

Let $\mathcal{G}_{u,p} = \{F; |||F|||_{u,p} < \infty\}$. Then $\mathcal{G}_{u,p}$ is a Banach space with norm $|||\cdot|||_{u,p}$.

Theorem 5.3. *Let $u \in C_{+,\log}$. Suppose $p > q$ is such that the inclusion mapping $i_{p,q} : \mathcal{E}_p \to \mathcal{E}_q$ is a Hilbert–Schmidt operator with $\|i_{p,q}\|_{\mathrm{HS}} \le e^{-1}$. Then*

$$\|F\|_{u,q} \le (1 - e^2 \|i_{p,q}\|_{\mathrm{HS}}^2)^{-1/2} |||F|||_{u,p}, \qquad \forall\, F \in \mathcal{G}_{u,p} . \tag{5.6}$$

Remark. Conditions (a) and (b) stated at the beginning of Sec. 1 imply that $\lim_{p\to\infty} \|i_{p,q}\|_{\mathrm{HS}} = 0$ for any $q \geq 0$. Hence for any given $q \geq 0$, there exists some $p > q$ such that $\|i_{p,q}\|_{\mathrm{HS}} \leq e^{-1}$. Therefore, it follows from the theorem that for any $q \geq 0$, there exists $p > q$ such that $\mathcal{G}_{u,p} \subset \mathcal{K}_{u,q}$ and the inclusion mapping is continuous by Eq. (5.6).

Proof. Suppose $F \in \mathcal{G}_{u,p}$. Then we have

$$|F(\xi)| \leq \|F\|_{u,p}\, u(|\xi|_{-p}^2)^{1/2}, \qquad \forall\, \xi \in \mathcal{E}_c\,. \tag{5.7}$$

Hence for q as specified in the theorem, we can apply Lemma 5.1 to show that $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\otimes n}\rangle$ with $f_n \in \mathcal{E}_{q,c}^{\hat{\otimes} n}$ and

$$|f_n|_q^2 \leq \|F\|_{u,p}^2 (e^2 \|i_{p,q}\|_{\mathrm{HS}}^2)^n \ell_u(n)\,.$$

Therefore,

$$\begin{aligned}\|F\|_{u,q}^2 &= \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_q^2 \leq \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} \|F\|_{u,p}^2 (e^2 \|i_{p,q}\|_{\mathrm{HS}}^2)^n \ell_u(n) \\ &= (1 - e^2 \|i_{p,q}\|_{\mathrm{HS}}^2)^{-1} \|F\|_{u,p}^2\,.\end{aligned}$$

This proves the inequality in Eq. (5.6). □

Theorem 5.4. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then for any $p \geq 1$, we have*

$$\|F\|_{u,p-1} \leq \sqrt{e}(2\rho^2 \log 1/\rho)^{-1/2}\, \|F\|_{u,p}, \qquad \forall\, F \in \mathcal{K}_{u,p}\,, \tag{5.8}$$

where the constant ρ is given in Condition (a) *at the beginning of Sec.* 1.

Remark. It follows from Eq. (5.8) that for any $p \geq 1, \mathcal{K}_{u,p} \subset \mathcal{G}_{u,p-1}$ and the inclusion mapping is continuous.

Proof. Let $F \in \mathcal{K}_{u,p}$ and $p \geq 1$. Since $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\hat{\otimes} n}\rangle$, we can derive that

$$\begin{aligned}|F(\xi)| &\leq \sum_{n=0}^{\infty} |f_n|_p |\xi|_{-p}^n = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{\ell_u(n)}} |f_n|_p \right) (\sqrt{\ell_u(n)} |\xi|_{-p}^n) \\ &\leq \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \ell_u(n)\, |\xi|_{-p}^{2n} \right)^{1/2} = \|F\|_{u,p} \mathcal{L}_u(|\xi|_{-p}^2)^{1/2}\,.\end{aligned} \tag{5.9}$$

Note that $|\xi|_{-p} \leq \rho |\xi|_{-p+1}$ and then apply Theorem 3.13(a) with $a = 1/\rho^2$ to get

$$\mathcal{L}_u(|\xi|_{-p}^2) \leq \mathcal{L}_u(\rho^2 |\xi|_{-p+1}^2) \leq e(2\rho^2 \log 1/\rho)^{-1} u(|\xi|_{-p+1}^2)\,. \tag{5.10}$$

Equations (5.9) and (5.10) imply the inequality in Eq. (5.8). □

Take a (log, exp)-convex function $u \in C_{+,\log}$. Let $\mathcal{K}_u$ and $\mathcal{G}_u$ be the projective limits of the families $\{\mathcal{K}_{u,p}; p \geq 0\}$ and $\{\mathcal{G}_{u,p}; p \geq 0\}$, respectively. By the remarks following each of Theorems 5.3 and 5.4 we see that $\mathcal{K}_u = \mathcal{G}_u$ and their respective topologies given by $\{\|\cdot\|_{u,p}; p \geq 0\}$ and $\{\|\!|\cdot|\!\|_{u,p}; p \geq 0\}$ coincide. In the forthcoming paper we will study the corresponding spaces of test and generalized functions and the characterization theorems.

Appendix A.

Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers. Let us extract the following list from Ref. 4:

(A1) $\alpha(0) = 1$ and $\inf_{n\geq 0} \alpha(n)\sigma^n > 0$ for some $\sigma \geq 1$.

(A2) $\lim_{n\to\infty} \left(\frac{\alpha(n)}{n!}\right)^{1/n} = 0\,.$

($\tilde{\text{A}}$2) $\lim_{n\to\infty} \left(\frac{1}{n!\alpha(n)}\right)^{1/n} = 0.$

(B1) $\limsup_{n\to\infty} \left(\frac{n!}{\alpha(n)} \inf_{r>0} \frac{G_\alpha(r)}{r^n}\right)^{1/n} < \infty.$

($\tilde{\text{B}}$1) $\limsup_{n\to\infty} \left(n!\alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n}\right)^{1/n} < \infty.$

(B2) The sequence $\gamma(n) = \frac{\alpha(n)}{n!}, n \geq 0$, is log-concave, i.e. for all $n \geq 0$,

$$\gamma(n)\gamma(n+2) \leq \gamma(n+1)^2\,.$$

($\tilde{\text{B}}$2) The sequence $\left\{\frac{1}{n!\alpha(n)}\right\}$ is log-concave.

(B3) The sequence $\{\alpha(n)\}$ is log-convex, i.e. for all $n \geq 0$,

$$\alpha(n)\alpha(n+2) \geq \alpha(n+1)^2\,.$$

(C1) There exists a constant c_1 such that for all $n \leq m$,

$$\alpha(n) \leq c_1^m \alpha(m)\,.$$

(C2) There exists a constant c_2 such that for all n and m,

$$\alpha(n+m) \leq c_2^{n+m}\alpha(n)\alpha(m)\,.$$

(C3) There exists a constant c_3 such that for all n and m,

$$\alpha(n)\alpha(m) \leq c_3^{n+m}\alpha(n+m)\,.$$

Cochran *et al.*[6] assumed condition (A1) with $\sigma = 1$. But our (A1) is strong enough to imply that the space of test functions is contained in the L^2-space of the white noise measure. In Ref. 6 conditions (A2), (B1), (B2) are considered. Condition (A2) is to assure that the function G_α is an entire function. Condition (B1) is used for the characterization theorem of generalized functions in Theorems 5.1 and 6.1.[6] Condition (B2) is shown to imply condition (B1) in Theorem 4.3.[6]

In the papers by Asai *et al.*,[1,2] conditions ($\tilde{\text{A}}$2), ($\tilde{\text{B}}$1), ($\tilde{\text{B}}$2), (B3) are considered. It can be easily checked that condition (A1) implies condition ($\tilde{\text{A}}$2). Condition ($\tilde{\text{A}}$2) is to assure that the function $G_{1/\alpha}$ is an entire function. In Ref. 2 condition ($\tilde{\text{B}}$1) is used for the characterization theorem of test functions. Condition ($\tilde{\text{B}}$2) implies condition ($\tilde{\text{B}}$1), while obviously condition (B3) implies condition ($\tilde{\text{B}}$2).

In the paper by Kubo *et al.*,[12] conditions (C1), (C2), (C3) are assumed in order to carry out the distribution theory for a CKS-space. As pointed out in Ref. 12, condition (C3) implies condition (C1).

An important example of $\{\alpha(n)\}$ is the sequence $\{b_k(n)\}$ of Bell's numbers of order $k \geq 2$. The sequence $\{b_k(n)\}$ satisfies conditions (A1), (A2), (B1) (as shown in Ref. 6), (B2), (B3) (as shown in Ref. 1) (C1), (C2), (C3) (as shown in Ref. 12). Therefore, Bell's numbers satisfy all conditions in the above list.

The essential conditions for distribution theory on a CKS-space are (A1), (A2), (B2), ($\tilde{\text{B}}$2), (C2) and (C3). All other conditions can be derived from these six conditions except for (B3). We have taken ($\tilde{\text{B}}$2) instead of (B3) for the following reason. The condition (B3) is rather strong and we do not know how to prove this condition for a growth function u. Fortunately, we do not need (B3) for white noise distribution theory.

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