Stochastic calculus with respect to continuous finite quadratic variation processes

Francesco RUSSO, Université Paris 13, Institut Galilée, Mathématiques, Avenue Jean-Baptiste Clément, F - 93430 Villetaneuse

Pierre VALLOIS, Université Henri Poincaré Nancy 1, Département de Mathématiques, Institut Elie Cartan, BP 239, F - 54506 Vandœuvre-lès-Nancy Cedex

Abstract. The quadratic variation of a continuous process (when it exists) is defined through a regularization procedure. A large class of finite quadratic variation processes is provided, with a particular emphasis on Gaussian processes. For such processes a calculus is developed with application to the study of some stochastic differential equations.

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Introduction.

Let $X = \big(X(t), t \geq 0\big)$, $Y = \big(Y(t), t \geq 0\big)$ be two stochastic processes ; X will be supposed to be continuous.

The objects of our interest are the forward integral $\int_0^{\cdot} Y d^- X$, the backward integral $\int_0^{\cdot} Y d^+ X$ and the covariation process $[X, Y]$ when they exist, see [RV2 ; RV4]. Our approach has three objectives.

a) We aim to develop a calculus which is relatively simple and as close as possible to a **pathwise approach**. A pathwise stochastic calculus has been initiated by Föllmer [F] with recent complements in [FPS].

Bertoin [B1 ; B2] and more recently [RV4 ; W1 ; W2 ; W3] have contibuted to a pathwise study of Dirichlet processes. The last papers make use of regularizing techniques and all the others implement Riemann sums. Given a semimartingale S, one interesting problem is to determine the class of f such that $\Big(f\big(S(t)\big), t \geq 0\Big)$ is a semimartingale or a Dirichlet process. In such a case it is useful to get the Fukushima decomposition ([Fu ; LZ]) that is to say to detect the local martingale part and the zero quadratic variation process. Elements of answer have been given in [BY ; B1 ; RV3 ; W1 ; W2 ; W3].

b) Our approach has the ambition to constitute a bridge between **causal and non-causal** stochastic calculus : in particular we wanted to relate enlargement of filtrations [J] and Skorohod integral, see [NP], [N]. A remark in this sense is in [RV1].

When the integrator is a Brownian motion and the integrand belongs to suitable Wiener analysis spaces, see [Ma], [N], [HKPS], [W], our forward integral coincides with the one of [AP], [BM], [KR]. In some cases it has been possible to give existence theorems for stochastic differential equations with anticipating initial conditions under quite weak assumptions, see for instance [RV1 ; RV3]. The uniqueness problem has been partially solved in this paper. An application to the stochastic heat equation has been performed by [T].

c) The third objective which is really the main one of this paper is to develop a calculus beyond the barrier of semimartingales which includes the case of Gaussian processes which have an infinite quadratic variation processes as fractional Brownian motion or other long memory Gaussian processes, see [CCM]. A calculus based on fractional calculus has been introduced by [Z1], [DU] making use of fractional calculus. Also stochastic differential equations driven by fractional Brownian motion have been treated by [Z2].

It is also significant to emphasize that our approach is connected with Colombeau theory of generalized functions for defining multiplication and non-linearity of distributions, see [C], [O] and [R] for connections with stochastic analysis.

The paper is organized as follows. In section 1, we recall some basic notions of forward and backward integration, in section 2 we list examples of processes having finite quadratic variation which motivate our study. We analyze in section 3 the class of Gaussian processes admitting a generalized bracket. Section 4 is devoted to complete our calculus with the introduction of the notion of (generalized) Itô processes. Section 5 studies existence and uniqueness of stochastic differential equation which are driven by a finite quadratic variation process ; we implement here Doss-Sussmann techniques (see [Do], [Su]).

1. General calculus.

For simplicity, we will concentrate on the case when the integrator is a continuous process.
$\mathcal{C}$ will denote the Fréchet space of continuous processes equipped with the metric topology of the uniform convergence in probability (ucp) on each compact interval. A metric on $\mathcal{C}$ can be given for instance by

$$\rho_{\mathcal{C}}(X,Y)=\sum_{n=0}^{\infty}2^{-n}E\left(\frac{\sup_{t\le n}|X(t)-Y(t)|}{1+\sup_{t\le n}|X(t)-Y(t)|}\right).$$

If X is continuous and $Y\in L^1_{\text{loc}}(dx)$ a.s. (i.e. $\int_0^t |Y_s|ds<\infty$, a.s. for any $t\ge 0$), we define $\int_0^t Yd^-X$ $\left(\text{resp. } \int_0^t Yd^+X\right)$ as the limit ucp of

$$\tag{1.1} \begin{gathered}\int_0^t Y(s)\frac{X(s+\varepsilon)-X(s)}{\varepsilon}ds\\ \left(\text{resp. } \int_0^t Y(s)\frac{X(s)-X\big((s-\varepsilon)\vee 0}{\varepsilon}ds\right)\end{gathered}$$

provided these integrals exist. If Y is continuous, we define $[X,Y]$ as the ucp limit of

$$\tag{1.2} C_\varepsilon(X,Y)(t)=\frac{1}{\varepsilon}\int_0^t \big(X(s+\varepsilon)-X(s)\big)\big(Y(s+\varepsilon)-Y(s)\big)ds.$$

when $\varepsilon\to 0^+$.

The following properties are direct consequences of the definition of generalized stochastic integrals and brackets.
(1.3) $[X,Y]=[Y,X]$.
(1.4) $[X,X]$ is an continuous increasing process.
A process such that $[X,X]$ exists will be called a **finite quadratic variation** process.

$$\tag{1.5} [X,Y](t)=\int_0^t Yd^+X-\int_0^t Yd^-X,$$

provided two of the previous objects exist.

An integration by parts formula holds :

$$\int_0^t Y d^{\mp} X = XY(t) - XY(0) - \int_0^t X d^{\pm} Y, \tag{1.6}$$

provided that one of the above integrals exists.

(1.7) If $[X,X]$ exists and $[Y,Y]=0$ then $[X,Y]=0$.

(Using Hölder inequality we easily prove

$$|C_\varepsilon(X,Y)(t)| \le \sqrt{C_\varepsilon(X,X)(t) C_\varepsilon(Y,Y)(t)}.$$

This implies (1.7)).

If Z is any random variable,

$$\int_0^t ZY(s) d^{\mp} X(s) = Z \int_0^t Y(s) d^{\mp} X(s). \tag{1.8}$$

A m-dimensional real process $(X_1, \dots, X_n)$ is said to have **all their mutual brackets** if $[X_i, X_j]$ exist for every $i,j = 1, \dots, n$. In this case

$$[X_i + X_j, X_i + X_j] = [X_i, X_i] + 2[X_i, X_j] + [X_j, X_j]. \tag{1.9}$$

Since $[X_i, X_j]$ is a difference of increasing processes, it is locally of bounded variation. The following property is proven in [RV2]. Let $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_m)$ having all their mutual brackets and $f, g \in C^1(\mathbb{R}^n)$.

Then $\big(f(X), g(Y)\big)$ has all its mutual brackets and,

$$[f(X), g(Y)](t) = \sum_{i,j=1}^{n} \int_0^t \partial_i f(X)(s) \partial_j g(Y)(s) d[X_i, Y_j](s) \tag{1.10}$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$.

Itô formula. This is a slightly different version with respect to the one stated in [RV2]. The proof is very similar. Let $X = (X_1, \dots, X_n)$ having all the mutual brackets and $V = (V_1, \dots, V_p)$ be a bounded variation continuous process.

$C^{2,1}(\mathbb{R}^{n+p})$ will stand for the set of functions $f \in C^1(\mathbb{R}^{n+p})$ such that $\partial_i f$ is C^1 for any $1 \le i \le n$. Let $f \in C^{2,1}(\mathbb{R}^{n+p})$. We suppose

$$\int_0^t \partial_i f\big(X(s), V(s)\big) d^- X_j(s) \text{ to exist, } 1 \le i \le n. \tag{1.11}$$

Then

$$
(1.12)\quad
\begin{aligned}
f\big(X(t)\big) &= f\big(X(0)\big) + \sum_{i=1}^{n}\int_0^t \partial_i f\big(X(s),V(s)\big)d^-X_i(s)\\
&+ \sum_{j=1}^{p}\int_0^t \partial_j f\big(X(s),V(s)\big)dV_j(s) + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \partial^2_{i,j} f\big(X(s),V(s)\big)d[X_i,X_j](s),
\end{aligned}
$$

with $\partial^2_{i,j} = \dfrac{\partial^2}{\partial x_i \partial x_j}$.

We remark that if $n = 1$, then (1.11) does not need to be assumed.

2. Examples.

We list here some examples of processes for which the theory we develop could be applied.

1) If X has locally bounded variation and Y is càd làg then $\int_0^t Y d^- X = \int_0^t Y d^+ X$ and it is the usual pathwise Stieljes integral (consequence of the proof of proposition 1.1 in [RV2]).

2) Let X and Y be continuous semimartingales with respect to some filtration $\mathcal{F}$, H a làg làd (i.e. H admits a.s. left and right limits at any time t) $\mathcal{F}$-previsible process.

Then $\int_0^t H d^- Y$ coincides with the Itô integral $\int_0^t H dY$ and the Fisk-Stratonovich integral $\int_0^t X \circ dY$ equals $\frac{1}{2}\int_0^t X d^- Y + \frac{1}{2}\int_0^t X d^+ Y$.

$[X, Y]$ is the usual covariation of X and Y denoted by $\langle X, Y\rangle$ (see essentially proposition 1.1 in [RV2]).

If M, N are given by

$$
M(t) = \int_0^t H(s) dX(s),\ N(t) = \int_0^t K(s) dY(s)
$$

for some previsible locally bounded processes H, K, then

$$
(2.1)\qquad [M,N](t) = \langle M,N\rangle(t) = \int_0^t H(s)K(s)d\langle X,Y\rangle(s).
$$

This property will be generalized in section 5.

3) We spend some words about the relation with Skorohod integral. Let B be a standard Brownian motion, $B_0 = 0$. First of all, if we replace ordinary product with Wick product

(see [RV1], [HOUZ]) then the ε-approximations converge to the Skorohod integral $\int_0^t H\delta B$ provided some suitable assumptions are made on H.
Moreover the forward and backward integrals are equal to Skorohod integral, plus a trace term of the Gross-Malliavin derivative DX, see [RV1 ; NP ; SU ; Z].
If

$$M(t) = \int_0^t U\delta B,\ N(t) = \int_0^t V\delta B,$$

where U, V fulfill the strong integral representation stated in [DN], then $[M,N](t) = \int_0^t U(s)V(s)ds$.
If U and V are adapted, the above identity corresponds to (2.1).
The Skorohod integral is one extension of Itô integral to the anticipating framework (of fonctional analysis nature). The forward integral is another extension of Itô integral. According to the considerations above, these two extensions do not coincide ; if the integrands are adapted, the trace term vanishes.

4) The forward integral does not depend either on the underlying probability measure or on filtrations. For this reason, it coincides with the integral which could be obtained by enlargement of filtrations, see eq [J]. This allows to relate the integral coming from enlargement of filtration and Skorohod integral.

5) **Substitution formulae.** They are ones of the main tools of this theory : it is often a replacement of the "enlargement" techniques.
They are useful for instance
a) in providing existence theorems to stochastic differential equations with anticipating initial condition, see [RV1], [RV3], [RV4],
b) in generalizing Itô formula for C^1 functions of stochastic flows, see [RV3].
Let $\big(X_i(t,x), t \ge 0, x \in \mathbb{R}^d\big)$, $i = 1,2$, be two family of continuous semimartingales with respect to some usual filtration $(\mathcal{F}_t)_{t\ge 0}$. Let $\big(H(t,x), t \ge 0, x \in \mathbb{R}^d\big)$ be a family of adapted processes depending on a parameter. Let α be a random variable taking its values in $\mathbb{R}^d$. Under suitable assumptions of Kolmogorov type on X_1, X_2 with respect to x we have
(2.2) $[X_1(.,\alpha), X_2(.,\alpha)]$ exists and equals $[X_1(.,x), X_2(.,x)]\big|_{x=\alpha}$.
This provides another class of examples of finite quadratic variation process, see [RV2].
Suppose that $X_1 = X_2 = X$, with additional assumptions on H, we have the existence of

$$\int_0^t H(s,\alpha)d^-X(s,\alpha) = \int_0^t H(s,x)d_sX(s,x)\big|_{x=\alpha}. \tag{2.3}$$

This result will be used in section 6. So we state more precisely it. It is a small modification of proposition 1.3, [RV3].

Proposition 2.1. *We suppose the following conditions*

i) $Y = M + V$, where $\big(M(t,x), t \geq 0, x \in \mathbb{R}^d\big)$ is a family of martingales depending on a parameter, and $\big(V(t,x), t \geq 0, x \in \mathbb{R}^d\big)$ a family of bounded variation continuous processes for every $x \in \mathbb{R}^d$.
ii) For any compact set $K \subset \mathbb{R}^d$, $T > 0$, we suppose the existence of $p > 0$, $\gamma = \gamma(p), \gamma' = \gamma'(p)$ both greater than d, a constant $c = c(p)$ such that

$$E\Big(\sup_{t\leq T} |H(t,x) - H(t,y)|^p\Big) \leq c\,|x-y|^\gamma$$
$$E\Big(\sup_{t\leq T} |H(t,0)|^p\Big) < \infty$$
$$E\big(|M(T,x) - M(T,y)|^p\big) \leq c\,|x-y|^{\gamma'}$$
$$\sup_{t\leq T} E\big(|M(t,0)|^p\big) < \infty.$$

Under such conditions, the substitution formula (2.3) holds.

6) **Delayed processes.** Let $\big(S(t), t \geq 0\big)$ be a continuous semimartingale with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$ and a continuous process $X = \big(X(t), t \geq 0\big)$ being $\big(\mathcal{F}_{(t-\tau)\vee 0}, t \geq 0\big)$-measurable for some delay $\tau > 0$.
One example can be provided by $X = \Big(\sigma\big(Y\big((s-\tau)\vee 0\big)\big), s \geq 0\Big)$ where Y is a solution of a stochastic delay equation

$$Y(t) = \int_0^t \sigma\big(Y\big((u-\tau)\vee 0\big)\big)dS(u).$$

For such kind of equations, see e.g. [M].

Proposition 2.2. *The covariation process $[X,S]$ vanishes. In particular the Itô integral $\int_0^t X d^- S$ equals the backward integral $\int_0^t Y d^+ S$.*

Proof. Let M the local martingale part of S. Through localization arguments we can suppose M to be a bounded continuous martingale, X and $[M,M]$ are bounded. Since S-M has locally bounded variation, it is enough to show that$[X,M] \equiv 0$. Let $\varepsilon \in]0,\tau[$, $\varepsilon \leq 1$. We set

$$C_\varepsilon(t) = C_\varepsilon(X,M) = \frac{1}{\varepsilon}\int_0^t \big(X(s+\varepsilon) - X(s)\big)\big(M(s+\varepsilon) - M(s)\big)ds.$$

We decompose the integral in two parts : C^1_ε (resp. C^2_ε) is defined by integrating from 0 (resp. $(t-\varepsilon)\vee 0$) to $(t-\varepsilon)\vee 0$ (resp. t). It is clear that,

$$|C^2_\varepsilon(t)| \leq \max_{0\leq s\leq t+1} \big|\big(X(s+\varepsilon) - X(s)\big)\big(M(s+\varepsilon) - M(s)\big)\big|.$$

Since X and M are continuous, they are uniformly continuous on any compact interval, hence C_ε^2 goes ucp to 0, as $\varepsilon \to 0_+$.
Since $\varepsilon < \tau$, the r.v. $X(s+\varepsilon) - X(s)$ is $\mathcal{F}_s$-measurable, therefore

$$\big(X(s+\varepsilon) - X(s)\big)\big(M(s+\varepsilon) - M(s)\big) = \int_s^{s+\varepsilon} \big(X(s+\varepsilon) - X(s)\big) dM(u).$$

We apply Fubini type theorem :

$$C_\varepsilon^1(t) = \int_0^t H_\varepsilon(u) dM(u),$$

where

$$H_\varepsilon(u) = \frac{1}{\varepsilon} \int_{(u-\varepsilon)\vee 0}^{u} \big(X(s+\varepsilon) - X(s)\big) ds.$$

X being continuous, H_ε goes ucp to 0.
Using Doob inequality we obtain

$$E\Big[\sup_{0\le t\le T} C_\varepsilon^1(t)^2\Big] \le 4E[C_\varepsilon^1(T)^2],$$

$$E[C_\varepsilon^1(T)^2] = E\Big[\int_0^T H_\varepsilon(u)^2 d\langle M, M\rangle(u)\Big].$$

Using the definition of H_ε, we immediatly obtain :

$$\sup_{0\le u\le T} |H_\varepsilon(u)| \le \sup_{0\le s\le T+1} |X(s+\varepsilon) - X(s)| \le 2 \sup_{0\le s\le T+1} |X(s)|.$$

This shows that H_ε is bounded. Consequently, the limit in $L^2(\Omega)$ of $C_\varepsilon^1(T)$ is null. This proves : $[X, M] = 0$. □

Next example is more complicated and it deserves a separate section: it investigates the case of Gaussian processes (respectively section 3 and 4).

3. The Gaussian case.

Let $\big(X(t)\big)_{t\ge 0}$ be a second order continuous process. We set

$$m(t) = E\big(X(t)\big),\ K(s,t) = \mathrm{Cov}\big(X(s), X(t)\big) = E\big(X(s)X(t)\big) - m(s)m(t)$$

for $0 \le s \le t < \infty$.
We observe that $[X, X]$ may not exist for every $t > 0$; it is enough to take a deterministic function $X \equiv m$ such that $[m, m]$ is infinite.

From now on we will suppose $[m,m]$ zero. In this case setting $Y(t) = X(t) - m(t)$, we have

$$[X,X] = [Y,Y] + 2[Y,m] + [m,m] = [Y,Y].$$

Property (1.7) implies that $[Y,m] = 0$.
So we will concentrate on a mean-zero process X. We will say that X has a **finite energy** if

(3.1) $E[C_\varepsilon(X,X)(.)]$ converges uniformly on each compact of $\mathbb{R}_+$, as ε goes to 0.

In this case the deterministic function (3.1) will be called **energy of** X and denoted by $\mathcal{E}n(X)$. Since $t \to E[C_\varepsilon(X,X)(t)]$ is a continuous function, $\mathcal{E}n(X)$ is continuous.
The object of this section is to find first necessary and sufficient conditions on the covariance K of X such that X has finite energy.
Among the finite energy processes we will give (almost) necessary and sufficient conditions on K so that $[X,X]$ exists and necessary and sufficient conditions on K so that $[X,X]$ is deterministic.
First of all we state and prove a lemma showing that the problem of the existence of $[X,X]$ (resp. $\mathcal{E}n(X)$) can be just reduced to a convergence in probability at each instant (resp. to a pointwise convergence).

Lemma 3.1. *Let $(Z_\varepsilon)_{\varepsilon>0}$ be a family of continuous processes. We suppose*

1) $\forall \varepsilon > 0$, $t \to Z_\varepsilon(t)$ *is increasing.*

2) *There is a continuous process $\big(Z(t)\big)_{t\geq 0}$ such that $Z_\varepsilon(t) \to Z(t)$ in probability when ε goes to zero.*

Then Z_ε converges to Z ucp.

Proof. Since Z is continuous and because of assumptions 1) and 2), it is clear that $\big(Z(t)\big)$ is an increasing process.
Let $T, \rho, \alpha > 0, N \in \mathbb{N}$. We set $t_i^N = \dfrac{iT}{N}$, $0 \leq i \leq N$.
We only consider processes on $[0,T]$.
For almost all ω in Ω,

$$\sup_i |Z(t_{i+1}^N) - Z(t_i^N)|(\omega) \leq \delta\Big(Z(.,\omega)\,;\frac{1}{N}\Big)$$

where $\delta\Big(Z(.,\omega)\,;\dfrac{1}{N}\Big)$ is the continuity modulus of $\big(Z(t,\omega), 0 \leq t \leq T\big)$.
Since $\delta\Big(Z(.,\omega)\,;\dfrac{1}{N}\Big) \underset{N\to\infty}{\longrightarrow} 0$, a.s., so it converges in probability: we choose N so that

$$P\Big\{\delta\Big(Z(.)\,;\frac{1}{N}\Big) > \frac{\alpha}{4}\Big\} \leq \rho/2.$$

We define

$$A=\Big\{\sup_{0\le t\le T}|Z_\varepsilon(t)-Z(t)|>\alpha\Big\}.$$

From now on, we simply write $t_i=t_i^N, 0\le i\le N$. Since

$$\sup_{0\le t\le T}|Z_\varepsilon(t)-Z(t)|=\sup_{0\le i\le N-1}\Big(\sup_{t\in[t_i,t_{i+1}]}|Z_\varepsilon(t)-Z(t)|\Big)$$

we decompose A as follows,

$$A=\bigcup_{i=0}^{N-1}A_i\ ;\ A_i=\Big\{\sup_{t\in[t_i,t_{i+1}]}|Z_\varepsilon(t)-Z(t)|>\alpha\Big\}\ ;\ 0\le i\le N-1.$$

Let $t\in[t_i,t_{i+1}]$. Process Z^ε and Z being increasing, we have

$$Z_\varepsilon(t)-Z(t)\le Z_\varepsilon(t_{i+1})-Z(t_i).$$

We modify the right hand-side of the above inequality,

$$Z_\varepsilon(t_{i+1})-Z(t_i)=Z_\varepsilon(t_{i+1})-Z(t_{i+1})+Z(t_{i+1})-Z(t_i).$$

Then

$$Z_\varepsilon(t_{i+1})-Z(t_i)\le Z_\varepsilon(t_{i+1})-Z(t_{i+1})+\delta\Big(Z(.)\,;\frac{1}{N}\Big),$$
$$Z_\varepsilon(t)-Z(t)\le Z_\varepsilon(t_{i+1})-Z(t_{i+1})+\delta\Big(Z(.)\,;\frac{1}{N}\Big).$$

Similarly

$$Z_\varepsilon(t)-Z(t)\ge Z_\varepsilon(t_i)-Z(t_i)-\delta\Big(Z(.)\,;\frac{1}{N}\Big).$$

Therefore,

$$|Z_\varepsilon(t)-Z(t)|\le 2\delta\Big(Z(.)\,;\frac{1}{N}\Big)+|Z_\varepsilon(t_{i+1})-Z(t_{i+1})|+|Z_\varepsilon(t_i)-Z(t_i)|\ ;\ \forall t\in[t_i,t_{i+1}].$$

So,

$$\sup_{t\in[t_i,t_{i+1}]}|Z_\varepsilon(t)-Z(t)|\le 2\delta\Big(Z(.)\,;\frac{1}{N}\Big)+|Z_\varepsilon(t_{i+1})-Z(t_{i+1})|+|Z_\varepsilon(t_i)-Z(t_i)|.$$

Consequently,

$$A_i\subset\Big\{\delta\Big(Z(.)\,;\frac{1}{N}\Big)>\frac{\alpha}{4}\Big\}\cup\widetilde{A}_i\cup\widetilde{A}_{i+1},$$

where

$$\widetilde{A}_i=\Big\{|Z_\varepsilon(t_i)-Z(t_i)|>\frac{\alpha}{4}\Big\}.$$

But $Z_\varepsilon(t_i)$ converges in probability to $Z(t_i)$, for any $0 \le i \le N-1$, ε going to 0. So that there exists $\varepsilon_0 > 0$ such that : $\varepsilon < \varepsilon_0 \Rightarrow P(\widetilde{A}_i) \le \dfrac{\rho}{4N}$.
Finally

$$P(A) \le \sum_{i=0}^{N-1} P(A_i) \le P\left(\delta\Big(Z(.)\,;\frac{1}{N}\Big) > \frac{\alpha}{4}\right) + 2\Big(\sum_{i=0}^{N-1} P(\widetilde{A}_i)\Big) \le \frac{\rho}{2} + \frac{\rho}{4N}2N = \rho.$$

Since $\rho > 0$ is arbitrary small, the result is established. □

We go on fixing some notations.
Let $X = \big(X(t), t \ge 0\big)$ be a mean-zero continuous process. We set

$$C_\varepsilon(t) = C_\varepsilon(X,X)(t) = \frac{1}{\varepsilon}\int_0^t \big(X(s+\varepsilon) - X(s)\big)^2 ds, \tag{3.2}$$

$$a_\varepsilon(t) = E\big(C_\varepsilon(t)\big). \tag{3.3}$$

Given a function $K : \mathbb{R}_+^2 \to \mathbb{R}$, $\varepsilon > 0$, we denote by

$$\Delta_{\varepsilon,\delta}K(s,t) = K(s+\varepsilon, t+\delta) + K(s,t) - K(s,t+\delta) - K(s+\varepsilon,t). \tag{3.4}$$

For $\gamma > 0$, we say that K has a γ-planar variation if

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}} \frac{1}{\varepsilon\delta}\int_{[0,t]^2} |\Delta_{\varepsilon,\delta}K(u,v)|^\gamma du dv \tag{3.5}$$

converges for any $t \ge 0$.

Remarks 3.2 : 1) If K is the covariance of a mean-zero process,

$$\Delta_{\varepsilon,\delta}K(s,t) = E\Big[\big(X(s+\varepsilon) - X(s)\big)\big(X(t+\delta) - X(t)\big)\Big]. \tag{3.6}$$

In particular

$$\Delta_{\varepsilon,\varepsilon}K(s,s) = E[(X(s+\varepsilon) - X(s))^2] = K(s+\varepsilon, s+\varepsilon) + K(s,s) - 2K(s,s+\varepsilon) \ge 0. \tag{3.7}$$

2) Suppose that K is the distribution function of a probability measure μ on $[0,+\infty[\times[0,+\infty[$ $\big($i.e. $K(s,t) = \mu(]0,s]\times]0,t])\big)$ then

$$\Delta_{\varepsilon,\delta}K(s,t) = \mu(]s,s+\varepsilon]\times]t,t+\delta]). \tag{3.8}$$

We apply the Fubini theorem, we obtain

$$\frac{1}{\varepsilon\delta}\int_{[0,t]^2}\Delta_{\varepsilon,\delta}K(u,v)dudv=\frac{1}{\varepsilon\delta}\int_{]0,t+\varepsilon]\times]0,t+\delta]}\big(x\wedge t-(x-\varepsilon)_+\big)\big(y\wedge t-(y-\delta)_+\big)\mu(dx,dy).$$

Consequently the 1-planar variation is equal to $\mu([0,t]\times[0,t])=K(t,t)$.

Proposition 3.3. *Suppose that X is a continuous process, with zero mean and covariance K.*

1) *The following properties are equivalent :*
X has a finite energy,

$$\lim_{\varepsilon\to 0_+}\frac{1}{\varepsilon}\int_0^t \Delta_{\varepsilon,\varepsilon}K(s,s)ds \text{ exists,} \tag{3.9}$$

$$\int_0^t \partial_2 K(s,s_+)ds := \lim_{\varepsilon\to 0_+}\int_0^t \frac{K(s,s+\varepsilon)-K(s,s)}{\varepsilon}ds \text{ exists,} \tag{3.10}$$

$$\int_0^t \partial_1 K(s_-,s)ds := \lim_{\varepsilon\to 0_+}\int_0^t \frac{K(s,s)-K\big((s-\varepsilon)\vee 0,s\big)}{\varepsilon}ds \text{ exists,} \tag{3.11}$$

We specify that all the convergence above are intended uniformly on compact intervals.

2) *If one the previous limits exists, then*

$$\mathcal{E}n(X)(t)=\lim_{\varepsilon\to 0_+}\frac{1}{\varepsilon}\int_0^t \Delta_{\varepsilon,\varepsilon}K(s,s)ds \tag{3.12}$$

$$\begin{aligned}\mathcal{E}n(X)(t)&=K(t,t)-K(0,0)-2\int_0^t \partial_2 K(s,s_+)ds\\&=-K(t,t)+K(0,0)+2\int_0^t \partial_1 K(s_-,s)ds.\end{aligned} \tag{3.13}$$

In particular,

$$K(t,t)-K(0,0)=\int_0^t \partial_2 K(s,s_+)ds+\int_0^t \partial_1 K(s_-,s)ds. \tag{3.14}$$

Remarks 3.4 : 1) If K admits a partial derivative with respect to the first (resp. second) variable at any (s,s) then

$$\lim_{\varepsilon\to 0}\frac{K(s,s)-K(s-\varepsilon,s)}{\varepsilon}=\partial_1 K(s,s)\Big(\text{resp. }\lim_{\varepsilon\to 0}\frac{K(s,s+\varepsilon)-K(s,s)}{\varepsilon}=\partial_2 K(s,s)\Big).$$

This justifies the notation of the limits in (3.11) (resp. (3.10)).
2) If one of the limits in (3.10), (3.11), (3.12) exists pointwise and it is continuous, then the convergence holds on each compact.

Proof of proposition 3.3.
a) We have

$$a_\varepsilon(t) = E[C_\varepsilon(t)] = \int_0^t \frac{E[(X_{s+\varepsilon} - X_s)^2]}{\varepsilon} ds = \frac{1}{\varepsilon} \int_0^t \Delta_{\varepsilon,\varepsilon} K(s,s) ds.$$

It is now obvious that X has a finite energy if and only if (3.9) holds.
Identity (3.7) implies

$$a_\varepsilon(t) = \frac{1}{\varepsilon}\Big(\int_0^t K(s+\varepsilon, s+\varepsilon) ds + \int_0^t K(s,s) ds - 2 \int_0^t K(s, s+\varepsilon) ds \Big).$$

The first integral in the above right hand-side is equal to $\displaystyle\int_\varepsilon^{t+\varepsilon} K(s,s)ds$, consequently,

$$a_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} K(s,s) ds - \frac{1}{\varepsilon} \int_0^\varepsilon K(s,s) ds - 2 \int_0^t \frac{K(s,s+\varepsilon) - K(s,s)}{\varepsilon} ds. \tag{3.15}$$

This shows that X has a finite energy if and only if (3.10) is satisfied, and the first equality in (3.13).
Part 2) of remark 3.4 will be a direct consequence of lemma 3.2, (3.15) and (3.7).
b) We note that

$$C_\varepsilon(t) = \int_\varepsilon^{t+\varepsilon} \frac{\big(X(s) - X(s-\varepsilon)\big)^2}{\varepsilon} ds.$$

Then the equivalence involving (3.11) will follow by a similar manipulation. □

We now consider mean-zero, Gaussian processes.
We are ready now to state conditions under which finite energy processes are finite quadratic variation processes.

Proposition 3.5. *Let X be a continuous, mean-zero, Gaussian and finite energy process. Then $\lim_{\varepsilon\to 0_+} C_\varepsilon(t)$ exists in probability for every $t \geq 0$ if and only if K has the 2-planar variation.*

Remark 3.6 : Previous statement means that the existence of the 2-planar variation is equivalent to the “weak” existence of the covariation.

The proof of proposition 3.5 is based on a preliminary result.

Lemma 3.7. *Let* (G_1, G_2) *be a two-dimensional mean-zero Gaussian vector. Then* $\mathrm{Cov}(G_1^2, G_2^2) = 2\theta^2$*, where* $\theta = \mathrm{Cov}(G_1, G_2)$*.*

Proof. We set $\sigma_i^2 = E(G_i^2)$, $i = 1, 2$. By linear regression we have $G_2 = \dfrac{\theta}{\sigma_1^2} G_1 + G_3$, where G_1 and G_3 are independent mean-zero Gaussian random variables.
This implies

$$E(G_2^2 | G_1) = \frac{\theta^2}{\sigma_1^4} G_1^2 + E(G_3^2). \tag{3.16}$$

On the other hand

$$\mathrm{Var}(G_2) = \frac{\theta^2}{\sigma_1^4}\sigma_1^2 + \mathrm{Var}\, G_3,$$

so

$$E(G_3^2) = \mathrm{Var}\, G_3 = \sigma_2^2 - \frac{\theta^2}{\sigma_1^2}. \tag{3.17}$$

Consequently using (3.16) and (3.17), we obtain

$$\begin{aligned} E(G_1^2 G_2^2) = E\big(G_1^2 E(G_2^2 | G_1)\big) &= \frac{\theta^2}{\sigma_1^4} E(G_1^4) + \Big(\sigma_2^2 - \frac{\theta^2}{\sigma_1^2}\Big) E(G_1^2) \\ &= 3\theta^2 + \sigma_1^2\sigma_2^2 - \theta^2 = 2\theta^2 + \sigma_1^2\sigma_2^2. \end{aligned}$$

Finally

$$\mathrm{Cov}(G_1^2, G_2^2) = E(G_1^2 G_2^2) - E(G_1^2)E(G_2^2) = 2\theta^2.$$

□

Proof of proposition 3.5.
a) For $\varepsilon, \delta > 0$ and $t \geq 0$, we first evaluate,

$$E\big[\big(C_\varepsilon(t) - C_\delta(t)\big)^2\big] = E[C_\varepsilon(t)^2] + E[C_\delta(t)^2] - 2E[C_\varepsilon(t)C_\delta(t)]. \tag{3.18}$$

We have,

$$E[C_\varepsilon(t)C_\delta(t)] = \frac{1}{\varepsilon\delta}\int_{[0,t]^2} E[G_\varepsilon(u)^2 G_\delta(v)^2] du dv, \tag{3.19}$$

where

$$G_\varepsilon(u) = X(u+\varepsilon) - X(u). \tag{3.20}$$

Since,

$$E[G_\varepsilon(u)^2 G_\delta(v)^2] = \mathrm{Cov}\big(G_\varepsilon(u)^2, G_\delta(v)^2\big) + E[G_\varepsilon(u)^2]E[G_\delta(v)^2],$$

(3.21) $$E[C_\varepsilon(t)C_\delta(t)] = I_1(\varepsilon,\delta) + I_2(\varepsilon,\delta),$$

with
$$I_1(\varepsilon,\delta) = \frac{1}{\varepsilon\delta}\int_{[0,t]^2} \operatorname{Cov}\big(G_\varepsilon(u)^2, G_\delta(v)^2\big)dudv,$$
$$I_2(\varepsilon,\delta) = \Big(\frac{1}{\varepsilon}\int_0^t E[G_\varepsilon(u)^2]du\Big)\Big(\frac{1}{\delta}\int_0^t E[G_\delta(v)^2]dv\Big).$$

Using lemma 3.7 and (3.6), we have

(3.22) $$I_1(\varepsilon,\delta) = \frac{2}{\varepsilon\delta}\int_{[0,t]^2} \big(\Delta_{\varepsilon,\delta}K(u,v)\big)^2 dudv,$$

(3.23) $$I_2(\varepsilon,\delta) = \Big(\frac{1}{\varepsilon}\int_0^t \Delta_{\varepsilon,\varepsilon}K(u,u)du\Big)\Big(\frac{1}{\delta}\int_0^t \Delta_{\delta,\delta}K(v,v)dv\Big).$$

Since X has a finite energy,

(3.24) $$\lim_{\varepsilon,\delta\to 0_+} I_2(\varepsilon,\delta) = \big(\mathcal{E}n(X)(t)\big)^2, \text{for any } t \geq 0.$$

b) We claim that $E[C_\varepsilon(t)^4]$ is uniformly bounded in $\varepsilon > 0$. Using the Cauchy-Schwarz inegality, we get

$$E[C_\varepsilon(t)^4] \leq \frac{1}{\varepsilon^4}\int_{[0,t]^4} \{E[G_\varepsilon(u_1)^8]E[G_\varepsilon(u_2)^8]E[G_\varepsilon(u_3)^8]E[G_\varepsilon(u_4)^8]\}^{1/4}du_1du_2du_3du_4.$$

$G_\varepsilon(u)$ being a mean-zero, Gaussian variable with variance $\Delta_\varepsilon K(u,u)$,

$$E\big[\big(G_\varepsilon(u)\big)^8\big] = \big(\Delta_\varepsilon K(u,u)\big)^4 E(G^8) = c\big(\Delta_\varepsilon K(u,u)\big)^4,$$

where G is a standard Gaussian variable $\big($i.e. $E(G) = 0$ and $\operatorname{Var} G = 1\big)$
Therefore,

(3.25) $$E[C_\varepsilon(t)^4] \leq c\Big(\frac{1}{\varepsilon}\int_0^t \Delta_\varepsilon K(u,u)du\Big)^4.$$

Since the upper bound converges to $\big(\mathcal{E}n(X)\big)^4$, then $\sup_{0<\varepsilon\leq 1} E[C_\varepsilon(t)^4] < \infty$.

c) Let us suppose that $C_\varepsilon(t)$ converges in probability for each $t \geq 0$. (3.25) says that $\big(C_\varepsilon^2(t)\,;\varepsilon > 0\big)$ is uniformly integrable, so that $C_\varepsilon(t)$ converges in $L^2(\Omega)$.
Then $E\big[\big(C_\varepsilon(t) - C_\delta(t)\big)^2\big]$ goes to zero, when $\varepsilon,\delta \to 0_+$.

Coming back to (3.18), we have,

$$\lim_{\varepsilon,\delta\to 0_+} E[C_\varepsilon(t)C_\delta(t)] = \lim_{\varepsilon\to 0_+} E[C_\varepsilon(t)^2]. \tag{3.26}$$

If we make use of (3.21), (3.24) and (3.22), then (3.26) implies that K has a 2-planar variation and

$$\mathcal{E}n(X)(t)^2 + \lim_{\varepsilon,\delta\to 0_+} \int_{[0,t]^2} \big(\Delta_{\varepsilon,\delta} K(u,v)\big)^2 du dv = \lim_{\varepsilon\to 0_+} E[C_\varepsilon(t)^2]. \tag{3.27}$$

d) Conversely, suppose K has a 2-planar variation. Then $I_1(\varepsilon,\delta)$ converges, (3.21), (3.24) both imply,

$$\lim_{\varepsilon,\delta\to 0_+} E[C_\varepsilon(t)C_\delta(t)] = \lim_{\varepsilon,\delta\to 0_+} \int_{[0,t]^2} \big(\Delta_{\varepsilon,\delta} K(u,v)\big)^2 du dv + \big(\mathcal{E}n(X)(t)\big)^2.$$

In particular, $E\big(C_\varepsilon(t)^2\big)$ converges to the same limit. This and (3.18) show that $C_\varepsilon(t)$ converges in $L^2(\Omega)$, hence in probability. ∎

Remark 3.8 : Suppose that X verifies one of the two equivalent properties of proposition 3.5. We have shown that,

(3.28) $C_\varepsilon(t)$ converges in $L^2(\Omega)$ and

$$\mathcal{E}n(X)(t)^2 + \lim_{\varepsilon,\delta\to 0_+} \int_{[0,t]^2} \big(\Delta_{\varepsilon,\delta} K(u,v)\big)^2 du dv = \lim_{\varepsilon\to 0_+} E[C_\varepsilon(t)^2]. \tag{3.29}$$

Under some more restrictive assumptions the quadratic variation of a Gaussian process is deterministic.

Proposition 3.9. *Let X be a continuous, mean-zero Gaussian process with finite energy. K denotes the covariance of X. Then $\lim_{\varepsilon\to 0} C_\varepsilon(t)$ (in probability) exists and is deterministic for every $t \geq 0$, if and only if, the 2-planar variation of K is zero. In this case $[X,X]$ exists and equals $\mathcal{E}n(X)$.*

Proof. Let $\varepsilon > 0$ and $t \geq 0$. We have

$$E\Big[\big(C_\varepsilon(t) - \mathcal{E}n(X)(t)\big)^2\Big] = E[C_\varepsilon(t)^2] - 2\mathcal{E}n(X)(t)E[C_\varepsilon(t)] + \big(\mathcal{E}n(X)(t)\big)^2. \tag{3.30}$$

Since X has a finite energy, $\lim_{\varepsilon\to 0} E[C_\varepsilon(t)] = \mathcal{E}n(X)(t)$.
Using moreover (3.21) (with $\delta = \varepsilon$), (3.22) and (3.24), we obtain :

$$\lim_{\varepsilon\to 0} E\Big[\big(C_\varepsilon(t) - \mathcal{E}n(X)(t)\big)^2\Big] = \lim_{\varepsilon,\delta\to 0_+} \int_{[0,t]^2} \big(\Delta_{\varepsilon,\delta} K(u,v)\big)^2 du dv. \tag{3.31}$$

Therefore if the 2-planar variation of K is zero, then $\lim_{\varepsilon\to 0_+} C_\varepsilon(t) = \mathcal{E}n(X)(t)$ in $L^2(\Omega)$.
Vice versa if $\lim_{\varepsilon\to 0_+} C_\varepsilon(t)$ is deterministic, by (3.28) we have,

$$\lim_{\varepsilon\to 0_+} E[C_\varepsilon(t)^2] = E\Big[\lim_{\varepsilon\to 0_+} C_\varepsilon(t)^2\Big] = \Big(\lim_{\varepsilon\to 0_+} C_\varepsilon(t)\Big)^2,$$

$$\mathcal{E}n(X)(t) = \lim_{\varepsilon\to 0} E[C_\varepsilon(t)] = E\Big[\lim_{\varepsilon\to 0} C_\varepsilon(t)\Big] = \lim_{\varepsilon\to 0} C_\varepsilon(t).$$

Consequently $\lim_{\varepsilon\to 0_+} C_\varepsilon(t) = \mathcal{E}n(X)(t)$.
Using (3.28) again, and (3.30) we have proved that the 2-planar variation of K vanishes. We remark that in that case, $[X, X]$ exists because $t \to \mathcal{E}n(X)(t)$ is a continuous function and because of lemma 3.1. $\square$

Proposition 3.10.
1) *Suppose that the covariance K of X belongs to $C^1(\Delta_+)$, where $\Delta_+ = \{(s,t) \in \mathbb{R}_+^2\,; 0 \le s \le t\}$. Then X has energy, and quadratic variation; moreover the 2-planar variation of K is zero.*
Moreover,

$$(3.32)\qquad \begin{aligned}[X, X](t) = \mathcal{E}n(X)(t) &= K(t,t) - K(0,0) - 2\int_0^t \partial_2 K(s, s_+)ds \\ &= K(0,0) - K(t,t) + 2\int_0^t \partial_1 K(s_-, s)ds\end{aligned}$$

where
(3.33)

$$\partial_2 K(s, s_+) = \lim_{\varepsilon\to 0_+} \frac{K(s, s+\varepsilon) - K(s,s)}{\varepsilon}\ ;\ \partial_1 K(s_-, s) = \lim_{\varepsilon\to 0_+} \frac{K\big((s-\varepsilon)\vee 0, s\big) - K(s,s)}{\varepsilon}.$$

Remark 3.11 : Suppose that K is of class $C^1(\Delta_+)$. Then $t \to K(t,t)$ is of class C^1, with derivative $\partial_1 K(s_-, s) + \partial_2 K(s, s_+)$.
This shows (3.14) and the second equality in (3.32).

Proof of proposition 3.10. We suppose that K is of class $C^1(\Delta_+)$.
a) In this case (3.11) holds and then X has finite energy. Relation (3.30) tells us that $C_\varepsilon(t)$ converges to $\mathcal{E}n(X)(t)$, in $L^2(\Omega)$ if

$$\lim_{\varepsilon\to 0} E[C_\varepsilon(t)^2] = \mathcal{E}n(X)(t)^2.$$

We have established $\big($see (3.21), (3.24)$\big)$ that $E[C_\varepsilon(t)^2] = I_1(\varepsilon,\varepsilon) + I_2(\varepsilon,\varepsilon)$ and $\lim_{\varepsilon\to 0_+} I_2(\varepsilon,\varepsilon) = \big(\mathcal{E}n(X)(t)\big)^2$, $I_1(\varepsilon,\varepsilon)$ being defined by (3.22).

We decompose $\Delta_{\varepsilon,\varepsilon}K(u,v)$ as follows :

$$\frac{1}{\varepsilon}\Delta_{\varepsilon,\varepsilon}K(u,v) = A_\varepsilon^{(1)}(u,v) - A_\varepsilon^{(2)}(u,v) - A_\varepsilon^{(3)}(u,v)$$

where

$$A_\varepsilon^{(1)}(u,v) = \frac{K(u+\varepsilon,v+\varepsilon)-K(u,v)}{\varepsilon}\,; A_\varepsilon^{(2)}(u,v) = \frac{K(u+\varepsilon,v)-K(u,v)}{\varepsilon}\,;$$
$$A_\varepsilon^{(3)}(u,v) = \frac{K(u,v+\varepsilon)-K(u,v)}{\varepsilon}.$$

K being of class $C^1(\Delta_+)$, if $u<v$, then,

$$\lim_{\varepsilon\to 0_+} A_\varepsilon^{(1)}(u,v) = \partial_1 K(u,v) + \partial_2 K(u,v)\,; \lim_{\varepsilon\to 0_+} A_\varepsilon^{(2)}(u,v) = \partial_1 K(u,v)\,;$$
$$\lim_{\varepsilon\to 0_+} A_\varepsilon^{(3)}(u,v) = \partial_2 K(u,v).$$

Consequently,

$$\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\Delta_{\varepsilon,\varepsilon}K(u,v) = 0, \text{ for any } u<v.$$

Using again the smoothness of K, if $u<v$

$$\sup\{|A_\varepsilon^{(i)}(u,v)|\,; 0<\varepsilon\le 1, 0\le u\le v\le t\} < \infty, i=1,3,$$
$$\sup\{|A_\varepsilon^{(2)}(u,v)|\,; 0<\varepsilon\le 1\,; 0\le u+\varepsilon\le v\le t\} < \infty.$$

When $0\le u\le v\le u+\varepsilon$, we decompose $A_\varepsilon^{(2)}(u,v)$ as follows,

$$A_\varepsilon^{(2)}(u,v) = \frac{K(v,u+\varepsilon)-K(u,v)}{\varepsilon} = \frac{K(v,u+\varepsilon)-K(v,v)}{\varepsilon} + \frac{K(v,v)-K(u,v)}{\varepsilon}.$$

Hence,

$$\sup\{|A_\varepsilon^{(2)}(u,v)|\,; 0<\varepsilon\le 1\,; 0\le u\le v\le u+\varepsilon\le t\} < \infty.$$

Since the function $\Delta_{\varepsilon,\varepsilon}K$ is symmetric, the dominated convergence theorem gives,

$$\lim_{\varepsilon\to 0_+} I_1(\varepsilon,\varepsilon) = \lim_{\varepsilon\to 0_+} 4\int_{[0,t]^2}\Big(\frac{\Delta_{\varepsilon,\varepsilon}K(u,v)}{\varepsilon}\Big)^2 1_{\{u<v\}}dudv = 0.$$

b) We know that the convergence in $L^2(\Omega)$ implies the convergence in probability. Therefore C_ε converges to the deterministic function $\mathcal{E}n(X)$. Using proposition 3.9, we deduce that $[X,X]$ exists and coincides with $\mathcal{E}n$, and the 2-planar variation is equal to 0. $\square$

Remark 3.12 : If we look carefully at the proof of the above proposition, we observe that if X has finite energy and

$$\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon^2} \int\int_{[0,t]} \left(\Delta_{\varepsilon,\varepsilon} K(u,v)\right)^2 du dv = 0, \text{ for any } t \geq 0, \tag{3.34}$$

then $[X,X]$ exists, $[X,X] = \mathcal{E}n(X)$ and the 2-planar variation of K vanishes.
It is not obvious to verify directly that if $\mathcal{E}n(X)$ exists and (3.34) holds, then the 2-planar variation is equal to 0. In other words, it is an open question to prove analytically, that the 2-planar-variation is equal to 0, is a consequence of (3.34) and (3.11).

We would like to briefly consider the multidimensional case. Assume that $X = (X_1, \ldots, X_n)$ is a n-dimensional mean zero Gaussian process. We denote by $K = (K_{i,j})_{1\leq i,j\leq n}$ its covariance matrix. We set

$$\mathcal{E}n(X_i, X_j)(t) = \lim_{\varepsilon\to 0_+} E[C_\varepsilon(X_i, X_j)(t)], \tag{3.35}$$

if the limit exists uniformly on each compact interval.
The analog of proposition 3.10 will be

Proposition 3.13. *Suppose that K is of class $C^1(\Delta_+)$ (i.e. $K_{i,j}$ is of class $C^1(\Delta_+)$, for any $1 \leq i,j \leq n$). Then $\mathcal{E}n(X_i, X_j)$ exists and the 2-planar variation of $K_{i,j}$ is zero, for any $1 \leq i,j \leq n$. Moreover $(X_1, \ldots, X_n)$ has mutual brackets and,*

$$\begin{aligned} [X_i, X_j](t) = \mathcal{E}n(X_i, X_j)(t) &= K_{i,j}(t,t) - K_{i,j}(0,0) - 2\int_0^t \partial_2 K_{i,j}(s, s_+) ds \\ &= K_{i,j}(0,0) - K_{i,j}(t,t) + 2\int_0^t \partial_1 K_{i,j}(s_-, s) ds. \end{aligned} \tag{3.36}$$

Proof. We remark that $X_i + X_j$ is a Gaussian process with covariance $K_{i,i} + 2K_{i,j} + K_{j,j}$. Proposition 3.13 is a direct consequence of bilinearity and proposition 3.9. ∎

Before ending this section, we would like to analyze the example of the fractional Brownian motion X_H. Recall that X_H is a mean zero Gaussian process, with covariance K_H, given by :

$$K_H(s,t) = \frac{1}{2}(|s|^H + |t|^H - |s-t|^H)\,;(s,t) \in \mathbb{R}^2, \tag{3.37}$$

where $0 < H < 2$.
We restrict X_H to $\mathbb{R}_+$.
If $H = 1$, X_1 is the Brownian motion and

$$K_1(s,t) = s \wedge t\,; \quad s,t \geq 0.$$

In our context, it is easy to define this notion of α-variation.

Definition. Let X be a continuous process. We say that X has a α-variation, $\alpha > 0$ if $\int_0^\cdot \frac{|X(s+\varepsilon) - X(s)|^\alpha}{\varepsilon} ds$ converges ucp. The limit will be noted $[X, X]^{(\alpha)}$.
It is obvious that if $\alpha = 2$, $[X, X]^{(\alpha)} = [X, X]$.

Proposition 3.14.
The fractional Brownian motion X_H has a $2/H$-variation and $[X, X]^{(2/H)}(t) = \rho_H t$ where $\rho_H = E[|G|^{2/H}]$, G is a centered Gaussian r.v. with unit variance.

Proof.
1) Using lemma 3.1, it is sufficient to check that for any $t > 0$, $C_\varepsilon^{(\alpha)}(t)$ converges to t, in $L^2(\Omega)$, when $\varepsilon \to 0_+$, where

$$C_\varepsilon^{(\alpha)}(t) = \frac{1}{\varepsilon} \int_0^t |X_H(s+\varepsilon) - X_H(s)|^\alpha ds, \ \alpha = 2/H.$$

A direct calculation shows that,

$$K_H(s+\varepsilon, s+\varepsilon) + K_H(s,s) - 2K_H(s, s+\varepsilon) = \varepsilon^H. \tag{3.38}$$

Consequently, $X_H(s+\varepsilon) - X_H(s)$ is a mean zero Gaussian variable with variance ε^H and

$$E[|X_H(s+\varepsilon) - X_H(s)|^\alpha] = (\varepsilon^H)^{\alpha/2} \rho_H = \varepsilon \rho_H.$$

Therefore

$$E[C_\varepsilon^{(\alpha)}(t)] = \rho_H t. \tag{3.39}$$

We have,

$$E[(C_\varepsilon^{(\alpha)}(t) - \rho_H t)^2] = E[C_\varepsilon^{(\alpha)}(t)^2] - 2\rho_H t E[C_\varepsilon^{(\alpha)}(t)] + \rho_H^2 t^2 = E[C_\varepsilon^{(\alpha)}(t)^2] - \rho_H^2 t^2.$$

$C_\varepsilon^{(\alpha)}(t)$ goes to $\rho_H t$, in $L^2(\Omega)$, if and only if,

$$\lim_{\varepsilon \to 0_+} E[C_\varepsilon^{(\alpha)}(t)^2] = \rho_H^2 t^2. \tag{3.40}$$

2) We claim that (3.40) holds.
We have

$$E[C_\varepsilon(t)^2] = \frac{2}{\varepsilon^2} \int_{[0,t]^2} \mu_\varepsilon 1_{\{u<v\}} du dv, \tag{3.41}$$

where

$$\mu_\varepsilon = E\Big[\big|\big(X_H(u+\varepsilon) - X_H(u)\big)\big(X_H(v+\varepsilon) - X_H(v)\big)\big|^H\Big].$$

Using linear regression (see the proof of lemma 3.7), we obtain,

$$\mu_\varepsilon = E\Big[|\varepsilon^{H/2}N_1|^\alpha \Big|\frac{\theta_\varepsilon}{\varepsilon^{H/2}}N_1 + \sqrt{\varepsilon^H - \frac{\theta_\varepsilon^2}{\varepsilon^H}}N_2\Big|^\alpha\Big]$$

where $\theta_\varepsilon = \Delta_{\varepsilon,\varepsilon}K(u,v)$, N_1 and N_2 being two independent, Gaussian r.v. with distribution $\mathcal{N}(0,1)$.

Then

$$\frac{\mu_\varepsilon}{\varepsilon^2} = E\Big[|N_1|^\alpha \Big|\frac{\theta_\varepsilon}{\varepsilon^H}N_1 + \sqrt{1 - \Big(\frac{\theta_\varepsilon}{\varepsilon^H}\Big)^2}N_2\Big|^\alpha\Big]. \tag{3.42}$$

A straightforward calculation shows that,

$$\Delta_{\varepsilon,\varepsilon}K_H(u,v) = \frac{1}{2}(|u+\varepsilon-v|^H + |v+\varepsilon-u|^H - 2|u-v|^H).$$

Let φ be the function,

$$\varphi(s) = (s+1)^H + |s-1|^H - 2s^H \quad s \geq 0.$$

φ is a continuous function. If $s > 1$,

$$\varphi(s) = s^H\left(\Big(1+\frac{1}{s}\Big)^H + \Big(1-\frac{1}{s}\Big)^H - 2\right).$$

Using the asymptotic expansion of $(1+x)^H$, when $x \to 0$, the behaviour of $\varphi(s)$, $s \to +\infty$, is

$$\varphi(s) \underset{s\to+\infty}{\sim} s^H H(H-1)\Big(\frac{1}{s}\Big)^2 = \frac{H(H-1)}{s^{2-H}}. \tag{3.43}$$

Recall that $2-H>0$, therefore $\lim\limits_{s\to+\infty}\varphi(s) = 0$.

Consequently $|\varphi|$ is bounded.

But,

$$\theta_\varepsilon = \Delta_{\varepsilon,\varepsilon}K_H(u,v) = \frac{\varepsilon^H}{2}\varphi\Big(\frac{v-u}{\varepsilon}\Big)\ ;\ u \leq v. \tag{3.44}$$

This implies that $\Big|\dfrac{\theta_\varepsilon}{\varepsilon^H}\Big| \leq C$, $u \leq v$.

Moreover (3.43) and (3.44) imply that $\lim\limits_{\varepsilon\to 0_+}\dfrac{\theta_\varepsilon}{\varepsilon^H} = 0$, $u < v$.

(3.40) is a consequence of (3.41), (3.42) and the dominated convergence theorem. ∎

4. (Generalized) Itô processes.

Let X be a continuous process such that $[X,X]$ exists. From section 2, we know that Itô formula holds for $f(X)$, $f \in C^2$: in particular $\int_0^t f'(X)d^{\mp}X$ exist. In fact it is possible to see that the existence of $[X,X]$ is also a **necessary** condition for the validity of Itô formula for any $f \in C^2$.

Proposition 4.1. *Let X be a continuous process. The following properties are equivalent.*
a) *$[X,X]$ exists.*
b) *Itô formula holds for any $f \in C^2$.*
c) *$\int_0^\cdot g(X)d^{\mp}X$ exists for any $g \in C^1$.*

Proof. a) $\Rightarrow$ b) has been the object of the introductory lines. If b) is assumed and $g \in C^1$, we consider G such that $G' = g$. By Itô formula

$$\int_0^t g(X)d^{\mp}X = G(X(t)) - G(X_0) \mp \frac{1}{2}\int_0^t g'(X)d[X,X] \tag{4.1}$$

so that b) $\Rightarrow$ c) is established.

c) $\Rightarrow$ a). If c) holds then $[g(X),X]$ exists since it is equal to the difference $\int_0^t g(X)d^+X - \int_0^t g(X)d^-X$, $g \in C^1$. Taking $g(x) = x$, we have a). $\square$

Let X is a multidimensional process, $(X^1,\dots,X^n)$ having its mutual brackets. This is not a sufficient condition for guaranteeing that $g(X)$ is a good integrand of X^i, $g \in C^1(\mathbb{R}^n)$.

Definition. *We say that $X = (X^1,\dots,X^n)$ is a Itô process if and only if*

$$\int_0^\cdot g(X)d^-X^i \text{ exists}, \ 1 \le i \le n, \ g \in C^1(\mathbb{R}^n). \tag{4.2}$$

Remark 4.2 : a) If (4.2) is verified then also the following holds

$$\int_0^\cdot g(X)d^+X^i \text{ exists}, \ 1 \le i \le n, \ g \in C^1(\mathbb{R}^n). \tag{4.3}$$

In fact by (1.10), $[g(X),X^i]$ exists and therefore (1.5) implies that (4.3) is verified. Obviously (4.3) and (4.2) are equivalent in the previous definition.
b) A continuous semimartingale $X = (X^1,\dots,X^n)$ is a Itô process. In fact (4.2) is verified since $g(X)$ is an adapted process.

c) Any one-dimensional finite quadratic variation process is a Itô process.

Let us introduce $\mathcal{T}_X$ the class of locally bounded processes Z such that

$$\int_0^\cdot Zd^\pm X^i \text{ exists, for any } i = 1, \ldots, n. \tag{4.4}$$

Clearly (4.2) can be rewritten by saying $\{g(X), g \in C^1(\mathbb{R}^n)\} \subset \mathcal{T}_X$.

Proposition 4.3. *Let $X = (X^1, \ldots, X^n)$ be a Itô process. We suppose $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ of class C^2 and we set $Y = \varphi(X) = \big(\varphi_1(X), \ldots, \varphi_m(X)\big)$. Let Z be a locally bounded process.*
If $Zf(X)$ belongs to $\mathcal{T}_X$ for any $f \in C^1(\mathbb{R}^n)$ then $Z \in \mathcal{T}_Y$ and

$$\int_0^t Zd^\mp Y^k = \sum_{i=1}^n \int_0^t Z\partial_i\varphi_k(X)d^\mp X^i \pm \frac{1}{2} \sum_{1\le i,j\le n} \int_0^t Z_s\partial^2_{i,j}\varphi_k(X_s)d[X^i, X^j](s) \tag{4.5}$$

for any $f \in C^1(\mathbb{R}^n)$, $1 \le k \le m$.
In particular Y is a Itô process.

Remark 4.4 : a) If $n = m = 1$, $Y = \varphi(X)$ is a Itô process only assuming φ of class C^1.
b) The goal of (4.5) is to give a precise meaning of the differential equality :

$$d^\mp Y^k = \sum_{i=1}^n \partial_i\varphi_k(X)d^\mp X^i \pm \frac{1}{2} \sum_{1\le i,j\le n} \partial^2_{i,j}\varphi_k(X(s))d[X^i, X^j](s). \tag{4.6}$$

We claim that (4.6) holds, if we choose an integrand belonging to $\mathcal{T}_Y$.
These differential equalities play a central role in the section 5, devoted to stochastic differential equation.

Proof of the proposition 4.3. For simplicity we suppose $m = 1$.
Let f be a fixed function belonging to $C^1(\mathbb{R})$. We have to check that $\int_0^\cdot Zd^\pm Y$ exist, and (4.5) holds, where $Y = \varphi(X)$. We only consider the forward integral, the approach of the backward integral being similar. We know that,

$$\int_0^\cdot Zd^- Y = \lim_{\varepsilon\to 0_+} \int_0^\cdot Z_s\frac{\varphi(X_{s+\varepsilon}) - \varphi(X_s)}{\varepsilon}ds.$$

If $a = (a^1, a^2, \ldots, a^n)$ and $b = (b^1, b^2, \ldots, b^n)$ belong to $\mathbb{R}^n$, since $\varphi \in C^2(\mathbb{R}^n)$, we have

$$\varphi(b) - \varphi(a) = \sum_{i=1}^n \partial_i\varphi(a)(b^i - a^i) + \sum_{1\le i,j\le n} (b^i - a^i)(b^j - a^j) \int_0^1 (1-r)\partial^2_{i,j}\varphi\big(a + r(b-a)\big)dr.$$

We rewrite previous integrals :

$$\int_0^1 (1-r)\partial^2_{i,j}\varphi\big(a+r(b-a)\big)dr = \frac{1}{2}\partial^2_{i,j}\varphi(a) + \int_0^1 (1-r)\big\{\partial^2_{i,j}\varphi\big(a+r(b-a)\big) - \partial^2_{i,j}\varphi(a)\big\}dr.$$

Consequently,

$$\int_0^{\cdot} Z_s \frac{\varphi(X_{s+\varepsilon}) - \varphi(X_s)}{\varepsilon} ds = I^1_\varepsilon + I^2_\varepsilon + R_\varepsilon,$$

where

$$I^1_\varepsilon = \sum_{1\le i\le n} \int_0^{\cdot} Z_s \partial_i\varphi(X_s) \frac{X^i_{s+\varepsilon} - X^i_s}{\varepsilon} ds$$

$$I^2_\varepsilon = \frac{1}{2} \sum_{1\le i,j\le n} \int_0^{\cdot} Z_s \partial^2_{i,j}\varphi(X_s) \frac{(X^i_{s+\varepsilon} - X^i_s)(X^j_{s+\varepsilon} - X^j_s)}{\varepsilon} ds$$

$$R_\varepsilon = \sum_{1\le i,j\le n} \int_0^{\cdot} Z_s \frac{(X^i_{s+\varepsilon} - X^i_s)(X^j_{s+\varepsilon} - X^j_s)}{\varepsilon} \times \Big[\int_0^1 (1-r)\big\{\partial^2_{i,j}\varphi\big(X_s + r(X_{s+\varepsilon} - X_s)\big) - \partial^2_{i,j}\varphi(X_s)\big\}dr\Big] ds.$$

Since $Zf(X)$ belongs to $\mathcal{T}_X$, the limits of I^1_ε and I^2_ε, as ε goes to 0_+, are respectively :

$$\sum_{1\le i\le n} \int_0^{\cdot} Z\partial_i\varphi(X)d^-X^i \quad \text{and} \quad \frac{1}{2} \sum_{1\le i,j\le n} \int_0^{\cdot} Z_s\partial^2_{i,j}\varphi(X_s)d[X^i, X^j](s).$$

Let us examine the convergence of R_ε. The functions $\partial^2_{i,j}\varphi$ and X being continuous, it is clear that $\int_0^{\cdot} (1-r)\big\{\partial^2_{i,j}\varphi\big(X_s + r(X_{s+\varepsilon} - X_s)\big) - \partial^2_{i,j}\varphi(X_s)\big\}dr$ goes to 0, when $\varepsilon \to 0_+$, uniformly with respect to $s \in [0,T]$.
Moreover,

$$\Big|\int_0^t \frac{(X^i_{s+\varepsilon} - X^i_s)(X^j_{s+\varepsilon} - X^j_s)}{\varepsilon} ds\Big| \le \Big\{\Big(\int_0^T \frac{(X^i_{s+\varepsilon} - X^i_s)^2}{\varepsilon} ds\Big)\Big(\int_0^T \frac{(X^j_{s+\varepsilon} - X^j_s)^2}{\varepsilon} ds\Big)\Big\}^{1/2}$$

for any $t \in [0,T]$, and

$$\int_0^T \frac{(X^i_{s+\varepsilon} - X^i_s)^2}{\varepsilon} ds \quad \text{converges to} \quad [X^i, X^i](T).$$

Hence R_ε goes to 0, with respect to the ucp convergence. □

Next result allows to give new examples of processes belonging to $\mathcal{T}_X$, X being a one-dimensional finite quadratic variation process X.

Proposition 4.5. *Suppose X is a one dimensional Itô process, and $A = (A_1, \dots, A_m)$ is a locally bounded variation vector process. Let $h : \mathbb{R}^{1+m} \to \mathbb{R}$ of class C^1, then $h(X, A) \in \mathcal{T}_X$. In particular (X, A) is a multidimensional Itô process.*

Proof. It is an immediate consequence of Itô formula, see (1.11) and (1.12) of section 1. Take for this $H(x,t) = \int_0^x h(y,t)dy$; then $H \in C^{2,1}(\mathbb{R}^{1+m})$. □

Remark : This result will appear as a consequence of proposition 4.1.

A finite quadratic variation process X produces examples of (multidimensional) Itô process.

Proposition 4.6. *Let X be a finite quadratic variation process, V a bounded variation continuous process, $f, g \in C^1(\mathbb{R}), h \in C^2(\mathbb{R})$.*
Then $\Big(\int_0^\cdot f(X)d^+X, \int_0^\cdot g(X)d^-X, V, h(X) \Big)$ is a $\mathbb{R}^4$-valued Itô process.

Proof. We set :

$$Y = \Big(\int_0^\cdot f(X)d^+X, \int_0^\cdot g(X)d^-X, V, h(X) \Big).$$

Let F and G be two primitives of f, respectively g :

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Since F and G belong to $C^2(\mathbb{R})$:

$$F(X_t) = F(X_0) + \int_0^t f(X)d^+X - \frac{1}{2}\int_0^t f'(X_s)d[X,X](s) \tag{4.7}$$

$$G(X_t) = G(X_0) + \int_0^t g(X)d^-X + \frac{1}{2}\int_0^t g'(X_s)d[X,X](s). \tag{4.8}$$

Using classical properties of covariation processes we easily get that Y has all its mutual brackets.
It remains to show that for $\varphi \in C^1(\mathbb{R}^4)$

$$\int_0^\cdot \varphi(Y)d^-Y^i \text{ exists for any } i = 1, \dots, 4. \tag{4.9}$$

We have,

$$\varphi(Y) = \tilde{\varphi}(Z), \ \tilde{\varphi} \in C^1(\mathbb{R}^4) \tag{4.10}$$

where

$$Z_t = (X_t, A_t),\ A_t = (V_t, V_t^1, V_t^2, V_t^3), \tag{4.11}$$

(V_t^i) being continuous processes, with locally bounded variation.
Remark that $\int_0^{\cdot} H d^- A = \int_0^{\cdot} H d^+ A$ coincides with usual Stieltjes integral. Consequently, using (4.7), (4.8) and (4.10), the validity of (4.9) holds provided that

$$\int_0^{\cdot} \psi(Z) d^- \rho(X) \text{ exists for any } \rho \in C^2(\mathbb{R}),\ \psi \in C^1(\mathbb{R}^4). \tag{4.12}$$

This is a consequence of propositions 4.3 and 4.11 which will follow.

□

Remark 4.7. If X is a continuous semimartingale, $f \in C^1(\mathbb{R})$, $h \in C^2(\mathbb{R})$ and A is adapted and has bounded variation, then the vector $Y = \left(\int_0^{\cdot} f(X)dX, A, h(X)\right)$ is a continuous semimartingale, therefore it is a Itô process. But if A is not adapted, $X + A$ is a Itô process and is not a priori a semimartingale.

If X is a finite quadratic variation process, $g \in C^1$, then

$$\int_0^{\cdot} g(X) d^- X = G\big(X(t)\big) - G\big(X(0)\big) + V(t)$$

where

$$G(x) = \int_0^x g(t)dt,\ V(t) = -\int_0^t g'\big(X(s)\big) d[X,X](s).$$

Since V has locally bounded variation,

$$\left[\int_0^{\cdot} g(X)d^-X, \int_0^{\cdot} g(X)d^-X\right](t) = [G(X), G(X)](t) = \int_0^t g^2(X(s))d[X,X](s). \tag{4.13}$$

Definition. *A process* $X = (X^1, X^2, \dots, X^n)$ *is called a* ***vector Itô process*** *if* $\left[\int_0^{\cdot} f(X)d^-X^i, \int_0^{\cdot} g(X)d^-X^j\right]$ *exists for any* f, g *in* $C^1(\mathbb{R}^n)$, $1 \le i, j \le n$, *and*

$$\left[\int_0^{\cdot} f(X)d^-X^i, \int_0^{\cdot} g(X)d^-X^j\right] = \int_0^{\cdot} f(X(s))g(X(s))d[X^i, X^j](s). \tag{4.14}$$

Remarks 4.8 : 1) A finite quadratic variation process X is a vector Itô process and a vector Itô process is a Itô process.

2) The analog of the concept of vector Itô process, in the classical stochastic calculus appears in [CS].

Using the bilinearity of the bracket and property (1.5), we easily established the validity of the following lemma.

Lemma 4.9. *Suppose X is a vector Itô process, $f_1, \dots, f_n$, $g_1, \dots, g_n$ are $2n$ fonctions of class C^1 then $\sum_{i=1}^{n} \int_0^{\cdot} f_i(X) d^{\pm} X^i$ and $\sum_{i=1}^{n} \int_0^{\cdot} g_i(X) d^{\pm} X^i$ have their mutual brackets. Moreover*

(4.15)

$$\Big[\sum_{i=1}^{n} \int_0^{\cdot} f_i(X) d^{\pm} X^i, \sum_{i=1}^{n} \int_0^{\cdot} g_i(X) d^{\pm} X^i\Big] = \sum_{1\le i,j\le n} \int_0^{\cdot} f_i(X(s)) g_j(X(s)) d[X^i, X^j](s).$$

Proposition 4.10. *Let X be a vector Itô process, $f_1, f_2, \dots, f_n$ (resp. $\varphi_1, \varphi_2, \dots, \varphi_m$) n (resp. m) functions of class C^1 (resp. C^2). We set*

$$Z = \sum_{i=1}^{n} \int_0^{\cdot} f_i(X) d^{\pm} X^i, \quad Y^i = \varphi_i(X), \quad 1 \le i \le m.$$

Then Z is a finite quadratic variation process and $Y = (Y^1, Y^2, \dots, Y^m)$ is a vector Itô process.

Proof. Lemma 4.9 implies that Z is a finite quadratic variation process.
We know (see proposition 4.3) that Y is a Itô process. Y is a vector Itô process if :

$$(4.16) \qquad \Big[\int_0^{\cdot} f(Y) d^- Y^i, \int_0^{\cdot} g(Y) d^- Y^j\Big] = \int_0^{\cdot} f(Y_s) g(Y_s) d[Y^i, Y^j](s), \; f, g \in C^1(\mathbb{R}^m).$$

We apply proposition 4.3 $\big($with $Z = f \circ \varphi(X)$ or $g \circ \varphi(X), \varphi = (\varphi_1, \dots, \varphi_m)\big)$

$$\int_0^{\cdot} f(Y) d^- Y^i = \sum_{k=1}^{n} \int_0^{\cdot} (f \circ \varphi)(X) \partial_k \varphi_i(X) d^- X^k + V_i,$$
$$\int_0^{\cdot} g(Y) d^- Y^j = \sum_{k=1}^{n} \int_0^{\cdot} (g \circ \varphi)(X) \partial_k \varphi_j(X) d^- X^k + W_j,$$

V_i, W_j being processes with locally bounded variation.
(4.16) is a direct consequence of lemma 4.9. ∎

An interesting preparatory result for next section be given in the following lines.

Proposition 4.11. *Let X be a finite quadratic variation process, $V = (V^1, \dots, V^p)$ be a $\mathbb{R}^p$-valued locally bounded variation continuous process. Then $(X, V^1, \dots, V^p)$ is a vector Itô process. In particular $h(X, V)$ belongs to $\mathcal{T}_X$ for any h of class C^1.*

Proof. First of all we observe that (X, V) is a Itô process and has all their mutual brackets. Let $g \in C^1(\mathbb{R}^{1+p})$. Obviously

$$\int_0^\cdot g(X, V) d^- V^i = \int_0^\cdot g(X, V) dV^i, \ 1 \leq i \leq p$$

exists and has local bounded variation.
On the other hand, setting $G(x) = \displaystyle\int_0^x g(y, v) dy$, $G \in C^{2,1}(\mathbb{R}^{1+p})$ and using Itô formula (see section 1), we get

$$\int_0^\cdot g(X, V) d^- X = G\big(X(t), V(t)\big) + A(t)$$

$$A(t) = - G\big(X(0), V(0)\big) - \sum_{j=2}^{p+1} \int_0^\cdot \partial_j G(X, V) dV_j - \frac{1}{2} \int_0^\cdot \partial_1 g(X, V) d[X, X].$$

$\big(A(t), t \geq 0\big)$ is a locally bounded variation process. This shows that (X, V) is a Itô process. Using (1.10), we easily obtain,

$$\Big[\int_0^\cdot g(X, V) d^- X, \int_0^\cdot g(X, V) d^- X\Big] = \int_0^t g^2\big(X(s), V(s)\big) d[X, X](s).$$

Consequently (X, V) is a vector Itô process. □

5. Stochastic differential equations.

Here we will not aim the biggest generality but we would like to show the method. It is the first time in the framework of our calculus that we study an uniqueness problem. Existence problems have been studied in [RV1] and [RV4] where we consider equations with anticipative initial condition. At our knowledge for such equations there are no good uniqueness result. On the other hand, our aim is also to study equations driven by a finite quadratic variation process. Our methods are similar to the ones developed by [Z2] when such a process is the fractional Brownian motion.
Let $\big(\xi(t), \geq 0\big)$ be a finite quadratic variation process, $\big(V(t), t \geq 0\big)$ be a locally bounded variation process. Both processes are supposed to be continuous and vanishing at zero.
Let $\sigma : \mathbb{R} \to \mathbb{R}$ be of class C^2, σ and σ' bounded, $\beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz function that is to say
- for any $x \in \mathbb{R}, t \mapsto \beta(t, x)$ is continuous,

- $x \mapsto \beta(t,x)$ is Lipschitz uniformly in t on each compact interval and $\big(\beta(t,0), t \geq 0\big)$ is locally bounded.
Let α be any random variable.
We are interested in the following stochastic differential equation :

$$(5.1)\qquad \begin{cases} d^- X(t) = \sigma\big(X(t)\big)d^-\xi(t) + \beta\big(t, X(t)\big)dV(t) \\ X(0) = \alpha. \end{cases}$$

First of all we need to specify the concept of solution.
A process $\big(X(t), t \geq 0\big)$ will be a solution to (5.1) if

$$(5.2)\qquad \begin{cases} \text{(i)} & (X, \xi, [\xi,\xi], V) \text{ is a vector Itô process,} \\ \text{(ii)} & \text{for any } \psi \in C^1(\mathbb{R}^4),\ Z(t) = \psi\big(X(t), \xi(t), V(t), [\xi,\xi](t)\big) \text{ verifies,} \\ & \displaystyle\int_0^\cdot Z(s)d^-X(s) = \int_0^\cdot Z(s)\Big(\sigma\big(X(s)\big)d^-\xi(s) + \beta\big(s, X(s)\big)dV(s)\Big), \\ \text{(iii)} & X(0) = \alpha. \end{cases}$$

Remarks 5.1 : 1) For $\psi \equiv 1$, we obtain the integral equation

$$(5.3)\qquad X(t) = \alpha + \int_0^t \Big\{\sigma\big(X(s)\big)d^-\xi(s) + \beta\big(s, X(s)\big)dV(s)\Big\}.$$

In our opinion taking only $Z \equiv 1$, deteriorates the information that we are truly interested in a “forward” equation.
2) If (ξ, V) is a $(\mathcal{F}_t)_{t\geq 0}$ continuous semimartingale, the differential form (5.1) is equivalent to (5.3) because for any adapted, locally bounded process H, then $\int_0^\cdot HdX$ exists, in other words H belongs to $\mathcal{T}_X$. In our general context, a priori such property does not hold,so we have to specify that the class $\mathcal{T}_X$ is rich enough, this is the meaning of (5.2) (ii).

In [RV1] we studied existence problem for equation (5.3) when ξ is a Brownian motion W and β is a deterministic function. Even in that case we did not have uniqueness result. That solution was expressed as following : $X(t) = \varphi_t(\alpha)$, where $\big(\varphi_t(x)\big)_{t\geq 0}$ is the solution to the non-anticipating equation

$$(5.4)\qquad \varphi_t(x) = x + \int_0^t \big\{\sigma\big(\varphi_s(x)\big)dW(s) + \beta\big(s, \varphi_s(x)\big)ds\big\}.$$

The technique was essentially a substitution theorem.

Proposition 5.2. *Let $X_t = \varphi_t(\alpha)$, then X is solution to (5.4) $\big($therefore to (5.2)$\big)$.*

Proof. It is the consequence of two substitution formulas, see section 2, 5). □

In particular our next result (proposition 5.3) will give a uniqueness statement for equation (5.4) under some suitable conditions on the coefficients.

We introduce the flow F generated by σ. This function of two variables plays a crucial role, since the solution X of (5.1) will be expressed through F (see theorem 5.4).
$F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is defined as a solution to

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) = \sigma\big(F(t,x)\big) \\ F(0,x) = x. \end{cases} \tag{5.5}$$

The classical theory of ordinary differential equations tells us that F defines a flow: σ being of class C^2, for any r, $F(r,.)$ is a C^2-diffeomorphism on $\mathbb{R}$. We set

$$H(r,x) = F^{-1}(r,x), \tag{5.6}$$

the inverse is taken with respect to the second variable x. H is again of class C^2 and

$$F\big(r, H(r,x)\big) = x, \quad H\big(r, F(r,x)\big) = x, \quad \forall r \in \mathbb{R} \quad x \in \mathbb{R}. \tag{5.7}$$

We prove a few relations involving F and its inverse H. These results will be used later on.
Deriving the first expression in (5.7) in r and x we get

$$\frac{\partial F}{\partial r}\big(r, H(r,x)\big) + \frac{\partial F}{\partial x}\big(r, H(r,x)\big)\frac{\partial H}{\partial r}(r,x) = 0, \tag{5.8}$$

$$\frac{\partial F}{\partial x}\big(r, H(r,x)\big)\frac{\partial H}{\partial x}(r,x) = 1. \tag{5.9}$$

We take the derivative with respect to s in $F\big(t, F(s,x)\big) = F(t+s,x)$, and take $s = 0$, then

$$\frac{\partial F}{\partial t}(t,x) = \sigma(x)\frac{\partial F}{\partial x}(t,x). \tag{5.10}$$

We have,

$$\frac{\partial}{\partial t}\Big(\frac{\partial F}{\partial x}(t,x)\Big) = \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial t}F(t,x)\Big) = \frac{\partial}{\partial x}\Big(\sigma\big(F(t,x)\big)\Big) = \sigma'\big(F(t,x)\big)\frac{\partial F}{\partial x}(t,x).$$

Integrating this linear ordinary differential equation, we have

$$\frac{\partial F}{\partial x}(t,x) = \exp\Big(\int_0^t \sigma'\big(F(s,x)\big)ds\Big). \tag{5.11}$$

We derive the second identity in (5.7) with respect to r, and we apply (5.5), we obtain,

$$\frac{\partial H}{\partial r}(r,x) = -\sigma(x)\frac{\partial H}{\partial x}(r,x). \tag{5.12}$$

Proposition 5.3. *There is at most a solution* $\big(X(t), t \geq 0\big)$ *to (5.1).*

Proof. 1) Let $\big(X(t), t \geq 0\big)$ be a solution to (5.1). We set

$$Y(t) = H\big(\xi(t), X(t)\big).$$

H is of class C^2, (ξ, X) is supposed to be a vector Itô process. Then we can apply the Itô formula :

$$\begin{aligned} Y(t) &= H\big(0, X(0)\big) + \int_0^t \frac{\partial H}{\partial r}\big(\xi(s), X(s)\big) d^-\xi(s) \\ &+ \int_0^t \frac{\partial H}{\partial x}\big(\xi(s), X(s)\big) d^- X(s) + \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial r^2}\big(\xi(s), X(s)\big) d[\xi,\xi](s) \\ &+ \int_0^t \frac{\partial^2 H}{\partial r \partial x}\big(\xi(s), X(s)\big) d[X,\xi](s) + \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial x^2}\big(\xi(s), X(s)\big) d[X,X](s). \end{aligned}$$

Assumption (5.2) (i) and lemma 4.9 imply that

$$\begin{aligned} d[\xi, X](s) &= \sigma\big(X(s)\big) d[\xi,\xi](s) \\ d[X,X](s) &= \sigma^2\big(X(s)\big) d[\xi,\xi](s). \end{aligned} \tag{5.13}$$

Using (5.12) , (5.2) (ii) and the former identities, we obtain

$$\begin{aligned} Y(t) &= \alpha + \int_0^t \beta\big(s, X(s)\big) \frac{\partial H}{\partial x}\big(\xi(s), X(s)\big) dV(s) \\ &+ \frac{1}{2}\int_0^t \Big\{ \frac{\partial^2 H}{\partial r^2}\big(\xi(s), X(s)\big) + 2\frac{\partial^2 H}{\partial r \partial x}\big(\xi(s), X(s)\big)\sigma\big(X(s)\big) \\ &+ \frac{\partial^2 H}{\partial x^2}\big(\xi(s), X(s)\big)\sigma^2\big(X(s)\big)\Big\} d[\xi,\xi](s). \end{aligned}$$

We take the partial derivative with respect to r in (5.12) :

$$\frac{\partial^2 H}{\partial r^2}(r,x) = -\sigma(x)\frac{\partial^2 H}{\partial r \partial x}(r,x). \tag{5.14}$$

Hence $\big(Y(t)\big)_{t\geq 0}$ is solution of the ordinary differential equation driven by bounded variation functions :

$$Y(t) = \alpha + \int_0^t \tilde{\beta}\big(s, Y(s)\big) dV(s) + \int_0^t \tilde{\sigma}\big(s, Y(s)\big) d[\xi,\xi](s) \tag{5.15}$$

where

$$\tilde{\beta}(s,y)=\beta\Big(s,F\big(\xi(s),y\big)\Big)\frac{\partial H}{\partial x}\Big(\xi(s),F\big(\xi(s),y\big)\Big) \tag{5.16}$$

$$\tilde{\sigma}(s,y)=\frac{1}{2}\frac{\partial^2 H}{\partial x^2}\Big(\xi(s),F\big(\xi(s),y\big)\Big)\sigma^2\Big(F\big(\xi(s),y\big)\Big) \tag{5.17}$$

$$+\frac{1}{2}\frac{\partial^2 H}{\partial r\partial x}\Big(\xi(s),F\big(\xi(s),y\big)\Big)\sigma\Big(F\big(\xi(s),y\big)\Big).$$

Formulae (5.9) and (5.11) imply,

$$\frac{\partial H}{\partial x}(t,x)=\exp\left(-\int_0^t \sigma'\Big(F\big(s,H(t,x)\big)\Big)ds\right). \tag{5.18}$$

Then

$$\frac{\partial H}{\partial x}\Big(\xi(s),F\big(\xi(s),y\big)\Big)=G\big(\xi(s),y\big), \tag{5.19}$$

where

$$G(t,y)=\exp\Big(-\int_0^t \sigma'\big(F(s,y)\big)ds\Big)=1\Big/\Big(\frac{\partial F}{\partial x}(t,y)\Big). \tag{5.20}$$

We have to calculate $\dfrac{\partial^2 H}{\partial x^2}$ and $\dfrac{\partial H}{\partial t\partial x}$.
We take the two partial derivatives in (5.18) :

$$\frac{\partial^2 H}{\partial t\partial x}(t,x)=-\sigma'(x)\frac{\partial H}{\partial x}(t,x)-\frac{\partial H}{\partial x}(t,x)\frac{\partial H}{\partial t}(t,x)\left(\int_0^t \sigma''\Big(F\big(s,H(t,x)\big)\Big)ds,\right), \tag{5.21}$$

$$\frac{\partial^2 H}{\partial x^2}(t,x)=-\Big(\frac{\partial H}{\partial x}(t,x)\Big)^2\left(\int_0^t \sigma''\Big(F\big(s,H(t,x)\big)\Big)ds\right). \tag{5.22}$$

Hence (5.12) implies,

$$\tilde{\sigma}(s,y)=-\frac{1}{2}(\sigma\sigma')\big(F(\xi(s),y)\big)G(\xi(s),y).$$

Consequently, Y solves,

$$Y(t)=\alpha+\int_0^t \beta\big(s,F(\xi(s),Y(s))\big)G(\xi(s),Y(s))dV(s) \tag{5.23}$$

$$-\frac{1}{2}\int_0^t (\sigma\sigma')\big(F(\xi(s),Y(s))\big)G(\xi(s),Y(s))d[\xi,\xi](s).$$

2) Let $T > 0$, $f_1, f_2, \dots, f_n : [0,T] \times \mathbb{R}^m \to \mathbb{R}^n$, and $g_1, \dots, g_n$ being continuous bounded variation functions.
We are interested by the solution $u : [0,T] \to \mathbb{R}^n$ of :

$$u(t) = x + \sum_{i=1}^{n} \int_0^t f_i\big(s, u(s)\big) dg_i(s). \tag{5.24}$$

Let $h : [0,T] \times \mathbb{R} \to \mathbb{R}$. We say that h belongs to the class LL if h is **L**ocally **L**ipschitz continuous and has linear growth :

$$|h(t,x) - h(t,y)| \le c_n |x-y| \ ; \ \forall t \in [0,T], \ \forall x, y, \ |x| \le n, \ |y| \le n, \tag{5.25}$$

$$|h(t,x)] \le c + c'|x| \ ; \ \forall t \in [0,T], \ \forall x \in \mathbb{R}. \tag{5.26}$$

Using the general results of Protter [P], we know that if $f_1, f_2, \dots, f_n$ belong to the class LL, there exists a unique solution of (5.24).
3) We have to prove that β_1 and σ_1 belong to the LL class, where

$$\beta_1(s,y) = \beta\big(s, F(\xi(s), y)\big) G(\xi(s), y), \ \sigma_1(s,y) = -\frac{1}{2}(\sigma\sigma')\big(F(\xi(s), y)\big) G(\xi(s), y).$$

The integral version of (5.5) being,

$$F(t,x) = x + \int_0^t \sigma\big(F(s,x)\big) ds,$$

therefore,

$$F(t,x) - F(t,y) = x - y + \int_0^t \Big(\sigma\big(F(s,x)\big) - \sigma\big(F(s,y)\big)\Big) ds.$$

σ is of class LL, then F too.
Our basic tool is the obvious result :
(5.27) Let $\sigma_1, \sigma_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$, of class LL, σ_1 bounded then $\sigma_1\sigma_2$ is a LL function.
We start analyzing β_1.
We remark that

$$G(\xi(s), y) = \exp\Big(- \int_0^{\xi(s)} \sigma'\big(F(u,y)\big) du \Big).$$

ξ being continuous on $[0,T]$, σ' is of class C^1 and bounded, therefore $\big(G(\xi(s), y) \, ; 0 \le s \le T, y \in \mathbb{R}\big)$ is of class LL and is bounded.
β and F being LL continuous functions, hence $(s,y) \to \beta\big(s, F(\xi(s), y)\big)$ and finally β_1 are of class LL.
Using a similar approach it is not difficult to verify that σ_1 is again of class LL. $\square$

Theorem 5.4. *There is a unique solution X to (5.1). Moreover $X_t = F\big(\xi(t), Y(t)\big)$ where Y is the unique solution of (5.23) and F is the function defined by (5.5).*

Proof. Let $\big(Y(t), t \geq 0\big)$ be the solution to (5.23). We set

$$X(t) = F\big(\xi(t), Y(t)\big).$$

$\big(Y(t), t \geq 0\big)$ is a bounded variation process, so by propositions 4.10 and 4.11, $\big(\xi, Y, [\xi,\xi], V\ ;\ t \geq 0\big)$ and $(X, \xi, [\xi,\xi], \eta)$ are vector Itô processes. As for the initial data,

$$X(0) = F(0, \alpha) = \alpha.$$

Let $\psi \in C^2(\mathbb{R}^4)$; we set $Z(t) = \psi\big(X(t), \xi(t), V(t), [\xi,\xi](t)\big)$; we have to show that

$$\int_0^{\cdot} Z(s) d^- X(s) = \int_0^{\cdot} Z(s)\Big(\sigma\big(X(s)\big) d^-\xi(s) + \beta\big(s, X(s)\big) dV(s)\Big). \tag{5.28}$$

We apply proposition 4.3 for getting

$$\int_0^t Z(s) d^- X(s) = \int_0^t Z(s) \frac{\partial F}{\partial t}\big(\xi(s), Y(s)\big) d^-\xi(s)$$
$$+ \int_0^t Z(s) \frac{\partial F}{\partial x}\big(\xi(s), Y(s)\big) d^- Y(s) + \frac{1}{2}\int_0^t Z(s) \frac{\partial^2 F}{\partial t^2}\big(\xi(s), Y(s)\big) d[\xi,\xi](s).$$

Identities (5.10), (5.20) and (5.23) yield,

$$\int_0^t Z(s) d^- X(s) = \int_0^t Z(s)\sigma\Big(F\big(\xi(s), Y(s)\big)\Big) d^-\xi(s) + \int_0^t Z(s)\beta(s, X(s)) dV(s)$$
$$+ \frac{1}{2}\int_0^t Z(s) H(s) d[\xi,\xi](s),$$

$$H(s) = -\frac{\partial F}{\partial x}(\xi(s), Y(s))(\sigma\sigma')(X(s)) G(\xi(s), Y(s)) + \frac{\partial^2 F}{\partial t^2}(\xi(s), Y(s)).$$

We apply the operator $\dfrac{\partial}{\partial t}$ to the first identity of (5.5) :

$$\frac{\partial^2 F}{\partial t^2}(t,x) = \sigma'\big(F(t,x)\big)\frac{\partial F}{\partial t}(t,x) = (\sigma\sigma')\big(F(t,x)\big).$$

Then using moreover (5.20) we can conclude that $H = 0$. This means that X solves (5.1). ∎

An interesting particular case is produced by the linear case.

Corollary 5.5. *The unique solution to*

$$\begin{cases} d^- X(t) = X(t) d^- \xi(t) \\ X(0) = \alpha \end{cases} \tag{5.29}$$

is given by

$$X(t) = \alpha \exp \Big\{ \xi(t) - \frac{1}{2} [\xi, \xi](t) \Big\}. \tag{5.30}$$

Proof. In this case $\sigma(x) = x$, $\beta = 0$, $V = 0$. Then $F(t, x) = xe^t$, $H(t, x) = xe^{-t}$ and $G(t, x) = e^{-t}$. Consequently Y solves,

$$dY(t) = -\frac{1}{2} Y(t) d[\xi, \xi](t),\, Y(0) = \alpha$$

and so

$$Y(t) = \alpha \exp \Big(-\frac{1}{2} [\xi, \xi](t) \Big).$$

(5.30) follows immediatly. □

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