d-dimensional Feynman Integrands as Hida Distributions

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Outline

Vector-valued White Noise

Consider as Gel'fand triple (e.g. [Wes95])

$$S_d(\mathbb{R}) \subset L^2(\mathbb{R}, dx)^d \subset S_d(\mathbb{R})'$$

where

$$S_d := S_d(\mathbb{R}) := \bigotimes_{n=1}^{d} S(\mathbb{R})$$

with underlying Hilbert space norm

$$\|\varphi\|^2 = \sum_{n=1}^{d} \|\varphi_n\|^2 = \sum_{n=1}^{d} \int_{\mathbb{R}} \varphi_n^2(x)\, dx$$

Consider the characteristic function

$$C(\varphi) = \exp\left(-\tfrac{1}{2}\|\varphi\|^2\right)$$

leading to the d-dimensional White Noise Measure μ, where

$$\exp\left(-\tfrac{1}{2}\|\varphi\|^2\right) = \int_{S'_d} \exp\left(i\langle\omega,\varphi\rangle_d\right)\, d\mu(\omega)$$

via Bochner-Minlos theorem, where naturally

$$\langle\omega,\varphi\rangle_d = \sum_{n=1}^{d} \langle\omega_n,\varphi_n\rangle$$

The Hida spaces (S_d) and $(S_d)'$

Hida test functions space and Hida distribution space in the Hida triple

$$(S_d) \subset L^2(S_d', \mu, \mathbb{C}) \subset (S_d)'$$

Examples for Hida test functions

exponential function: $\exp\left(\langle \omega, \varphi \rangle_d\right)$

wick-ordered exponential function: $: \exp\left(\langle \omega, \varphi \rangle_d\right) :$
$= \exp\left(\langle \omega, \varphi \rangle_d\right) \exp\left(-\frac{1}{2}\langle \varphi, \varphi \rangle_d\right)$

Example for square-intagrable function

d-dimensional Brownian Motion: $B_{t_0,t}(\omega) := \left(\langle \omega_j, \mathbb{1}_{[t_0,t)} \rangle\right)_{j=1,\cdots,d}$

Examples for Hida distributions

Donsker's Delta: $\delta^d\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle_d - (\boldsymbol{y}-\boldsymbol{y_0})\right)$

with path $\boldsymbol{x}(\tau) = \boldsymbol{y_0} + B_{t_0,\tau}(\cdot)$, $t_0 \leqslant \tau \leqslant t$, fixed at endpoint $\boldsymbol{y}$ at time t.

Normalized Exponential: $\text{Nexp}\left(\frac{i+1}{2}\int_{t_0}^{t}\sum_{n=1}^{d}\omega_n^2(\tau)\,d\tau\right)$

For d-dimensional Donsker's Delta note ([LLSW94])

$$\delta_{\boldsymbol{a}}^d(\boldsymbol{B}_t)(\boldsymbol{\omega}) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(i\boldsymbol{x}(\boldsymbol{B}_t(\boldsymbol{\omega})-\boldsymbol{a})\right)d\boldsymbol{x}$$

$$= \left(\frac{1}{2\pi}\right)^d \prod_{n=1}^{d}\int_{\mathbb{R}} \exp\left(ix(\langle\omega_n, \mathbb{1}_{[t_0,t)}\rangle - a_n)\right)dx$$

Definition (S- and T-transform)

Since $:\exp(\langle \cdot, \varphi \rangle_d):, \exp(i\langle \cdot, \varphi \rangle_d) \in (S_d)$ for all $\varphi \in S_{d,\mathbb{C}}$, we can define for all $\boldsymbol{\Phi} \in (S_d)'$:

$$S_{d,\mathbb{C}} \ni \varphi \mapsto S\boldsymbol{\Phi}(\varphi) := \langle\langle \boldsymbol{\Phi}, :\exp(\langle \cdot, \varphi \rangle_d): \rangle\rangle_d \in \mathbb{C}$$

and

$$S_{d,\mathbb{C}} \ni \varphi \mapsto T\boldsymbol{\Phi}(\varphi) := \langle\langle \boldsymbol{\Phi}, \exp(i\langle \cdot, \varphi \rangle_d) \rangle\rangle_d \in \mathbb{C},$$

the S-transform, T-transform respectively, of $\boldsymbol{\Phi}$.

e.g. [PS91], [KLP+96], [HKPS93]

Theorem (Characterization Theorem)

$\Phi \in (S_d)'$ iff $T\Phi$ is a U-functional, i.e.,

$$(i) \quad \mathbb{R} \ni x \mapsto T\Phi(\varphi + x\psi) \in \mathbb{C} \quad \text{is analytic} \quad \forall \varphi, \psi \in S_d,$$

$$(ii) \quad \exists K_1, K_2 > 0,\ p \in \mathbb{N}_0 : \quad |T\Phi(z\varphi)| \leqslant K_1 \exp(K_2 |z|^2 \|\varphi\|_p^2) \quad \forall \varphi \in S_d,\ z \in \mathbb{C}.$$

Theorem (Convergence Theorem)

Let $(F_n)_n$ be an sequence of U-functionals with

$$(i) \quad (F_n(\varphi))_n \ \text{is Cauchy} \ \forall \varphi \in S_d$$

$$(ii) \quad \exists K_1, K_2 > 0,\ p \in \mathbb{N}_0 : \quad |F_n(z\varphi)| \leqslant K_1 \exp(K_2 |z|^2 \|\varphi\|_p^2) \quad \forall \varphi \in S_d,\ z \in \mathbb{C}.$$

Then $\left(S^{-1}(F_n)\right)_n$ and $\left(T^{-1}(F_n)\right)_n$ converge in $(S_d)'$.

Theorem (Integration Theorem I)

Let $(\Omega, \mathcal{F}, m)$ *be a measure space and* $\boldsymbol{\Phi} : \Omega \to (S_d)'$. *Assume* $U := S\boldsymbol{\Phi}$ $(T\boldsymbol{\Phi}$ *respectively*$)$, *where* $U(x, \varphi) := (S(\boldsymbol{\Phi}(x)))(\varphi)$ *(or with* T *respectively), satisfies:*

1. $\forall \varphi \in S_d$ *the mapping* $\Omega \ni x \mapsto U(x, \varphi) \in \mathbb{C}$ *is measurable.*
2. *There exist* $A, B : \Omega \to [0, \infty)$ *measurable and* $p \in \mathbb{N}_0$ *such that*

$$|U(x, z\varphi)| \leqslant A(x) \exp(B(x)|z|^2 \|\varphi\|_p^2), \forall x \in \Omega, z \in \mathbb{C}, \varphi \in S_d$$

and $A \in L^1(\Omega, m)$, $B \in L^\infty(\Omega, m)$.

Theorem (Integration Theorem II)

Then there exists $p' \in \mathbb{N}_0$ *such that* $\boldsymbol{\Phi} \in L^1(\Omega, (\mathcal{H}_{-p',d}), m)$*, i.e.,* $\boldsymbol{\Phi}$ *is Bochner integrable with values in* $(\mathcal{H}_{-p',d})$*, thus in particular*

$$\int_{\Omega} \boldsymbol{\Phi}(x)\, dm(x) \in (\mathcal{H}_{-p',d}) \subset (S_d)'$$

and

$$S\left(\int_{\Omega} \boldsymbol{\Phi}(x)\, dm(x)\right)(\varphi) = \int_{\Omega} S\boldsymbol{\Phi}(x)(\varphi)\, dm(x), \;\; \varphi \in S_d,$$

$$T\left(\int_{\Omega} \boldsymbol{\Phi}(x)\, dm(x)\right)(\varphi) = \int_{\Omega} T\boldsymbol{\Phi}(x)(\varphi)\, dm(x), \;\; \varphi \in S_d,$$

respectively.

Generalized Gauss Kernels ([HKPS93], [GS99])

Let $\mathcal{B}$ be the set of all continuous bilinear mappings $B : S_d \times S_d \to \mathbb{C}$. Then the functions

$$\exp\left(-\tfrac{1}{2}B(\varphi,\varphi)\right), \quad \varphi \in S_d, \quad B \in \mathcal{B},$$

are U-functionals.
Thus the inverse T-transforms of these functions

$$\Phi_B = T^{-1}\exp\left(-\tfrac{1}{2}B\right)$$

are elements of $(S_d)'$ by the characterization theorem for Hida distributions.

Definition

The set of *generalized Gauss kernels* (GGK) is defined by

$$GGK := \{\Phi_B,\ B \in \mathcal{B}\}$$

Example

Let K *be a symmetric trace class operator on* $L^2_d(\mathbb{R})$ *such that* $\sigma K \subset (-\frac{1}{2}, 0)$*.*
Then $\omega \mapsto \langle \omega, K\omega \rangle_d, \ \omega \in S'_d$ *is measurable with respect to the* σ*-algebra* $\mathcal{B}_\sigma(S'_d)$*.*
By direct computation one obtains

$$\int_{S'_d} exp\left(-\langle \omega, K\omega \rangle_d\right) \, d\mu(\omega) = \left(\det(Id + 2K)\right)^{-\frac{1}{2}} < \infty.$$

The above expression makes sense if $\det(Id + 2K) \neq 0$*.*
It is $g := exp(-\frac{1}{2}\langle \omega, K\omega \rangle_d)$ *square-integrable and its T-transform is given by*

$$Tg(\varphi) = \left(\det(Id + K)\right)^{-\frac{1}{2}} exp\left(-\frac{1}{2}(\varphi, (Id + K)^{-1}\varphi)\right), \quad \varphi \in S_d.$$

Therefore $\left(\det(Id + K)\right)^{\frac{1}{2}} g \in GGK$*.*

Corollary

Let $\boldsymbol{K}$ be an $d \times d$ operator matrix with symmetric trace class operators as matrix entries, i.e. $\boldsymbol{K}$ is an operator on $L^2_d(\mathbb{R})$ and $\boldsymbol{K} := \begin{pmatrix} K_{1,1} & \dots & K_{1,d} \\ \vdots & \ddots & \vdots \\ K_{d,1} & \dots & K_{d,d} \end{pmatrix}$,
where $K_{i,j}$ are symmetric trace class operators on $L^2(\mathbb{R})$.
Then $\omega \mapsto \langle \omega, \boldsymbol{K}\omega \rangle_d$, $x \in S'_d$ is measurable with respect to the σ- algebra $B_\sigma(S'_d)$ and

$$T(exp(-\tfrac{1}{2}\langle \cdot, \boldsymbol{K} \cdot \rangle_d))(\varphi) = \left(\det(\boldsymbol{Id} + \tilde{\boldsymbol{K}})\right)^{-\frac{1}{2}} exp\left(-\tfrac{1}{2}(\varphi, (\boldsymbol{Id} + \tilde{\boldsymbol{K}})^{-1}\varphi)\right), \quad \varphi \in S_d,$$

with $\tilde{\boldsymbol{K}}_{i,j} = \begin{cases} \frac{K_{i,j}+K_{j,i}}{2}, & \text{if } j \neq i \\ K_{i,j} & \text{otherwise} \end{cases}$
if $\tilde{\boldsymbol{K}}$ fullfils the assumptions of the afore-mentioned example.

The Normalized Exponential

Nexp

If $B_{\boldsymbol{K}} := (\cdot, (\boldsymbol{Id} + \boldsymbol{K})^{-1} \cdot)$ *is a continuous bilinear map on* $S_d \times S_d$*, but the prefactor does not exist, we can still define the so-called normalized exponential*

$$\boldsymbol{\Phi}_{B_{\boldsymbol{K}}} = \mathit{Nexp}(-\tfrac{1}{2}\langle \cdot, \boldsymbol{K} \cdot \rangle_d) \in GGK$$

$$\text{by} \quad T\boldsymbol{\Phi}_{B_{\boldsymbol{K}}}(\varphi) := \exp(-\tfrac{1}{2}\langle \varphi, (\boldsymbol{Id} + \tilde{\boldsymbol{K}})^{-1}\varphi \rangle_d), \quad \varphi \in S_d.$$

Multiplication of Nexp and exp

For sufficiently "nice" operators $\boldsymbol{K}$ and $\boldsymbol{L}$ on S_d' we can define the product

$$\textit{Nexp}\left(-\tfrac{1}{2}\langle\cdot,\boldsymbol{K}\cdot\rangle_d\right)\,\textit{exp}\left(-\tfrac{1}{2}\langle\cdot,\boldsymbol{L}\cdot\rangle_d\right)$$

of two square-integrable functions by defining its T-transform via

$$T\left(\textit{Nexp}\left(-\tfrac{1}{2}\langle\cdot,\boldsymbol{K}\cdot\rangle_d\right)\textit{exp}\left(-\tfrac{1}{2}\langle\cdot,\boldsymbol{L}\cdot\rangle_d\right)\right)(\varphi)$$
$$=\sqrt{\frac{\det(\boldsymbol{Id}+\tilde{\boldsymbol{K}})}{\det(\boldsymbol{Id}+\tilde{\boldsymbol{K}}+\tilde{\boldsymbol{L}})}}\,\textit{exp}\left(-\tfrac{1}{2}\langle\varphi,(\boldsymbol{Id}+\tilde{\boldsymbol{K}}+\tilde{\boldsymbol{L}})^{-1}\varphi\rangle_d\right),\quad \varphi\in S_d,$$

where the numerator of the prefactor is the renormalization of the normalized exponential.

Definition

The point-wise product of a Hida distribution $\boldsymbol{\Phi} \in (S_d)'$ with a Hida test function $G \in (S_d)$ is defined via

$$\langle\langle \boldsymbol{\Phi} \cdot G, F \rangle\rangle_d := \langle\langle \boldsymbol{\Phi}, G \cdot F \rangle\rangle_d, \text{ for } F \in (S_d)$$

Definition ([GS99])

The point-wise product of a Hida distribution $\boldsymbol{\Phi} \in (S_d)'$ with an exponential of a linear term, i.e.

$$\boldsymbol{\Phi} \cdot \exp(i\langle \cdot, \mathbf{g} \rangle_d + c), \quad \mathbf{g} \in L^2_d(\mathbb{R})_{\mathbb{C}}, \, c \in \mathbb{C}$$

is given by

$$T(\boldsymbol{\Phi} \cdot \exp(i\langle \cdot, \mathbf{g} \rangle_d + c))(\boldsymbol{\varphi}) = T\boldsymbol{\Phi}(\boldsymbol{\varphi} + \mathbf{g})\exp(c), \quad \boldsymbol{\varphi} \in S_d,$$

if $T\boldsymbol{\Phi}$ has a continuous extension to $\boldsymbol{\varphi} + \mathbf{g}$. Furthermore the term on the right-hand side is a U-functional.

Example (Φ = Nexp)

Considering $\Phi_{B_K} = \text{Nexp}(-\frac{1}{2}\langle \cdot, K \cdot \rangle_d)$ *we formally get*

$$T(\Phi_{B_K} \cdot \exp(i\langle \cdot, g\rangle_d))(\varphi) = T\Phi_{B_K}(\varphi + g)$$
$$= \exp\Big(-\tfrac{1}{2}(\varphi + g, (Id + K)^{-1}(\varphi + g))\Big), \quad \varphi \in S_d.$$

This product ist well-defined if, for example, -1 *is in the resolvent set of* K.

Definition (Product with Donsker's Delta (in addition [LLSW94]))

Let $\boldsymbol{a} \in \mathbb{R}^d$, $-\infty < t_0 < t < \infty$, $\gamma_\alpha := \{se^{i\alpha} \mid s \in \mathbb{R}\}$, $\alpha \in (-\pi, \pi)$, $\gamma_\alpha^d := \times_{n=1}^d \gamma_\alpha$. Assume $\gamma_\alpha^d \ni \boldsymbol{\lambda} \to \exp(-i\boldsymbol{\lambda}\boldsymbol{a})\boldsymbol{\Phi}\exp(i\boldsymbol{\lambda}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle_d) \in (S_d)'$ fulfills the conditions of the integration theorem for all $\alpha \in D \subseteq (-\pi, \pi)$ such that $0 \in \overline{D}$. Then we define

$$\boldsymbol{\Phi}\delta_{\boldsymbol{a}}^d(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle_d) := \lim_{\alpha \to 0} \left(\frac{1}{2\pi}\right)^d \int_{\gamma_\alpha^d} \exp(-i\boldsymbol{\lambda}a)\boldsymbol{\Phi}\exp(i\boldsymbol{\lambda}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle_d)\, d\boldsymbol{\lambda}$$

in the case that the limit exists in $(S_d)'$.

Physical Framework

Lagrangian

$$\mathcal{L}(x,\dot{x},t) = \mathcal{L}_0(\dot{x}) - \mathcal{L}_1(x,\dot{x},t) = -\tfrac{1}{2}m\dot{x}^2 - V(x,\dot{x},t)$$

$$\mathcal{L}_H(x,\dot{x},t) = -\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

Action

$$S(x) = S_0(x) - \int_{t_0}^{t} V\left(x(\tau),\dot{x}(\tau),\tau\right)\, d\tau.$$

$$S_H(x) = S_0(x) - \tfrac{1}{2}k\int_{t_0}^{t} x^2\, d\tau.$$

Aim:

Model the d-dimensional Feynman integrand with exactly the affections of an possibly acting potential in each dimension.

Formal Ansatz

$$I_V = \text{Nexp}\left(\frac{i+1}{2}\sum_{n=1}^{d}\int_{t_0}^{t}\omega_n^2(\tau)\,d\tau\right)\text{exp}\left(-i\int_{t_0}^{t}\boldsymbol{V}\left(x(\tau),\dot{x}(\tau),\tau\right)d\tau\right)$$
$$\times\ \delta^d\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle_d-(\boldsymbol{y}-\boldsymbol{y}_0)\right)$$

The Free Feynman Path Integral

For the ansatz we choose

$$\boldsymbol{K} = \begin{pmatrix} -(1+i)P_{[t_0,t)} & 0 & \dots & 0 \\ 0 & -(1+i)P_{[t_0,t)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -(1+i)P_{[t_0,t)} \end{pmatrix}$$

as a $d \times d$ symmetric operator matrix that is trace class.
With this we get $B_{\boldsymbol{K}} := (\cdot, (\boldsymbol{Id}+\boldsymbol{K})^{-1}\cdot)$ to be a continuous bilinear form with

$$(\boldsymbol{Id}+\boldsymbol{K})^{-1} = \begin{pmatrix} P_{[t_0,t)^c} + iP_{[t_0,t)} & 0 & \dots & 0 \\ 0 & P_{[t_0,t)^c} + iP_{[t_0,t)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & P_{[t_0,t)^c} + iP_{[t_0,t)} \end{pmatrix}$$

Realization as Hida Distribution

Thus $\boldsymbol{\Phi}_{B_{\boldsymbol{K}}} = \mathrm{Nexp}(-\frac{1}{2}\langle \cdot, \boldsymbol{K}\cdot\rangle_d) = \mathrm{Nexp}\left(\frac{i+1}{2}\int_{t_0}^{t}\sum_{n=1}^{d}\omega_n^2(\tau)\,d\tau\right)$ is well-defined as GGK. This justifies to state the ansatz for the free Feynman Integrand

$$I_0 = \mathrm{Nexp}\left(\frac{i+1}{2}\int_{t_0}^{t}\sum_{n=1}^{d}\omega_n^2(\tau)\,d\tau\right)\delta^d\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle_d - (\boldsymbol{y}-\boldsymbol{y_0})\right)$$

with formal T-transform

$$TI_0(\varphi) = \int_{S_d'} \mathrm{Nexp}(-\tfrac{1}{2}\langle \boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d)\,\delta^d_{y-y_0}(\langle \boldsymbol{\omega}, \mathbb{1}_{[t_0,t)}\rangle_d)\exp(i\langle \boldsymbol{\omega}, \boldsymbol{\varphi}\rangle_d)\,d\boldsymbol{\mu}(\boldsymbol{\omega})$$

Rigorous Definition of the T-Transform

Naive transformation justifies to define this T-transform via

$$\int_{S'_d} \text{Nexp}\left(-\tfrac{1}{2}\langle \boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d\right) \delta^d_{y-y_0}\left(\langle \boldsymbol{\omega}, \mathbb{1}_{[t_0,t)}\rangle_d\right) \exp\left(i\langle \boldsymbol{\omega}, \boldsymbol{\varphi}\rangle_d\right) d\boldsymbol{\mu}(\boldsymbol{\omega})$$

$$:= \left(\tfrac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \int_{S'_d} \text{Nexp}\left(-\tfrac{1}{2}\langle \boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d\right) \exp\left(i\langle \boldsymbol{\omega}, \boldsymbol{x}\mathbb{1}_{[t_0,t)} + \boldsymbol{\varphi}\rangle_d\right) d\boldsymbol{\mu}(\boldsymbol{\omega}) \exp\left(-i(\boldsymbol{y}-\boldsymbol{y_0})\boldsymbol{x}\right) d\boldsymbol{x}$$

$$:= \lim_{\alpha\to 0} \left(\tfrac{1}{2\pi}\right)^d \int_{\gamma^d_\alpha} T\left(\text{Nexp}\left(-\tfrac{1}{2}\langle \boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d\right)\right)\left(\boldsymbol{\lambda}\mathbb{1}_{[t_0,t)} + \boldsymbol{\varphi}\right) \exp\left(-i\boldsymbol{\lambda}(\boldsymbol{y}-\boldsymbol{y_0})\right) d\boldsymbol{\lambda}$$

Lemma

$T(Nexp(-\frac{1}{2}\langle\boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d)(\boldsymbol{\lambda}\mathbb{1}_{[t_0,t)} + \cdot)exp(-i\boldsymbol{\lambda}(\mathbf{y}-\mathbf{y_0}))) : S_d \to \mathbb{C}$ is a U-functional for every choice of $\boldsymbol{\lambda} \in \gamma_\alpha^d$.

We can apply the integration theorem to its corresponding Hida Distribution by checking the assumptions for

$$T(\text{Nexp}(-\frac{1}{2}\langle\boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d)(\boldsymbol{\lambda}\mathbb{1}_{[t_0,t)} + \cdot)\text{exp}(-i\boldsymbol{\lambda}(\mathbf{y}-\mathbf{y_0})))$$
$$= \text{exp}(-\frac{1}{2}\sum_{n=1}^{d}\left(i\int_{[t_0,t)}(\varphi_n(p)+\lambda_n)^2\,dp + \int_{[t_0,t)^c}\varphi_n^2(p)\,dp - i\lambda_n(y_n+y_{0,n})\right))$$

while restricting $\alpha \in (0, \frac{\pi}{2})$.
Further restriction of α to $\alpha \in (0, \frac{\pi}{4})$ allows the calculation

$$\int_{\gamma_\alpha^d} T(\mathrm{Nexp}(-\tfrac{1}{2}\langle \boldsymbol{\omega}, K\boldsymbol{\omega}\rangle_d)(\boldsymbol{\lambda}\mathbb{1}_{[t_0,t)} + \boldsymbol{\varphi})\exp(-i\boldsymbol{\lambda}(\boldsymbol{y}-\boldsymbol{y_0}))\, d\boldsymbol{\lambda}$$

$$=\exp(-\tfrac{1}{2}\sum_{n=1}^{d}\left(i\int_{[t_0,t)}\varphi_n^2(p)\,dp + \int_{[t_0,t)^c}\varphi_n^2(p)\,dp\right))$$

$$\times\int_{\gamma\alpha}\cdots\int_{\gamma\alpha}\exp(-\tfrac{1}{2}\sum_{n=1}^{d}\left(2i\lambda_n\int_{[t_0,t)}\varphi_n(p)dp - i\lambda_n(y_n+y_{0,n}) + i(t-t_0)\lambda_n^2\right))\,d\lambda_1\cdots d\lambda_d$$

$$=\exp(-\tfrac{1}{2}\sum_{n=1}^{d}\left(i\int_{[t_0,t)}\varphi_n^2(p)\,dp + \int_{[t_0,t)^c}\varphi_n^2(p)\,dp\right))$$

$$\times\left(\frac{2\pi}{i(t-t_0)}\right)^{\frac{d}{2}}\exp\left(\frac{i}{2(t-t_0)}\sum_{n=1}^{d}\left(\int_{[t_0,t)}\varphi_n(p)\,dp - (y_n - y_{0,n})\right)^2\right)$$

Obviously the integral does not depend on α anymore, thus the limit $\alpha \to 0$ exists and is equal to the above.

Furthermore it is

$$\mathbb{E}_{\mu}(I_0) = TI_0(0) = \left(\frac{1}{2\pi i(t-t_0)}\right)^{\frac{d}{2}} \exp\left(\frac{i}{2(t-t_0)}\|\mathbf{y}-\mathbf{y_0}\|^2\right),$$

concluding the construction.

Theorem (Free Feynman Integrand in d Dimensions)

The free Feynman Integrand

$$I_0 := Nexp\left(\frac{i+1}{2}\int_{t_0}^{t}\sum_{n=1}^{d}\omega_n(\tau)\,d\tau\right)\delta^d\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle_d-(\mathbf{y}-\mathbf{y_0})\right)\in(S_d)',$$

for all $-\infty < t_0 < t < \infty$ *and* $\mathbf{y_0},\mathbf{y}\in\mathbb{R}^d$, *where for* $\varphi\in S_d$

$$TI_0(\varphi)=\left(\frac{1}{2\pi i(t-t_0)}\right)^{\frac{d}{2}} exp\left(-\frac{1}{2}\sum_{n=1}^{d}\left(i\int_{[t_0,t)}\varphi_n^2(p)\,dp+\int_{[t_0,t)^c}\varphi_n^2(p)\,dp\right)\right)$$
$$\times\, exp\left(\frac{i}{2(t-t_0)}\sum_{n=1}^{d}\left(\int_{[t_0,t)}\varphi_n(p)\,dp-(y_n-y_{0,n})\right)^2\right)$$

Underlying Theorem

Theorem (Hida, Streit, Grothaus)

Let L be of trace class and K such that $Id+K$ and $N=Id+K+L$ have bounded inverse. Furthermore assume that $L(Id+K)^{-1}$ is diagonalizable. Let e be the unit vector $t^{-\frac{1}{2}}\mathbb{1}_{[0,t)}\in L^2(\mathbb{R},\mathbb{C})$ and either $\Re(e,N^{-1}e)>0$ or $\Re(e,N^{-1}e)=0$ and $\Im(e,N^{-1}e)\neq 0$. Then

$$Nexp\left(-\tfrac{1}{2}\langle\omega,K\omega\rangle\right) exp\left(-\tfrac{1}{2}\langle\omega,L\omega\rangle\right) exp\left(i\langle\omega,g\rangle\right)\delta\left(\langle\omega,\mathbb{1}_{[0,t)}\rangle-y\right),$$

$g\in L^2(\mathbb{R},\mathbb{C})$, $t>0$, $y\in\mathbb{R}$, is a Hida distribution. Its T-transform at $\varphi\in S(\mathbb{R})$ is given by

$$(2\pi t(e,N^{-1}e)\det(Id+L(Id+K)^{-1}))^{-\frac{1}{2}} exp\Big(-\frac{1}{2}(f+g,N^{-1}(f+g)) - \frac{1}{2(e,N^{-1}e)}\Big(\frac{y}{\sqrt{t}}-\frac{i}{2}((e,N^{-1}(f+g))+(f+g,N^{-1}e))^2\Big).$$

Choose

$$\boldsymbol{K} = \begin{pmatrix} -(1+i)P_{[0,t)} & 0 & \dots & 0 \\ 0 & -(1+i)P_{[0,t)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -(1+i)P_{[0,t)} \end{pmatrix}$$

as a $d \times d$ operator matrix and additionally $\boldsymbol{L}$ as $d \times d$-matrix with operators $\boldsymbol{L}_{lm}\omega(s) = 0$ for $l \neq m$ and

$$\boldsymbol{L}_{lm} f(s) = iOf(s) P_{[0,t)} f(s) := ik \int_s^t \int_0^\tau f(r)\, dr d\tau\, P_{[0,t)} f(s)$$

for $l = m$.

With above $\boldsymbol{L}$ it is

$$\exp\left(-\tfrac{1}{2} ik \int_0^t \langle \boldsymbol{\omega}, 1\!\!1_{[0,\tau)} \rangle_d^2 \, d\tau\right) = \exp\left(-\tfrac{1}{2} \langle \boldsymbol{\omega}, \boldsymbol{L}\boldsymbol{\omega} \rangle_d\right)$$

Formally it is

$$N := \boldsymbol{Id} + \boldsymbol{K} + \boldsymbol{L} = -i(\boldsymbol{Id} - \boldsymbol{O})\boldsymbol{P}_{[0,t)} + \boldsymbol{P}_{[0,t)^c}$$

$$\boldsymbol{N}^{-1} = i(\boldsymbol{Id} - \boldsymbol{O})^{-1}\boldsymbol{P}_{[0,t)} + \boldsymbol{P}_{[0,t)^c}$$

Direct Computation yields

$$\det\left(\boldsymbol{Id} + \boldsymbol{L}(\boldsymbol{Id} + \boldsymbol{K})^{-1}\right) = \prod_{n=1}^{\infty}(1 - o_n)^d$$

$$''\left(\boldsymbol{e}, \boldsymbol{N}^{-1}\boldsymbol{e}\right)'' = \prod_{n=1}^{d}\sum_{n=1}^{\infty}\left(\frac{i}{1 - o_n}\right)(e, e_n)^2$$

where

$(e_n)_n$ and $(o_n)_n$ ONS of eigenvectors and corresponding eigenvalues to O and $\boldsymbol{e} = \left(\frac{1}{\sqrt{t}}\mathbb{1}_{[0,t)}, \cdots, \frac{1}{\sqrt{t}}\mathbb{1}_{[0,t)}\right)$

Eigenvectors and Eigenvalues of O

With

$$Of(s) = k \int_s^t \int_0^\tau f(r)\, dr d\tau$$

we get

$$e_n(s) = \cos\left(\frac{s}{t}(n - \tfrac{1}{2})\pi\right)$$

as we need the eigenvectors to be 2-periodic w.r.t. integration and $e_n'(0) = 0$. Explicit calculation yields

$$o_n e_n(s) = k \left(\frac{t}{(n - \frac{1}{2})\pi}\right)^2 \cos\left(\frac{s}{t}(n - \tfrac{1}{2})\pi\right)$$

With concrete values

$$\det\left(\boldsymbol{Id} + \boldsymbol{L}(\boldsymbol{Id} + \boldsymbol{K})^{-1}\right) = \cos^d\left(\sqrt{k}t\right)$$

$$\left(\boldsymbol{e}, \boldsymbol{N}^{-1}\boldsymbol{e}\right) = \left(i\frac{\tan\left(\sqrt{k}t\right)}{\sqrt{k}t}\right)^d$$

Multiplied

$$\left(\left(\boldsymbol{e}, \boldsymbol{N}^{-1}\boldsymbol{e}\right)\det\left(\boldsymbol{Id} + \boldsymbol{L}(\boldsymbol{Id} + \boldsymbol{K})^{-1}\right)\right)^{-\frac{1}{2}} = \left(\frac{\sqrt{k}}{2\pi i \sin\left(\sqrt{k}t\right)}\right)^{\frac{d}{2}}$$

Lemma (Feynman Integrand w.r.t. the Harmonic Oscillator)

It is

$$I_{SC} := N\exp\left(-\tfrac{1}{2}\langle \boldsymbol{\omega}, \boldsymbol{K}\boldsymbol{\omega}\rangle_d\right) \exp\left(-\tfrac{1}{2}\langle \boldsymbol{\omega}, \boldsymbol{L}\boldsymbol{\omega}\rangle_d\right) \delta^d\left(\langle \omega, \mathbb{1}_{[0,t)}\rangle - \boldsymbol{y}\right) \in (S_d)'$$

for all $0 < t < \infty$ *and* $\boldsymbol{y} \in \mathbb{R}^d$*, where for* $\boldsymbol{\varphi} \in S_d$

$$TI_{SC}(\boldsymbol{\varphi}) = \left(\frac{\sqrt{k}}{2\pi i \sin\left(\sqrt{k}t\right)}\right)^{\frac{d}{2}} \exp\left(-\tfrac{1}{2}\sum_{n=1}^{d}\left(\varphi_n, N^{-1}\varphi_n\right)\right)$$
$$\times \exp\left(\frac{i\sqrt{k}}{2}\cot\left(\sqrt{k}t\right)\sum_{n=1}^{d}\left(y_n + \int_0^t (Id - O)^{-1}\varphi_n(p)\,dp\right)^2\right)$$

Finally

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\mu}}(I_0) &= TI_{SC}(0) \\ &= \left(\frac{\sqrt{k}}{2\pi i \sin\left(\sqrt{k}t\right)}\right)^{\frac{d}{2}} \exp\left(\frac{i\sqrt{k}}{2}\cot\left(\sqrt{k}t\right)\|\boldsymbol{y}\|^2\right)\end{aligned}$$

concluding the construction.

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