Meixner's class of non-commutative generalized stochastic processes with freely independent values

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Based on

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Meixner polynomials

Meixner searched for all probability measures μ on $\mathbb{R}$ with infinite support whose system of monic orthogonal polynomials $(p^{(n)})_{n=0}^{\infty}$ has an (exponential) generating function of the exponential type:

$$\sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = \exp(x\Psi(z) + \Phi(z)).$$

Meixner discovered that this holds, for a centered measure μ, if and only if there exist

$$\lambda \in \mathbb{R}, \quad \eta \geq 0, \quad k > 0$$

such that the polynomials $(p^{(n)})_{n=0}^{\infty}$ satisfy the recursive relation

$$xp^{(n)}(x) = p^{(n+1)}(x) + \lambda n p^{(n)}(x) + (kn + \eta n(n-1))p^{(n-1)}(x).$$

Meixner polynomials

Set $k = 1$.

Case 1. $\lambda = \eta = 0$: μ is Gaussian measure, $(p^{(n)})_{n=0}^{\infty}$ are Hermite polynomials.

Case 2. $\eta = 0$, $\lambda \neq 0$: μ is centered Poisson measure, $(p^{(n)})_{n=0}^{\infty}$ are Charlier polynomials.

Case 3. $\eta > 0$, $\lambda^2 = 4\eta$: μ is (centered) Gamma measure, $(p^{(n)})_{n=0}^{\infty}$ are Laguerre polynomials.

Case 4. $\eta > 0$, $\lambda^2 > 4\eta$: μ is negative binomial (Pascal) measure, $(p^{(n)})_{n=0}^{\infty}$ are Meixner polynomials of the first kind.

Case 5. $\eta > 0$, $\lambda^2 < 4\eta$: μ is Meixner measure (the density involves the square of the complex gamma function), $(p^{(n)})_{n=0}^{\infty}$ are Meixner polynomials of the second kind, or the Meixner–Pollaczek polynomials.

Meixner polynomials

We introduce in $L^2(\mathbb{R},\mu)$ creation (raising) and annihilation (lowering) operators through

$$\partial^\dagger p^{(n)} := p^{(n+1)}, \qquad \partial p^{(n)} := n p^{(n-1)}.$$

Then the action of the operator of multiplication by x in $L^2(\mathbb{R},\mu)$ has a representation

$$x\cdot = \partial^\dagger + \lambda \partial^\dagger \partial + \partial + \eta \partial^\dagger \partial \partial.$$

Cumulants

Denote by $\mathscr{P}(n)$ the collection of all set partitions of the set $\{1, 2, \dots, n\}$.
For any random variables $X_1, \dots, X_n$ with finite moments, the cumulant of $X_1, \dots, X_n$, denoted by $C_n(X_1, \dots, X_n)$, is defined recurrently through the formula

$$\mathbb{E}(X_1 \cdots X_n) = \sum_{\pi \in \mathscr{P}(n)} \prod_{A \in \pi} C_A(X_1, \dots, X_n),$$

where for any $A = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$

$$C_A(X_1, \dots, X_n) := C_k(X_{i_1}, \dots, X_{i_k}).$$

Cumulants

For example:

$$\mathbb{E}(X_1) = C_1(X_1),$$

$$\mathbb{E}(X_1X_2) = C_1(X_1)C_1(X_2) + C_2(X_1, X_2),$$

$$\mathbb{E}(X_1X_2X_3) = C_1(X_1)C_1(X_2)C_1(X_3) + C_2(X_1, X_2)C_1(X_3) + C_2(X_1, X_3)C_1(X_2) + C_1(X_1)C_2(X_2, X_3) + C_3(X_1, X_2, X_3)$$

Cumulants

The cumulant generating function of a random variable X:

$$C(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} C_n(X, \dots, X).$$

In fact,

$$C(z) = \log\big(\mathbb{E}(\exp(zX))\big).$$

Meixner polynomials — Generating function

$$\sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = \exp\big(x\Psi(z) \underbrace{-C(\Psi(z))}_{+\Phi(z)}\big).$$

Here $C(\cdot)$ is the cumulant generating function of the measure of orthogonality μ

Meixner polynomials — Generating function

Define numbers $\alpha, \beta \in \mathbb{C}$ through the equation

$$1 + \lambda t + \eta t^2 = (1 - \alpha t)(1 - \beta t).$$

Meixner polynomials — Generating function

$$\sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = \exp\big(x\Psi(z) - C(\Psi(z))\big).$$

Here

$$C(z) = \sum_{m=1}^{\infty} \frac{(\alpha\beta)^{m-1}}{m} \left[\sum_{n=2}^{\infty} \frac{(-z)^n}{n!} \Big(\beta^{n-2} + \beta^{n-3}\alpha + \cdots + \alpha^{n-2}\Big)\right]^m,$$

$$\Psi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \Big(\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}\Big),$$

$$C(\Psi(z)) = \sum_{n=2}^{\infty} \frac{z^n}{n} \Big(\alpha^{n-2} + \alpha^{n-3}\beta + \cdots + \beta^{n-2}\Big).$$

Lévy processes

T — complete, connected, oriented C^∞ manifold (e.g. $T = \mathbb{R}^d$ or a subset of $\mathbb{R}^d$),
dt — volume measure

$$\mathcal{D} = C_0^\infty(T) \subset L^2(T, dt) \subset \mathcal{D}'.$$

$\mathscr{C}$ — the cylinder σ-algebra on $\mathcal{D}'$

Lévy processes

Lévy process: a probability measure μ on $(\mathcal{D}', \mathscr{C})$ with Fourier transform:

$$\int_{\mathcal{D}'} \exp[i\langle\omega, \varphi\rangle]\,\mu(d\omega) = \exp\left[\int_T \int_{\mathbb{R}} \left(e^{is\varphi(t)} - 1 - is\varphi(t)\right)\,\frac{1}{s^2}\,\nu(ds)\,dt\right],$$

where $\varphi \in \mathcal{D}$.
Here ν is a probability measure on $\mathbb{R}$ about which we assume that

$$\int_{\mathbb{R}} e^{\epsilon|s|}\,\nu(ds) < \infty \quad \text{for some } \epsilon > 0.$$

Lévy processes

Examples:
If $\nu = \delta_0$, then μ is Gaussian white noise:

$$\int_{\mathcal{D}'} \exp[i\langle \omega, \varphi \rangle] \, \mu(d\omega) = \exp\left[-\frac{1}{2} \int_T \varphi^2(t) \, dt \right].$$

If $\nu = \delta_1$, then μ is centered Poisson white noise:

$$\int_{\mathcal{D}'} \exp[i\langle \omega, \varphi \rangle] \, \mu(d\omega) = \exp\left[\int_T \left(e^{i\varphi(t)} - 1 - i\varphi(t) \right) \, dt \right].$$

Isomorphism with symmetric Fock space

Theorem

There exists a unique unitary isomorphism

$$I : L^2(\mathcal{D}', \mu) \to \mathcal{F}_{\mathrm{sym}}(L^2(T \times \mathbb{R}, dt\,\nu(ds)))$$

such that

$$I1 = \Omega \qquad (\Omega \text{ being the vacuum vector})$$

and for each $\varphi \in \mathcal{D}$ *the image of the operator of multiplication by* $\langle \omega, \varphi \rangle$ *in* $L^2(\mathcal{D}', \mu)$ *is the following operator in the Fock space:*

$$\begin{aligned} X(\varphi) &= a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + a^0(\varphi \otimes s) \\ &= a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + \Gamma(M_\varphi \otimes M_s). \end{aligned}$$

Chaos expansion

CP: Continuous polynomials, functions on $\mathcal{D}'$ of the form

$$\sum_{k=0}^{n} \langle \omega^{\otimes k}, f^{(k)} \rangle, \qquad f^{(k)} \in \mathcal{D}^{\odot k} = C^{\infty}_{0,\,\mathrm{sym}}(T^k).$$

CP is a dense subset of $L^2(\mathcal{D}', \mu)$

$\mathbf{CP}^{(n)}$: continuous polynomials of order $\le n$

$\mathbf{MP}^{(n)}$: measurable polynomials of order $\le n$ — the closure of $\mathbf{CP}^{(n)}$ in $L^2(\mathcal{D}', \mu)$

$\mathbf{OP}^{(n)}$: orthogonal polynomials of order n:

$$\mathbf{OP}^{(n)} = \mathbf{MP}^{(n)} \ominus \mathbf{MP}^{(n-1)}$$

Thus

$$L^2(\mathcal{D}', \mu) = \bigoplus_{n=0}^{\infty} \mathbf{OP}^{(n)}$$

Chaos expansion

For $f^{(n)} \in \mathcal{D}^{\odot n}$, we take the orthogonal projection of the monomial $\langle \omega^{\otimes n}, f^{(n)} \rangle$ onto $\mathbf{OP}^{(n)}$:

$$P(f^{(n)}, \omega) = \langle P^{(n)}(\omega), f^{(n)} \rangle$$

Theorem (L., 2003)

For any $f^{(n)}, g^{(n)} \in \mathcal{D}^{\odot n}$, we have

$$\Big(\langle P^{(n)}(\cdot), f^{(n)} \rangle, \langle P^{(n)}(\cdot), g^{(n)} \rangle \Big)_{L^2(\mathcal{D}', \mu)} = (f^{(n)}, g^{(n)})_{L^2(T^n, \sigma^{(n)})}.$$

Here $\sigma^{(n)}$ is a measure on T^n which can be explicitly calculated, and which depends on ν.

Chaos expansion

Corollary

Denote

$$\mathcal{F}_{\text{ext}} := \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(T^n, \sigma^{(n)}).$$

We have a unitary isomorphism

$$U : \mathcal{F}_{\text{ext}} \to L^2(\mathcal{D}', \mu)$$

given through

$$\mathcal{D}^{\odot n} \ni f^{(n)} \mapsto (Uf^{(n)})(\omega) = \langle P^{(n)}(\omega), f^{(n)} \rangle.$$

Lévy processes of Meixner's type

Theorem (L. 2003; Berezansky, L., Mierzejewski, 2003))

For each $f^{(n)} \in \mathcal{D}^{\odot n}$,

$$\langle P^{(n)}(\cdot), f^{(n)} \rangle \in \mathbf{CP}$$

if and only if there exist $\lambda \in \mathbb{R}$ *and* $\eta \geq 0$ *such that* ν *is the measure of orthogonality of polynomials* $(q^{(n)})_{n=0}^{\infty}$ *satisfying*

$$sq^{(n)}(s) = q^{(n+1)}(s) + \lambda(n+1)q^{(n)}(s) + \eta n(n+1)q^{(n+1)}(s).$$

Lévy processes of Meixner's type

Theorem (continuation)

In the latter case, we have:

$$\langle\omega,\varphi\rangle\langle P^{(n)}(\omega),f^{\odot n}\rangle=\langle P^{(n+1)}(\omega),\varphi\odot f^{\odot n}\rangle + n\langle P^{(n)}(\omega),(\lambda\varphi f)\odot f^{\odot(n-1)}\rangle+n\langle\varphi,f\rangle\langle P^{(n-1)}(\omega),f^{\odot(n-1)}\rangle + n(n-1)\langle P^{(n-1)}(\omega),(\eta\varphi f^2)\odot f^{\odot(n-2)}\rangle.$$

Lévy processes of Meixner's type

Theorem (continuation)

Thus, the operator of multiplication by $\langle\omega,\varphi\rangle$ *in* $L^2(\mathcal{D}',\mu)$ *admits the representation*

$$\langle\omega,\varphi\rangle = \int_T dt\,\varphi(t)\omega(t),$$

where

$$\omega(t) = \partial_t^\dagger + \lambda\partial_t^\dagger\partial_t + \partial_t + \eta\partial_t^\dagger\partial_t\partial_t.$$

Here, in the realization in the $\mathcal{F}_{\mathrm{ext}}$ *space*

$$\partial_t^\dagger f^{(n)} = \delta_t \odot f^{(n)},$$

$$(\partial_t f^{(n)})(t_1,\dots,t_{n-1}) = n f^{(n)}(t,t_1,\dots,t_{n-1}).$$

Processes with independent values of Meixner's type

It is possible to generalize the above result to the case of generalized stochastic processes with independent values.
Main changes: the measure ν now depends on t, i.e., $\nu(t, ds)$ and

$$\omega(t) = \partial_t^\dagger + \lambda(t)\partial_t^\dagger\partial_t + \partial_t + \eta(t)\partial_t^\dagger\partial_t\partial_t,$$

where $\lambda(\cdot)$ and $\eta(\cdot)$ are smooth functions on T.

Processes with independent values of Meixner's type

The cumulant generating function of μ:

$$C(\varphi) = \log\left(\int_{\mathcal{D}'} e^{\langle \omega, \varphi \rangle} \mu(d\omega)\right), \quad \varphi \in \mathcal{D}.$$

$$C(\varphi) = \int_T C_{\lambda(t),\eta(t)}(\varphi(t))\, dt.$$

Processes with independent values of Meixner's type

The generating function of the orthogonal polynomials:
In the one-dimensional case:

$$\sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = \exp\big(x\Psi(z) \underbrace{-C(\Psi(z))}_{+\Phi(z)}\big).$$

In the infinite-dimensional case

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, \langle P^{(n)}(\omega), \varphi^{\otimes n}\rangle$$
$$= \exp\left[\big\langle \omega(\cdot), \Psi_{\lambda(\cdot),\eta(\cdot)}(\varphi(\cdot))\big\rangle - \int_T C_{\lambda(t),\eta(t)}\big(\Psi_{\lambda(t),\eta(t)}(\varphi(t))\big)\, dt\right]$$

q-numbers

For $q \in [-1, 1]$, one defines

$$[0]_q := 0, \quad [n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

$$[n]_q! := [1]_q[2]_q \times \cdots \times [n]_q.$$

Free probability: $q = 0$, so that

$$[n]_0 = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n \in \mathbb{N}, \end{cases} \qquad [n]_0! = 1.$$

Free analog of $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Free Meixner polynomials

Saitoh, Yoshida, 2001, Anshelevich, 2003: orthogonal polynomials $(p^{(n)})_{n=0}^{\infty}$ with generating function of the resolvent type:

$$\sum_{n=0}^{\infty} p^{(n)}(x)\, z^n = \frac{1}{1-(x\Psi(z)+\Phi(z))}.$$

This holds, for a centered measure μ, if and only if there exist

$$\lambda \in \mathbb{R}, \quad \eta \geq 0, \quad k > 0$$

such that the polynomials $(p^{(n)})_{n=0}^{\infty}$ satisfy the recursive relation

$$xp^{(n)}(x) = p^{(n+1)}(x) + \lambda[n]_0 p^{(n)}(x) + (k[n]_0 + \eta[n]_0[n-1]_0)p^{(n-1)}(x).$$

Othogonality measure

$$\mu(dx) = \frac{\sqrt{4(1+\eta)-(x-\lambda)^2}}{2\pi(1+\lambda x+\eta t^2)}\,\chi_{[\lambda-2\sqrt{1+\eta},\lambda+2\sqrt{1+\eta}]}\,dx + \text{0,1,2 atoms}$$

In particular:
For $\lambda=\eta=0$,

$$\mu(dx) = \frac{\sqrt{4-x^2}}{2\pi}\,\chi_{[-2,2]}(x)\,dx$$

Wiegner's semicircular distribution.

For $\eta=0$, $\lambda\neq 0$, μ is Marchenko–Pastur distribution (has one atom).

Free Meixner polynomials

Set $k = 1$.

We introduce in $L^2(\mathbb{R}, \mu)$ creation (raising) and annihilation (lowering) operators through

$$\partial^\dagger p^{(n)} := p^{(n+1)}, \qquad \partial p^{(n)} := [n]_0 p^{(n-1)}.$$

Then the action of the operator of multiplication by x in $L^2(\mathbb{R}, \mu)$ has a representation

$$x\cdot = \partial^\dagger + \lambda \partial^\dagger \partial + \partial + \eta \partial^\dagger \partial \partial.$$

Free cumulants

Consider a von Neumann algebra $\mathscr{A}$ with a normal, faithful, tracial state

$$\tau : \mathscr{A} \to \mathbb{C}.$$

[I.e., τ is linear, , $\tau(ab) = \tau(ba)$, $\tau(\mathrm{id}) = 1$, $\tau(aa^*) \geq 0$, and $\tau(aa^*) = 0$ implies $a = 0$.]
A non-commutative random variable X is a self-adjoint element of $\mathscr{A}$.
τ is understood as expectation on $\mathscr{A}$.

Denote by $\mathcal{NC}(n)$ the collection of all non-crossing partitions of $\{1, \dots, n\}$, i.e., all set partitions $\pi = \{A_1, \dots, A_k\}$, $k \geq 1$, of $\{1, \dots, n\}$ such that there do not exist $A_i, A_j \in \pi$, $A_i \neq A_j$, for which the following inequalities hold:

$$x_1 < y_1 < x_2 < y_2$$

for some $x_1, x_2 \in A_i$ and $y_1, y_2 \in A_j$.

Free cumulants

For any non-commutative random variables $X_1, \dots, X_n$, the free cumulant of $X_1, \dots, X_n$, denoted by $C_n(X_1, \dots, X_n)$, is defined recurrently through the formula

$$\tau(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{NC}(n)} \prod_{A \in \pi} C_A(X_1, \dots, X_n),$$

where for any $A = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$

$$C_A(X_1, \dots, X_n) := C_k(X_{i_1}, \dots, X_{i_k}).$$

Free cumulants

The free cumulant generating function of a random variable X:

$$C(z) = \sum_{n=1}^{\infty} z^n C_n(X, \ldots, X).$$

If μ is a probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^n \rho(dx) = \tau(X^n),$$

we say that $C(z)$ is the free cumulant generating function of μ.

Non-commutative random variables $X_1, \ldots, X_n$ are called free independent if all mixed free cumulants of $X_1, \ldots, X_n$ vanish.

Free Meixner polynomials — Generating function

$$\sum_{n=0}^{\infty} p^{(n)}(x)\, z^n = \frac{1}{1 - x\Psi(z) + \underbrace{C(\Psi(z))}_{-\Phi(z)}}.$$

Here $C(\cdot)$ is the free cumulant generating function of the measure of orthogonality μ.

Meixner polynomials — Generating function

$$\sum_{n=0}^{\infty} p^{(n)}(x)\, z^n = \frac{1}{1 - x\Psi(z) + C(\Psi(z))}.$$

Here

$$C(z) = \frac{2z^2}{1 - \lambda z + \sqrt{(1-\lambda z)^2 - 4z^2\eta}},$$

$$\Psi(z) = \frac{z}{1 + \lambda z + \eta z^2},$$

$$C(\Psi(z)) = \frac{z^2}{1 + \lambda z + \eta z^2}.$$

Process with freely independent values

From now on $\mathcal{D} = C_0(T)$,

For each $t \in T$ let $\nu(t,\cdot)$ be a probability measure on $\mathbb{R}$ with compact support.

Full Fock space

$$\mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t, ds))).$$

For each $\varphi \in \mathcal{D}$

$$\begin{aligned} X(\varphi) &= a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + a^0(\varphi \otimes s) \\ &= a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + \Gamma(M_\varphi \otimes M_s). \end{aligned}$$

$\mathscr{A}$ — the real algebra generated by $(X(\varphi))_{\varphi \in \mathcal{D}}$.

For $A \in \mathscr{A}$, the expectation of A is defined as

$$\tau(A) := (A\Omega, \Omega)_{\mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t,ds)))}$$

Processes with freely independent values

Proposition

Let $\varphi_1, \dots, \varphi_n \in \mathcal{D}$ be such that

$$\varphi_i \varphi_j = 0 \quad \textit{a.e. if } i \neq j.$$

Then $X(\varphi_1), \dots, X(\varphi_n)$ are freely independent.

Thus, if

$$X(\varphi) = \int_T dt\, \varphi(t)\omega(t) = \langle \omega, \varphi \rangle,$$

then $\omega(\cdot)$ may be thought of as a non-commutative generalized stochastic process with freely independent values.

The space $L^2(\tau)$

We define the Hilbert space $L^2(\tau)$ as the closure of $\mathscr{A}$ in the norm generated by the scalar product

$$(A_1, A_2)_{L^2(\tau)} := \tau(A_1 A_2), \quad A_1, A_2 \in \mathscr{A}.$$

The space $L^2(\tau)$

Theorem (Bożejko, L. 2009)

The set

$$\{A\Omega :\ A \in \mathscr{A}\}$$

is dense in $\mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t, ds)))$.

There exists a unique isomorphism

$$I : L^2(\tau) \to \mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t, ds)))$$

such that

$$I1 = \Omega$$

and for each $\varphi \in \mathcal{D}$ *the image of the operator of left multiplication by* $\langle \omega, \varphi \rangle$ *in* $L^2(\tau)$ *is the operator* $\langle \omega, \varphi \rangle$ *in the Fock space.*

Chaos expansion

$\mathbf{CP}$: Continuous polynomials, operators of the form

$$\sum_{k=0}^{n} \langle \omega^{\otimes k}, f^{(k)} \rangle, \qquad f^{(k)} \in \mathcal{D}^{(k)} = C_0(T^k).$$

$\mathbf{CP}$ is a dense subset of $L^2(\tau)$

Thus

$$L^2(\mathcal{D}', \mu) = \bigoplus_{n=0}^{\infty} \mathbf{OP}^{(n)}$$

Chaos expansion

For $f^{(n)} \in \mathcal{D}^{(n)}$, we denote by

$$P(f^{(n)}, \omega) = \langle P^{(n)}(\omega), f^{(n)} \rangle$$

the orthogonal projection of the monomial $\langle \omega^{\otimes n}, f^{(n)} \rangle$ onto $\mathbf{OP}^{(n)}$.

Theorem

For any $f^{(n)}, g^{(n)} \in \mathcal{D}^{(n)}$, we have

$$\Big(\langle P^{(n)}(\cdot), f^{(n)} \rangle, \langle P^{(n)}(\cdot), g^{(n)} \rangle \Big)_{L^2(\tau)} = (f^{(n)}, g^{(n)})_{L^2(T^n, \sigma^{(n)})}.$$

Here $\sigma^{(n)}$ is a measure on T^n which can be explicitly calculated.

Chaos expansion

Corollary

Denote

$$\mathcal{F}_{\text{ext}} := \bigoplus_{n=0}^{\infty} L^2(T^n, \sigma^{(n)}).$$

We have a unitary isomorphism

$$U : \mathcal{F}_{\text{ext}} \to L^2(\tau)$$

given through

$$\mathcal{D}^{(n)} \ni f^{(n)} \mapsto U f^{(n)} = \langle P^{(n)}(\omega), f^{(n)} \rangle.$$

Free processes of Meixner's type

Theorem

For each $f^{(n)} \in \mathcal{D}^{(n)}$

$$\langle P^{(n)}(\omega), f^{(n)} \rangle \in \mathbf{CP}$$

if and only if there exist $\lambda, \eta \in C(T)$ *with* $\lambda(t) \in \mathbb{R}$, $\eta(t) \geq 0$ *and for each* $t \in T$ $\nu(t, \cdot)$ *is the measure of orthogonality of polynomials* $(q^{(n)})_{n=0}^{\infty}$ *satisfying*

$$sq^{(n)}(s) = q^{(n+1)}(s) + \lambda(t)q^{(n)}(s) + \eta(t)q^{(n+1)}(s).$$

Lévy processes of Meixner's type

Theorem (continuation)

In the latter case, we have the following recursive relation for orthogonal polynomials:

$$\begin{aligned}\langle\omega,\varphi\rangle\langle P^{(n)}(\omega), f_1\otimes\cdots\otimes f_n\rangle &= \langle P^{(n+1)}(\omega),\varphi\otimes f_1\otimes\cdots\otimes f_n\rangle\\ &+ [n]_0\langle P^{(n)}(\omega),(\lambda\varphi f_1)\otimes f_2\otimes\cdots\otimes f_n\rangle\\ &+ [n]_0\langle\varphi, f_1\rangle\langle P^{(n-1)}(\omega), f_2\otimes\cdots\otimes f_n\rangle\\ &+ [n]_0[n-1]_0\langle P^{(n-1)}(\omega),(\eta\varphi f_1 f_2)\otimes f_3\otimes\cdots\otimes f_n\rangle.\end{aligned}$$

Thus,

$$\omega(t)=\partial_t^\dagger+\lambda(t)\partial_t^\dagger\partial_t+\partial_t+\eta(t)\partial_t^\dagger\partial_t\partial_t.$$

Cumulant generating function

Proposition

The cumulant generating function of each $\langle \omega, \varphi \rangle$ is

$$C(\varphi) = \int_T C_{\lambda(t),\eta(t)}(\varphi(t))\, dt.$$

Generating function of free Meixner polynomials

Recall that in the classical case we had the generating function

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega), \varphi^{\otimes n} \rangle,$$

where $\varphi \in \mathcal{D}$. However, if ω is a field of non-commuting operators such an expression would not characterize it. Instead φ should be an operator-valued function whose values do not commute with each other but they do commute with all $\omega(t)$.

Generating function of free Meixner polynomials

Let $\mathcal{G}$ be a Hilbert space. We consider the tensor product

$$\mathcal{G} \otimes \mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t, ds))).$$

We associate any operators

$$A \in \mathscr{L}(\mathcal{G}), \quad B \in \mathscr{L}(\mathcal{F}(L^2(T \times \mathbb{R}, dt\, \nu(t, ds))))$$

with the operators

$$A \otimes \mathbf{1}, \quad \mathbf{1} \otimes B$$

in the tensor product.
We define a class $\mathcal{Z}$ of operator-valued functions

$$T \ni t \mapsto Z(t) \in \mathscr{L}(\mathcal{G})$$

with compact support.

Generating function of free Meixner polynomials

Theorem (Bożejko, L.)

For each $Z \in \mathcal{Z}$,

$$\sum_{n=0}^{\infty} \int_{T^n} P^{(n)}(\omega)(t_1, \dots, t_n) Z(t_1) \cdots Z(t_n)\, dt_1 \cdots dt_n$$

$$= \left(\mathbf{1} - \langle \omega(\cdot), \Psi_{\lambda(\cdot),\eta(\cdot)}(Z(\cdot)) \rangle + \int_T C_{\lambda(t),\eta(t)}(\Psi_{\lambda(t),\eta(t)}(Z(t)))\, dt \right)^{-1}$$

$$= \left(\mathbf{1} - \left\langle \omega, \frac{Z}{\mathbf{1} + \lambda Z + \eta Z^2} \right\rangle + \int_T \frac{Z(t)^2}{\mathbf{1} + \lambda(t) Z(t) + \eta(t) Z(t)^2}\, dt \right)^{-1}$$

Annihilation operator

Classical one-dimensional case: Recall

$$\sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = \exp(x\Psi(z) + \Phi(z)).$$

and

$$\partial p^{(n)} = np^{(n-1)}.$$

Hence

$$\partial \sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n = z \sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!}\, z^n,$$

hence

$$\partial \exp[x\Psi(z)] = z \exp[x\Psi(z)].$$

Let D denote the operator of differentiation. Then

$$D \exp[x\Psi(z)] = \Psi(z) \exp[x\Psi(z)].$$

Annihilation operator

Hence

$$\partial = \Psi^{-1}(D).$$

For example, if $\eta = 0$,

$$\partial = \frac{1}{\lambda}\left(e^{\lambda D} - 1\right) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} D^n.$$

Annihilation operator

Classical infinite-dimensional case:

$$\partial_t = \Psi^{-1}_{\lambda(t),\eta(t)}(D_t),$$

where D_t is the Hida–Malliavin derivative, i.e., derivative in direction δ_t.

If $\eta(t) = 0$.

$$\partial_t = \frac{1}{\lambda(t)}(e^{\lambda(t)D_t} - 1) = \sum_{n=1}^{\infty} \frac{\lambda(t)^{n-1}}{n!} D_t^n.$$

Annihilation operator

Free infinite-dimensional case:

Free differentiation at t:

$$D_t\langle \omega^{\otimes n}, f_1 \otimes f_2, \otimes \cdots \otimes f_n\rangle = [n]_0 f_1(t)\,\langle \omega^{\otimes (n-1)}, f_2, \otimes \cdots \otimes f_n\rangle.$$

Theorem (Bożejko, L.)

Let $\eta \equiv 0$. For each $t \in T$, the operator ∂_t acting on $\mathbf{CP}$ has the following representation:

$$\partial_t = \Psi^{-1}_{\lambda(t),0}(D_t\mathbb{G}) = \frac{D_t\mathbb{G}}{\mathbf{1} - \lambda(t)D_t\mathbb{G}} = \sum_{k=1}^{\infty} \lambda(t)^{k-1}(D_t\mathbb{G})^k.$$

where $\mathbb{G}$ is a 'global' operator, independent of t. In particular, if $\lambda(t) = 0$,

$$\partial_t = D_t\mathbb{G}.$$