A new Wick formula for products of White Noise distributions and application to Feynman path integrands

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Outline

White Noise

Literatur

- T.Hida, H.-H. Kuo, J.Potthoff and L.Streit. **White Noise - An Infinite Dimensional Calculus.** Kluwer Academic Publisher, Dordrecht, Boston, London 1993.
- N.Obata. **White Noise Calculus and Fock space**. volume 1577 of LNM. Springer Verlag, Berlin, Heidelberg, New York, 1994.
- H.-H. Kuo. **White Noise Distribution Theory**. CRC Press, Boca Raton, New York, London, Tokyo, 1996.
- W. Westerkamp. **Recent Results in Infinite Dimensional Analysis and Applications to Feynman Integrals**. PhD thesis, University of Bielefeld, 1995.

White Noise spaces

Hida and regular test and generalized functions:

$$(S) \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' \subset (S)'$$

The dual pairing between $(S)'$ and (S) is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

Wiener-Itô-chaos decomposition of $\Phi \in (S)'$:

$$\Phi = \sum_{n=0}^{\infty} I(\Phi^{(n)}) = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \Phi^{(n)} \rangle, \quad \Phi^{(n)} \in \left(\widehat{S'_{\mathbb{C}}(\mathbb{R})}\right)^{\otimes n}.$$

For $\Phi \in \mathcal{G}'$ one has that $\Phi^{(n)} \in \left(\widehat{L^2_{\mathbb{C}}(\mathbb{R})}\right)^{\otimes n}$.

S- and T-transform of $\Phi \in (S)'$:

$$S\Phi(g) = \langle\langle \Phi, :\exp(\langle \cdot, g \rangle):\rangle\rangle \quad T\Phi(g) = \langle\langle \Phi, \exp(i\langle \cdot, g\rangle)\rangle\rangle \qquad g \in S(\mathbb{R})$$

Remember the characterizations of $(S)'$ and $\mathcal{G}'$ and the corresponding sequence and integral theorems, see [PS91], [KLP+96] and [GKS97]

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Important examples

- Brownian motion and Brownian bridge

$$B_t(\cdot) := \langle \cdot, \mathbf{1}_{[t_0,t)} \rangle = \int_{t_0}^{t} \omega(s)\, ds, \quad 0 < t_0 < t < \infty,$$

$$B^{0\to 0}_{t_0,t,s}(\cdot) := \langle \cdot, \mathbf{1}_{[t_0,s)} \rangle - \frac{s}{t} \langle \cdot, \mathbf{1}_{[t_0,t)} \rangle, \quad 0 < t_0 \leqslant s \leqslant t < \infty.$$

- Donskers Delta: $\delta(\langle \cdot, h \rangle - a) \in (S)'$, $h \in L^2(\mathbb{R})$, $a \in \mathbb{R}$, z.B. für $h = \mathbf{1}_{[t_0,t)}$, $0 < t < \infty$ und $a = x - x_0$ gilt:

$$S(\delta(\langle \cdot, \mathbf{1}_{[t_0,t)} \rangle - a))(g) = \frac{1}{\sqrt{2\pi(t-t_0)}} \exp\left(-\frac{1}{2(t-t_0)} ((g, \mathbf{1}_{[t_0,t)}) - (x - x_0))^2 \right)$$

- Normalized exponential:

$$\Phi = \mathrm{Nexp}\left(\frac{1}{2}(1+i) \int_{t_0}^{t} \omega(\tau)^2\, d\tau \right)$$

$$S(\Phi)(g) = \exp\left(-\frac{1}{2}(1-i) \int_{t_0}^{t} g^2(r)\, dr \right), \quad g \in S(\mathbb{R})$$

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Scaling and localized scaling

Skalierungsoperator:

For $\varphi \in (S)$ the scaling in $\sqrt{i}$ of φ by

$$(\sigma_{\sqrt{i}}\varphi)(\omega) = \varphi(\sqrt{i}\omega), \quad \omega \in S'(\mathbb{R}).$$

$$\sigma_{\sqrt{i}}\varphi = \sum_{n=0}^{\infty} \left\langle :\omega^{\otimes n}:, \sqrt{i}^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{i-1}{2}\right)^k \mathrm{tr}^k \varphi^{(n+2k)} \right\rangle. \qquad (1)$$

where $\mathrm{tr}^{k,(n+2k)}$ is defined by

$$\mathrm{tr}^k \varphi^{(n+2k)} := \left(\mathrm{Tr}^{\otimes k}, \varphi^{(n+2k)}\right)_{L^2(\mathbb{R})^{\otimes 2k}} \in S'(\mathbb{R})^{\hat{\otimes} n}, \quad \mathrm{Tr}(\xi \otimes \eta) = (\xi, \eta)_{L^2(\mathbb{R})}, \ \xi, \eta \in S(\mathbb{R})$$

Properties:

- We call $A_{\sqrt{i}} \subset (S)'$ its domain, i.e. the set of $\Phi \in (S)'$ for which (1) converge in $(S)'$
- For $\Phi, \Psi \in A_{\sqrt{i}}$ with $\Phi\Psi \in A_{\sqrt{i}}$, one has that $\sigma_{\sqrt{i}}(\Phi\Psi) = \sigma_{\sqrt{i}}\Phi\,\sigma_{\sqrt{i}}\Psi$.
- $$\mathrm{Nexp}\left(\frac{i+1}{2}\int_{\mathbb{R}} \omega(\tau)^2\, d\tau\right)\Phi = \sigma^{\dagger}_{\sqrt{i}}\sigma_{\sqrt{i}}\Phi \in (S)'.$$

Lokalized scaling

$$\sigma_{\sqrt{i},t_0,t}\varphi = \sum_{n=0}^{\infty}\left\langle :\omega^{\otimes n}:, \sqrt{i}^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(\frac{i-1}{2}\right)^k \mathrm{tr}_{t_0,t}{}^k \varphi^{(n+2k)}\right\rangle. \tag{2}$$

$0 < t_0 < t$, where $\mathrm{tr}^{\mathrm{k}}_{\mathrm{t_0,t}}{}^{,(\mathrm{n+2k})}$ is defined by

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$$\mathrm{Tr}_{\mathrm{t_0,t}}(\xi\otimes\eta) = (\xi, \mathbf{1}_{[t_0,t)}\eta)_{L^2(\mathbb{R})},\ \xi,\eta\in S(\mathbb{R})$$

Properties:

- Analogously to the scaling we define its domain $A_{\sqrt{i},t_0,t}\subset(S)'$
- If $\Phi\in A_{\sqrt{i},t_0,t}$ with kernels $\Phi^{(n)},\ n\in\mathbb{N}$ fulfills $\mathbf{1}^{\otimes n}_{[t_0,t)}\Phi^{(n)} = \Phi^{(n)}$ then $\sigma_{\sqrt{i},t_0,t}\Phi = \sigma_{\sqrt{i}}\Phi$. E.g. $\Phi = \delta(\langle\omega,\mathbf{1}_{[t_0,t)}\rangle + x_0 - x)$
- $$\mathrm{Nexp}\left(\frac{i+1}{2}\int_{t_0}^{t}\omega(\tau)^2\,d\tau\right)\Phi = \sigma^{\dagger}_{\sqrt{i},t_0,t}\sigma_{\sqrt{i},t_0,t}\Phi\in(S)',\quad \Phi\in A_{\sqrt{i},t_0,t}.$$

Wick Formula
How to realize products of regular distributions with Donsker's Delta?

The Wick formula:

orthogonal projection

$$P_h : \mathcal{G} \to \mathcal{G},\ \varphi \mapsto \varphi\left(\cdot - \langle \cdot, h\rangle h\right), \quad h \in L^2(\mathbb{R}),\ \|h\|_{L^2} = 1$$

translation

$$\tau_\eta : \mathcal{G} \to \mathcal{G},\ \varphi \mapsto \varphi\left(\cdot + \eta\right), \quad \eta \in L^2_{\mathbb{C}}(\mathbb{R})$$

Theorem

The product of $\Phi \in \mathcal{G}'$ with $\delta\left(\langle \cdot, h\rangle - a\right)$, $h \in L^2(\mathbb{R})$, $\|h\|_{L^2} = 1$, and $a \in \mathbb{C}$, exists in $\mathcal{G}'$ if and only if the orthogonal projection $P_h\Phi \in \mathcal{G}'$. Furthermore

$$\delta\left(\langle \cdot, h\rangle - a\right)\Phi = \delta\left(\langle \cdot, h\rangle - a\right) \diamond \tau_{ah} P_h \Phi,$$

where the Wick product is an independent pointwise product.

Example: $\Phi = x_0 + \langle \cdot, \mathbf{1}_{[t_0,s)}\rangle$, $h = \mathbf{1}_{[t_0,t)}$, $a = x - x_0$, then:

$$\delta\left(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0)\right)\Phi = \frac{1}{\sqrt{t-t_0}}\delta\left(\left\langle \cdot, \frac{\mathbf{1}_{[t_0,t)}}{\sqrt{t-t_0}}\right\rangle - \frac{x-x_0}{\sqrt{t-t_0}}\right)\left(x_0 + \langle \cdot, \mathbf{1}_{[t_0,s)}\rangle\right)$$

$$= \delta\left(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle - (x - x_0)\right) \diamond \left(x_0 - \frac{s-t_0}{t-t_0}(x-x_0) + \left\langle \cdot, \mathbf{1}_{[t_0,s)} - \frac{s-t_0}{t-t_0}\mathbf{1}_{[t_0,t)}\right\rangle\right)$$

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Idea of the proof:

- **Lemma (Westerkamp 1995)**

Let $\varphi \in \mathcal{G}$, $h \in L^2(\mathbb{R})$, $\|h\|_{L^2} = 1$, *and* $a \in \mathbb{C}$, *then:*

$$\langle\langle \delta\left(\langle \cdot, h\rangle - a\right), \varphi\rangle\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} E\left(\varphi\left(\cdot + \left(a - \langle \cdot, h\rangle\right)h\right)\right)$$

- Using this Lemma one can show that the Wick formula in $\mathcal{G}$ holds:

$$\delta\left(\langle \cdot, h\rangle - a\right)\varphi = \delta\left(\langle \cdot, h\rangle - a\right) \diamond \varphi\left(\cdot + \left(a - \langle \cdot, h\rangle\right)h\right)$$

- the translation τ_{ah} is a continuous mapping from $\mathcal{G}'$ into itself.
- Let $\Phi \in \mathcal{G}'$ with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$. Then $P_h\Phi \in \mathcal{G}'$ if and only if

$$\lim_{M\to\infty} P_h\varphi_M \in \mathcal{G}', \quad \varphi_M = \sum_{n=0}^{M} \langle : \cdot^{\otimes n} :, \Phi^{(n)}\rangle \in \mathcal{G}.$$

In this case $P_h\Phi = \lim_{M\to\infty} P_h\varphi_M$.

- With Step 2 and 3:

$$\Phi\delta\left(\langle \cdot, h\rangle - a\right) = \lim_{M\to\infty} \varphi_M\delta\left(\langle \cdot, h\rangle - a\right) = \lim_{M\to\infty} \tau_{ah}P_h\varphi_M \diamond \delta\left(\langle \cdot, h\rangle - a\right)$$

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Idea of the proof:

- **Lemma (Westerkamp 1995)**

  Let $\varphi \in \mathcal{G}$, $h \in L^2(\mathbb{R})$, $\|h\|_{L^2} = 1$, and $a \in \mathbb{C}$, then:

$$\langle\langle \delta\left(\langle \cdot, h\rangle - a\right), \varphi \rangle\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} E\left(\varphi\left(\cdot + \left(a - \langle \cdot, h\rangle\right) h\right)\right)$$

- Using this Lemma one can show that the Wick formula in $\mathcal{G}$ holds:

$$\delta\left(\langle \cdot, h\rangle - a\right)\varphi = \delta\left(\langle \cdot, h\rangle - a\right) \diamond \varphi\left(\cdot + \left(a - \langle \cdot, h\rangle\right) h\right)$$

- the translation τ_{ah} is a continuous mapping from $\mathcal{G}'$ into itself.
- Let $\Phi \in \mathcal{G}'$ with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$. Then $P_h\Phi \in \mathcal{G}'$ if and only if

$$\lim_{M \to \infty} P_h \varphi_M \in \mathcal{G}', \quad \varphi_M = \sum_{n=0}^{M} \langle : \cdot^{\otimes n} :, \Phi^{(n)} \rangle \in \mathcal{G}.$$

  In this case $P_h\Phi = \lim_{M \to \infty} P_h \varphi_M$.

- With Step 2 and 3:

$$\Phi\delta\left(\langle \cdot, h\rangle - a\right) = \lim_{M \to \infty} \varphi_M \delta\left(\langle \cdot, h\rangle - a\right) = \lim_{M \to \infty} \tau_{ah} P_h \varphi_M \diamond \delta\left(\langle \cdot, h\rangle - a\right)$$

$$= \tau_{ah}\left(\lim_{M \to \infty} P_h \varphi_M\right) \diamond \delta\left(\langle \cdot, h\rangle - a\right) = \tau_{ah} P_h \Phi \diamond \delta\left(\langle \cdot, h\rangle - a\right) \in \mathcal{G}'$$

Idea of the proof:

- **Lemma (Westerkamp 1995)**

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    $$\begin{aligned}\Phi\delta\left(\langle \cdot, h\rangle - a\right) &= \lim_{M\to\infty} \varphi_M\delta\left(\langle \cdot, h\rangle - a\right) = \lim_{M\to\infty} \tau_{ah}P_h\varphi_M \diamond \delta\left(\langle \cdot, h\rangle - a\right) \\ &= \tau_{ah}\left(\lim_{M\to\infty} P_h\varphi_M\right) \diamond \delta\left(\langle \cdot, h\rangle - a\right) = \tau_{ah}P_h\Phi \diamond \delta\left(\langle \cdot, h\rangle - a\right) \in \mathcal{G}'\end{aligned}$$

Applications to Feynman path integrals

- R. P. Feynman. **Space-time approach to non-relativistic quantum mechanics.** *Rev. Modern Physics*, 20:367–387, 1948.
- T. Hida and L. Streit. **Generalized brownian functionals and the feynman integral.** *Stoch. Proc. Appl.*, 16:55–69, 1983.

The classical approach and the Khandekar-Streit class

- D.C. Khandekar and L. Streit. **Constructing the Feynman integrand.** *Ann. Physik*, 1:46–55, 1992.
- A. Lascheck, P. Leukert, L. Streit, and W. Westerkamp. **Quantum mechanical propagators in terms of Hida distributions.** *Rep. Math. Phys.*, 33:221–232, 1993.

General White Noise Ansatz for Feynman integrands

Feynman Integrand in White Noise

$$I_V = \mathrm{Nexp}\left(\frac{i}{2}\int_{t_0}^{t} \dot{x}(\tau)^2\, d\tau + \frac{1}{2}\int_{t_0}^{t} \dot{x}(\tau)^2\, d\tau\right) \exp\left(-i\int_{t_0}^{t} V(x(\tau))\, d\tau\right) \delta(x(t)-x)$$

$$I_V = I_0 \exp\left(-i\int_{t_0}^{t} V(x(r))\, dr\right) = \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n} V(x_j) I_0 \delta(x(r_j) - x_j)\, dr_j dx_j,$$

where $\Lambda_n = \{(r_1, \dots, r_n) \mid t_0 = r_0 < r_1 < \dots < r_n < t\}$. For a special class of potentials I_V is constructed as a Hida distribution in [KS92] and [LLSW93].

The Khandekar-Streit class

The interactions:

$$V_2(x) = \int_{\mathbb{R}} \delta(x-y)\, dm(y), \tag{3}$$

where $dm(y) := v_2(y)dy$ is a finite Borel measure of compact support, K. For simplicity we assume that $v_2 : \mathbb{R} \to \mathbb{R}$ to be a continuous function with compact support.

The Feynman integrand :

$$I_V = I_0 + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \left(I_0 \prod_{j=1}^{n} \delta(x(r_j) - x_j) \right) v_2(x_j)\, dr_j dx_j \in (S)'$$

$T_0 \leqslant t_0 < t \leqslant T$ and $x, x_0 \in \mathbb{R}$.

T-transform of I_V:

$$T(I_V)(g) = I_0 + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} T \left(I_0 \prod_{j=1}^{n} \delta(x(r_j) - x_j) \right)(g)\, v_2(x_j)\, dr_j dx_j,$$

for all $g \in S(\mathbb{R})$.

The complex scaling approach

- R. H. Cameron. **A family of integrals serving to connect the Wiener and Feynman integrals.** *J. Math. and Phys.*, 39:126–140, 1960/1961.
- H. Doss. **Sur une resolution stochastique de l'equation de schrödinger à coefficients analytiques.** *Communications in Mathematical Physics*, 73:247–264, Oktober 1980.
- M. Grothaus, L. Streit, and A. Vogel. **The Complex Scaled Feynman-Kac Formula for Singular Initial Distributions.** 2010. Accepted for publication in Proceedings of the "International Conference on Stochastic Analysis and Applications", Hammamet,Tunisia, October 12-17 2009 , Stochastics, An International Journal of Probability and Stochastic Processes.

Informal ansatz

$$
\begin{aligned}
I_V &= \mathrm{Nexp}\left(\frac{i}{2}\int_{t_0}^{t} \dot{x}(\tau)^2\,d\tau + \frac{1}{2}\int_{t_0}^{t} \dot{x}(\tau)^2\,d\tau\right)\exp\left(-i\int_{t_0}^{t} V(x(\tau))\,d\tau\right)\delta(x(t)-x) \\
&= \sigma^{\dagger}_{\sqrt{i},t_0,t}\,\sigma_{\sqrt{i},t_0,t}\left(\exp\left(-i\int_{t_0}^{t} V(x(\tau))\,d\tau\right)\delta(x(t)-x)\right) \\
&= \sigma^{\dagger}_{\sqrt{i},t_0,t}\left(\sigma_{\sqrt{i}}\left(\tau_{\frac{x-x_0}{(t-t_0)}1\!\!1_{[t_0,t)}}\,P_{\frac{1}{\sqrt{t-t_0}}\vec{1\!\!1}_{[t_0,t)}}\left(-i\int_{t_0}^{t} V(x(\tau))\,d\tau\right)\right)\diamond\sigma_{\sqrt{i}}\delta(x(t)-x)\right)
\end{aligned}
$$

For which kind of potentials this formula is useful?

The Doss class

The Domain

For $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we consider the set

$$\mathcal{D} = \{z = x + \sqrt{i}y \in \mathbb{C} | x \in \mathcal{O}, y \in \mathbb{R}\} \subset \mathbb{C}.$$

The potentials

We consider $V : \mathcal{D} \to \mathbb{C}$ analytic fulfilling suitable integrability conditions

Examples

$$V : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto x^6$$

(or more general a potential of degree $4n + 2$, $n \in \mathbb{N}$)

For $\mathcal{O} = \mathbb{R} \setminus \{b\}$, $b \in \mathbb{R}$

$$V_1 : \mathcal{D} \to \mathbb{C} \qquad V_2 : \mathcal{D} \to \mathbb{C}$$
$$x \mapsto \frac{a}{|x-b|^n}, \qquad x \mapsto \frac{a}{(x-b)^n}$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$

The Doss class

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For $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we consider the set

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where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$

Theorem

Let V be a potential as above, $0 \leqslant t_0 < t \leqslant T$ and $x, x_0 \in \mathcal{O}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$ for all $t_0 < r < t$. Then the corresponding Feynman integrand exists as a Hida distribution and is given by

$$I_V = \sigma^{\dagger}_{\sqrt{i},t_0,t}\Bigg(\sigma_{\sqrt{i}} \exp\Bigg(-i\int_{t_0}^{t} V(x_0 + \frac{s-t_0}{t-t_0}(x-x_0) \\ + \langle \cdot, \mathbf{1}_{[t_0,r)}\rangle - \frac{s-t_0}{t-t_0}\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle)\, dr\Bigg) \diamond \sigma_{\sqrt{i}}\delta(x(t) - x)\Bigg),$$

where the Wick product inside is an independent product.

Theorem

Let V be a potential from the Doss class, $0 \leqslant t_0 < t \leqslant T < \infty$, $x, x_0 \in \mathbb{R}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$ for all $t_0 < r < t$. Then for all $g \in S(\mathbb{R})$ we get that

$$
\begin{aligned}
K_V^{(\dot{g})}(x, t; x_0, t_0) &:= \exp\left(-ixg(t) + ix_0 g(t_0) + \frac{1}{2}|g_{\Lambda^c}|_0^2\right) TI_V(g) \\
&= E\left(\sigma_{\sqrt{i}} \exp\left(-i\int_{t_0}^{t} V(B_{t_0,t,s}^{x_0 \to x}) + \dot{g}(r) B_{t_0,t,s}^{x_0 \to x}\, dr\right)\right. \\
&\qquad \left. \times E\left(\sigma_{\sqrt{i}} \delta(\langle \cdot, \mathbf{1}_{[t_0,t)}\rangle + x_0 - x)\right)\right)
\end{aligned}
$$

where $\Lambda = [t_0, t]$ and g_{Λ^c} is the restriction of g to the complement of Λ. Moreover $K_V^{(\dot{g})}(x, t; x_0, t_0)$ solves the Schrödinger equation

$$
\left(i\frac{\partial}{\partial t} - \frac{1}{2}\Delta - V_{\dot{g}}(t, x)\right) K_V^{(\dot{g})}(x, t; x_0, t_0) = i\delta_{x_0}\delta_{t_0}. \tag{4}
$$

Here $V_{\dot{g}}(t, x) = V(x) + g(t)x$, for all $0 < t < T$ and all $x \in \mathbb{R}$.

Theorem

Let V be defined as above, $t_0 < t_1 < \ldots < t_{n+1} = t$ and $x_j \in \mathcal{O}$ such that there exists a convex set $\mathcal{A} \subset \mathcal{O}$ with $x_j \in \mathcal{A}$ for all $0 \leqslant j \leqslant n+1$. Then

$$T\left(I_V \prod_{j=1}^{n} \delta(x(t_j) - x_j)\right)(g) = \exp\left(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0\right) \times \prod_{j=1}^{n+1} K_V^{(\dot{g})}(x_j, t_j; x_{j-1}, t_{j-1}),$$

for all $g \in S(\mathbb{R})$.

Combination of both approaches

Theorem

Let $V = V_1 + V_2$, where $V_1 : \mathbb{C} \to \mathbb{C}$ is in the Doss class and V_2 is defined as in (3). Then the corresponding Feynman integrand is given as a Hida distribution by

$$I_V = \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} I_{V_1} \prod_{j=1}^{n} \delta(x(r_j) - x_j) v_2(x_j)\, dr_j\, dx_j, \tag{5}$$

for all $x, x_0 \in \mathbb{R}$ and all $0 \leqslant t_0 < t \leqslant T < \infty$.

Idea of the proof Use the characterization theorem for sequences and integrals and the previous knowledge... and the following Lemma

Lemma

Let $V : \mathbb{C} \to \mathbb{C}$ be a potential from the Doss class,
$0 \leqslant t_0 = r_0 < r_1 < \ldots < r_n < r_{n+1} = t$ and $x_j \in \mathbb{R}$, $j = 1, \ldots, n$ for $n \in \mathbb{N}$. Then there exists some constants $0 < C < \infty$ such that

$$\left| S\left(\exp\left(-i \sum_{j=1}^{n+1} \int_{r_{j-1}}^{t_j} V\left(x_{j-1} + \frac{r - r_{j-1}}{r_j - r_{j-1}} (x_j - x_{j-1}) \right.\right.\right.\right. \\ \left.\left.\left.\left. + \sqrt{i}\langle \cdot, 1_{[r_{j-1}, r)} \rangle - \sqrt{i} \frac{r - r_{j-1}}{r_j - r_{j-1}} \langle \cdot, 1_{[r_{j-1}, r_j)} \rangle \right) dr \right) \right)(zg) \right| \\ \leqslant \sup_{y \in K_n} B(y) C \exp\left(\frac{1}{2} |z|^2 |g|_0^2 \right) \qquad (6)$$

for all $z \in \mathbb{C}$ and all $g \in S(\mathbb{R})$. Here $B : \mathbb{R} \to \mathbb{R}$ locally bounded and K_n denotes the convex hull of $\{x_1, \ldots, x_{n+1}\}$.

Follows directly by the integrability conditions on V

Theorem

Let $V = V_1 + V_2$*, where* $V_1 : \mathbb{C} \to \mathbb{C}$ *is in the Doss class and* V_2 *is defined as in* (3), $0 \leqslant t_0 < t \leqslant T < \infty$ *and* $x, x_0 \in \mathbb{R}$*. Then for all* $g \in S(\mathbb{R})$ *we get that*

$$K_V^{(\dot{g})}(x, t; x_0, t_0) := \exp\left(-ixg(t) + ix_0 g(t_0) + \frac{1}{2}|g_{\Lambda^c}|_0^2\right) TI_V(g)$$

solves the Schrödinger equation for all $x, x_0 \in \mathbb{R}$, $0 \leqslant t_0 \leqslant t \leqslant T$*, i.e.*

$$\left(i\frac{\partial}{\partial t} - \frac{1}{2}\Delta - V_{\dot{g}}(t, x)\right) K_V^{(\dot{g})}(x, t; x_0, t_0) = i\delta_{x_0}\delta_{t_0}. \tag{7}$$

Here $V_{\dot{g}}(t, x) = V_1(x) + V_2(x) + g(t)x$*, for all* $0 < t < T$ *and all* $x \in \mathbb{R}$*.*

Idea of the proof I

- $$K_V^{(\dot{g})}(x,t;x_0,t_0) = \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n+1} K_{V_1}^{(\dot{g})}(x_j,t_j;x_{j-1},t_{j-1}) \prod_{j=1}^{n} v_2(x_j)\, dr_j\, dx_j,$$

- $$K_V^{(\dot{g})}(x,t;x_0,t_0) = K_{V_1}^{(\dot{g})}(x,t;x_0,t_0) - i \int\int K_{V_1}^{(\dot{g})}(x,t;y,r) K_V^{(\dot{g})}(y,r;x_0,t_0)\, v_2(y)\, dy\, dr, \tag{8}$$

- Set $\hat{L}_{V_1} = i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + V_1(x) - \dot{g}(t)x$ and denoted its adjoint by $\hat{L}^*_{V_1}$.

Idea of the proof II

- Using (8) and Fubini's Theorem we get for $\varphi \in D(\Omega)$ that

$$\left\langle \hat{L}_{V_1} K_V^{(\dot{g})}, \varphi \right\rangle$$

$$= \left\langle K_{V_1}^{(\dot{g})}(x,t;x_0,t_0) - i \int\int K_{V_1}^{(\dot{g})}(x,t;y,r) K_V^{(\dot{g})}(y,r;x_0,t_0)\, v_2(y)\, dy\, dr, \hat{L}_{V_1}^* \varphi \right\rangle$$

$$= \left\langle K_{V_1}^{(\dot{g})}(x,t;x_0,t_0), \hat{L}_{V_1}^* \varphi \right\rangle$$

$$- i \int\int K_V^{(\dot{g})}(y,r;x_0,t_0) \int\int K_{V_1}^{(\dot{g})}(x,t;y,r) \hat{L}_{V_1}^* \varphi(x,t)\, dx\, dt\, v_2(y)\, dy\, dr.$$

- Since $K_{V_1}^{(\dot{g})}$ is the Green's function of $\hat{L}_{V_1}$ it follows that

$$i\varphi(x_0,t) + \int\int K^{(\dot{g})}(y,r;x_0,t_0)\varphi(y,r) v_2(y)\, dy\, dr = \langle i\delta_{x_0}\delta_{t_0}, \varphi\rangle + \left\langle v_2 K_V^{(\dot{g})}, \varphi \right\rangle.$$

Thanks for your attention!

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J. Potthoff and L. Streit.
A characterization of Hida distributions.
J. Funct. Anal., 101:212–229, 1991.

Assumption

Let $0 < T < \infty$. We assume that the potential $V : \mathcal{D} \to \mathbb{C}$ is analytic and that there exist a constant $0 < A < \infty$, a locally bounded functions $B : \mathcal{O} \to \mathbb{R}$ and some $\varepsilon < \frac{1}{8T}$ such that for all $x_0 \in \mathcal{O}$ and $y \in \mathbb{R}$ one has that

$$|\exp(-iV(x))| \leqslant A \exp\left(\varepsilon x^2\right) \quad \textit{and} \quad \left|\exp\left(-iV\left(x_0 + \sqrt{i}y\right)\right)\right| \leqslant B(x_0) \exp\left(\varepsilon y^2\right).$$

Assumption

Let $0 < T < \infty$ and $V : \mathcal{D} \to \mathbb{C}$ such that Assumption 9 is fulfilled. Then we require that there exist a locally bounded function $C : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ and some $0 < \varepsilon < \frac{1}{8T}$ such that for all $x_0, x_1 \in \mathcal{O}$ and $y \in \mathbb{R}$ one has that

$$\left| V\left(x_0 + \sqrt{i}y\right) \exp\left(-iV\left(x_1 + \sqrt{i}y\right)\right) \right| \leqslant C(x_0, x_1) \exp\left(\varepsilon y^2\right)$$

and

$$\left| \frac{\partial}{\partial z} V\left(x_0 + \sqrt{i}y\right) \exp\left(-iV\left(x_1 + \sqrt{i}y\right)\right) \right| \leqslant C(x_0, x_1) \exp\left(\varepsilon y^2\right).$$

Here $\frac{\partial}{\partial z}$ denotes the derivative of $z \mapsto V(z)$.