

Madeira Math Encounters XXXV

**Markov evolutions in the continuum:  
The Multicomponent case**

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- **The configuration space  $\Gamma^2$**

$$\Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty, \forall \text{ compact } K \subset \mathbb{R}^d\}$$

Each  $\gamma \in \Gamma$  is identified with a Radon measure:

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \quad (\text{configuration})$$

$\delta_x$  := the Dirac measure with mass at  $x$

$$\boxed{\Gamma^2 := \{(\gamma^+, \gamma^-) \in \Gamma^+ \times \Gamma^- : \gamma^+ \cap \gamma^- = \emptyset\}}$$

Each  $(\gamma^+, \gamma^-) \in \Gamma^2$  may be regarded as a marked one-configuration for the space of marks  $\{+, -\}$

- **The finite configuration space  $\Gamma_0^2$**

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \{\gamma : |\gamma| = n\}$$

We define

$$\boxed{\Gamma_0^2 := \{(\eta^+, \eta^-) \in \Gamma_0^+ \times \Gamma_0^- : \eta^+ \cap \eta^- = \emptyset\}}$$

## Combinatorial harmonic analysis

Given a  $G : \Gamma_0^2 \rightarrow \mathbb{C}$  one associates a mapping  
 $\mathcal{K}G : \Gamma^2 \rightarrow \mathbb{C}$

$$(\mathcal{K}G)(\gamma^+, \gamma^-) := \sum_{\substack{\eta^+ \subset \gamma^+ \\ |\eta^+| < \infty}} \sum_{\substack{\eta^- \subset \gamma^- \\ |\eta^-| < \infty}} G(\eta^+, \eta^-)$$

- For coherent states:

$$\left( \mathcal{K} \underbrace{\prod_{x \in \cdot} \theta^+(x)}_{e_\lambda(\theta^+, \theta^-)} \prod_{x \in \cdot} \theta^-(x) \right)(\gamma) = \underbrace{\prod_{x \in \gamma^+} (1 + \theta^+(x))}_{(Ke_\lambda(\theta^+))(\gamma^+)} \underbrace{\prod_{x \in \gamma^-} (1 + \theta^-(x))}_{(Ke_\lambda(\theta^-))(\gamma^-)}$$

- $|G| \leq C \mathbb{1}_{\bigsqcup_{n=0}^{N^+} \Gamma_{\Lambda^+}^{(n)} \times \bigsqcup_{n=0}^{N^-} \Gamma_{\Lambda^-}^{(n)}}$  ( $G \in B_{bs}(\Gamma_0^2)$ ):

$$\begin{aligned} |(\mathcal{K}G)(\gamma^+, \gamma^-)| &= |(\mathcal{K}G)(\gamma_{\Lambda^+}^+, \gamma_{\Lambda^-}^-)| \\ &\leq C (1 + |\gamma_{\Lambda^+}^+|)^{N^+} (1 + |\gamma_{\Lambda^-}^-|)^{N^-} \end{aligned}$$

Moreover,

$$\mathcal{K} : B_{bs}(\Gamma_0^2) \rightarrow \mathcal{K}(B_{bs}(\Gamma_0^2)) := \mathcal{F}\mathcal{P}(\Gamma^2)$$

is a linear isomorphism:

$$(\mathcal{K}^{-1} F)(\eta^+, \eta^-) := \sum_{\xi^+ \subset \eta^+} \sum_{\xi^- \subset \eta^-} (-1)^{|\eta^+ \setminus \xi^+| + |\eta^- \setminus \xi^-|} F(\xi^+, \xi^-)$$

## Correlation measures

$\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma^2)$  := probability measure on  $\Gamma^2$  s.t.

$$\int_{\Gamma^2} d\mu(\gamma) |\gamma_\Lambda^+|^n |\gamma_\Lambda^-|^n < \infty, \quad n \in \mathbb{N}, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

**Correlation measure  $\rho_\mu$  corresponding to  $\mu$ :**

Measure defined on  $\Gamma_0^2$  by

$$\int_{\Gamma_0^2} d\rho_\mu(\eta) G(\eta) = \int_{\Gamma^2} d\mu(\gamma) (\mathcal{K}G)(\gamma)$$

for all  $G \in \mathcal{B}_{\text{bs}}(\Gamma_0^2)$

- $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma^2) \implies \underbrace{\rho_\mu((\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}) \cap \Gamma_0^2) < \infty}_{\mathcal{M}_{\text{lf}}(\Gamma_0^2)}$

- $\mathcal{B}_{\text{bs}}(\Gamma_0^2) \subset L^1(\Gamma_0^2, \rho_\mu)$ . Moreover,

$$\|\mathcal{K}G\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)} \implies \boxed{\mathcal{K} : L^1(\Gamma_0^2, \rho_\mu) \rightarrow L^1(\Gamma^2, \mu)}$$

bounded linear operator

**Example:** Poisson/Lebesgue-Poisson measure

$$\rho_{\pi_\sigma} = \lambda_\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)} \quad (\text{on } \Gamma_0)$$

$\sigma = dx$  (Lebesgue measure on  $\mathbb{R}^d$ ).

- $\lambda^2 := \lambda_{dx} \otimes \lambda_{dx}$  is the correlation measure corresponding to the product measure  $\pi_{dx} \otimes \pi_{dx}$

$$\langle F, \mu \rangle = \int_{\Gamma} d\mu(\gamma) F(\gamma)$$

$$\boxed{\frac{\partial}{\partial t} F_t = L F_t}$$

$$F$$

$$\boxed{\frac{d}{dt} \mu_t = L^* \mu_t}$$

$$\mu$$

$$K^*$$

$$K$$

$$\boxed{\frac{\partial}{\partial t} G_t = \hat{L} G_t}$$

$$G$$

$$\langle G, \rho_\mu \rangle = \int_{\Gamma_0} d\rho_\mu(\eta) G(\eta)$$

$$\rho_\mu$$

$$\boxed{\frac{\partial}{\partial t} k_t = \hat{L}^* k_t}$$

$$\int_{\Gamma_0} ((\underbrace{K^{-1} L K}_{\hat{L}}) G) k_t d\lambda$$

$$\int_{\Gamma} L(KG) d\mu_t = \int_{\Gamma} \underbrace{L(KG)}_{\hat{L}} d\mu_t =$$

$$\frac{d}{dt} \int_{\Gamma_0} \underbrace{G k_t d\lambda}_{d\rho_t} = \frac{d}{dt} \int_{\Gamma} KG d\mu_t$$

## Potts birth-and-death dynamics

$$L = L^+ + L^-$$

where

$$\begin{aligned} & (L^+ F)(\gamma^+, \gamma^-) \\ &:= \sum_{x \in \gamma^+} d^+(x, \gamma^+ \setminus x, \gamma^-) (F(\gamma^+ \setminus x, \gamma^-) - F(\gamma^+, \gamma^-)) \\ &\quad + \int_{\mathbb{R}^d} dx b^+(x, \gamma^+, \gamma^-) (F(\gamma^+ \cup x, \gamma^-) - F(\gamma^+, \gamma^-)) \end{aligned}$$

and

$$\begin{aligned} & (L^- F)(\gamma^+, \gamma^-) \\ &:= \sum_{y \in \gamma^-} d^-(y, \gamma^+, \gamma^- \setminus y) (F(\gamma^+, \gamma^- \setminus y) - F(\gamma^+, \gamma^-)) \\ &\quad + \int_{\mathbb{R}^d} dy b^-(y, \gamma^+, \gamma^-) (F(\gamma^+, \gamma^- \cup y) - F(\gamma^+, \gamma^-)) \end{aligned}$$


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For  $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma^2)$  s.t. for all  $n \in \mathbb{N}_0$ ,  $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\begin{aligned} & \int_{\Gamma^2} d\mu(\gamma) |\gamma_\Lambda^+|^n |\gamma_\Lambda^-|^n \left\{ \sum_{x \in \gamma_\Lambda^+} d^+(x, \gamma^+ \setminus x, \gamma^-) + \int_{\Lambda} dx b^+(x, \gamma^+, \gamma^-) \right\} \\ & \int_{\Gamma^2} d\mu(\gamma) |\gamma_\Lambda^-|^n |\gamma_\Lambda^-|^n \left\{ \sum_{y \in \gamma_\Lambda^-} d^-(y, \gamma^+, \gamma^- \setminus y) + \int_{\Lambda} dy b^-(y, \gamma^+, \gamma^-) \right\} < \infty \end{aligned}$$


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- $L(\underbrace{\mathcal{FP}(\Gamma^2)}_{\mathcal{K}(B_{bs}(\Gamma_0^2))}) \subset L^1(\Gamma^2, \mu)$

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We want to consider the case

$$b^\pm(x, \cdot^+, \cdot^-) = \mathcal{K} B_x^\pm \geq 0, \quad d^\pm(x, \cdot^+, \cdot^-) = \mathcal{K} D_x^\pm \geq 0$$

For this purpose:

- $B_x^\pm, D_x^\pm \in L^1(\Gamma_0^2, \rho_\mu), x \in \mathbb{R}^d$
  - $\int_{\Gamma^2} d\mu(\gamma) |\gamma_\Lambda^+|^n |\gamma_\Lambda^-|^n \left\{ \sum_{x \in \gamma_\Lambda^+} (\mathcal{K}|D_x^+|)(\gamma^+ \setminus x, \gamma^-) + \int_\Lambda dx (\mathcal{K}|B_x^+|)(\gamma) \right\} +$
  - $\int_{\Gamma^2} d\mu(\gamma) |\gamma_\Lambda^-|^n |\gamma_\Lambda^-|^n \left\{ \sum_{y \in \gamma_\Lambda^-} (\mathcal{K}|D_y^-|)(\gamma^+, \gamma^- \setminus y) + \int_\Lambda dy (\mathcal{K}|B_y^-|)(\gamma) \right\} < \infty$
- 

$$(G_1 \star G_2)(\eta) := \sum_{\substack{(\eta_1^+, \eta_2^+, \eta_3^+) \in \mathcal{P}_3(\eta^+) \\ (\eta_1^-, \eta_2^-, \eta_3^-) \in \mathcal{P}_3(\eta^-)}} G_1(\eta_1^+ \cup \eta_2^+, \eta_1^- \cup \eta_2^-) G_2(\eta_2^+ \cup \eta_3^+, \eta_2^- \cup \eta_3^-)$$

One has

$$\mathcal{K}(G_1 \star G_2) = (\mathcal{K}G_1) \cdot (\mathcal{K}G_2)$$


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This yields the following expression for  $\widehat{L} := \mathcal{K}^{-1} L \mathcal{K}$ :

$$\begin{aligned} (\widehat{L}G)(\eta^+, \eta^-) &= \\ &- \sum_{x \in \eta^+} (D_x^+ \star G(\cdot \cup x, \cdot)) (\eta^+ \setminus x, \eta^-) + \int_{\mathbb{R}^d} dx (B_x^+ \star G(\cdot \cup x, \cdot)) (\eta) \\ &- \sum_{y \in \eta^-} (D_y^- \star G(\cdot, \cdot \cup y)) (\eta^+, \eta^- \setminus y) + \int_{\mathbb{R}^d} dy (B_y^- \star G(\cdot, \cdot \cup y)) (\eta) \\ &= (\widehat{L^+}G)(\eta^+, \eta^-) + (\widehat{L^-}G)(\eta^+, \eta^-), \end{aligned}$$

for all  $G \in \mathcal{B}_{bs}(\Gamma_0^2)$ .

## Integrability

- $G \in B_{bs}(\Gamma_0^2) \implies |G| \leq C \mathbb{1}_{\bigsqcup_{n=0}^{N^+} \Gamma_{\Lambda^+}^{(n)} \times \bigsqcup_{n=0}^{N^-} \Gamma_{\Lambda^-}^{(n)}}$

$$\begin{aligned} & \left\| \widehat{L^+} \mathbb{1}_{\bigsqcup_{n=0}^{N^+} \Gamma_{\Lambda^+}^{(n)} \times \bigsqcup_{n=0}^{N^-} \Gamma_{\Lambda^-}^{(n)}} \right\|_{L^1(\rho_\mu)} \\ & \leq \sum_{n=0}^{N^+-1} \sum_{m=0}^{N^-} \left[ \int_{\Gamma_0^2} d\rho_\mu(\eta) \left\{ \sum_{x \in \eta_{\Lambda^+}^+} \left( |D_x^+| * \mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \right) (\eta^+ \setminus x, \eta^-) \right. \right. \\ & \quad \left. \left. + \int_{\Lambda^+} dx \left( |B_x^+| * \mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \right) (\eta^+, \eta^-) \right\} \right] \end{aligned}$$

with

$$\boxed{\square} = \int_{\Gamma^2} d\mu(\gamma) \left\{ \sum_{x \in \gamma_{\Lambda^+}^+} (\mathcal{K}|D_x^+|)(\gamma^+ \setminus x, \gamma^-) \left( \mathcal{K} \mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \right) (\gamma^+ \setminus x, \gamma^-) \right. \\ \left. + \int_{\Lambda^+} dx (\mathcal{K}|B_x^+|)(\gamma^+, \gamma^-) \left( \mathcal{K} \mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \right) (\gamma^+, \gamma^-) \right\}$$

$$\mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \in B_{bs}(\Gamma_0^2) \implies \left( \mathcal{K} \mathbb{1}_{\Gamma_{\Lambda^+}^{(n)} \times \Gamma_{\Lambda^-}^{(m)}} \right) (\gamma) \leq (1 + |\gamma_{\Lambda^+}^+|)^n (1 + |\gamma_{\Lambda^-}^-|)^m$$

## Consequences:

- $\widehat{L^+}(B_{bs}(\Gamma_0^2)) \subset L^1(\Gamma_0^2, \rho_\mu)$
- The integrability condition assumed is the weakest possible one to derive this inclusion
- If  $B_x^\pm, D_x^\pm \in L^1(\Gamma_0^2, \rho)$  for some  $\rho \in \mathcal{M}_{lf}(\Gamma_0^2)$  and  $\boxed{\rho_\mu \leftrightarrow \rho} < \infty$  for all  $n, m \in \mathbb{N}_0$  and all  $\Lambda^+, \Lambda^- \in \mathcal{B}_c(\mathbb{R}^d)$ , then

$$\widehat{L^+}(B_{bs}(\Gamma_0^2)) \subset L^1(\Gamma_0^2, \rho)$$

- Similar conclusions hold for  $\widehat{L^-}$ , and thus for  $\widehat{L}$

## In terms of correlation functions

$$\int_{\Gamma_0^2} d\lambda^2(\eta) (\hat{L}^* k)(\eta) G(\eta) = \int_{\Gamma_0^2} d\lambda^2(\eta) (\hat{L} G)(\eta) k(\eta)$$

**Proposition:** Let  $k : \Gamma_0^2 \rightarrow \mathbb{R}_0^+$  be such that

$$\int_{\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}} d\lambda^2(\eta) k(\eta) < \infty, \quad \forall n, m \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d).$$

If  $B_x^\pm, D_x^\pm \in L^1(\Gamma_0^2, k\lambda^2)$  and

$$\begin{aligned} & \int_{\Gamma_0^2} d\lambda^2(\eta) k(\eta) \left\{ \sum_{x \in \eta_\Lambda^+} \left( |D_x^+| * \mathbb{1}_{\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}} \right) (\eta^+ \setminus x, \eta^-) \right. \\ & \quad \left. + \int_\Lambda dx \left( |B_x^+| * \mathbb{1}_{\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}} \right) (\eta) \right\} \\ & + \int_{\Gamma_0^2} d\lambda^2(\eta) k(\eta) \left\{ \sum_{y \in \eta_\Lambda^-} \left( |D_y^-| * \mathbb{1}_{\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}} \right) (\eta^+, \eta^- \setminus y) \right. \\ & \quad \left. + \int_\Lambda dy \left( |B_y^-| * \mathbb{1}_{\Gamma_\Lambda^{(n)} \times \Gamma_\Lambda^{(m)}} \right) (\eta) \right\} < \infty \end{aligned}$$

for all  $n, m \in \mathbb{N}_0$  and all  $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ , then

$$\begin{aligned} & (\hat{L}^* k)(\eta) = \\ & - \int_{\Gamma_0^2} d\lambda^2(\zeta) k(\eta^+ \cup \zeta^+, \eta^- \cup \zeta^-) \sum_{x \in \eta^+} \sum_{\substack{\xi^+ \subset \eta^+ \setminus x \\ \xi^- \subset \eta^-}} D_x^+(\zeta^+ \cup \xi^+, \zeta^- \cup \xi^-) \\ & + \int_{\Gamma_0^2} d\lambda^2(\zeta) \sum_{x \in \eta^+} k(\zeta^+ \cup (\eta^+ \setminus x), \eta^- \cup \zeta^-) \sum_{\substack{\xi^+ \subset \eta^+ \setminus x \\ \xi^- \subset \eta^-}} B_x^+(\zeta^+ \cup \xi^+, \zeta^- \cup \xi^-) \\ & - \int_{\Gamma_0^2} d\lambda^2(\zeta) k(\eta^+ \cup \zeta^+, \eta^- \cup \zeta^-) \sum_{y \in \eta^-} \sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^- \setminus y}} D_y^-(\zeta^+ \cup \xi^+, \zeta^- \cup \xi^-) \\ & + \int_{\Gamma_0^2} d\lambda^2(\zeta) \sum_{y \in \eta^-} k(\zeta^+ \cup \eta^+, \zeta^- \cup (\eta^- \setminus y)) \sum_{\substack{\xi^+ \subset \eta^+ \\ \xi^- \subset \eta^- \setminus y}} B_y^-(\zeta^+ \cup \xi^+, \zeta^- \cup \xi^-), \end{aligned}$$

for  $\lambda^2$ -a.a.  $\eta = (\eta^+, \eta^-) \in \Gamma_0^2$ .

## The Glauber-Potts dynamics:

$$\begin{aligned}
 d^\pm &\equiv 1 = \mathcal{K}e_\lambda(0, 0), \\
 b^+(x, \gamma^+, \gamma^-) &= e^{-E^+(x, \gamma^+)} = \left( \mathcal{K}e_\lambda(e^{-\phi^+(x-\cdot)} - 1, 0) \right) (\gamma^+, \gamma^-), \\
 b^-(y, \gamma^+, \gamma^-) &= e^{-E^-(y, \gamma^-)} = \left( \mathcal{K}e_\lambda(0, e^{-\phi^-(y-\cdot)} - 1) \right) (\gamma^+, \gamma^-).
 \end{aligned}$$

## Generators:

- $(\hat{L}G)(\eta) = -(|\eta^+| + |\eta^-|)G(\eta)$   
 $+ \sum_{\xi^+ \subset \eta^+} \int_{\mathbb{R}^d} dx e^{-E^+(x, \xi^+)} G(\xi^+ \cup x, \eta^-) e_\lambda(e^{-\phi^+(x-\cdot)} - 1, \eta^+ \setminus \xi^+)$   
 $+ \sum_{\xi^- \subset \eta^-} \int_{\mathbb{R}^d} dy e^{-E^-(y, \xi^-)} G(\eta^+, \xi^- \cup y) e_\lambda(e^{-\phi^-(y-\cdot)} - 1, \eta^- \setminus \xi^-),$
- $(\hat{L}^*k)(\eta) = -(|\eta^+| + |\eta^-|)k(\eta)$   
 $+ \sum_{x \in \eta^+} e^{-E^+(x, \eta^+ \setminus x)} \int_{\Gamma_0} d\lambda(\zeta^+) e_\lambda(e^{-\phi^+(x-\cdot)} - 1, \zeta^+) \cdot k(\zeta^+ \cup (\eta^+ \setminus x), \eta^-)$   
 $+ \sum_{y \in \eta^-} e^{-E^-(y, \eta^- \setminus y)} \int_{\Gamma_0} d\lambda(\zeta^-) e_\lambda(e^{-\phi^-(y-\cdot)} - 1, \zeta^-) \cdot k(\eta^+, \zeta^- \cup (\eta^- \setminus y))$

## The conflict model:

$$d^+(x, \gamma^+, \gamma^-) = \sum_{x' \in \gamma^+} d_+(x - x') = \mathcal{K}(d_+(x - \cdot)e_\lambda(0))(\gamma^+, \gamma^-),$$

$$b^+(x, \gamma^+, \gamma^-) = \sum_{x' \in \gamma^+} b_+(x - x') = \mathcal{K}(b_+(x - \cdot)e_\lambda(0))(\gamma^+, \gamma^-),$$

and

$$d^-(y, \gamma^+, \gamma^-) = \sum_{y' \in \gamma^-} d_-(y - y') = \mathcal{K}(e_\lambda(0)d_-(y - \cdot))(\gamma^+, \gamma^-),$$

$$b^-(y, \gamma^+, \gamma^-) = \sum_{y' \in \gamma^-} b_-(y - y') = \mathcal{K}(e_\lambda(0)b_-(y - \cdot))(\gamma^+, \gamma^-).$$

## Generators:

- $-(\hat{L}G)(\eta) =$ 
  - $- \sum_{x \in \eta^+} \sum_{x' \in \eta^+ \setminus x} d_+(x - x') (G(\eta^+ \setminus x', \eta^-) + G(\eta^+, \eta^-))$
  - $- \sum_{y \in \eta^-} \sum_{y' \in \eta^- \setminus y} d_-(y - y') (G(\eta^+, \eta^- \setminus y') + G(\eta^+, \eta^-))$
  - $+ \sum_{x' \in \eta^+} \int_{\mathbb{R}^d} dx b_+(x - x') (G((\eta^+ \setminus x') \cup x, \eta^-) + G(\eta^+ \cup x, \eta^-))$
  - $+ \sum_{y' \in \eta^-} \int_{\mathbb{R}^d} dy b_-(y - y') (G(\eta^+, (\eta^- \setminus y') \cup y) + G(\eta^+, \eta^-))$ ,

and

- $(\hat{L}^* k)(\eta) =$

$$\begin{aligned}
 & -k(\eta) \left\{ \sum_{x \in \eta^+} \sum_{x' \in \eta^+ \setminus x} d_+(x - x') + \sum_{y \in \eta^-} \sum_{y' \in \eta^- \setminus y} d_-(y - y') \right\} \\
 & - \int_{\mathbb{R}^d} dx' k(\eta^+ \cup x', \eta^-) \sum_{x \in \eta^+} b_+(x - x') \\
 & - \int_{\mathbb{R}^d} dy' k(\eta^+, \eta^- \cup y') \sum_{y \in \eta^-} d_-(y - y') \\
 & + \sum_{x \in \eta^+} k(\eta^+ \setminus x, \eta^-) \sum_{x' \in \eta^+ \setminus x} b_+(x - x') \\
 & + \sum_{y \in \eta^-} k(\eta^+, \eta^- \setminus y) \sum_{y' \in \eta^- \setminus y} b_-(y - y') \\
 & + \int_{\mathbb{R}^d} dx' \sum_{x \in \eta^+} k((\eta^+ \setminus x) \cup x', \eta^-) b_+(x - x') \\
 & + \int_{\mathbb{R}^d} dy' \sum_{y \in \eta^-} k(\eta^+, (\eta^- \setminus y) \cup y') b_-(y - y')
 \end{aligned}$$

## Hopping particles: general cases

$$(L_1 F)(\gamma) \\ := \sum_{x \in \gamma^+} \int_{\mathbb{R}^d} dx' c_1^+(x, x', \gamma^+, \gamma^-) (F(\gamma^+ \setminus x \cup x', \gamma^-) - F(\gamma^+, \gamma^-)) \\ + \sum_{y \in \gamma^-} \int_{\mathbb{R}^d} dy' c_1^-(y, y', \gamma^+, \gamma^-) (F(\gamma^+, \gamma^- \setminus y \cup y') - F(\gamma^+, \gamma^-))$$

or

$$(L_2 F)(\gamma) \\ := \sum_{x \in \gamma^+} \int_{\mathbb{R}^d} dy c_2^+(x, y, \gamma^+, \gamma^-) (F(\gamma^+ \setminus x, \gamma^- \cup y) - F(\gamma^+, \gamma^-)) \\ + \sum_{y \in \gamma^-} \int_{\mathbb{R}^d} dx c_2^-(x, y, \gamma^+, \gamma^-) (F(\gamma^+ \cup x, \gamma^- \setminus y) - F(\gamma^+, \gamma^-))$$

with

$$c_i^+(x, y, \gamma^+, \gamma^-) = (\mathcal{K}C_{i,x,y}^+) (\gamma^+ \setminus x, \gamma^-) \geq 0, \quad i = 1, 2,$$

$$c_1^-(x, y, \gamma^+, \gamma^-) = (\mathcal{K}C_{1,x,y}^-) (\gamma^+, \gamma^- \setminus x) \geq 0,$$

$$c_2^-(x, y, \gamma^+, \gamma^-) = (\mathcal{K}C_{2,x,y}^-) (\gamma^+, \gamma^- \setminus y) \geq 0.$$