Taming the Untamable:

Divergence-free Nonrenormalizable Models

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Nonrenormalizable Scalar Fields

- Classical starting model
- Quartic interactions for $n > 4$
- Ambiguity of quantization
- Choice of counter terms
- Lattice formulation
- Continuum limit
- Classical limit

Conventional Viewpoint (1)

Lattice point $k \in Z^n$; lattice spacing a

$$I = \Sigma_k [\tfrac{1}{2}(\phi_{k^*} - \phi_k)^2 a^{n-2} + \tfrac{1}{2} m_0^2 \phi_k^2 a^n + \lambda_0 \phi_k^4 a^n]$$

Lattice limit $(n \geq 5)$

$$\lim \quad M \int e^{\Sigma_k h_k \phi_k a^n - I} \Pi_k d\phi_k = e^{-\int h(x) A(x-y) h(y) d^n x d^n y}$$

A nonlinear theory has become linear!

Conventional Viewpoint (2)

Regularized perturbation theory leads to infinitely many distinct counter terms of ever increasing degree of divergence.

How should one interpret this?

By itself this story is incomplete.

Strategy for Divergences

Dealing with divergences :

Re- active : Repair every outbreak

Pro - active : Remove the source

Each approach requires different counter terms

Basic Background (1)

Periodic spacetime lattice ; size $N = L^n$

Points $k = (k_0, k_1, \ldots, k_s)$; spacing a

$$I_p(A) = \int [\Sigma_k \phi_k^2]^p \; e^{-A\Sigma_k \phi_k^2} \; \Pi_k d\phi_k$$

$$= \int \kappa^{2p} e^{-A\kappa^2} \kappa^{N-1} d\kappa \; d\Omega_{N-1}$$

$$I_p(A) = O((N/A)^p) \; I_0(A) \quad ; \quad \Delta = A - 1$$

$$I_1(A) = I_1(1) - \Delta \, I_2(1) + \Delta^2 I_3(1)/2 - \cdots$$

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$$I_1(A) = I_1(1) - \Delta\, I_2(1) + \Delta^2 I_3(1)/2 - \cdots$$

Basic Background (2)

Periodic spacetime lattice ; size $N = L^n$

Points $k = (k_0, k_1, \ldots, k_s)$; spacing a

$$I'_p(A) = \int \kappa^{2p} e^{-A\kappa^2} d\kappa \; d\Omega_{N-1}$$

$$I'_p(A) = O((1/A)^p) \; I'_0(A) \quad ; \quad \Delta = A - 1$$

$$I'_1(A) = I'_1(1) - \Delta \, I'_2(1) + \Delta^2 I'_3(1)/2 - \cdots$$

Basic Background (3)

Elementary quantum mechanics

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{\lambda(\lambda+1)}{x^2} + \frac{1}{2}x^2 - E_0 \quad , \quad (x \neq 0)$$

$$H\psi(x) = 0 \quad ; \qquad H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\psi(x)}\frac{d^2\psi(x)}{dx^2}$$

$$\psi(x) = C\,|\,x\,|^{-\lambda}\, e^{-x^2/2} \quad ; \quad E_0 = \frac{1}{2}(1-2\lambda)$$

Least energy : $0 < \lambda < 1/2$

`Suggested Lattice Action'

$$I(\phi,a,\hbar)=\tfrac{1}{2}\Sigma_k(\phi_{k^*}-\phi_k)^2a^{n-2}+\tfrac{1}{2}m_0^2\Sigma_k\phi_k^2a^n$$

$$+\lambda_0\Sigma_k\phi_k^4a^n+\tfrac{1}{2}\hbar^2\boxed{F\,\Sigma_k\phi_k^{-2}}a^n$$

$$F=a^{-2s}(\tfrac{3}{4}-0)=a^{-2s}(\tfrac{1}{4}+\tfrac{1}{2}-0)\quad,$$

`Idealized version'

Proposed Lattice Action (1)

$$I(\phi,a,\hbar)=\tfrac{1}{2}\Sigma_k(\phi_{k^*}-\phi_k)^2a^{n-2}+\tfrac{1}{2}m_0^2\Sigma_k\phi_k^2a^n$$
$$+\lambda_0\Sigma_k\phi_k^4a^n+\tfrac{1}{2}\hbar^2\Sigma_k\mathsf{F}_k(\phi)a^n$$

$$\mathsf{F}_k=a^{-2s}D^{-1}D_{,kk}\quad,\quad D=\Pi'_k[\Sigma'_lJ_{k,l}\phi_l^2]^{-\gamma}$$

$$\gamma=(N'-1)/4N'\quad,\quad N'=L^s$$

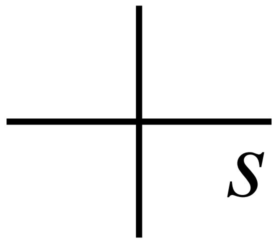
No new parameters; scales as kinetic term

Proposed Lattice Action (2)

$$\mathsf{F}_k(\phi) = \tfrac{1}{4}[(N'-1)/N']^2 \Sigma'_{r,t} \frac{a^{-2s} J_{r,k} J_{t,k} \phi_k^2}{[\Sigma'_l J_{r,l} \phi_l^2][\Sigma'_m J_{t,m} \phi_m^2]}$$

$$+ \tfrac{1}{2}(N'-1)/N' \; \Sigma'_t \left\{ 2 \frac{a^{-2s} J_{t,k}^2 \phi_k^2}{[\Sigma'_m J_{t,m} \phi_m^2]^2} - \frac{a^{-2s} J_{t,k}}{[\Sigma'_m J_{t,m} \phi_m^2]} \right\}$$

$$J_{k,l} = \frac{1}{2s+1} \delta_{k,l \in \{k \cup k_{nn}\}}$$



$$s = 2$$

Basic Ingredients

$$S(h) = M \int e^{Z^{-1/2}\Sigma_k h_k \phi_k a^n/\hbar - I(\phi,a,\hbar)/\hbar} \Pi d\phi_k$$

$$H = -\tfrac{\hbar^2}{2}\Sigma'_k \frac{\partial^2}{\partial \varphi_k^2} a^{-s} + \tfrac{1}{2}\Sigma'_k (\phi_{k^*} - \phi_k)^2 a^{s-2} - E_0$$

$$+ \tfrac{1}{2} m_0^2 \Sigma'_k \phi_k^2 a^s + \lambda_0 \Sigma'_k \phi_k^4 a^s + \tfrac{\hbar^2}{2}\Sigma'_k \mathrm{F}_k(\phi) a^s$$

$$H\Psi(\phi) = 0 \quad ; \quad \Psi(\phi) = \frac{e^{-U(\phi,a,\hbar)/2}}{\Pi'_k [\Sigma'_k J_{k,l} \phi_l^2]^{(N'-1)/4N'}}$$

Taming the Divergences

$$\Psi_I(\phi)^2 = \frac{K\, e^{-\Sigma'_{k,l}\phi_k A_{k-l}\phi_l a^{2s}/\hbar}}{\Pi'_k[\Sigma'_l J_{k,l}\phi_l^2]^{(N'-1)/2N'}}$$

$$\phi_k = \kappa\eta_k\,, \;\; \Sigma'_k\phi_k^2 = \kappa^2,\;\; \Sigma'_k\eta_k^2 = 1$$

$$\int[\Sigma'_k\phi_k^2 a^s]^p\,\Psi_I(\phi)^2\,\Pi' d\phi_k$$

$$= K\int\kappa^{2p}a^{sp}\,\frac{e^{-\kappa^2\Sigma'_{k,l}\eta_k A_{k-l}\eta_l a^{2s}/\hbar}}{\Pi'_k[\Sigma'_l J_{k,l}\eta_l^2]^{(N'-1)/2N'}}\,d\kappa\; d\mu(\eta)$$

Bounds on Lattice Averages

$$\langle(\cdot)\rangle \equiv M\int(\cdot)\ e^{-I/\hbar}\ \Pi_k d\phi_k$$

$$\left\langle[\Sigma_0 F(\phi,a)a]^p\right\rangle = \tilde{\Sigma}_0 a^p \left\langle F(\phi_1,a)\cdots F(\phi_p,a)\right\rangle$$

$$|\left\langle[\Sigma_0 F(\phi,a)a]^p\right\rangle| \le \tilde{\Sigma}_0 a^p\ |\left\langle F(\phi_1,a)\cdots F(\phi_p,a)\right\rangle|$$

$$\le \tilde{\Sigma}_0 a^p\ |\left\langle F(\phi_1,a)^p\right\rangle\cdots\left\langle F(\phi_p,a)^p\right\rangle|^{1/p}$$

$$\left\langle F(\phi,a)^p\right\rangle = \int F(\phi,a)^p\ \Psi(\phi)^2\ \Pi'_k d\phi_k$$

Fixing the Constants (1)

To fix Z : $Z^{-p}\int[\Sigma'_k h_k\phi_k a^s]^{2p}\,\Psi(\phi)^2\,\Pi'_k d\phi_k$

To fix m_0^2 : $\int[m_0^2\Sigma'_k\phi_k^2 a^s]^p\,\Psi(\phi)^2\,\Pi'_k d\phi_k$

To fix λ_0 : $\int[\lambda_0\Sigma'_k\phi_k^4 a^s]^p\,\Psi(\phi)^2\,\Pi'_k d\phi_k$

Result : $\left\langle[m_0^2\Sigma_k\phi_k^2 a^n]^p\right\rangle < \infty$

Result : $\left\langle[\lambda_0\Sigma_k\phi_k^4 a^n]^p\right\rangle < \infty$

Fixing the Constants (2)

$$\int [m_0^2 \Sigma_k' \phi_k^2 a^s]^p \; \Psi(\phi)^2 \; \Pi_k' d\phi_k$$

$$= K \int m_0^{2p} a^{sp} \kappa^{2p} \frac{e^{-\kappa^2 \sum_{k,l}' \eta_k A_{k-l} \eta_l a^{2s} / \hbar}}{\Pi_k' [\Sigma_l' J_{k,l} \eta_l^2]^{(N'-1)/2N'}} \, d\kappa \; d\mu(\eta)$$

$$\propto m_0^{2p} a^{sp} / (N' a^{(s-1)})^p \qquad ; \quad m_0^2 = N'(qa)^{-1} m^2$$

Result : $\left\langle [m_0^2 \Sigma_k \phi_k^2 a^n]^p \right\rangle < \infty$

Fixing the Constants (3)

$$Z^{-p} \int [\Sigma'_k h_k \phi_k a^s]^{2p} \; \Psi(\phi)^2 \; \Pi'_k d\phi_k \; : Z = N'^{-2} (qa)^{1-s}$$

$$\int [m_0^2 \Sigma'_k \phi_k^2 a^s]^p \; \Psi(\phi)^2 \; \Pi'_k d\phi_k \; : m_0^2 = N' (qa)^{-1} m^2$$

$$\int [\lambda_0 \Sigma'_k \phi_k^4 a^s]^p \; \Psi(\phi)^2 \; \Pi'_k d\phi_k \; : \lambda_0 = N'^3 (qa)^{s-2} \lambda$$

Result : $\left\langle [m_0^2 \Sigma_k \phi_k^2 a^n]^p \right\rangle < \infty$

Result : $\left\langle [\lambda_0 \Sigma_k \phi_k^4 a^n]^p \right\rangle < \infty$

Conclusions

- Add a single counter term that is h-bar dependent and scales as kinetic term
- Determine parameters by self consistency in continuum limit
- Leads to divergence-free perturbative formalism about pseudofree model which contains the counter term