Kawasaki dynamic in continuum

Math. Encounters XXXV, Madeira

In honor of Ludwig Streit

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Definitions I

- Configuration space

$$\Gamma := \{\gamma \subset \mathbb{R}^d : \ |\gamma \cap B_R(0)| < \infty \text{ for all } R\},$$

where $B_R(0)$ ball of radius R.

- Empirical field

$$\langle f, \gamma \rangle := \sum_{x \in \gamma} f(x)$$

- $f = \frac{1}{\mathrm{Vol}(\Lambda)} \mathbb{1}_\Lambda$ for $\Lambda \subset \mathbb{R}^d$.

Generator

- Free Kawasaki dynamics: $0 \le a \in L^1(\mathbb{R}^d)$

$$(LF)(\gamma) := \sum_{x\in\gamma} \int dy a(x-y) \Big(F(\gamma \cup \{y\} \setminus \{x\}) - F(\gamma) \Big)$$

- Jump process: Exponential clock with rate $\int a(x)dx$; probability of jump $x \to y$

$$\frac{a(x-y)}{\int a(z)dz}$$

- Independent jumps

Independent infinite particle process

Kondratiev, Lytvynov , Röckner: Independent infinite Markovian particles as an Markov process on the configuration space

- Construct infinite product process on $\left(\mathbb{R}^d\right)^{\mathbb{N}}$
- Projection:

$$\begin{array}{ccc} \left(\mathbb{R}^d\right)^{\mathbb{N}} & \rightarrow & \Gamma \\ (x_n)_{n\in\mathbb{N}} & \mapsto & \{x_n\}_{n\in\mathbb{N}} \end{array}$$

- Corresponding time homogeneous *cadlag* Markov process $\mathbf{X}_t$ with law $\mathbf{P}_\gamma$ for initial value $\gamma \in \Theta$.
- Admissible configurations

$$\Theta := \left\{ \gamma \in \Gamma : \limsup_{R\to\infty} R^{-d} |\gamma \cap B_R(0)| < \infty \right\}$$

Reduction to one particle dynamics

- Special class of functions: Bogoliubov exponentials

$$e_B(f)(\gamma) := \prod_{x\in\gamma}(1+f(x)).$$

Proper exponential for configuration space

$$e_B(f) = \exp\left(\langle \ln(1+f), \gamma\rangle\right)$$

- Time development of exponentials

$$\int e_B(f)(\mathbf{X}_t(\omega))\, \mathbf{P}_\gamma(d\omega) = e_B(e^{tA}f)(\gamma)$$

- Operator A Markov generator on $C_\infty(\mathbb{R}^d)$

$$(Af)(x) := \int_{\mathbb{R}^d} a(y)(f(y+x) - f(x))$$

Equilibrium dynamics

- Initial measure π_z with z constant
- a symmetric: Dirichlet form, Kondratiev, Lytvynov, Röckner

$$\int_\Gamma \pi_z(d\gamma) \sum_{x\in\gamma} \int dy a(x-y)\Big(F(\gamma \setminus x \cup y) - F(\gamma)\Big)^2$$

- Second quantization, (also asymmetric). Unique extension.

Local equilibrium dynamics

- Poisson random field π_z: general intensity $0 \le z \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$

$$\int_\Gamma e_B(f)(\gamma)\pi_z(d\gamma) = \exp\left(\int f(x)z(x)dx\right)$$

- Time-development of initial distribution π_z

$$\int F(\gamma)\mathbf{P}_{\pi_z,t}(d\gamma) := \int F(\mathbf{X}_t(\omega)) \int_\Gamma \mathbf{P}_\gamma(d\omega)\pi_z(d\gamma)$$

- Solution

$$\mathbf{P}_{\pi_z,t} = \pi_{z_t}$$

with

$$z_t := e^{tA^*} z$$

- Invariant measures: Poisson random fields with constant intensity $z(x) = z_0$.

One particle operator

- Operator A Markov generator on $C_\infty(\mathbb{R}^d)$
$$(Af)(x) := \int_{\mathbb{R}^d} a(y)(f(y+x) - f(x))$$
- Jump process: Exponential clock with rate $\int a(x)dx$; probability of jump $x \to y$
$$\frac{a(x-y)}{\int a(z)dz}$$
- Easy form in Fourier variables
$$\widehat{Af}(k) = (2\pi)^{d/2}(\hat{a}(k) - \hat{a}(0))\hat{f}(k)$$
and for the semi-group
$$\left(e^{tA}f\right)(x) = \frac{1}{(2\pi)^{d/2}} \int dk e^{ikx} e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}\hat{f}(k)$$

Large Time Asymptotic I

- Invariant measures:

$$\mathbf{P}_{\pi_z,t} = \pi_z$$

Poisson random fields with constant intensity $z(x) = z_0$.

- Large time: for $t \to \infty$

$$\lim_{t\to\infty} \mathbf{P}_{\pi_z,t} = \lim_{t\to\infty} \pi_{z_t} = \pi_{\lim_{t\to\infty} e^{tA^*} z}$$

- Reduce to one particle

$$\lim_{t\to\infty} \int e^{tA} f(x) z(x) dx = \lim_{t\to\infty} \int e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))} \hat{f}(k)\hat{z}(k) dk.$$

Dominated by 1

$$e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))} \to \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Large Time Asymptotic II

- If $z \in L^1(\mathbb{R}^d)$ then $\hat{z}(k) \in C_\infty(\mathbb{R}^d)$

$$\lim_{t\to\infty} \int e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}\hat{f}(k)\hat{z}(k)dk = \int_{\{0\}} \hat{f}(k)\hat{z}(k)dk = 0.$$

- If $z := z_0 + \Delta z$ with z_0 constant $\Delta z \in L^1(\mathbb{R}^d)$ then

$$\hat{z}(k) = z_0\delta(k) + \widehat{\Delta z}(k)$$

Consequently,

$$\lim_{t\to\infty} \int e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}\hat{f}(k)\hat{z}(k)dk$$
$$= \int_{\{0\}} \hat{f}(k)\left(z_0\delta(k) + \hat{\Delta} z(k)\right) dk = z_0.$$

- General argument $\hat{z}(k)dk$ signed measure. Then

$$\lim_{t\to\infty} \int e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}\hat{f}(k)\hat{z}(dk)$$
$$= \int_{\{0\}} \hat{f}(k)\hat{z}(dk) = \hat{f}(0)\hat{z}(\{0\}).$$

Large Time Asymptotic III

- Concluding

$$\lim_{t\to\infty} \int e^{tA} f(x) z(x) dx = \hat{z}(\{0\}) \int_{\mathbb{R}^d} f(x) dx$$

- Define constant by

$$\mathrm{mean}(z) := \lim_{R\to\infty} \frac{1}{\mathrm{Vol}(B_R(0))} \int_{B_R(0)} z(x) dx.$$

- $\forall \varphi \in L^1(\mathbb{R}^d)$ holds

$$\lim_{R\to\infty} R^{-d} \int_{\mathbb{R}^d} dx \varphi(x/R) z(x) = \mathrm{mean}(z) \int_{\mathbb{R}^d} dx \varphi(x)$$

Large Time Asymptotic IV

- $\forall \varphi \in L^1(\mathbb{R}^d)$ holds

$$\lim_{R\to\infty} R^{-d} \int_{\mathbb{R}^d} dx \varphi(x/R) z(x) = \text{mean}(z) \int_{\mathbb{R}^d} dx \varphi(x)$$

- If $\hat{z}(k)dk$ signed measure. Then $\forall \varphi \in L^1(\mathbb{R}^d)$ holds

$$\begin{aligned} & \lim_{R\to\infty} R^{-d} \int_{\mathbb{R}^d} dx \varphi(x/R) z(x) \\ = \quad & \lim_{R\to\infty} \int_{\mathbb{R}^d} dx \hat{\varphi}(kR) \hat{z}(dk) = \hat{\varphi}(0) \hat{z}(\{0\}) \end{aligned}$$

- Hence

$$\lim_{t\to\infty} \mathbf{P}_{\pi_z,t} = \pi_{\text{mean}(z)}.$$

Equilibrium vs. Non-equilibrium, Scales

Equilibrium vs. non-equilibrium

- Equilibrium: π_z with z constant
- Near equilibrium: density w.r.t. π_z with z constant
- Local equilibrium: π_z with z slowly varying
- Far from equilibrium: no density w.r.t any Poisson measure

Scales

- System scale: infinity
- Space scale (observation)
- Time scale (observation)
- Interaction scale
- Initial data

Examples

- Depend only on $|x| \to \infty$.

$$\text{mean}(z) := \begin{cases} 0, & \text{if z goes to 0} \\ 0, & \text{if } z \in L^p(\mathbb{R}^d), p \in [1,2] \\ z_0, & \text{if } z(x) = z_0 \\ z_0, & \text{if } z(x) = z_0(1 - \alpha \sin(x)) \\ z_0/2, & \text{if } z(x) = z_0 \mathbb{1}_{(\infty,0]}(x) \end{cases}$$

- Last case: $\hat{z}$ not signed measure.

General result

- Hypothesis:

$$\lim_{t\to\infty} \mathbf{P}_{\pi_z,t} = \pi_{\text{mean}(z)}.$$

  if and only if mean(z) exists.

- If for z exists $t_n \to \infty$ such that

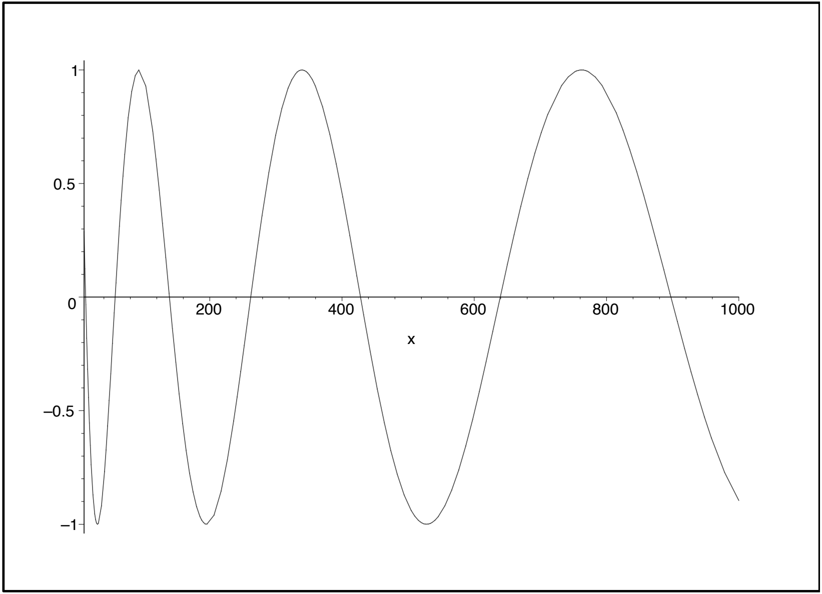
$$\lim_{n\to\infty} \int e^{t_n A} f(x) z(x) dx$$

  the limit is $C \int f(x) dx$.

- Mean does not exist for all z

$$z(x) := \begin{cases} 1, & \text{if } 2^{2k} \leq |x| \leq 2^{2k+1} \\ 0, & \text{otherwise} \end{cases}$$

No overall density



Non-equilibrium

- Initial measure: μ not Poisson
- Cumulants, Ursel functions, truncated moments

$$\ln\left(\int e^{\langle f,\gamma\rangle}\mu(d\gamma)\right)$$

$$=: \quad \sum_{n=1}^{\infty}\int_{\mathbb{R}^{dn}}\prod_{i=1}^{n}(e^{f(x_i)}-1)u_{\mu}^{(n)}(x_1,\ldots,x_n)d^{dn}x$$

- Decay of correlation: some "mixing condition"

$$\sup_x\sum_{k=0}^{\infty}\int_{\mathbb{R}^{dk}}\left|u_{\mu}^{(k)}(\{x,y_1,\ldots,y_k\})\right|dy_1\ldots dy_n<\infty.$$

- Density: $\rho_{\mu}^{(1)}=u_{\mu}^{(1)}$

Large time asymptotic

- Large times

$$\lim_{t\to\infty} \mathbf{P}_{\mu,t} \to \pi_{\mathrm{mean}(\rho_\mu^{(1)})}$$

where

$$\rho_\mu^{(1)} := \lim_{R\to\infty} \frac{1}{\mathrm{Vol}(B_R(0))} \int_\Gamma \sum_{x\in\gamma} \mathbb{1}_{B_R(0)}(x)\mu(d\gamma).$$

- $\widehat{\rho_\mu}$ measure.

Future projects

- Time asymptotic for general initial condition
- Front propagation: Velocity, Shape
- Current
- Kawasaki with interaction
- Glauber plus Kawasaki
- Further variants of interaction