

A transformation of a Feynman–Kac formula for holomorphic families of type B

Olaf Wittich^{a)}

Fachbereich Mathematik, Universität Kaiserslautern, Erwin Schrödinger Straße, 67633 Kaiserslautern, Germany and IBB, GSF–National Research Center for Environment and Health, 85764 Neuherberg, Germany

(Received 14 April 1999; accepted for publication 14 September 1999)

A transformation formula for resolvents of families of Schrödinger operators $H(\xi) = -\frac{1}{2}\Delta + \xi Q$, which are assumed to be holomorphic of type B, is proved. It can be used to derive the well-known correspondence between three-dimensional Coulomb problem and four-dimensional harmonic oscillator. © 2000 American Institute of Physics. [S0022-2488(99)03112-6]

I. INTRODUCTION

It is well known (see Ref. 1) that the heat equation semigroup generated by a Schrödinger operator bounded from below (and therefore its resolvent) can be represented probabilistically as an expectation value of a functional of some stochastic process. One simple consequence of the mere existence of those Feynman–Kac formulas is that the stochastic process is determined only up to *version*, i.e., you can use any other process with the same law. Surprisingly this may yield nontrivial results. In this paper we aim to illustrate this fact by a well-known example: the correspondence between the harmonic oscillator in dimension 4 and the Coulomb problem in dimension 3. To obtain this result the following transformation formula (Theorem 6.3) is proved using a Feynman–Kac formula for holomorphic families of type B.

Given a proper and surjective harmonic morphism $\Phi: M \rightarrow N$ between complete and orientable Riemannian manifolds without boundary and a holomorphic family of type B,

$$H(\xi) = -\frac{1}{2}\Delta + \xi Q,$$

of Schrödinger operators on $L^2(N)$, the resolvent family can be lifted to the resolvent family of a corresponding holomorphic family,

$$G(\kappa) = -\frac{1}{2}\Delta + \kappa\lambda,$$

on $L^2(M)$, provided $\Phi^\Delta Q(x) := \lambda(x)Q \circ \Phi(x) \equiv C$ equals a constant and $\lambda: M \rightarrow \mathbb{R}$ is the square of the dilatation of the harmonic morphism Φ . The correspondence is given for compactly supported $f \in L^2(N)$ by

$$R(\zeta, H(\xi))f(x) = R(-C\xi, G(-\zeta))\Phi^\Delta f(y),$$

for $y \in \Phi^{-1}(x)$.

The proof uses a version of Brownian motion constructed by Csink and Oksendal: suitably time transformed Brownian motion on M , mapped to N by a harmonic morphism, coincides in law with Brownian motion on M .² This property generalizes the classical scale invariance due to Lévy. Once the Feynman–Kac formula is taken for granted the proof reduces to the transformation formula for integrals on some infinite-dimensional measure space. It is believed that other con-

^{a)}Electronic mail: wittich@gsf.de

structions from stochastic calculus can be used in a similar way and that looking for symmetries of some underlying stochastic process provides a common point of view upon analogous correspondences between quantum mechanical systems.

The paper is organized as follows: In Sec. II we summarize some facts about operators that are used in the sequel. Section III mainly consists of a proof of a Feynman–Kac formula for a holomorphic family of generators and a representation of the corresponding resolvent. Since in the proof of (Theorem 6.3) the unique continuation property of holomorphic families of type B is used, in Sec. IV we summarize some facts about the domain of holomorphicity of the resolvent. In Sec. V we deal with well-known facts about harmonic morphisms; of special importance is Proposition 5.5, which is the invariance property mentioned above. In Sec. VI the transformation formula is proved and in Sec. VII it is applied to the well-known^{3,4} correspondence between the Coulomb problem in dimension 3 and the harmonic oscillator in dimension 4. Finally, Sec. VIII contains the corresponding transformation formula for the kernel of the resolvent.

II. SOME FACTS ABOUT SECTORIAL OPERATORS

For the convenience of the reader, some facts from functional analysis are summarized. Almost all of them can be found in the classical book of Kato. In the sequel, M denotes a complete and oriented smooth Riemannian manifold without a boundary and $V=L^2(M)$.

If the measurable real function $Q \in L^1_{loc}(M)$ is bounded from below, i.e. $\text{ess\,inf } Q = C \in \mathbb{R}$, pointwise multiplication with Q yields a sectorial operator \hat{Q} from a dense domain $\mathbb{D}(\hat{Q}) \subset V$ to V (Ref. 5, Example 1.5, p. 312). It can be assumed that \hat{Q} is closed (Ref. 5, Example 1.15, p. 315). The same holds for $\xi\hat{Q}$ if $\xi \in \mathbb{C}$ with $\text{Re}(\xi) > 0$. The domains of the operators $\xi\hat{Q}$ coincide and are equal to $\mathbb{D}(\hat{Q})$. The associated quadratic forms $\xi q(f) := \xi(f, \hat{Q}f)$ are as well sectorial and closed with domain $\mathbb{D}(q) \subset V$, independent of ξ .

Let us now denote $\Delta := -d^*d$ the Laplacian on functions on M and let $\delta(f) := -\frac{1}{2}(f, \Delta f)$ the associated quadratic form with domain $\mathbb{D}(\delta) \subset V$. Then, since the Laplacian is self-adjoint (Ref. 6, Theorem 5.7, p. 117) on V , δ is a densely defined closed sectorial form.

By the Friedrichs construction (Ref. 5 Theorem 2.1, p. 322, Theorem 2.23, p. 331) there are uniquely determined closed operators associated to the forms ξq and δ . They are also denoted by $-\frac{1}{2}\Delta$ and $\xi\hat{Q}$, respectively. These operators turn out to be m -sectorial (Ref. 5, Sec. V.10, p. 280). By Ref. 5, Theorem 1.31, p. 319, the sum

$$h(\xi) := \delta + \xi q,$$

with common domain $\mathbb{D} = \mathbb{D}(\delta) \cap \mathbb{D}(q)$, is closed and sectorial for $\text{Re}(\xi) > 0$. In other words, $h(\xi), \text{Re}(\xi) > 0$ is a *holomorphic family of type (a)* (Ref. 5, Sec. VII.2, p. 395).

Therefore the m -sectorial operators $H(\xi), \text{Re}(\xi) > 0$, associated to $h(\xi)$ by the Friedrichs construction, i.e., the *form sums*,

$$H(\xi) = -\frac{1}{2}\Delta + \xi\hat{Q},$$

form a *holomorphic family of type B* (Ref. 5, Theorem 4.2, p. 395). For fixed ξ , the *numerical range*

$$\Theta(h(\xi)) := \{(f, H(\xi)f) : \|f\| = 1\},$$

is contained in a *sector*,

$$S(\rho, \gamma) := \{z \in \mathbb{C} : \arg|z - \rho| < \gamma\},$$

$\rho \in \mathbb{R}, \gamma < \pi/2$ that contains ξC .

Crucial for all the applications in the sequel is the following result due to Simon,⁷ who also emphasized the fact that the form sum is most suitable in connection with the Feynman–Kac formula.⁸

Theorem 2.1: *Let A, B be m -sectorial operators in V with associated closed sectorial forms a, b . Let P the orthogonal projection onto the closure $V' \subset V$ of the intersection of domains $D(a) \cap D(b)$. Then*

$$\lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n f = e^{-t(A+B)} P f,$$

for each $f \in V$.

It should be noted that, by the Hille–Yoshida theorem, that m -sectorial operators generate strongly continuous semigroups.

III. A FEYNMAN–KAC FORMULA FOR HOLOMORPHIC FAMILIES OF TYPE B

Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ denote the space of continuous paths in (a suitable compactification of) M and B^x Brownian motion with starting point $x \in M$. Brownian motion on M is, in general, only defined up to an *explosion time* denoted by T_M . The present section is devoted to a proof of the following statement.

Proposition 3.1: *Let $Q \in L^1_{loc}(M)$ be a potential such that*

$$H(\xi) = -\frac{1}{2}\Delta + \xi\hat{Q}$$

is a holomorphic family of type B for $\xi \in U$, where $U \subset \{\xi: \text{Re}(\xi) > 0\}$ is some open connected set such that $U \cap \mathbb{R} \neq \emptyset$. Then the Feynman–Kac formula holds, i.e., for $t > 0$ and $f \in V$,

$$e^{-tH(\xi)} f = E \left[f(B_t) \exp \left(-\xi \int_0^t ds Q(B_s) \right) \chi_{\{t < \widehat{T}_M\}} \right],$$

where the domain $D = D(H(\xi))$ is dense in V .

Proof: The proof consists of three steps.

- (1) The formula is proved for bounded continuous potentials

$$Q: M \rightarrow \mathbb{R},$$

i.e., $|Q| \leq C$.

- (2) By a first approximation argument the formula is extended to measurable potentials $Q \in L^1_{loc}(M)$ bounded from below.
- (3) By a further approximation argument for quadratic forms, the formula is extended to potentials $Q \in L^1_{loc}(M)$, subject to the condition that the operators $H(\xi)$ defined above actually form a holomorphic family of type B.

(1) Assume first Q to be continuous. Then for fixed $(\omega, x) \in \Omega \times M$ the approximation by Riemann sums,

$$F_n^x(\omega) := f(B_t^x(\omega)) \exp \left(-\frac{t\xi}{n} \sum_{k=1}^n Q(B_{kt/n}^x(\omega)) \right) \chi_{\{t < \widehat{T}_M\}}(\omega),$$

converges to

$$F^x(\omega) = f(B_t^x(\omega)) \exp \left(-\xi \int_0^t ds Q(B_s^x(\omega)) \right) \chi_{\{t < \widehat{T}_M\}}(\omega),$$

as n tends to infinity. Let vol_M denote the *volume form* on M . Then, for $f \in V$,

$$\begin{aligned}
 \|E(F_n^x - F^x)\|_V^2 &= \int_M \text{vol}_M(dx) |E(F_n^x - F^x)|^2 \leq \int_M \text{vol}_M(dx) (E|F_n^x - F^x|)^2 \\
 &= \int_M \text{vol}_M(dx) E(F_n^x - F^x) \overline{E(F_n^x - F^x)} \\
 &= \int_{M \times \Omega \times \Omega'} \text{vol}_M(dx) P(d\omega) P(d\omega') (F_n^x - F^x)(\omega) \overline{(F_n^x - F^x)(\omega')} \\
 &\leq \int_{M \times \Omega \times \Omega'} \text{vol}_M(dx) P(d\omega) P(d\omega') |f(B_t^x(\omega))| |f(B_t^x(\omega'))| e^{-2C \text{Re}(\xi)t} \\
 &= e^{-2C \text{Re}(\xi)t} \int_M \text{vol}_M(dx) E|f(B_t^x)| E|f(B_t^x)| \\
 &= e^{-2C \text{Re}(\xi)t} \|e^{t\Delta/2} f\|_V^2 \leq e^{-2C \text{Re}(\xi)t} \|f\|_V^2,
 \end{aligned}$$

since $e^{t\Delta/2}$ is bounded with a norm of less than one. Therefore

$$|f(B_t^x(\omega))| |f(B_t^x(\omega'))| e^{-2C \text{Re}(\xi)t} \geq (F_n^x - F^x)(\omega) \overline{(F_n^x - F^x)(\omega')}$$

is a $\lambda \times P \times P$ -integrable majorant. By the above, for all (ω, x) , $F_n^x - F^x$ converges to zero. That implies by Lebesgue’s dominated convergence,

$$\|EF_n^x - EF^x\|_V^2 \rightarrow 0,$$

as $n \rightarrow \infty$.

Since continuous potentials are in $L^1_{\text{loc}}(M)$, smooth functions with compact support are contained in the domains of δ and ξq . Since the manifold was assumed to be complete these functions form a dense subset of V . Therefore we may set $P = 1$ in (2.1). The statement now follows from the equality

$$E \left[f(B_t) \exp \left(-\frac{t\xi}{n} \sum_{k=1}^n Q(B_{kt/n}) \right) \chi_{\{t < T_M\}} \right] = (e^{t\Delta/2n} e^{-t\xi \hat{Q}/n})^n f.$$

(2) First of all, it has to be shown that the Feynman–Kac formula only depends on the equivalence class of $Q \in L^1_{\text{loc}}(M)$. This follows from the fact that since the transition density of Brownian motion possesses a C^∞ density with respect to the volume form, the *expected occupation time*,

$$\mu_t^x(A) := E \int_0^{t \wedge T_M} \chi_A(B_s^x) ds,$$

for a Borel set $A \subset M$ yields a finite Borel measure on M of total mass less or equal to t that possesses a density with respect to the volume form as well. From that the following can be directly concluded.

Lemma 3.2: Let $Q \in L^1_{\text{loc}}(M)$ be bounded with $\text{esssup } |Q| = 0$. Then

$$E \int_0^{t \wedge T_M} Q(B_s^x) ds = \int_M Q(y) \mu_t^x(dy) = 0.$$

Now for a general measurable potential Q with $\text{esssup } |Q| = 0$, we have by monotone convergence,

$$\begin{aligned} & E \left| \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) - 1 \right| \chi_{\{t < T_M\}} \\ &= \lim_{N \rightarrow \infty} E \left| \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) - 1 \right| \chi_{\{t < T_M\}} \\ &\leq \lim_{N \rightarrow \infty} |\xi| e^{|\xi|N} \chi_{\{t < T_M\}} E \int_0^t Q(B_s^x) ds = 0. \end{aligned}$$

Therefore the probability

$$P \left(\left| \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) - 1 \right| \chi_{\{t < T_M\}} \neq 0 \right) = 0,$$

which implies

$$E \left[f(B_t) \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) \chi_{\{t < T_M\}} \right] = E[f(B_t^x) \chi_{\{t < T_M\}}],$$

and the expectation of the Feynman–Kac functional only depends on the class of $Q \in L^1_{loc}(M)$. Moreover, the Feynman–Kac formula for Brownian motion is continuous in $Q \in L^\infty(M)$.

Let now $Q \in L^1_{loc}(M)$ be bounded from below. Then $Q \wedge n \in \mathcal{L}^\infty(M)$ and there is a sequence of bounded continuous potentials $Q_{n,k}$ such that

$$\lim_{k \rightarrow \infty} \|Q_{n,k} - Q \wedge n\|_\infty = 0.$$

By the above mentioned continuity;

$$\lim_{k \rightarrow \infty} E \left[f(B_t) \exp \left(-\xi \int_0^t Q_{n,k}(B_s^x) ds \right) \chi_{\{t < T_M\}} \right] = E \left[f(B_t^x) \exp \left(-\xi \int_0^t Q \wedge n(B_s^x) ds \right) \chi_{\{t < T_M\}} \right].$$

On the other hand, since $Q_{n,k} - Q \wedge n \in L^\infty(M)$ is bounded as multiplication operator on V , the domains of

$$H_n(\xi) = -\frac{1}{2}\Delta + \xi(Q \wedge n)$$

and

$$H_{n,k}(\xi) = -\frac{1}{2}\Delta + \xi Q_{n,k}$$

coincide, and for all $f \in D(\delta) \cap D(Q \wedge n)$,

$$\|(H_{n,k}(\xi) - H_n(\xi))f\|_V \leq \|Q_{n,k} - Q \wedge n\|_\infty \|f\|_V,$$

which implies *generalized strong convergence* in the sense of Ref. 5, Sec. VIII 1, p. 427. By Ref. 5, Theorem 2.16, p. 504, this finally implies uniform convergence of the corresponding semi-groups in any finite subinterval of the positive real axis. Therefore (3.1) is proven for potentials $Q \wedge n$.

Now let n tend to infinity. The convergence of the corresponding Feynman–Kac functionals follows by monotone convergence and the fact that for $\text{Re}(\xi) > 0$,

$$\left| f(B_t^x) \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) \chi_{\{t < T_M\}} \right| \leq |f(B_t^x)| e^{-\text{Re}(\xi)Ct},$$

is P integrable.

The convergence of the corresponding operator semigroups follows by the fact mentioned in Sec. II, that for $\text{Re}(\xi) > 0$ the form sums $H_n(\xi)$ and $H(\xi)$ are m -sectorial with a common sector $S(\rho, \gamma)$ that contains $\xi \in \mathbb{C}$. Since $\mathcal{D}(H(\xi))$ is a common core for these operators and $(Q \wedge n) \times(x) \rightarrow Q(x)$, Ref. 5, Theorem 1.5, p. 429 implies generalized strong convergence $H_n(\xi) \rightarrow H(\xi)$ for $\text{Re}(\xi) > 0$ and therefore again by Ref. 5, Theorem 2.16, p. 504 convergence of the corresponding semigroups.

(3) For a general potential $Q \in L^1_{\text{loc}}(M)$ such that the operators $H(\xi)$ form a holomorphic family of type B, the Feynman–Kac formula is obtained by a criterion for convergence of sectorial forms from above (Ref. 5, Theorem 3.6, p. 455). Consider

$$H_m(\xi) := -\frac{1}{2}\Delta + \xi(Q \vee -m).$$

The associated quadratic forms are densely defined and sectorial for $\text{Re}(\xi) > 0$ and by the above, the Feynman–Kac formula holds for the corresponding semigroups. By Sec. II the operators $H_m(\xi)$ form a holomorphic family of type B for $\text{Re}(\xi) > 0$.

(a) For real parameter $\xi \in U \cap \mathbb{R}$, the associated quadratic forms decrease, i.e.,

$$h_m(\xi)(f) \geq h(\xi)(f) \geq c,$$

for some $c \in \mathbb{R}$, since $h(\xi)$ was assumed to be sectorial. This implies

$$\mathcal{D}(h_m(\xi)) \subset \mathcal{D}(h(\xi)),$$

for real ξ , but since the operators $h_m(\xi), h(\xi)$ form holomorphic families of type (a) this statement does not depend on $\xi \in U$.

(b) For $f \in \mathcal{D}(h_m(\xi))$ and $h_m^o(f) := h_m(\xi)(f) - h(\xi)(f)$,

$$\begin{aligned} |\text{Im}(h_m^o(f))| &= |\text{Im}(\xi)| (f, (Q \vee (-m) - Q)f) = \frac{|\text{Im}(\xi)|}{\text{Re}(\xi)} \text{Re}(\xi)(f, (Q \vee (-m) - Q)f) \\ &= K \text{Re}(h_m^o(f)), \end{aligned}$$

since $Q \vee (-m) - Q \geq 0$.

(c) Smooth functions $C_0^\infty(M)$ with compact support form a common core of $h(\xi)$ and $h_m(\xi)$ for all m . By monotone convergence,

$$(f, Q \vee (-m)f) \rightarrow (f, Qf),$$

for $f \in C_0^\infty(M)$ as m tends to infinity.

Now (a), (b), and (c) are the conditions under which the convergence criterion for quadratic forms mentioned above can be applied. It yields generalized strong convergence,

$$H_m(\xi) \rightarrow H(\xi),$$

for $\xi \in U$ as m tends to infinity.

By Ref. 5, Theorem 1.2, p. 427 generalized strong convergence of the generators implies strong convergence of the resolvents for $\text{Re}(\zeta) < \rho(\xi)$. Again, by Ref. 5, Theorem 2.16, p. 504, this implies convergence of the corresponding semigroups.

On the other hand,

$$\lim_{m \rightarrow \infty} E \left[f(B_t) \exp \left(-\xi \int_0^t (Q \vee (-m))(B_s^x) ds \right) \chi_{\{t < T_M\}} \right] = E \left[f(B_t) \exp \left(-\xi \int_0^t Q(B_s^x) ds \right) \chi_{\{t < T_M\}} \right],$$

compactly for real $\xi \in U \cap \mathbb{R}$ by monotone convergence. For general $\xi \in U$ convergence follows by Vitali's theorem (Ref. 9, p. 154). \square

Corollary 3.3: Under the assumptions above the resolvent of $H(\xi)$ can be expressed by

$$(H(\xi) - \zeta)^{-1}f(x) = E^x \left[\int_0^\infty dt f(B_t) \exp \left(\int_0^t (\zeta - \xi Q(B_s)) ds \right) \chi_{\{t < T_M\}} \right],$$

for $\text{Re}(\zeta) < \rho(\xi)$, where $S(\rho(\xi), \gamma(\xi))$ is a sector corresponding to $H(\xi)$.

Remark 3.4: The Feynman–Kac formula admits the following generalization: Instead of the Laplacian, it could be taken by any symmetric differential operator densely defined on V that is bounded from below and that generates a uniquely determined Markov semigroup with a transition probability that possesses a C^∞ density with respect to the volume form.

Remark 3.5: It should also be noted that if Q is Δ bounded with relative bound $b < \frac{1}{2}$, the operators

$$H(\xi) := -\frac{1}{2}\Delta + \xi Q$$

form a holomorphic family of type B for $|\xi| < 1/2b$ (Ref. 5, Sec. VII.4, Theorem 4.16, p. 403). By (3.1) this implies the validity of the Feynman–Kac formula as well and especially yields the Feynman–Kac Formula for

$$H := -\frac{1}{2}\Delta + Q.$$

IV. DOMAIN OF HOLOMORPHY FOR THE RESOLVENT

The results above for a single ξ in the parameter space of $H(\xi)$ also hold uniformly for parameters varying in a compact set. This will now be made precise.

Proposition 4.1 Let: $T(\xi)$ be any holomorphic family of type B for $\xi \in U \subset \mathbb{C}$ and $K \subset \subset U$ a relatively compact subset. Then we have the following.

- (1) All numerical ranges of $T(\xi), \xi \in K$ are contained in a common sector $S(\rho_K, \gamma_K)$.
- (2) The set,

$$U_K := (\mathbb{C} - S(\rho_K, \gamma_K)) \times K,$$

is contained in the domain of holomorphy of the resolvent,

$$R(\zeta, \xi) = (T(\xi) - \zeta)^{-1}.$$

- (3) For $(\zeta, \xi) \in U_K$ with $\text{Re}(\zeta) < \rho_K$ the resolvent is given by the Laplace integral,

$$R(\zeta, \xi)f = \int_0^\infty dt e^{\zeta t - tT(\xi)} f.$$

Proof:

- (1) This property is called *local uniform sectoriality* in Ref. 5, Theorem 4.2, p. 395.
- (2) See the remark after the definition of *m*-sectoriality, Ref. 5, Chap. V, p. 280.
- (3) It follows from the fact that $\zeta - T(\xi)$ for $\text{Re}(\zeta) < \rho_K$ generates a contraction semigroup. \square

Remark 4.2: Since by the above for $Q \in L^1_{\text{loc}}(M)$ with $Q \geq 0$ the form sum

$$H(\xi) = -\frac{1}{2}\Delta + \xi Q$$

yields a holomorphic family of type B for $\text{Re}(\xi) > 0$, and for each $K \subset \subset \{\xi: \text{Re}(\xi) > 0\}$ a common sector can be chosen with vertex $\rho_K = 0$, the corresponding resolvent $R(\zeta, \xi)$ can be represented by the Laplace transform for each $\text{Re}(\zeta) < 0$.

The following remark is important for the proof of the transformation formula of Theorem 6.3.

Remark 4.3: Holomorphic families of type B and their resolvent functions enjoy the unique continuation property, i.e., if the resolvent functions of two holomorphic families coincide on some open set $U \subset \mathbb{C}^2$, the two families coincide as well.

Proof: Reference 5, Remark 1.6, p. 368. □

V. HARMONIC MORPHISMS AND BROWNIAN MOTION

Harmonic morphisms are twice continuously differentiable mappings between Riemannian manifolds such that composition with harmonic functions on the target manifold yields a harmonic function on the preimage.

Harmonic morphisms can as well be characterized by their geometric and stochastic properties. The following definitions and results can be found in Refs. 10 and 2.

Definition 5.1 (harmonic morphism): Let $(M, g_M), (N, g_N)$ be Riemannian manifolds and Δ_M, Δ_N their Laplace–Beltrami operators. A twice continuously differentiable map $\Phi: M \rightarrow N$ is called harmonic morphism if the pullback of germs of harmonic functions on N yields germs of harmonic functions on M , i.e.,

$$\Delta_N f_{\Phi(x)} = 0 \Rightarrow \Delta_M (f \circ \Phi)_x = 0.$$

Definition 5.2 (horizontally conformal map): A C^2 mapping $\Phi: M \rightarrow N$ between Riemannian manifolds is called horizontally conformal, if for every $x \in M$ such that $T_x \Phi \neq 0$, the restriction of the tangent map,

$$T_x \Phi|_{K_x^\perp}: K_x^\perp \rightarrow T_{\Phi(x)} N,$$

to the orthogonal complement of $K_x := \ker(T_x \Phi) \subset T_x M$ is surjective and conformal,

$$d_\Phi(x) := \begin{cases} \|T_x \Phi\|, & \text{if } x \text{ is a regular point,} \\ 0, & \text{otherwise,} \end{cases}$$

is called the dilation of Φ .

Theorem 5.3 (geometric characterization): For a C^2 map $\Phi: M \rightarrow N$ are equivalent.

- (1) Φ is a harmonic morphism.
- (2) Φ is harmonic and horizontally conformal.

Proof: Reference 10 Theorem, p. 123. □

Remark 5.4: (1) By the semiconformality of harmonic morphisms, the tangent map is equal to zero for each nonregular point.

(2) The set of nonregular points for a nonconstant harmonic morphism $\Phi: M \rightarrow N$ can be covered by a countable collection of submanifolds of M of codimension less or equal to two and therefore is polar in M (compare Ref. 10, p. 116, Remark).

The following proposition is a direct consequence of the stochastic characterization of harmonic morphisms in Ref. 2 and will be the main tool to prove the transformation formula. To avoid the construction of “ τ welding,” the harmonic morphism is assumed to be surjective.

Proposition 5.5: Let $\Phi: M \rightarrow N$ be a harmonic morphism of dilation d_Φ . Let further be $\lambda(x) := d_\Phi^2(x)$ and B^N, B^M be Brownian motion on N, M , respectively. Consider now the time transform

$$\tau(t) := \int_0^t \lambda(B_s^M) ds,$$

and its inverse,

$$t(\tau) := \inf\{s \geq 0 : \tau(s) = \tau\}.$$

Then the image under Φ of time-transformed Brownian motion $Z_\tau(\omega) := B_{t(\tau)}^M(\omega)$ coincides in law with Brownian motion on N , i.e.,

$$\Phi(Z_\tau^y(\omega)) \sim B_\tau^{N,x}(\omega),$$

for any $y \in \Phi^{-1}(x)$.

Remark 5.6 (stochastic characterization): Harmonic morphisms can as well be characterized by a slightly modified stochastic property as in Proposition (5.5) (see Ref. 2, Theorem 1, p. 224).

VI. THE TRANSFORMATION FORMULA

The transformation formula just consists of inserting special harmonic morphisms into the Feynman–Kac formula for the resolvent. The harmonic morphisms used in the sequel are assumed to be proper in order to be able to lift distributions. In the sequel the explosion times of N, M are denoted by T_N, T_M , respectively.

Lemma 6.1: Let $\Phi: M \rightarrow N$ be a proper and surjective harmonic morphism. Then, the domain $\mathbb{D}(\Phi^\Lambda)$ of the linear map,

$$\Phi^\Lambda: \mathbb{D}(\Phi^\Lambda) \rightarrow L^2(M),$$

with

$$\Phi^\Lambda f(x) := \lambda(x) \cdot (f \circ \Phi)(x);$$

$(\lambda = d_\Phi^2)$ is dense in $L^2(N)$.

Proof: Compactly supported functions are dense in $L^2(N)$. By a proper map they are lifted to compactly supported functions in $L^2(M)$. That means $C_0^\infty(N) \subset \mathbb{D}(\Phi^\Lambda)$. \square

Proposition 6.2: Let Φ be as above and $Q \in L_{loc}^1(N)$ be a measurable potential such that

$$H(\xi) = -\frac{1}{2}\Delta_n + \xi Q$$

is a holomorphic family of type B on $L^2(N)$ for $\xi \in U \subset \mathbb{C}$, U open. Then for $\xi \in K \subset U$ with corresponding uniform sector $S(\rho_K, \gamma_K)$ and $\text{Re}(\xi) < \rho_K$ the resolvent equals

$$R(\zeta, \xi)f(x) = E^y \int_0^\infty dt \Phi^\Lambda f(B_t^M) \exp\left(\int_0^t ds \Phi^\Lambda(\zeta - \xi Q)(B_s^M)\right) \chi_{\{t < T_M\}},$$

for any $y \in \Phi^{-1}(x)$.

Proof: By Corollary 3.3 the resolvent can be represented by a Feynman–Kac formula,

$$R(\zeta, \xi)f(x) = E^x \int_0^\infty d\tau f(B_\tau^N) \exp\left(\int_0^\tau d\sigma(\zeta - \xi Q)(B_\sigma^N)\right) \chi_{\{\tau < T_N\}}.$$

Since $\Phi(Z_\tau^y) \sim B_\tau^x$ by Proposition 5.5,

$$R(\zeta, \xi)f(x) = E^y \int_0^\infty d\tau f \circ \Phi(Z_\tau) \exp\left(\int_0^\tau d\sigma(\zeta - \xi Q)(\Phi(Z_\sigma))\right) \chi_{\{\tau < T_N\}}.$$

By the time transform $\tau(t, \omega) = \int_0^t ds \lambda(B_s^M)$, i.e.,

$$d\tau = \lambda(B_t^M) dt,$$

and by $Z_\tau \sim B_{t(\tau)}^M$:

$$\begin{aligned} R(\zeta, \xi)f(x) &= E^y \int_0^{t(\tau, \omega)} dt \lambda(B_t^M) \cdot f \circ \Phi(B_t^M) \exp\left(\int_0^t ds \lambda(B_s^M)(\zeta - \xi Q)(\Phi(B_s^M))\right) \chi_{\{\tau < T_N\}} \\ &= E^y \int_0^\infty dt \Phi^\Lambda f(B_t^M) \exp\left(\int_0^t ds \Phi^\Lambda(\zeta - \xi Q)(B_s^M)\right) \chi_{\{t < T_M\}}, \end{aligned}$$

since $t(T_N) = T_M$ almost surely. □

If $\Phi^\Lambda Q \equiv C$ equals a constant, then

$$\Phi^\Lambda(\zeta - \xi Q)(x) = \zeta \lambda(x) - C \xi.$$

This is interesting because, in that case, by the uniqueness of analytic continuation, the transformed expectation value can again be interpreted as a resolvent, namely the resolvent of the operator

$$G(\kappa) = -\frac{1}{2}\Delta + \kappa\lambda,$$

where the coupling parameter and the resolvent parameter change place. Since λ is non-negative and continuous, the domain of $G(\kappa)$ contains $C_0^\infty(M)$ and forms a holomorphic family of type B for $\text{Re}(\kappa) > 0$ (Remark 4.2).

Theorem 6.3 (transformation formula): *Let*

$$H(\xi) = -\frac{1}{2}\Delta + \xi Q$$

be a holomorphic family of type B on $L^2(N)$ on an open subset $U \subset \mathbb{C}$ such that

$$U^+ := U \cap \{z : \text{Re}(z) > 0\}$$

and

$$U^- := U \cap \{z : \text{Re}(z) < 0\},$$

are both nonempty. Let $K \subset U$. Then, the resolvent $R_H(\zeta, \xi)$ is holomorphic in U_K . Let $\Phi : M \rightarrow N$ be a proper and surjective harmonic morphism with $\Phi^\Lambda Q \equiv C \neq 0$.

Then

$$G(\kappa) = -\frac{1}{2}\Delta + \kappa\lambda$$

is a holomorphic family of type B for $\text{Re}(\kappa) > 0$ on $L^2(M)$ and

$$R_H(\zeta, \xi)f(x) = R_G(-C\xi, -\zeta)\Phi^\Lambda f(y),$$

for all $y \in \Phi^{-1}(x)$.

Proof: Choose K such that

$$K^+ := K \cap \{z : \text{Re}(z) > 0\}$$

and

$$K^- := K \cap \{z : \text{Re}(z) < 0\}$$

are both nonempty. By (Proposition 6.2),

$$\begin{aligned}
 R_H(\zeta, \xi)f(x) &= E^y \int_0^\infty dt \Phi^\Lambda f(B_t^M) \exp\left(\int_0^t ds \Phi^\Lambda(\zeta - \xi Q)(B_s^M)\right) \chi_{\{t < T_M\}} \\
 &= E^y \int_0^\infty dt \Phi^\Lambda f(B_t^M) \exp\left(\int_0^t ds (-C\xi + \zeta\lambda)(B_s^M)\right) \chi_{\{t < T_M\}} \\
 &= E^y \int_0^\infty dt \Phi^\Lambda f(B_t^M) \exp\left(\int_0^t ds (-C\xi - (-\zeta)\lambda(B_s))\right) \chi_{\{t < T_M\}} \\
 &= R_G(-C\xi, -\zeta)\Phi^\Lambda f(y).
 \end{aligned}$$

The above equality holds for $(\zeta, \xi) \in U_K$, therefore $\text{Re}(\zeta) < \rho_K \wedge 0$, which implies that the intersection of $\text{Re}(-\zeta) > -\rho_K \vee 0$ with that part of the parameter space, where $G(\kappa)$ forms a holomorphic family of type B, is an open set. On the other hand, by Remark 4.2, the last equality holds for those values of ξ , where the Laplace integral can indeed be interpreted as a resolvent. Depending on the sign of C this is the case for $\xi \in K^+$ if $C < 0$ and for $\xi \in K^-$ if $C > 0$. Now since the identity holds on some open subset of $U_K \subset \mathbb{C}^2$ both resolvent functions coincide by Remark 4.3.

VII. EXAMPLE

By the above formula, a well-known correspondence^{3,4} between the harmonic oscillator in dimension 4 and the Coulomb System in dimension 3 can be established.

Consider Coulomb’s potential,

$$Q(x) := -\frac{1}{|x|} : \mathbb{R}^3 \rightarrow \mathbb{R}.$$

The family $H(\xi) := -\frac{1}{2}\Delta - \xi/r$ is of type B for every $\xi \in \mathbb{C}$, since Q is in the Kato class.

Definition 7.1 (Kuustanheimo–Stiefel transform): The mapping

$$\Phi = (\Phi^1, \Phi^2, \Phi^3) : \mathbb{R}^4 \rightarrow \mathbb{R}^3,$$

given by quadratic forms,

$$w^i := \Phi^i(\mathbf{x}) = (\mathbf{x}, e_i \mathbf{x}), \quad i = 1, 2, 3,$$

with

$$e_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is called the Kuustanheimo–Stiefel transform. Its restriction to $S^3 \subset \mathbb{R}^4$ is the Hopf map.

Φ is a harmonic morphism of the so-called Clifford type,¹¹ since

$$\{e_i, e_j\} := e_i e_j + e_j e_i = 2\delta_{ij},$$

which means that the matrices e_1, e_2, e_3 yield an irreducible representation of the Clifford algebra Cl_3^* (see Ref. 6).

The square of the dilation of Φ is given by

$$\lambda(\mathbf{x}) = (\nabla \Phi^i(\mathbf{x}), \nabla \Phi^j(\mathbf{x})) = 4\|e_i \mathbf{x}\|^2 = 4\|\mathbf{x}\|^2,$$

since the matrices e_i are orthogonal.

Φ is proper since $\|\Phi(\mathbf{x})\| = \|\mathbf{x}\|^2$ and surjective by computing explicitly that the preimage of a point $\mathbf{x} \in \mathbb{R}^3 - \{0\}$ is given by a one-sphere.

Finally, by

$$\Phi^\Lambda\left(-\frac{1}{|\mathbf{x}|}\right) = -\frac{4|\mathbf{x}|^2}{\|\Phi(\mathbf{x})\|} = -4,$$

and $\lambda(x) = 4\|\mathbf{x}\|^2$ the correspondence

$$R\left(\zeta; -\frac{1}{2}\Delta - \frac{\xi}{|\mathbf{w}|}\right)f(\mathbf{w}) = R\left(-4\xi; -\frac{1}{2}\Delta + 4\zeta|\mathbf{x}|^2\right)\Phi^\Lambda f(\mathbf{x})$$

is obtained.

VIII. THE DUAL SEMIGROUP. COMPUTATION OF THE RESOLVENT KERNEL

Throughout this section the assumptions of Theorem 6.3 remain valid. The resolvent and semigroup kernels for $H(\xi)$, $G(\kappa)$, respectively, are denoted by upper indices M, N corresponding to the underlying manifold. If some statement holds for both of them, the index is omitted. Remember that the vertex of the common sector for $H(\xi)$, $\xi \in K$ is denoted by ρ_K (Proposition 4.1).

Denote by $k_t^N(\xi; x, dy)$ the kernel of the Feynman–Kac semigroup generated by $H(\xi)$, $\xi \in K$. The resolvent kernel,

$$R_H(\xi, \zeta)f(x) = \int_n \rho^N(\xi, \zeta; x, dy)f(y),$$

can, for $\text{Re}(\zeta) < \rho_K$ be computed by the Laplace transform,

$$\rho^N(\xi, \zeta; x, dy) \stackrel{w}{=} \int_0^\infty dt e^{\zeta t} k_t^N(\xi; x, dy),$$

where $\stackrel{w}{=}$ means that this equality is, in general, valid only in the weak sense, i.e., for each continuous function φ with compact support,

$$\int_N \rho^N(\xi, \zeta; x, dy)\varphi(y) = \int_0^\infty dt e^{\zeta t} k_t^N(\xi; x, dy)\varphi(y).$$

In the case of existing densities, this statement can be reformulated as follows.

Lemma 8.1: If the transition kernel possesses a density with respect to the volume form, i.e.,

$$k_t(x, dy) = k_t(x, y)\text{vol}_N(dy),$$

then the resolvent kernel possesses a density with respect to the volume form as well. This density $\rho(\xi; x, y)$ is also obtained by the Laplace transform

$$\rho(\xi, \zeta; x, y) \stackrel{w}{=} \int_0^\infty dt e^{\zeta t} k_t(\xi; x, y).$$

The following fact is a consequence of the weak continuity of the dual semigroup.

Proposition 8.2: The Feynman–Kac kernel is obtained by

$$k_t(\xi; x, y) \stackrel{w}{=} E^x \left[\delta_y(B_t) \exp \left[-\xi \int_0^t Q(B_s) ds \right] \chi_{\{T_M > t\}} \right],$$

where δ_y denotes the Delta distribution with support $\{y\}$.

By using Brownian bridges, the integration about the position of Brownian motion at time t can be carried out explicitly.

Definition 8.3: A Brownian bridge b_s with initial point $x = b_0(\omega)$ and final point $y = b_t(\omega)$ is the stochastic process,

$$b: [0, t] \times \{\omega \in \Omega: \omega(0) = x, \omega(t) = y\} \rightarrow N,$$

given by $b_s(\omega) := \omega(s)$ and distributed by the conditional probability

$$Q^{x,y,t}(d\omega) := P^x(d\omega | B_t(\omega) = y, T_N(\omega) > t),$$

where P^x denotes the probability distribution of Brownian motion on the manifold with starting point x .

The same construction works for M . Brownian motion on M, N , respectively, is not distinguished by notation. It will be clear from the context which one is meant.

This yields the following expression for the kernel.

Proposition 8.4: The Feynman–Kac Kernel is given by

$$k_t(\xi; x, dy) = p_t(x, dy) E^{x,y,t} \left[\exp - \xi \int_0^t Q(b_s) ds \right],$$

where $E^{x,y,t}$ denotes expectation with respect to $Q^{x,y,t}$ and, as above,

$$p_t(x, dy) := P^x(B_t(\omega) \in dy, T(\omega) > t),$$

where T denotes the explosion time on M, N , respectively.

Proof: The computation is carried out for N . By (Proposition 8.2),

$$\begin{aligned} k_t^N(\xi; x, dy) &= E^x \left[\delta_y(B_t) \exp \left[- \xi \int_0^t Q(B_s) ds \right] \chi_{\{T_N > t\}} \right] \\ &= \int_N P^x(B_t \in du, T_N > t) E^x \left[\delta_y(B_t) \exp \left[- \xi \int_0^t Q(B_s) ds \right] \chi_{\{T_N > t\}} \middle| B_t = u, T_N > t \right] \\ &= \int_N P_t^N(x, du) \delta_y(u) E^x \left[\exp \left[- \xi \int_0^t Q(B_s) ds \right] \chi_{\{T_N > t\}} \middle| B_t = u, T_N > t \right] \\ &= p_t^N(x, dy) E^{x,y,t} \left[\exp \left[- \xi \int_0^t Q(b_s) ds \right] \right]. \end{aligned}$$

□

Corollary 8.5: If $p_t(x, dy) = p_t(x, y) \text{vol}(dy)$ possesses a density with respect to the volume form on M, N , respectively, then

$$k_t(\xi; x, y) = p_t(x, y) E^{x,y,t} \left[\exp \left[- \xi \int_0^t Q(b_s) ds \right] \right].$$

Under the same assumptions the resolvent kernel can (Lemma 8.1) be expressed by

$$\rho(\xi, \zeta; x, y) = \int_0^\infty dt p_t(x, y) E^{x,y,t} \left[\exp \left[\int_0^t (\zeta - \xi Q)(b_s) ds \right] \right],$$

for $\text{Re}(\zeta) < \rho_K$ in the case of $H(\xi), \xi \in K$ and for $\text{Re}(\zeta) < 0$ in the case of $G(\xi), \text{Re}(\xi) > 0$.

By these considerations the Feynman–Kac formula for holomorphic families of type B (3.1) and the transformation formulas Proposition (6.2) and Theorem (6.3) can be transferred to the corresponding kernels.

Corollary 8.6: *The following reformulations of the relevant formulas are valid.*

- (1) *Under the assumptions of Proposition (3.1), the resolvent kernel $\rho(\zeta, \xi; x, dy)$ of the holomorphic family can be expressed by*

$$\rho(\zeta, \xi; x, dy) = \int_0^\infty dt p_t(x, dy) E^{x, y, t} \left[\exp \left[\int_0^t (\zeta - \xi Q(b_s)) ds \right] \right].$$

- (2) *Under the assumptions of Proposition (6.2), the resolvent kernel is obtained by*

$$\rho^N(\zeta, \xi; x, dy) = E^z \left[\int_0^\infty dt \Phi^\Lambda \delta_y(B_t) \exp \left[\int_0^t ds \Phi^\Lambda (\zeta - \xi Q(B_s)) \chi_{\{T_M > t\}} \right] \right],$$

- (3) *where $z \in \Phi^{-1}(x)$.*

- (4) *If, furthermore, as in Theorem 6.3,*

$$\Phi^\Lambda Q \equiv C,$$

then

$$\int_n \rho^N(\zeta, \xi; x, du) f(u) = \int_M \rho^M(-C\xi, -\zeta; z, dv) \lambda(z) f \circ \Phi(v)$$

for f continuous with compact support.

In the case of existing densities this can be made more explicit by a decomposition of the volume form along the fibers of the harmonic morphism, a procedure that is well known and summarized in the following Lemma.

Lemma 8.7: *Let $*$ denote the Hodge–Star operator and $\Phi: M \rightarrow N$ a harmonic morphism. Consider points $z \in \Phi^{-1}(y)$, where $y \in N$ is such that all points in the fiber are regular values of Φ . Then we have the following.*

- (1) *There is an open neighborhood $U(y) \in N$ and a diffeomorphism,*

$$\theta: U(y) \times F_y \xrightarrow{\sim} \Phi^{-1}(U(y)),$$

$F_y := \Phi^{-1}(y)$. For simplicity, the composition $\Phi \circ \theta$ is again denoted by Φ .

- (2) *For $z = \theta(u, f)$,*

$$\text{vol}_M(dz) = \Phi_z^* \text{vol}_N \wedge \frac{\Phi_z^* \text{vol}_N}{\|\Phi_z^* \text{vol}_N\|^2} (dz) = \Phi_z^* \text{vol}_N(du) \wedge \lambda^{-n}(u, f) * \Phi_z^* \text{vol}_N(df).$$

The last equality holds because by horizontal semiconformality, the square of the determinant of the cotangent mapping equals λ^n . This yields the following modification of (3) of Corollary 8.6.

Corollary 8.8: *Let $n = \dim(N)$ and $y \in N$ a point such that all preimages are regular values of the harmonic morphism. Then, under the assumptions of Proposition (6.2),*

$$\rho^N(\xi, \zeta; x, y) = \int_{F_y} \Phi_z^* \text{vol}_N(df) \lambda^{(2-n)/2}(y, f) \rho^M(-C\xi, -\zeta; z, (y, f)).$$

Proof: By the transformation formula

$$\begin{aligned}
 \rho^N(\zeta, \xi; x, y) &= \int_0^\infty dt \int_M p_t^M(z, dz') \Phi^\Lambda \delta_y(z') E^{z, z', t} \left[\exp \left[\int_0^t \Phi^\Lambda(\zeta - \xi Q)(b_s) ds \right] \right] \\
 &= \int_0^\infty dt \int_M \text{vol}_M(dz') p_t^M(z, z') \Phi^\Lambda \delta_y(z') E^{z, z', t} \left[\exp \left[\int_0^t \Phi^\Lambda(\zeta - \xi Q)(b_s) ds \right] \right] \\
 &= \int_{(u, f) \in U(y) \times F_y} \Phi^* \text{vol}_N(du) \wedge * \Phi^* \text{vol}_N(df) \lambda^{(2-n)/2}(u, f) \delta_y(u) \\
 &\quad \times \int_0^\infty dt p_t^M(z, (u, f)) E^{z, (u, f), t} \left[\exp \left[\int_0^t \Phi^\Lambda(\zeta - \xi Q)(b_s) ds \right] \right] \\
 &= \int_{f \in F_y} * \Phi^* \text{vol}_N(df) \lambda^{(2-n)/2}(y, f) \int_0^\infty dt p_t^N(z, (y, f)) E^{z, (y, f), t} \\
 &\quad \times \left[\exp \left[\int_0^t \Phi^\Lambda(\zeta - \xi Q)(b_s) ds \right] \right] \\
 &= \int_{f \in F_y} * \Phi^* \text{vol}_N(df) \lambda^{(2-n)/2}(y, f) \rho(-C\xi, -\zeta; z, (y, f)).
 \end{aligned}$$

Example 8.9: In the case of the example of Sec. VII, the preimage of a point $y \in \mathbb{R}^3$ can be computed explicitly. If in spherical coordinates,

$$y = r(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),$$

the preimage is given by

$$\Phi^{-1}(y) = \{U(r, \theta, \phi, \gamma) \mid \gamma \in [0, 4\pi)\},$$

where

$$\begin{aligned}
 U(r, \theta, \phi, \gamma) &= \sqrt{r} \left(\cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi + \gamma}{2}\right), \right. \\
 &\quad \left. - \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\phi + \gamma}{2}\right), \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi - \gamma}{2}\right), \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\phi - \gamma}{2}\right) \right).
 \end{aligned}$$

Now by (8.8) and the fact that the volume element is given by

$$\Phi_z^* \text{vol}_N = \frac{1}{2} |x|^3 d\gamma,$$

the kernel can be expressed by

$$\rho^N(\xi, \zeta; x, y) = \frac{1}{2} |y| \int_0^{4\pi} d\gamma \rho^M(-C\xi, -\zeta; z, U(r, \theta, \phi, \gamma)),$$

where $z \in \Phi^{-1}(x)$ and r, θ, ϕ as above. The kernel on the right-hand side is given by Mehler's formula.

ACKNOWLEDGMENTS

The author wishes to thank the University of Göttingen for hospitality during the winter term 97/98 and the F. Thyssen-Stiftung for financial support.

- ¹B. Simon, *Functional Integration and Quantum Physics* (Academic, London, 1979).
- ²L. Cszink and B. Oksendal, “Stochastic harmonic morphisms: Functions mapping the paths of one diffusion into the paths of another,” *Ann. Inst. Fourier* **33**, 219–240 (1983).
- ³P. Blanchard, “Transformation of Wiener integrals and the desingularization of the Coulomb problem,” in *Proceedings of the Workshop on Stochastic Processes in Quantum Theory and Statistical Physics*, Marseille 1981, Springer Lecture Notes in Physics (Springer-Verlag, Berlin, 1982), Vol. 173, pp. 19–28.
- ⁴I. H. Duru and H. Kleinert, *Phys. Lett. B* **84**, 30 (1979).
- ⁵T. Kato, *Perturbation Theory for Linear Operators*, corr. pr. 2nd ed. (Springer-Verlag, Berlin, 1980).
- ⁶H. B. Lawson and M. L. Michelsohn, *Spin Geometry* (Princeton University Press, Princeton, NJ, 1989).
- ⁷T. Kato, *Trotter’s Product Formula for an Arbitrary Pair of Self-Adjoint Contraction Semigroups*, in: *Advances in Math. Supplementary Studies 1978*, edited by Gohberg and Kac (Academic, New York, 1978), Vol. 3, pp. 185–195.
- ⁸B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms* (Princeton University Press, Princeton, NJ, 1971).
- ⁹J. B. Conway, *Functions of One Complex Variable*, 2nd ed. (Springer-Verlag, New York, 1978).
- ¹⁰B. Fuglede, “Harmonic morphisms between Riemannian manifolds,” *Ann. Inst. Fourier* **28**, 107–144 (1978).
- ¹¹P. Baird, “Harmonic morphisms and circle actions on 3- and 4-manifolds,” *Ann. Inst. Fourier* **40**, 177–212 (1990).