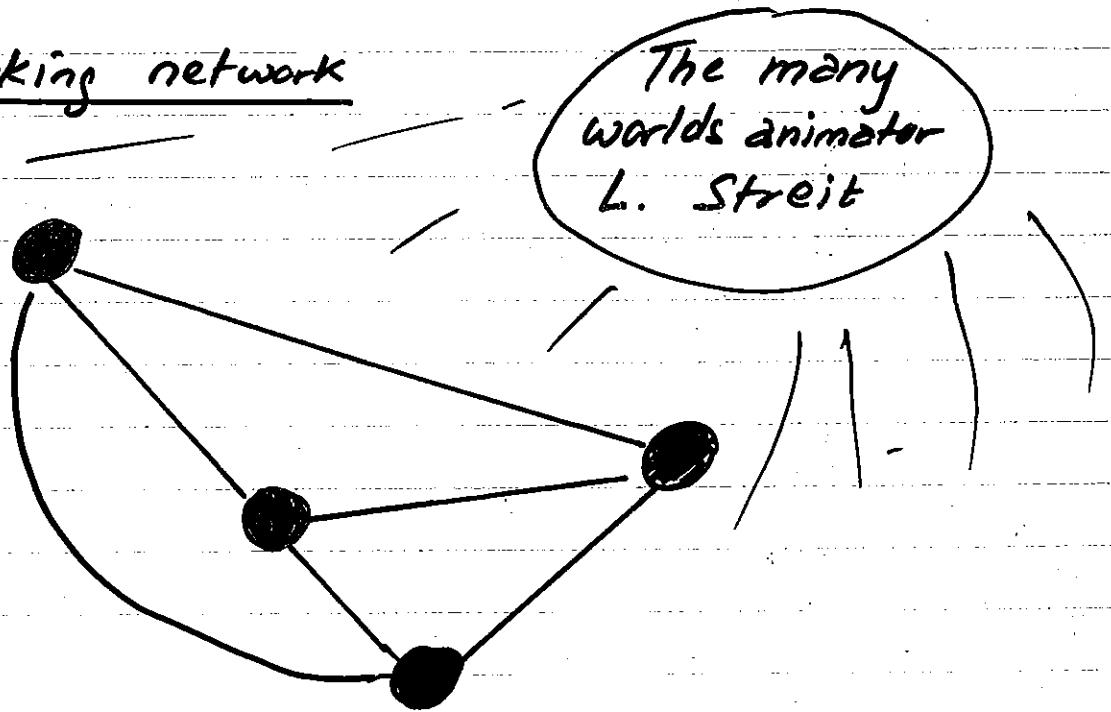


Synchronization in agents' Networks.

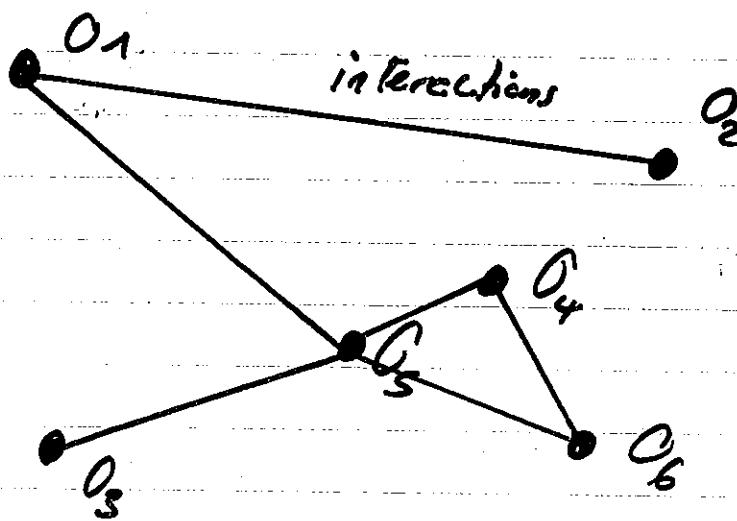
working network



- Ph. Blanchard (Bielefeld)
- R. Filliger (Bern-Biel)
- J. Rodriguez (EPFL Lausanne)
- M.-O. Hongler (EPF-Lausanne)

- 1) Network of oscillators with adaptive frequencies.
- 2) Swarm dynamics and super-diffusive noise.

1. Network of oscillators



- $\{O_k\}$ a collection of stable oscillators

Synchronization \Rightarrow w_c consensual frequency
and learning

"Plasticity"

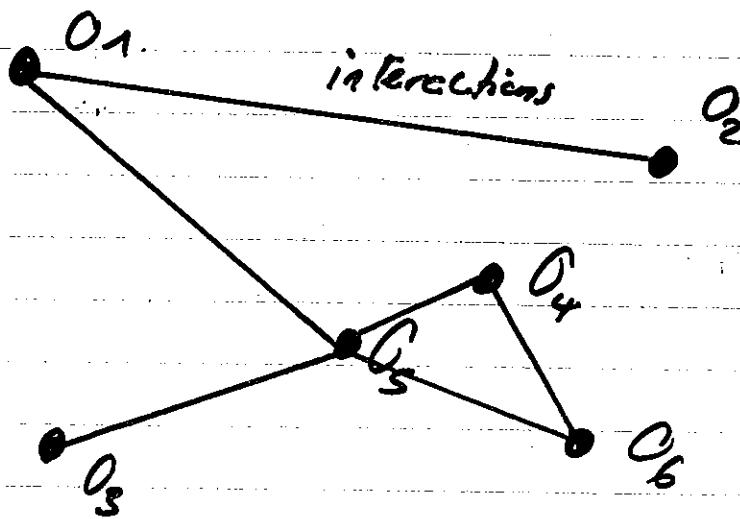
After w_c has been obtained, every oscillator retain what they did learn



w_c remains even if interactions are removed.

Applications: Clock synchronization with permanent auto-corrections.

1. Network of oscillators



$\{O_k\}$ a collection of stable oscillators

Synchronization \Rightarrow w_c consensual frequency
and learning

"Plasticity"

After w_c has been attained, every
oscillators retain what they did learn



w_c remains even if interactions are removed.

Applications: Clock synchronization with permanent auto-corrections.

Questions (not exhaustive!!!)

- . Can we calculate the consensual frequency of w_c ?
- . Does w_c depend on the network topology?
- . How does the network topology affect the convergence to w_c ?
-

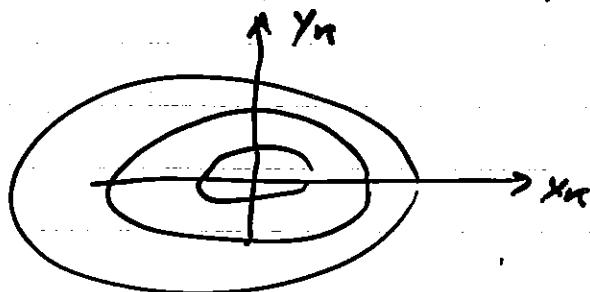
Network dynamics, Mixed canonical-dissipative

$$\dot{x}_k = w_k H_{y_k} + g_k(H(x_k, y_k)) H_{x_k} + C_k z,$$

$$\dot{y}_k = -w_k H_{x_k} + g_k(H(x_k, y_k)) H_{y_k} + C_k y,$$

$$\dot{w}_k = K_k [Dy C_k x - Dx C_k y], \quad k=1, \dots, N.$$

- w_k — learning mechanism via coupling
- canonical drift (circulate on the limit cycle)
- dissipative mechanism (find a limit cycle)
- $H(x_k, y_k) = \text{const} \Leftrightarrow$ closed curves in phase sp.



- Diffusive coupling

$$C_k x = \sum_{j=1}^N L_{kj} x_j \quad C_k y = \sum_{j=1}^N L_{kj} y_j$$

$L = A - D$ diagonal degree matrix

A adjacency matrix

Laplacian matrix

- $Dy = \sum_{j=1}^N H_{y_j}, \quad Dx = \sum_{j=1}^N H_{x_j}$

Illustration with circular symmetry.

$$H(x, y) = x^2 + y^2 = r^2$$

$$g(H) = 1 - H(x, y) = 1 - r^2 \quad (\text{Hopf oscillator})$$

Dynamics $k=1, \dots, N$ oscillators.

$$\dot{r}_k = 2(1-r_k^2)r_k + 2 \sum L_{kj} \cos(\phi_k - \phi_j)$$

$$\dot{\phi}_k = -2\omega_k - \frac{2}{r_k} \sum_{j=1}^N L_{kj} \sin(\phi_k - \phi_j)$$

$$\dot{\omega}_k = 4K_k \sum_{j=1}^N L_{kj} \sin(\phi_k - \phi_j)$$

→ Limit cycle ($r_k = 1$) generator.

→ Kuramoto's dynamics.

→ Learning mechanism (K_k learning rate).

Prop 1.

$$J = \sum_{k=1}^N \frac{\omega_k(t)}{K_k}$$

is a constant of
the motion.

• Proof.

$$\begin{aligned} \sum_{k=1}^N \frac{\dot{\omega}_k}{K_k} &= \sum_{k=1}^N D_y G_k x - \sum_{k=1}^N D_x C_k y \\ &= D_y \sum_{k=1}^N \sum_{j=1}^N L_{kj} x_j - D_x \sum_{k=1}^N \sum_{j=1}^N L_{kj} y_j \\ &= D_y \sum_{j=1}^N x_j \sum_{k=1}^N L_{kj} - D_x \sum_{j=1}^N y_j \sum_{k=1}^N L_{kj} = 0 \end{aligned}$$

Prop 2. Assume $H_k \equiv H$ (Homogeneous network)
of oscillators

Then $S = (x_s(t), y_s(t), w_s; \dots, x_s(t), y_s(t), w_s)$
is a solution of the dynamics with
 $g(H(x_s, y_s)) \equiv 0$.

• Proof. By model construction.

Corollary

The consensual frequency w_c is given by

$$w_c = \frac{\sum_{j=1}^N w_j(0)}{\sum_{j=1}^N \frac{1}{k_j}}$$

Proof. Due to the existence of J a constant of the motion

Corollary w_c is indep. of the network

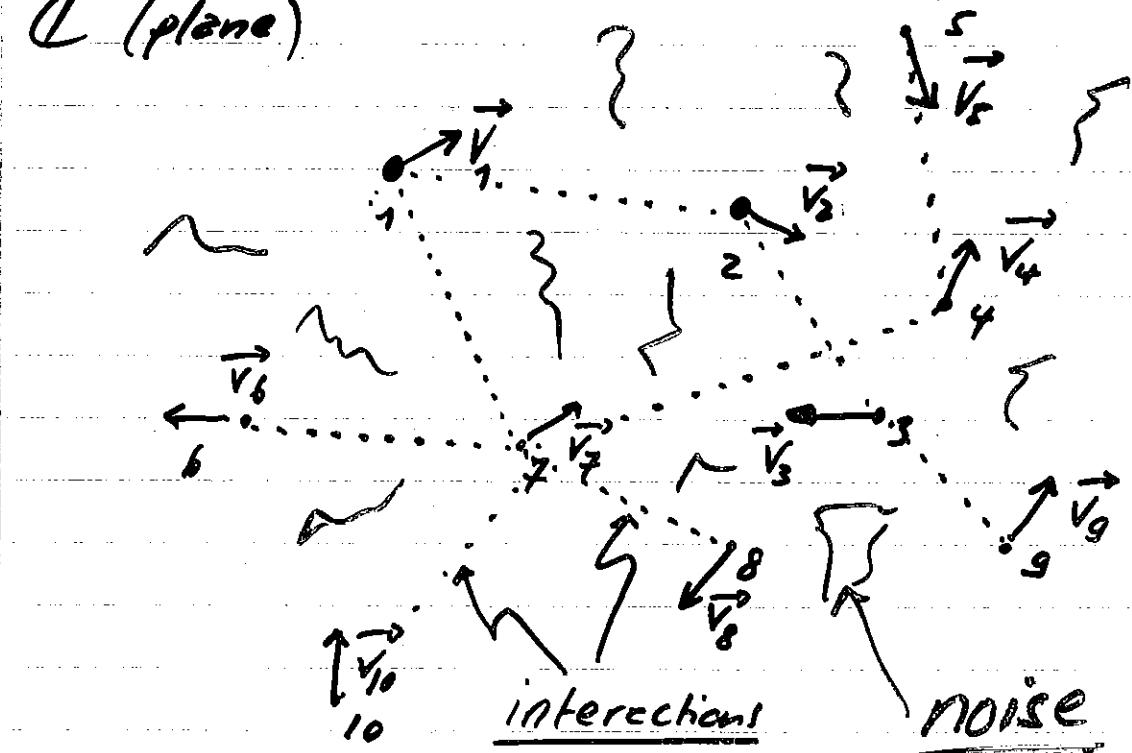
But the convergence rate depends on L_{K_j} :

Linear stability analysis \Rightarrow

("mass gap"
Fiedler number controls the
convergence rate)

2. Swarm dynamics. on the plane

\mathcal{C} (plane)



$$\dot{\vec{r}}_k = \vec{v}_k \quad \text{velocity}$$

Dynamics

$$\left\{ \begin{array}{l} \dot{\vec{r}}_k = \vec{v}_k \\ \vec{v}_k = f(\vec{r}_k, \vec{v}_k) + \xi_k \end{array} \right. \quad \begin{array}{l} \text{(additive noise)} \\ \text{(in general)} \end{array}$$

- F. Cucker & Smale S. (IEEE Trans. Autom. Control 2007)
Gives the clustering condition in term of the interaction range)

Oscillators models & collective motion.

(D. PALEY et al. (IEEE control system magazine 2007))

$$r_i(t) \in \mathbb{C} \quad i=1, 2, \dots, N$$

$$\|\dot{r}_i(t)\| = 1 \quad (\text{constant velocity})$$

Dynamics

$$\dot{r}_i(t) = e^{i\theta_i(t)}$$

$$\dot{\theta}_i(t) = u_i(\vec{r}, \vec{\theta}, \xi_i)$$

↑ noise
control function

Exemples. (phase controls only)

1) Grégoire & Chaté (PRL 2004)

$$u_i(\vec{\theta}_i, \xi_i(t)) = \text{angle} \left\{ N^{-1} \sum_{k=1}^{N_i} e^{i\theta_k(t)} \right\} + \sqrt{D} \xi_i(t)$$

2) Aldana et al. (PRL 2007)

$$u_i(\vec{\theta}_i, \xi_i(t)) = \text{angle} \left\{ N^{-1} \sum_{k=1}^{N_i} e^{i\theta_k(t) + iD\xi_i(t)} \right\}$$

✓ N_i : neighbouring agents

Swarm dynamics \leftrightarrow barycenter motion

$$G(t) = \frac{1}{N} \sum_{k=1}^N r_k(t)$$

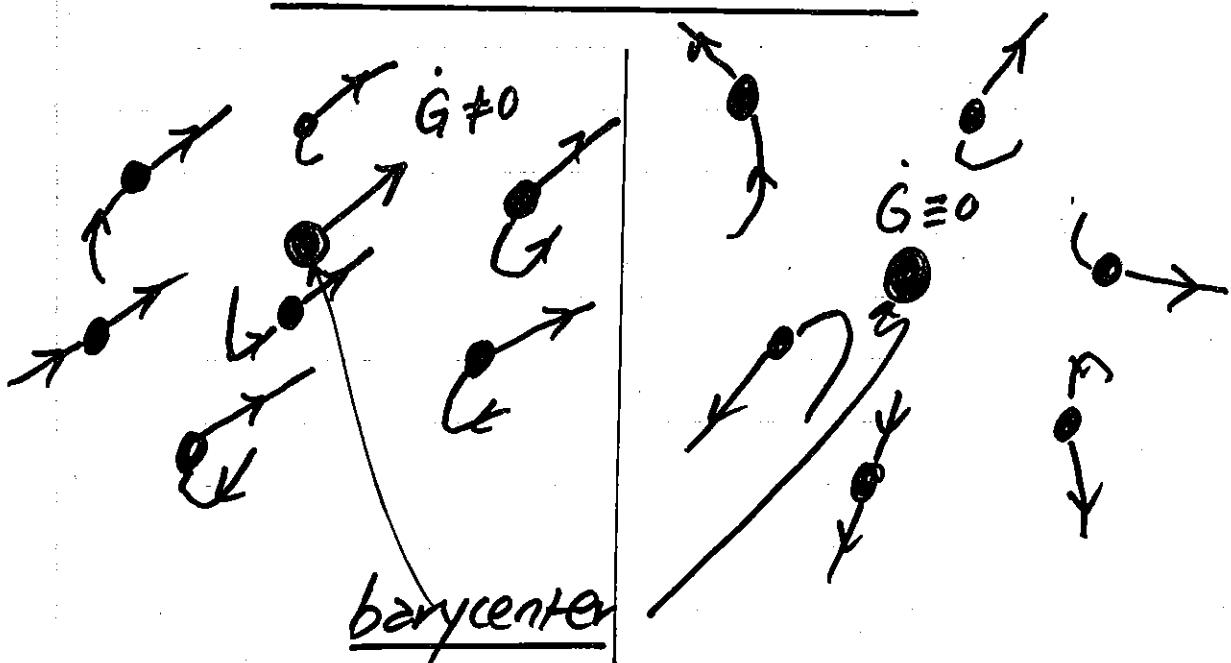
$$\dot{G}(t) = \frac{1}{N} \sum_{k=1}^N \dot{r}_k(t) = \frac{1}{N} \sum_{k=1}^N \ell^{i\theta_k(t)} = R(t) e^{i\psi(t)}$$

order param.
(OP)

$R(t) \in [0, 1]$ (range of the OP)

$R(t) = 1$ fully synchronized st.

$R(t) = 0$ balanced phases \leftrightarrow de-synchronized st.



Synchronized swarm

de-synchronized swarm

Kuramoto dynamics

$$\dot{\theta}_i(t) = \omega_i + (\vec{\theta}(t), \xi_i(t)) = \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) + \sqrt{2D} \xi_i(t)$$

.1) $\xi_i(t)$ WGN (Gaussian)

.2) $\xi_i(t)$ Ornstein-Uhlenbeck (Gaussian)

.3) $\xi_i(t)$ Super-diffusive noise (R. Filliger, P. Blanchard, M-O.H, J. Rodriguez) (2008)
(non-Gaussian)

→ $N \rightarrow \infty$ Mean field regime (MFR)

$$R(t) e^{i\psi(t)} = \int_0^{2\pi} e^{i\theta} \rho(\theta, t) d\theta$$

fraction of oscillators with phase $\theta \in [\theta, \theta + d\theta]$.

Fokker-Planck

$$\frac{\partial \rho(\theta, t)}{\partial t} = - \frac{\partial}{\partial \theta} [V(\theta, t) \rho(\theta, t)] + D \frac{\partial^2}{\partial \theta^2} \rho(\theta, t).$$

. $\rho(\theta, 0) = \rho_0(\theta)$ (initial dist.)

. $V(\theta, t) = K \sin(4\theta - \theta)$ (drift) (MFR)

. $\int_{-\pi}^{\pi} \rho(\theta, t) d\theta \equiv 1$ (normalization)

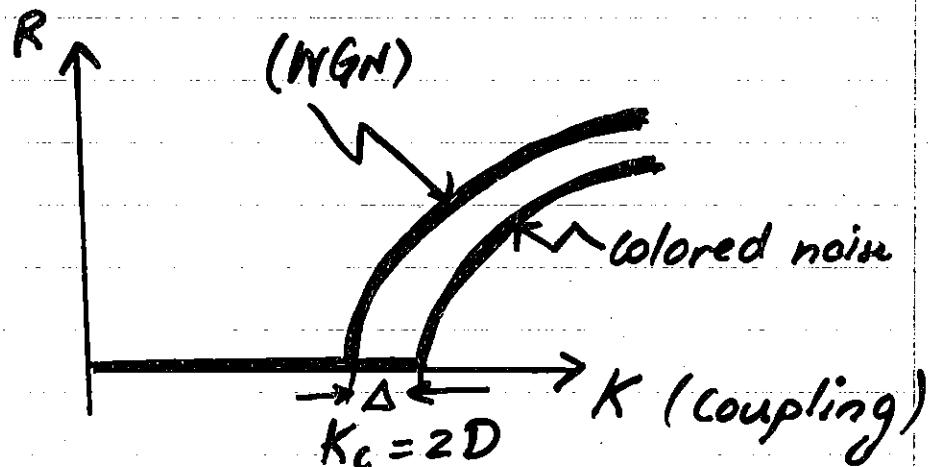
Stationary regime.

$$R_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt$$

$R_{\infty} = 0$, $\rho(\theta) = (2\pi)^{-1}$ (Uniform densit)

Decoherent motion.

2nd-ord. phase transition



$$R_{\infty} = 0 \Rightarrow \dot{G}(t) = 0$$

$R_{\infty} > 0 \Rightarrow \dot{G}(t) > 0$ organized motion.

Note: $R_{\infty} = 1$ (fully organized) occurs only when $D = 0$ (no noise)

ORNSTEIN - UHLENBECK Noise (Rougemont, P.R. 2006)
 $\langle N_{\text{def}} \rangle$

$\Delta \uparrow$ when correlations \uparrow

→ no qualitative change

Super-diffusive noise - (Non Gaussian proc.)

$$\left\{ \begin{array}{l} \dot{\theta}_i = K \sum_{j=1}^N \sin(\theta_j - \theta_i) + \sqrt{2D} \zeta_i(t) \\ \zeta_i(t) = dZ_i(t) = \beta t g h(\beta Z_i(t)) dt + dW_t \end{array} \right.$$

Wiener

Non-Gaussian noise generator

$$P(z_i, t | 00) = e^{-\frac{\beta t}{2}} \cosh(\beta z_i) \frac{e^{-\frac{z_i^2}{2t}}}{\sqrt{2\pi t}}$$

$$\langle Z_i(t) \rangle = 0, \quad \langle Z_i^2(t) \rangle = t + \underbrace{\beta t^2}_{\text{g}}$$

"ballistic" super-diffusion

$$\rightarrow P(z_i, t | 00) = G_+(z_i, t | 00) + G_-(z_i, t | 00),$$

$$G_\pm = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x_\pm - \beta t)^2}{2t}}$$

$$dZ_i(t) = (\beta B) dt + dW_t, \quad Z_i(0)$$

$\uparrow \beta \in \{-1, +1\}$ is a Bernoulli
symmetric variable

Remark : Super-diffusive ballistic noise

State space $\{-1, +1\} \times \mathbb{R}$

$$dx = \beta dt + dW_t$$

$$\rightarrow \mathbb{R}$$

$$dx = -\beta dt + dN_t$$

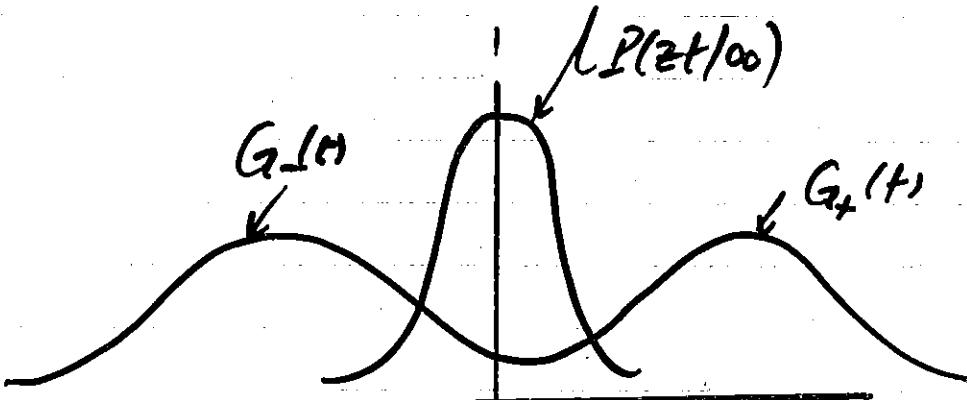
$$\rightarrow \mathbb{R}$$

Markov function
"lumpability"

Rogers & Pitman
(1981) Ann Prob

Benjamini & Lee
(1997)

"Conditioned diffusion
which are Brownian Bridges"



$$P(zt|0)$$

$$G_-(t)$$

$$G_+(t)$$

$$-\log[\cosh x]$$

$$dx = \beta \tanh(\beta x) dt + dW_t$$

Use as a Noise source in SDE.

Ph. Blanchard, R. Filliger & M.C.H.
(Physica 2007)

Effects on Swarm dynamics.

Gaussian noise, (WGN),

$$d\theta_i = \left[\frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \right] dt + \sqrt{2D} dW_i(t)$$

Super-diffusive noise: $d\theta_i = (\dots) dt + \sqrt{2D} dZ_i(t)$

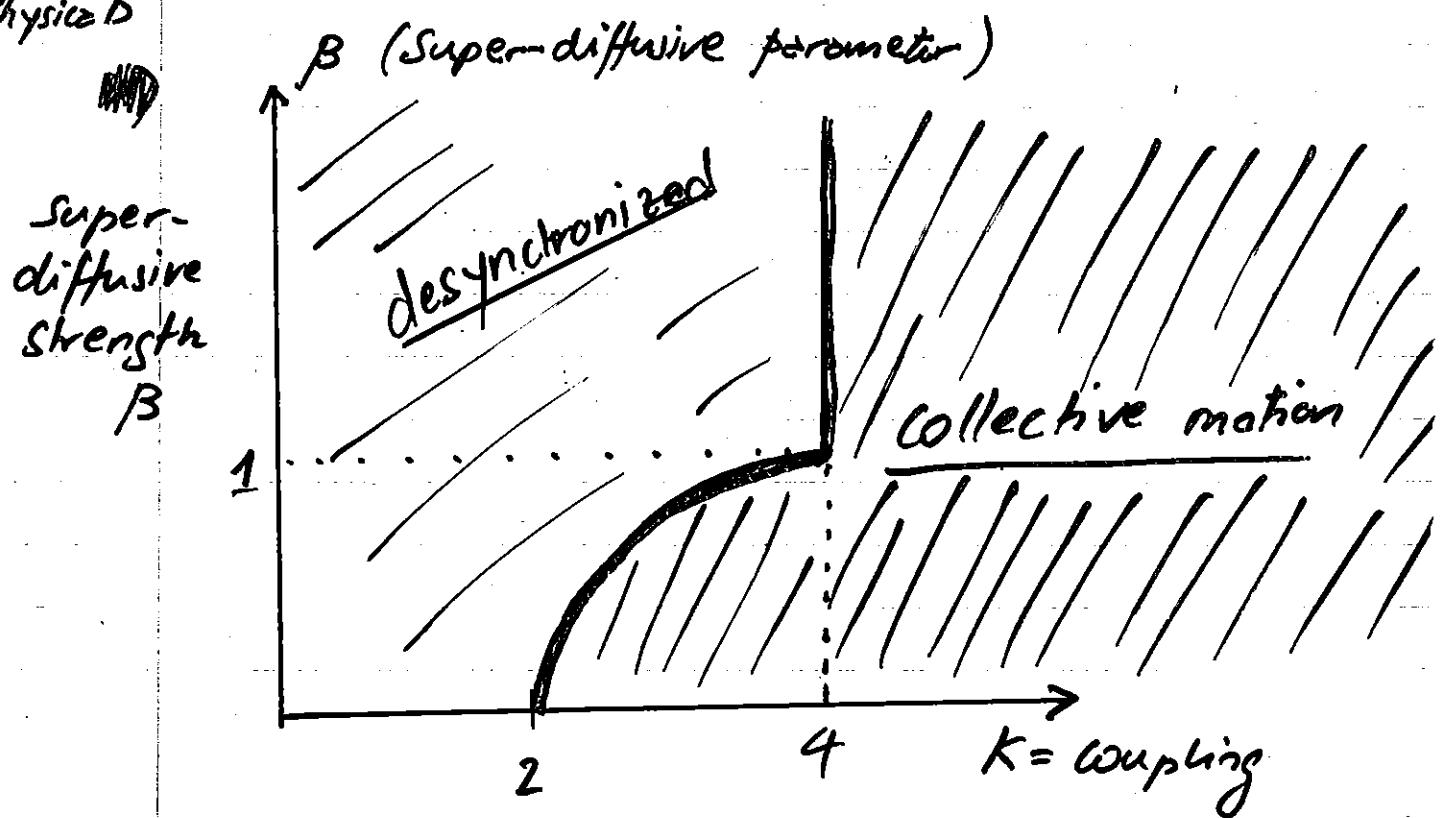
$$d\theta_i = (W_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)) dt + \sqrt{2D} dW_i(t)$$

w_i randomly distributed: law $g(w)$

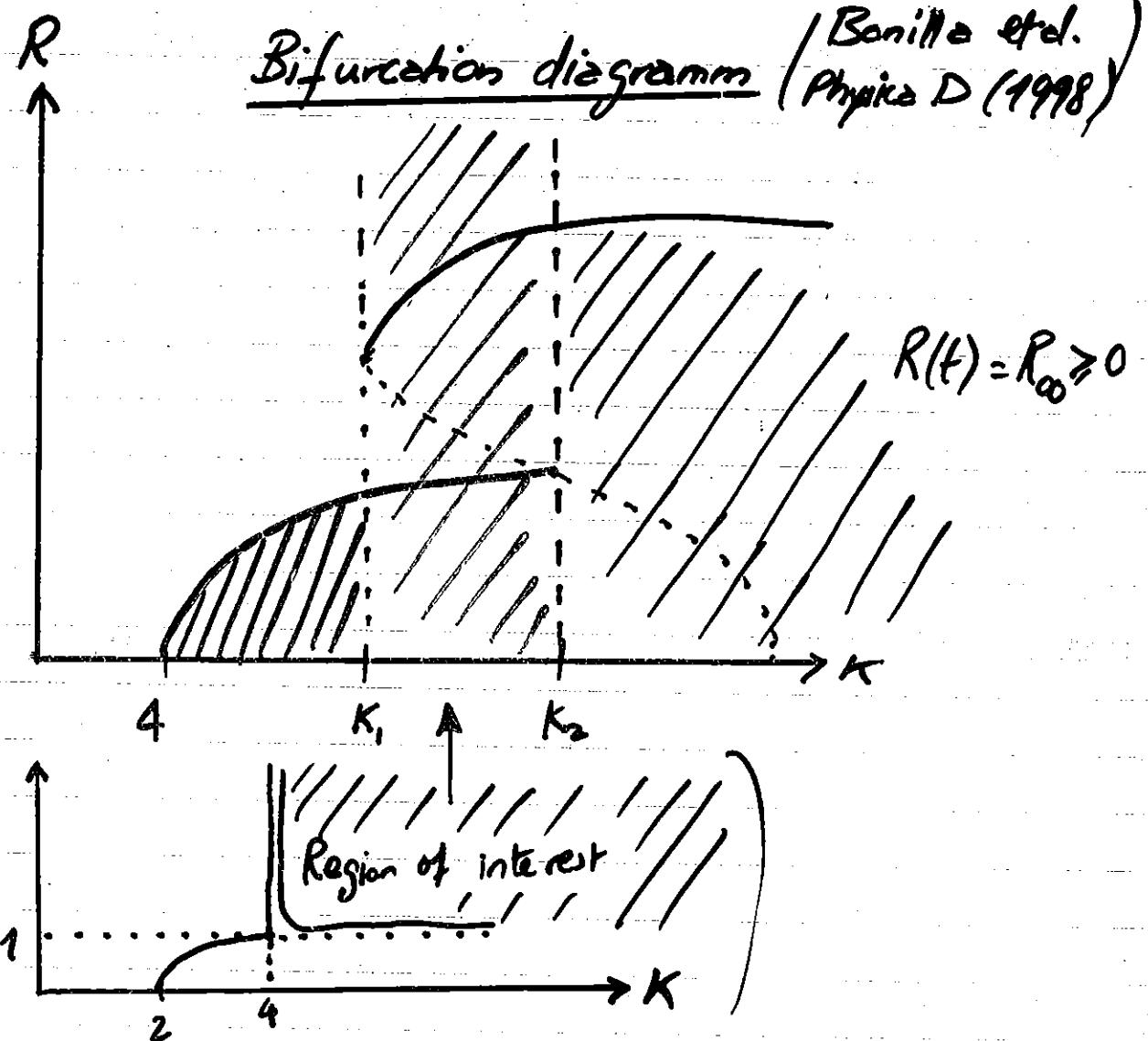
$$\rightarrow g(w) = \frac{1}{2}\delta(w - \beta\sqrt{2D}) + \frac{1}{2}\delta(w + \beta\sqrt{2D})$$

Bonilla
et al. (1998)
Physica D

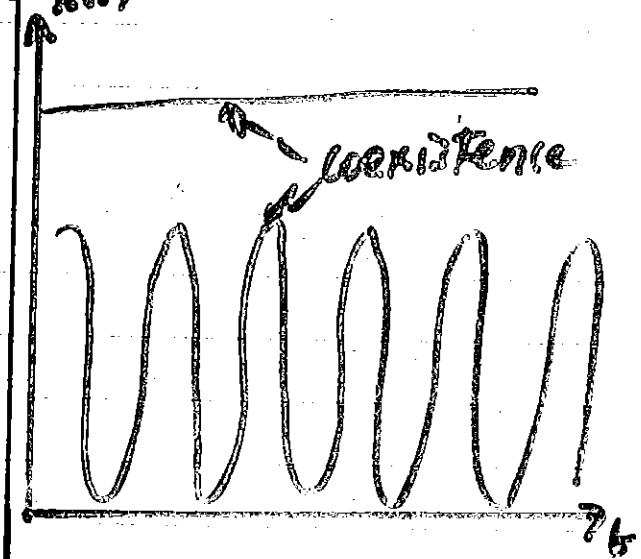
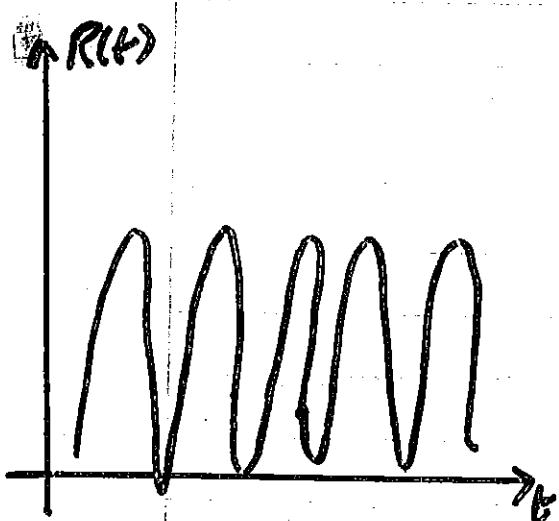
Noise-induced structural modification.



order
param



oscillations of R | Coexistence of 2 regions



\Downarrow

noise-induced oscillating behavior
at the swim center.