Interacting Particle systems in Continuum: Continuous vs. Discrete

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AGENDA

1. Introduction: individual based models in continuum
2. Mathematical framework
3. Glauber dynamics I: spectral gap
4. Glauber dynamics II: non-equilibrium case
5. IBM in spatial ecology
6. IBM in economics
7. Mutation-selection models
8. Ecological processes in random environment
9. Concluding remarks and discussions

MACRO and MICRO models

PDE and SPDE

as macroscopic descriptions coming from
microscopic models via, in particular,

– scaling limits
(e.g., for densities)

– scaling of fluctuations
(normal or abnormal, equilibrium or non-equilibrium)

– closure of (infinite linear) moments systems
(leading to non-linear but finite systems of PDEs)

– hierarchical chains (BBGKY etc.)

– heuristic arguments
(e.g., chemotaxis, porous media, mathematical finance, ...)

MACRO and MICRO models

Some qualitative predictions
based on the use of PDE and SPDE

may be considered as approximations (in a sense)
to possible behaviors of microscopic systems,

which are mathematical caricatures of real world models,
which are .....

"All models are caricatures of reality."

Mark Kac

$\mathrm{MICRO} \Rightarrow \mathrm{MACRO}$?

$\mathrm{MACRO} \Rightarrow \mathrm{MICRO}$?
Inverse problems, calibration, data assimilation, etc.

Complex Systems: BIO sciences

S.Levin (Princeton)
"Complex Adaptive Systems: Exploring the Known, the Unknown and the Unknowable"
Bull. AMS, 2002:
(1) diversity and individuality of components
(2) localized interactions among components
(3) the outcomes of interactions used for replication or enhancement of components

Complex Systems: SOCIO sciences

Philip Ball "Critical Mass" (2004): PHYSICS OF SOCIETY

– Complexity theory seeks to understand how the order and stability arise from the interactions of many agents
– We can make predictions about society even in the face of individual free will, and perhaps even illuminate the limits of that free will
– It is a science of humans collective behavior

Thomas Hobbes, "Leviaphan" (1651):
WE MUST ASK NOT JUST **HOW** THINGS HAPPEN IN SOCIETY, BUT **WHY**.

Statistical Mechanics for CS

R.Gomory:
The central problem is to develop an appropriate statistical mechanics that allows one to separate the knowable unknown from the truly unknowable.
Such mechanics will have to deal with heterogeneous ensembles of interacting agents and with the continual refreshment of that ensemble by novel and unpredictable types.

Statistical Mechanics for CS

The shift from Newtonian determinism to statistical science is what makes a physics of society possible.

Society itself is fundamentally a statistical phenomenon.

Statistical Mechanics for Physics

Equilibrium StatPhys

Non-equilibrium StatPhys

Hamiltonian dynamics

Stochastic dynamics
(e.g., Glauber, Kawasaki, Metropolis, ...)

$$\boxed{(\text{Math. StatPhys}) \subset (\text{ID Analysis})}$$

Interacting Particle Systems

IPS as models in

condensed matter physics

chemical kinetics

population biology, ecology (individual based models=IBM)

sociology, economics (agent based models=ABM)

Lattice (or) (and) (vs.) Continuous

Individual Based Models

R. Law et al., *Ecology*, **84**(2003):
IBM is a stochastic (Markov) process
with events comprising
birth,
death,
and movement.

Ecological models:
Bolker/Pacala, 1997, ...
Dickmann/Law, 2000, ...
....
Birch/Young, 2006
Kondratiev/Srorokhod, 2006
Meleard et al., 2007
Finkelshtein/Kondratiev/Kutovyi, 2007

We will consider

INDIVIDUAL BASED DYNAMICAL COMPETITION MODELS

rather than

COEXISTENCE REGULATION MECHANISMS,
C.F.,
K/Minlos/Zhizhina, '07 (economics)
K/Kuna/Ohlerich, '07 (genetics)

Configuration spaces

X = locally compact Polish space

(e.g., $X = \mathbb{R}^d$ below)

$\sigma(dx)$ intensity measure $(= z\,dx)$

$\Gamma = \Gamma(X) \ni \gamma$, $\gamma \subset X$ locally finite configuration

$F : \Gamma \to \mathbb{R}$ observables

General facts and notations

The configuration space:

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty \text{ for all compact } \Lambda \subset \mathbb{R}^d \right\}.$$

$|\cdot|$ - cardinality of the set.

Vague topology $O(\Gamma)$:
the weakest topology
s.t. all functions

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

are continuous for all $f \in C_0(\mathbb{R}^d)$.

Remark: Γ is a Polish space.

n-point configuration space:

$$\Gamma^{(n)} := \left\{ \eta \subset \mathbb{R}^d \mid |\eta| = n \right\}, \quad n \in \mathbb{N}_0.$$

The space of finite configurations:

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}.$$

Classes of functions: $L^0(\Gamma_0)$: measurable functions on Γ_0,
$L^0_{\mathrm{ls}}(\Gamma_0)$: measurable with local support on Γ_0
$G \in L^0_{\mathrm{ls}}(\Gamma_0) \Leftrightarrow \exists\ \Lambda \in \mathcal{B}_b(\mathbb{R}^d) : G \restriction_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$,
$B_{\mathrm{bs}}(\Gamma_0)$: bounded with bounded support on Γ_0
$G \in B_{\mathrm{bs}}(\Gamma_0) \Leftrightarrow$ bounded $\& \ \exists N \in \mathbb{N}, \exists\ \Lambda \in \mathcal{B}_b(\mathbb{R}^d)$:

$$G \restriction_{\Gamma_0 \setminus \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}} = 0.$$

Cylinder functions on Γ:
$\mathcal{F}L^0(\Gamma)$: $G \in L^0(\Gamma)$, s.t. for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

$$F(\gamma) = F(\gamma_\Lambda).$$

K-transform

Combinatorial Fourier transform (Lenard; Kondratiev/ Kuna):

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi),$$

$$\gamma \in \Gamma, \;\; G \in L^0_{\mathrm{ls}}(\Gamma_0);$$

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi),$$

$$\eta \in \Gamma_0, \; F \in \mathcal{F}L^0(\Gamma).$$

Convolution (Kondratiev/Kuna):

$$(G_1 \star G_2)(\eta) := \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1\cup\xi_2)G_2(\xi_2\cup\xi_3),$$

with property

$$K(G_1 \star G_2) = KG_1 \cdot KG_2,$$
$$G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0).$$

Correlation measure

$$\mathcal{M}^1_{\mathrm{fm}}(\Gamma) \quad = \quad \underline{\textit{probability measures}} \text{ with finite local moments.}$$

$$\mathcal{M}_{\mathrm{lf}}(\Gamma_0) \quad = \quad \underline{\textit{locally finite measures}} \text{ on } \Gamma_0.$$

One can define
$K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$:

$\forall \mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

$$\rho_\mu := K^*\mu$$

is called the **correlation measure**.

Theorem

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be given. For any $G \in L^1(\Gamma_0, \rho_\mu)$ we define

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta),$$

where the later series is μ-a.s. absolutely convergent. Furthermore, we have $KG \in L^1(\Gamma, \mu)$,

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\, \mu(d\gamma).$$

Lebesgue-Poisson measure

$\sigma \quad = \quad$ Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
For any $n \in \mathbb{N}$ measure $\sigma^{\otimes n}$ can be considered on $\widetilde{(\mathbb{R}^d)^n}$.

$\sigma^{(n)} \quad = \quad$ projection on $\Gamma^{(n)}$.
The **Lebesgue-Poisson measure** λ_z, $z > 0$ on Γ_0:

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

The restriction of λ_z to Γ_Λ we also denote by λ_z.

Poisson measure

The **Poisson measure** π_z on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where π_z^Λ is the measure on Γ_Λ defined by

$$\pi_z^\Lambda := e^{-z\sigma(\Lambda)} \lambda_z.$$

Correlation functions

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called

locally absolutely continuous

w.r.t. π_z iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$

is absolutely continuous with respect to $\pi_z^\Lambda = \pi_z \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

In this case $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t λ_z.

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_z}(\eta),\ \eta \in \Gamma_0.$$

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}^+$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) :=$$

$$k_\mu(\{x_1, \ldots, x_n\})$$

correlation functions.

Definition

A measure $\rho \in \mathcal{M}_{lf}(\Gamma_0)$ is called positive definite if

$$\int_{\Gamma_0} (G \star \overline{G})(\eta)\rho(d\eta) \geq 0, \ \ \forall G \in B_{bs}(\Gamma_0),$$

where $\overline{G}$ is a complex conjugate of G. The measure ρ is called normalized iff $\rho(\{\emptyset\}) = 1$.

Theorem (Kondratiev/Kuna)

Let $\rho \in \mathcal{M}_{lf}(\Gamma_0)$ be given. Assume that ρ is positive definite, normalized and that for each bounded open $\Lambda \subset \mathbb{R}^d$, for every $C > 0$ there exists $D_{\Lambda, C} > 0$ s.t.

$$\rho(\Gamma_\Lambda^n) \leq D_{\Lambda, C} C^n, \quad n \in \mathbb{N}_0.$$

Then there exists a unique measure $\mu \in \mathcal{M}_{fm}^1(\Gamma)$ with $\rho = K^\star \mu$.

Remark: A sufficient condition for the bound in the theorem:

$$\rho(\Gamma_\Lambda^{(n)}) \leq (n!)^{-\varepsilon_\Lambda} (C_\Lambda)^n.$$

BAD generators

Birth-and-death processes in continuum

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\setminus x)[F(\gamma\setminus x) - F(\gamma)] +$$

$$\int_X b(x,\gamma)[F(\gamma\cup x) - F(\gamma)]\, dx$$

BIO (ecology) processes are specified by:

$b(x,\gamma) = \sum_{y\in\gamma} B_y(x,\gamma)$

$b(x,\emptyset) = 0$

Existence problem

$$\left.\begin{array}{l}\text{K/Lytvynov}\\ \text{K/Lytvynov/Röckner}\end{array}\right\}\text{Dirichlet forms}$$

$$\left.\begin{array}{l}\text{K/Kutoviy/Zhizhina}\\ \text{K/Kutoviy/Minlos}\end{array}\right\}\text{Glauber dynamics}$$

$$\left.\begin{array}{l}\text{Holley/Stroock}\\ \text{Ferrari/ Garcia}\\ \text{K/Skorokhod}\\ \text{Garcia/Kurtz}\end{array}\right\}\text{particular classes}$$

Correlation equations

(= moment equations = hierarchical equations)

μ_0 = initial distribution

$X_t^{\mu_0} \in \Gamma$ Markov process with initial distribution μ_0

$\mu_t \in \mathcal{M}^1(\Gamma)$ distribution at time $t > 0$

Correlation functions

$f^{(n)}(x_1, \ldots, x_n)$ symmetric function on X^n,
$\gamma = \{x_1, x_2, \ldots\} \subset X$

$$\int_\Gamma \sum_{\{x_{i_1}, \ldots, x_{i_n}\} \subset \gamma} f^{(n)}(x_{i_1}, \ldots, x_{i_n})\, d\mu(\gamma)$$
$$= \frac{1}{n!} \int_{X^n} f^{(n)}(y_1, \ldots, y_n) k_\mu^{(n)}(y_1, \ldots, y_n)\, d\sigma(y_1) \cdots d\sigma(y_n)$$

$\mu \rightleftarrows (k_\mu^{(n)})_{n=0}^{\infty}$
Lenard
K/Kuna
Berezansky/K/Kuna/Lytvynov

Correlation functions dynamics

In components:

$$\frac{\partial k_t^{(n)}}{\partial t} = (L_{CF} k_t)^{(n)}, n \geq 0$$

$$\boxed{\frac{\partial k_t}{\partial t} = L_{CF} k_t}$$

L_{CF}: CF evolution generator

General theory of CF generators for BAD processes
and several particular models: [Finkelstein/K/Oliveira '07]

Compare: BBGKY-hierarchy etc.

Spatial birth-and-death processes

We consider a Markov pre-generator on Γ, the action of which is given by

$$(LF)(\gamma) := (L_{b,d}F)(\gamma) =$$

$$= \sum_{x\in\gamma} d(x,\gamma\setminus x)D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma) D_x^+ F(\gamma)dx,$$

$$D_x^- F(\gamma) = F(\gamma\setminus x) - F(\gamma),$$

$$D_x^+ F(\gamma) = F(\gamma\cup x) - F(\gamma).$$

Our systems are <u>defined</u> via stochastic dynamics.
Symmetrizing measures?

A Gibbs measure

$$\mu \in \mathcal{G}(\beta, z)$$

is reversible w.r.t. the stationary Markov process associated with the generator L in $L^2(\Gamma, \mu)$ (i.e. L is symmetric in $L^2(\Gamma, \mu)$) iff:

$$b(x, \gamma) = ze^{-\beta E(x, \gamma)} d(x, \gamma),$$

(detailed balance condition)

where $E(x, \gamma)$ is the relative energy of interaction between a particle located at x and the configuration γ:

$$E(x, \gamma) := \sum_{y \in \gamma} \phi(x - y)$$

Gibbs measures

A measure $\mu \in \mathcal{M}^1(\Gamma)$ is called a Gibbs measure iff

$$\int_\Gamma \sum_{x\in\gamma} F(\gamma, x)\mu(d\gamma) =$$

$$= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z dx \exp\left[-\beta E(x, \gamma)\right] F(\gamma \cup x, x)$$

for any measurable function

$$F : \Gamma \times \mathbb{R}^d \to [0, +\infty).$$

$\mathcal{G}(z, \beta) =$ all (grand canonical) Gibbs measures.

Glauber dynamics in continuum

Bertini/Cancrini/Cesi '02 (finite volume systems)

$$b(x,\gamma) = ze^{-\beta E(x,\gamma)}, \quad d(x,\gamma) = 1.$$

[Kondratiev/Lytvynov,'03]:
under general conditions on the potential ϕ and the parameters β, z there exists a Markov process on Γ with the stationary measure $\mu \in \mathcal{G}(z,\beta)$.
The corresponding Markov generator L has form:

$$(LF)(\gamma) = \sum_{x\in\gamma} (F(\gamma \setminus x) - F(\gamma))$$

$$+z \int_{\mathbb{R}^d} \exp(-\beta E(x,\gamma))(F(\gamma \cup x) - F(\gamma))dx$$

We will discuss properties of the operator

$$H = -L \geq 0$$

in $L^2(\Gamma, \mu)$ for

$$\mu \in \mathcal{G}(z, \beta)$$

We will say that the potential ϕ satisfies the integrability condition (I), if

$$\forall \beta > 0 \quad C(\beta) := \int |1 - e^{-\beta\phi(x)}| dx < \infty.$$

Theorem (Kondratiev/Lytvynov)

Suppose that the potential $\phi \geq 0$ and satisfies the integrability condition. For any $\mu \in \mathcal{G}(z, \beta)$ the operator H is essentially self-adjoint in $L^2(\Gamma, \mu)$.

Poincaré inequality for Glauber generator

$$(HF, F)_{L^2(\mu)} = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) |(D_x^- F)(\gamma)|^2$$

Theorem (Poincaré Inequality)

Assume additionally

$$\delta := zC(\beta) < \frac{1}{e}.$$

Then for the unique Gibbs measure $\mu \in \mathcal{G}(z, \beta)$ holds

$$(HF, F)_{L^2(\mu)} \geq (1 - \delta) \int_\Gamma (F(\gamma) - < F >_\mu)^2 d\mu(\gamma),$$

for all $F \in D(H)$.

Remark: all known proofs

Bertini/Cancrini/Cesi 2002 (finite volume)

Wu 2003 (finite volume)

Kondratiev/Lytvynov 2003

Kondratiev/Minlos/Zhizhina 2004
(+ one-particle subspaces etc.)

use the condition $\phi \geq 0$.

Definition (IbP formula)

The function $r : X \times \Gamma \longrightarrow \mathbb{R}_+$ which fulfills

$$\int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} f(x,\gamma\setminus x) = \int_X \nu(dx) \int_\Gamma \mu(d\gamma) r(x,\gamma) f(x,\gamma)$$

for all measurable functions $f : X \times \Gamma \to \mathbb{R}_+$ is called Papangelou intensity (PI) of a measure μ on $(\Gamma, \mathcal{B}(\Gamma))$.

Glauber dynamics of continuous particle systems

Consider the Dirichlet form of the Glauber dynamics on $L^2(\Gamma,\mu)$, where

$$\mathcal{E}(F,G) :=$$

$$\int\limits_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)(D_x^- F)(\gamma)(D_x^- G)(\gamma)$$

$$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$$
$$D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma).$$

The bilinear form $\mathcal{E}$ is closable on $L^2(\Gamma, \mu)$ and its closure is a Dirichlet form.
The generator $(-L, D(L))$ of the form $(\mathcal{E}, D(\mathcal{E}))$ is given by

$$(LF)(\gamma) = \sum_{x\in\gamma}(F(\gamma \setminus x) - F(\gamma))$$

$$+ \int_{\mathbb{R}^d} r(x,\gamma)(F(\gamma \cup x) - F(\gamma))dx$$

Theorem

Under mild conditions on Papangelou intensity, there exists a Hunt process

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma})$$

on Γ which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$.
$\mathbf{M}$ is up to μ-equivalence unique. In particular, $\mathbf{M}$ is μ-symmetric and has μ as an invariant measure.

“Carré du champ” and “Carré du champ itéré”

Define the "carré du champ" corresponding to L as

$$\square(F,G) := \frac{1}{2}(L(FG) - FLG - GLF).$$

$$\square(F,G) = \frac{1}{2}\sum_{x\in\gamma} D_x^- F(\gamma) D_x^- G(\gamma) + \frac{1}{2}\int r(x,\gamma) D_x^+ F(\gamma) D_x^+ G(\gamma) dx.$$

Iterating the definition of "carré du champ" introduce “carré du champ itéré” $\square_2$

$$2\square_2(F,F) := L\square(F,F) - 2\square(F,LF).$$

Coercivity identity

Through direct calculations we obtain

$$\mathcal{E}(F,F) = \int_\Gamma \Box(F,F)(\gamma)\mu(d\gamma).$$

$$\int_\Gamma (LF)^2(\gamma)\mu(d\gamma) = \int_\Gamma \Box_2(F,F)(\gamma)\mu(d\gamma)$$

Theorem

$$\int_\Gamma (LF)^2(\gamma)\mu(d\gamma) = \int_\Gamma \Box_2(F,F)(\gamma)\mu(d\gamma)$$

$$= \int_\Gamma \Box(F,F)(\gamma)\mu(d\gamma) + \int_\Gamma \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} \left(D_x^- D_y^- F\right)^2(\gamma)\mu(d\gamma)$$

$$+ \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma) D_y^+ F(\gamma) D_x^+ F(\gamma)\, dy\, dx\, \mu(d\gamma)$$

Sufficient condition for spectral gap

The Poincaré inequality

$$c \int \left(f - \int f d\mu \right)^2 d\mu \leq \mathcal{E}(f, f).$$

The largest possible c gives the spectral gap of the generator H.
Coercivity inequality: for a nonnegative (essentially self-adjoint) operator H (generator of $\mathcal{E}$)

$$\int_\Gamma (HF)^2(\gamma)\mu(d\gamma) \geq c\mathcal{E}(F, F), \quad c > 0$$

If fulfilled, then $(0, c)$ does not belong to the spectrum of H.

Rewriting in terms of the "carré du champ" and $\square_2$

$$\int_\Gamma \square_2(F,F)(\gamma)\mu(d\gamma) \geq c\int_\Gamma \square(F,F)(\gamma)\mu(d\gamma).$$

The following inequality is sufficient:

$$\begin{aligned}(1-c)&\int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)(D_x^+F)^2(\gamma)dx\mu(d\gamma)\\ + &\int_\Gamma\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma)D_x^+r(y,\cdot)(\gamma)D_y^+F(\gamma)D_x^+F(\gamma)dydx\mu(d\gamma) \geq 0\end{aligned}$$

Definition

A (generalized) function $B : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ is called a positive definite kernel if for all $\psi \in C_0^\infty(\mathbb{R}^d)$ holds

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} B(x,y)\psi(x)\overline{\psi(y)}dxdy (= \langle B, \psi \otimes \overline{\psi} \rangle) \geq 0.$$

Theorem

If for μ-a.a. γ the kernel

$$r(x,\gamma)(r(y,\gamma) - r(y,\gamma \cup x)) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y)$$

is positive definite then the coercivity inequality holds for H with constant c.

Coercivity inequality for Gibbs measures

Theorem

Let μ be a Gibbs measure for a pair potential ϕ and activity z. If for each fixed γ the kernel

$$e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}z(1-e^{-\phi(x-y)})+(1-c)\delta(x-y)$$

is positive definite then the coercivity inequality holds for H with constant c.

The condition above with $c=1$ holds if

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1-e^{-\phi(x-y)})\psi(y)\overline{\psi}(x)dxdy \geq 0 \qquad (1)$$

for all $\psi \in C_0(\mathbb{R}^d)$.

Definition

A measurable function $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called positive definite if for all $\psi \in C_0(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x-y)\psi(x)\overline{\psi(y)}dxdy \geq 0.$$

Condition (1) means that $x \mapsto 1 - e^{-\phi(x)}$ is a positive definite function.

Examples

$\phi(x)$	$f(x)$	Parameters
$-\ln(1-e^{-tx^2}\cos(ax)),$	$e^{-tx^2}\cos(ax),$	$t>0, a\in\mathbb{R}$
$-\ln(1-e^{-t\lvert x\rvert}\cos(ax)),$	$e^{-t\lvert x\rvert}\cos(ax),$	$t>0, a\in\mathbb{R}$
$-\ln\left(1-\dfrac{\cos(ax)}{1+\sigma^2x^2}\right),$	$\dfrac{1}{1+\sigma^2x^2}\cos(ax),$	$\sigma>0, a\in\mathbb{R}$

In all examples above one can change $\cos(ax)$ to $\frac{\sin(ax)}{ax}$.

We can also give d-dimensional examples.

Theorem

Let f be a continuous positive definite function which is (R). Define $\phi := -\ln(1-f)$. Then ϕ is (SS) and (R). For every tempered Gibbs measure μ with potential ϕ the generator H fulfills the coercivity inequality for $c = 1$.

Now we consider which properties a potential necessarily has which fulfills condition (1).

Theorem

Let ϕ be a potential which fulfills condition (1) and is (S), (R), and continuous. Then it is of the form $\phi := -\ln(1-f)$ and hence also (SS). Furthermore, ϕ is integrable, itself positive definite in the sense of generalized functions, and

$$\lim_{x\to 0} \frac{\phi(x)}{-2\ln(x)} \le 1.$$

Lemma

Let $f:\mathbb{R}\to[0,1]$, $f\in C^2(\mathbb{R})$, *even function, decreasing and convex on* $\mathbb{R}_+$. *Denote* $\phi(x)=-\ln(1-f(x))$. *Then* $f_\beta=1-e^{-\beta\phi(x)}$ *is also positive definite for all* β *such that* $0\le\beta\le1$.

In the d-dimensional case we can give following examples:
$\phi(x) = -\ln(1 - f(x))$.

$f(x)$	Parameters
$e^{-t\|x\|^2}\cos(a\cdot x)$	$x \in \mathbb{R}^d, t > 0, a \in \mathbb{R}^d$
$e^{-t\|x\|^2}\prod_{j=1}^{d}\frac{\sin(a_j x_j)}{a_j x_j}$	$x \in \mathbb{R}^d, t > 0$
$(\frac{r}{\|x\|})^{n/2} J_{n/2}(r\|x\|)$	$r \geq 0,\ n > 2d-1$
$\frac{2^{n/2}\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \cdot \frac{t}{(\|x\|^2+t^2)^{\frac{n+1}{2}}}$	$t > 0,\ n > d-1$

where $J_{n/2}$ is the Bessel function of the first kind.

Dirichlet forms $\mapsto$ equilibrium Markov processes

Possible initial states:
absolute continuous w.r.t. reversible measures

Non-equilibrium dynamics: general initial states

Problem: how far from equilibrium stochastic dynamics makes sense?

Mathematical formulation: admissible classes of initial measures

References:
K/Kutoviy/Zhizhina, J.Math.Phys., 2006
K/Kutoviy/Minlos, J.Funct.Anal., 2008

General scheme of construction of non-equilibrium process

- Let $\rho : \Gamma_0 \to \mathbb{R}$ be arbitrary positive function. Denote
$$\mathcal{M}^1_\rho(\Gamma) := \left\{\mu \in \mathcal{M}^1(\Gamma) \mid k_\mu \leq const \cdot \rho\right\}.$$
- Let L be a Markov pre-generator defined on some set of functions $\mathcal{F}(\Gamma)$ given on the configuration space Γ.
Comments:
$$\frac{\partial F_t}{\partial t} = LF_t, \quad (KE)$$
$$F_t = e^{tL} F_0 \quad (Markov\, semigroup)$$
$$\frac{\partial \mu_t}{\partial t} = L^\star \mu_t, \quad (DKE = FPE)$$
$$\mu_t = e^{tL^\star} \mu_0 \quad (state\, evolution)$$

- Let

$$\hat{L} := K^{-1}LK$$

be a formal K- *transform* or *symbol* of the operator L (our starting object).

- We consider

$$\hat{L} : D(\hat{L}) \subset \mathcal{L} \to \mathcal{L}$$

in a Banach space

$$\mathcal{L} := L^1(\Gamma_0, \rho\, d\lambda_1) = \bigoplus_{n=0}^{\infty} L^1\left(\Gamma^{(n)},\, \rho^{(n)}\sigma^{(n)}\right).$$

Suppose that domain of this operator is such that it is closed and densely defined in $\mathcal{L}$.

- Suppose that $(\hat{L},\, D(\hat{L}))$ is a generator of a semigroup in $\mathcal{L}$:

$$\hat{L} \to \hat{U}_t,\ _{t\geq 0}$$

- Introducing duality between Banach spaces $\mathcal{L}$ and

$$\mathcal{K}(\rho) := \left\{k : \Gamma_0 \to \mathbb{R} \mid k \cdot \rho^{-1} \in L^{\infty}(\Gamma_0, \lambda_1)\right\} :$$

$$<< G, k >>:=$$

$$= \int_{\Gamma_0} G \cdot k d\lambda_1 = \int_{\Gamma_0} G \cdot \frac{k}{\rho} \cdot \rho d\lambda_1, \; G \in \mathcal{L},$$

  we construct semigroup $\hat{U}_t^{\star}, \; _t \geq 0$ on $\mathcal{K}(\rho)$.

- Suppose that function ρ in the definition of $\mathcal{K}(\rho)$ satisfies Ruelle-type bound. Let $k \in \mathcal{K}(\rho)$ is a correlation function (i.e. the corresponding correlation measure is normalized, positive definite) and let

$$k_t := \hat{U}_t^{\star} k, \; t \geq 0$$

  denotes an evolution of function k.

- Assume that for any $t \geq 0$, $k_t \in \mathcal{K}(\rho)$ is positive definite, normalized function.

By the main CF theorem one can easily construct a semigroup on $\mathcal{M}_\rho^1$:

$$k \quad \rightarrow \quad \mu,$$

$$k_t = \hat{U}_t^\star k \quad \rightarrow \quad U_t^\star \mu, \;\; t \geq 0,$$

$$\mu_t = U_t^\star \mu \in \mathcal{M}_\rho^1.$$

The existence of semigroup $U_t^\star, \; _{t\geq 0}$ on $\mathcal{M}_\rho^1$ implies the existence of process $(X_t^\mu)_{t\geq 0}$ associated with generator L for any initial distribution $\mu \in \mathcal{M}_\rho^1$.

Spatial birth-and-death processes

We consider a Markov pre-generator on Γ, the action of which is given by

$$(LF)(\gamma) := (L_{b,d}F)(\gamma) =$$

$$= \sum_{x\in\gamma} d(x, \gamma \setminus x) D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x, \gamma) D_x^+ F(\gamma) dx,$$

$$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma),$$

$$D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma).$$

It is known that a Gibbs measure

$$\mu \in \mathcal{G}(\beta, z)$$

is reversible w.r.t. the stationary Markov process associated with the generator L in $L^2(\Gamma, \mu)$ (i.e. L is symmetric in $L^2(\Gamma, \mu)$) iff:

$$b(x, \gamma) = ze^{-\beta E(x, \gamma)} d(x, \gamma),$$

where $E(x, \gamma)$ is the relative energy of interaction between a particle located at x and the configuration γ:

$$E(x, \gamma) := \sum_{y \in \gamma} \phi(x - y)$$

Gibbs measures

A measure $\mu \in \mathcal{M}^1(\Gamma)$ is called a Gibbs measure iff

$$\int_\Gamma \sum_{x\in\gamma} F(\gamma, x)\mu(d\gamma) =$$

$$= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z dx \exp\left[-\beta E(x, \gamma)\right] F(\gamma \cup x, x)$$

for any measurable function

$$F : \Gamma \times \mathbb{R}^d \to [0, +\infty).$$

$\mathcal{G}(z, \beta) \quad = \quad$ all Gibbs measures.

Glauber dynamics $(G_{\pm})$

Consider as above the case

$$b(x,\gamma) = ze^{-\beta E(x,\gamma)}, \quad d(x,\gamma) = 1.$$

The corresponding generator: $L^{+} := L$.

Glauber dynamics (G_{-}):

$$b(x,\gamma) = z, \quad d(x,\gamma) = e^{\beta E(x,\gamma)}.$$

The corresponding generator: L^{-}.

The symbol of Glauber generator on the space of finite configurations

Potential: $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ Borel, even function.
(I) Integrability: $\forall\, \beta > 0$

$$c(\beta) := \int_{\mathbb{R}^d} |1 - e^{-\beta\phi(x)}| dx < \infty,$$

(P) Positivity: $\phi(x) \geq 0$ for all $x \in \mathbb{R}^d$.
The image of L under the K-transform has the following form:

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta) = -|\eta|G(\eta) +$$

$$+z \sum_{\xi \subseteq \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \prod_{y \in \eta \setminus \xi} (e^{-\beta\phi(x-y)} - 1) e^{-\beta E(x,\,\xi)} dx =$$

$$= (L_0 G)(\eta) + z(L_1 G)(\eta), \;\; G \in B_{bs}(\Gamma_0).$$

Non-equilibrium dynamics (G_+). Construction of a semigroup.

$\lambda \quad = \quad$ Lebesgue-Poisson measure on Γ_0 with $z = 1$.
For fixed $C > 0$ and $\beta > 0$, we consider operator $\widehat{L}$ in

$$\mathcal{L}_{C,\,\beta} := L^1(\Gamma_0, C^{|\eta|}e^{-\beta E(\eta)}d\lambda(\eta)).$$

Let $\kappa := z > 0$ be the parameter of the considering model. Then

$$(\widehat{L}G)(\eta) = (\widehat{L_\kappa}G)(\eta) = (L_0G)(\eta) + \kappa(L_1G)(\eta),$$

$$G \in D(L_1) = D(L_0) =$$
$$= \left\{G \in \mathcal{L}_{C,\,\beta} \,\middle|\, |\eta|G(\eta) \in \mathcal{L}_{C,\,\beta}\right\}.$$

Theorem

For any $C > 0$, and for all κ, $\beta > 0$ which satisfy

$$\kappa \exp\left(C(\beta)C\right) < C, \tag{2}$$

the operator $\widehat{L_\kappa}$ is a generator of a holomorphic semigroup in $\mathcal{L}_{C,\beta}$.

Construction of non-equilibrium Markov process

Fix any triple C, κ and β: $\kappa \exp\left(C(\beta)C\right) < C$.

$\widehat{U_t}(C, \kappa, \beta)$ be holomorphic semigroup generated by $\widehat{L_\kappa}$

$$\mathcal{K}_{C,\,\beta} := \left\{ k : \Gamma_0 \to \mathbb{R}_+ \mid k(\cdot)\, C^{-|\cdot|} e^{\beta E(\cdot)} \in L^\infty(\Gamma_0, \lambda) \right\}$$

the space for possible correlation functions.

Duality between quasi-observables $G \in \mathcal{L}_{C,\,\beta}$ *and functions* $k \in \mathcal{K}_{C,\,\beta}$:

$$\langle\langle G,\, k \rangle\rangle := \langle G,\, k \rangle_{L^2(\Gamma_0,\, \lambda)}\,.$$

Remark: 1. $\langle G,\, k\rangle_{L^2(\Gamma_0,\,\lambda)} =$

$$= \int_{\Gamma_0} G(\eta) C^{|\eta|} e^{-\beta E(\eta)} k(\eta) C^{-|\eta|} e^{\beta E(\eta)} d\lambda(\eta) < \infty.$$

2. $k(\cdot)\, C^{-|\cdot|} e^{\beta E(\cdot)} \in L^\infty(\Gamma_0, \lambda)$ implies

$$k(\eta) \leq const\, C^{|\eta|} e^{-\beta E(\eta)}.$$

Duality determines semigroup on $\mathcal{K}_{C,\,\beta}$:

$$\widehat{U_t}(C, \kappa, \beta) \longleftrightarrow \widehat{U_t^\star}(C, \kappa, \beta).$$

Lemma

Let positive constants C, κ and β which satisfy

$$\kappa \exp\left(C(\beta)C\right) < C$$

be arbitrary and fixed. The semigroup $\widehat{U_t^\star}(C, \kappa, \beta)$ on $\mathcal{K}_{C,\,\beta}$ preserves positive definiteness, i.e.

$$\left\langle\left\langle G \star G,\, \widehat{U_t^\star}(C, \kappa, \beta) k \right\rangle\right\rangle \geq 0, \qquad \forall\, G \in B_{bs}(\Gamma_0)$$

if

$$\langle\langle G \star G,\, k \rangle\rangle \geq 0,$$

for any $G \in B_{bs}(\Gamma_0)$.

Let $\mathcal{M}_{C,\,\beta}$ - the set of all probability measures on Γ, locally absolutely continuous with respect to Poisson measure, with locally finite moments, whose correlation functions satisfy bound

$$k(\eta) \leq const\, C^{|\eta|} e^{-\beta E(\eta)}.$$

Theorem

Suppose that conditions (I) and (P) are satisfied. For any triple of positive constants C, κ and β which satisfy

$$\kappa \exp\left(C(\beta)C\right) < C$$

and any $\mu \in \mathcal{M}_{C,\,\beta}$ there exists Markov process $X_t^\mu \in \Gamma$ with initial distribution μ associated with generator L_κ.

Glauber dynamics G_-

$$b(x,\gamma) = z, \quad d(x,\gamma) = e^{\beta E(x,\gamma)}.$$

For arbitrary and fixed $C > 0$ we consider $\hat{L}^-$ in

$$\mathcal{L}_C := L_1(\Gamma_0, C^{|\eta|} d\lambda(\eta)).$$

For the potential ϕ assume:

(S) Stability: $\exists B \geq 0$, s.t. $\forall \eta \in \Gamma_0$

$$E(\eta) := \sum_{\{x,y\}\subset\eta} \phi(x-y) \geq -B|\eta|,$$

(SI) Strong Integrability:

$$C_{st}(\beta) := \int_{\mathbb{R}^d} |1 - e^{\beta\phi(x)}| dx < \infty$$

Theorem

For any $C > 0$, and for all κ, $\beta > 0$ which satisfy

$$e^{C_{st}(\beta)} + \kappa e^{2B\beta} C^{-1} < 2$$

the operator $\widehat{L^-}$ is a generator of a holomorphic semigroup in $\mathcal{L}_C$.

Stochastic growth

BIO: independent growth (plants)

Dispersion kernel:

$a^+(x-y)\,dx$
$0 \le a^+ \in L^1(\mathbb{R}^d)$ even.

Generator:

$$(L_{IG}F)(\gamma) = \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$$

Stochastic growth

SOCIO-ECO: free development=independent birth
(creation by an outer free will)

$$(L_{IB}F)(\gamma) = z \int_{\mathbb{R}^d} (F(\gamma \cup x) - F(\gamma))dx$$

GENETICS: generalized mutation models

Evans/Steinsalz/Wichtner, 2005

Density of population

Independent Growth: $k_t(x) \sim Ce^{\lambda t}$, $t \to \infty$

Independent Birth: $k_t(x) \sim Ct, t \to \infty$

$$\boxed{Free\ \ Evolution\ \ Models}$$

Problem: to analyze stochastic evolution models in the presence of global regulations and local competitions

IG with mortality = Contact Model

Generator:

$$(LF)(\gamma) = \sum_{x\in\gamma} m[F(\gamma\setminus x) - F(\gamma)] +$$
$$\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^{+}(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$$

$m =$ global mortality intensity

Existence of Markov process:

[K/Skorokhod, 06] finite range a^{+}

[Finkelstein/K/Skorokhod, 07] general case a^{+}

Correlation equations for CM

[K/Kutoviy/Pirogov, 07]

What can happen?

Take translation-invariant initial condition

$k_t(x) \equiv \rho_t$.
$\frac{\partial \rho_t}{\partial t} = -(m - \langle a^+ \rangle)\rho_t$
$m > \langle a^+ \rangle \Rightarrow \rho_t \to 0$
$m < \langle a^+ \rangle \Rightarrow \rho_t \to +\infty$
$m = \langle a^+ \rangle \Rightarrow \rho_t \equiv \rho_0$
$m = \langle a^+ \rangle$ critical value of mortality

Possible invariant state: for $m = \langle a^+ \rangle$
$\boxed{m = m_{\text{cr}}}$

CF time evolution

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) =$$

$$-nmk_t^{(n)}(x_1,\ldots,x_n)+$$

$$\sum_{i=1}^{n}\Big[\sum_{j\neq i} a^+(x_i - x_j)\Big]k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n)+$$

$$\sum_{i=1}^{n}\int_{\mathbb{R}^d} a^+(x_i - y)k_t^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)\,dy$$

Theorem

Let $k_0^{(n)} \leq C^n n!$, $n \geq 0$.
Then $k_t^{(n)}(x_1, \ldots, x_n) \leq A^n (C+t)^n e^{n(<a^+>-m)t} n!$.

Remark:

actually for a Poisson initial state and $x_1, ..., x_n$ inside a small ball

$$k_t^{(n)}(x_1, \ldots, x_n) \simeq C_t^n n!, \ \ t > 0, n \to \infty,$$

that means

STRONG CLUSTERING!

Invariant measures

Let $d \geq 3$.

Stationary equation:

$$\frac{\partial k_t}{\partial t} = 0$$

Theorem

Assume a^+*:* $\int_{\mathbb{R}^d} |x|^2 a^+(x)\, dx < \infty$.

$\forall \rho > 0\ \exists!$ *solution* $(k^{(n),\rho})_{n=0}^{\infty}$ *corresponding to a measure* $\mu^\rho \in \mathcal{M}^1(\Gamma)$ *with* $k^{(1),\rho}(x) = \rho$.

We have $k^{(n),\rho}(x) \leq C(\rho)^n (n!)^2$, $n \geq 1$.

For $d \leq 2$ μ^ρ does not exist!

The point:

$$\int_{|p|\leq 1} \frac{dp}{\tilde{a}(0) - \tilde{a}(p)} < \infty$$

necessary condition for the existence of $k^{(2),\rho}(x,y)$.

CM + Kawasaki dynamics (plankton model)

K/Kutoviy/Struckmeier '08
Equilibrium state for CM needs $d \geq 3$.
$d = 2$?
Consider a CM with a motion of individuals:
take into account different time scales
Generator in the bio-time scale:

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma} \int_{\mathbb{R}^d} \varkappa(x-y)[F(\gamma \setminus x \cup y) - F(\gamma)]\, dy + (L_{\mathsf{CM}}F)(\gamma) \\ &= ((L_{\mathsf{K}} + L_{\mathsf{CM}})F)(\gamma)\end{aligned}$$

$$\boxed{L = L_{\mathsf{K}}(\varkappa) + L_{\mathsf{CM}}(m, a^+)}$$

Critical value $m = \langle a^+ \rangle$.

Assume long tail jumps:

$$\int_{|p|\leq 1} \frac{1}{1-\tilde{\varkappa}(p)}\, dp < \infty$$

For example:

$\tilde{\varkappa}(p) = e^{-|p|^\alpha}$,

$1 \leq \alpha < 2$ (stable distributions)

Theorem

$\forall \rho > 0\ \exists!$ *invariant measure* μ^ρ *for CM with Kawasaki dynamics.*

INTERPRETATION: super-diffusive stochastic dynamics of individuals in bio-time scale.

Independent birth with mortality

Surgailis process:

$(LF)(\gamma) = m\sum_{x\in\gamma}(F(\gamma\setminus x) - F(\gamma)) +$
$z\int_{\mathbb{R}^d}(F(\gamma\cup x) - F(\gamma))dx$

Unique invariant measure is the Poisson measure on Γ with intensity

$$\frac{z}{m}$$

ANY non-zero mortality stabilizes the system!

Bolker-Pacala model

(= CM + density dependent mortality)

$$L = L_{CM} + L(a^-)$$

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(m + \sum_{x'\in\gamma\setminus x} a^-(x-x')\right)[F(\gamma\setminus x) - F(\gamma)]$$

$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup x) - F(\gamma)]dx$$

$a^+(x-y)$ – dispersion kernel
$a^-(x-y)$ – competition kernel
$0 \le a^\pm \in L^1(\mathbb{R}^d)$

CF evolution

[Finkelshtein/K/Kutoviy, 2007]
Generator on correlation functions:

$$(L_{CF}k)(\eta) = -k(\eta)\left[m|\eta| + \varkappa^- E^{a^-}(\eta)\right] -$$

$$\varkappa^- \int_{\mathbb{R}^d} \sum_{y\in\eta} a^-(x-y)k(\eta\cup x)dx +$$

$$+\varkappa^+ \sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in\eta\setminus x} a^+(x-y) +$$

$$\varkappa^+ \int_{\mathbb{R}^d} \sum_{x\in\eta} a^+(x-y)k(\eta\setminus x\cup y)dy.$$

Ecological problem

May the competition mechanism lead to

THE "REGULARLY DISTRIBUTED" IN SPACE POPULATION

with bounded in time density?

Necessary condition: big enough mortality m
(that follows from an accretivity assumption)

Competition assumptions (CA):

$$\exists C > 0 : \ a^{+}(x) \leq C a^{-}(x)$$

$$m > C\langle a^{-}\rangle + \langle a^{+}\rangle$$

that means strong enough competition and mortality

Theorem

Let (CA) fulfilled. For any initial CF s.t.

$$k_0^{(n)} \leq C^n, n \in \mathbb{N}$$

there exists the unique solution of the CF equation $k_t^{(n)}, t \geq 0, n \in \mathbb{N}$, *satisfying the same bound*

This gives a sub-Poissonian bound,

i.e., the strong enough competition

destroys the clustering in CM!

CM with establishment

[Finkelshtein/K. 2007]

$a^+(x-y) \;\longrightarrow\; a^+(x-y)e^{-E^\phi(x,\gamma)}$

$e^{-E^\phi(x,\gamma)} = e(x,\gamma)$ establishment rate

$E^\phi(x,\gamma) = \sum_{x'\in\gamma} \phi(x-x')$

competition for survival

Bolker-Pacala establishment rate

$$e(x,\gamma) = \frac{1}{1+E^\phi(x,\gamma)}$$

Generator

$$(LF)(\gamma) =$$

$$\sum_{x\in\gamma} m[F(\gamma \setminus x) - F(\gamma)] +$$

$$\sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y)e(x,\gamma)[F(\gamma \cup x) - F(\gamma)]dx$$

Stability condition: $m > 0$, $\exists \beta > 0 : a^+(x) \leq \beta\phi(x)$

$L = L(m, a^+, \phi)$

Theorem

Under stability condition there exists a unique Markov process on Γ for $L(m, a^+, \phi)$ (for certain class of initial configurations).

Proof is based on a delicate use of

a [Garcia/Kurtz, 2006] existence result

Correlation functions in CM with establishment

Time evolution of CF

Theorem (K/Finkelstein '07)

Assume stability condition is true and initial CFs satisfy a sub-Poisson bound $k_0^{(n)} \leq C^n, n \geq 1.$
Then there exists $C_1 > 0$ *s.t. for the time dependent CFs holds*

$$k_t^{(n)} \leq C_1^n, n \geq 1.$$

Remark:
The establishment mechanism is more effective comparing with density dependent mortality:

we do not need to have a big enough mortality to organize control over time-space behavior of the population

Density dependent fecundity

$$b(x,\gamma) = \sum_{y\in\gamma} a^{+}(x-y) f(y, \gamma \setminus y)$$

,

$$f(y,\gamma \setminus y) = e^{-E^{\phi}(y,\gamma\setminus y)}$$

Open problem:
produces the fecundity an effective regulation of the population?

Dieckmann-Law model

[Finkelshtein/Kondratiev, '08] (in preparation)

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(m + \sum_{x'\in\gamma\setminus x} a^{-}(x-x')\right)[F(\gamma\setminus x) - F(\gamma)]$$

$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^{+}(x-y)\left(1+\sum_{y'\in\gamma\setminus y} b(y-y')\right)[F(\gamma\cup x) - F(\gamma)]dx$$

Dieckmann-Law model without competition

i)DLM without local competition($a^- \equiv 0$).

Assume existence of corresponding MP $X_t \in \Gamma$.

For $\Lambda \subset \mathbb{R}^d$ bounded put

$$n_t^\Lambda = \mathbb{E}(|X_t \cap \Lambda|)$$

Then $\exists t_0$:

$$n_t^\Lambda \to +\infty$$

$t \to t_0-$ (explosion).

Dieckmann-Law model: non-explosion

ii) DLM: non-explosion via competition $d \geq 2$
Construction of a Lyapunov functional
(cf. [K/Skorokhod, '06]

$$e_\delta(x) = \frac{1}{1+|x|^\delta}, \;\; \delta > d$$

$$\Psi_\delta(x,y) = e_\delta(x)e_\delta(y)\frac{|x-y|+1}{|x-y|}1\!\!1_{\{x\neq y\}}$$

$$\mathbb{L}_\delta(\gamma) = \langle e_\delta, \gamma\rangle$$

$$\mathbb{E}_\delta(\gamma) = \sum_{\{x,y\}\subset\gamma} \Psi_\delta(x,y)$$

$$\mathbb{V}_\delta(\gamma) = \mathbb{L}_\delta + \mathbb{E}_\delta$$

Assume

$$a^+(x) \le \frac{A}{(1+|x|)^{2\delta}}$$

Introduce $A_\delta = AC(\delta, d) > 0$

Theorem

Assume

$$a^-(x) \ge 2A_\delta b(x)$$

$\Rightarrow \exists C > 0:$

$$L\mathbb{V}_\delta(\gamma) \le C\mathbb{V}_\delta(\gamma).$$

If, additionally, $m \ge 2A_\delta\langle a^+\rangle \Rightarrow$

$$L\mathbb{V}_\delta(\gamma) \le 0$$

Independent development with competition

$b(x,\gamma) = z > 0$ (independent birth with constant intensity)

(i) $d(x,\gamma\setminus x) = e^{\beta E(x,\gamma\setminus x)} \Longrightarrow$ generator L^-

$E(x,\gamma\setminus x) = \sum_{y\in\gamma\setminus x} \phi(x-y)$

(ii) $d(x,\gamma\setminus x) = m + \sum_{y\in\gamma\setminus x} a^-(x-y) \Longrightarrow$ generator L_m

(iii) $d(x,\gamma\setminus x) = \sum_{y\in\gamma\setminus x} a^-(x-y) \Longrightarrow$ generator L_0

(i) Glauber dynamics G^-

Symmetrizing measures for L^- are

grand canonical Gibbs measures $\mathcal{G}_{gc}(\beta, \phi, z)$

$L^- \longmapsto X_t \in \Gamma$ equilibrium Glauber dynamics via the Dirichlet form method [K/Lytvynov, '05]

Non-equilibrium GD:

[K/Kutovyi/Minlos,'07]

via CF evolution (see Lecture 2).

(ii) Surgailis process with competition

$$(L_m F)(\gamma) = \sum_{x\in\gamma}(m + \sum_{y\in\gamma\setminus x} a^-(x-y))(F(\gamma\setminus x) - F(\gamma)) + z\int (F(\gamma\cup x) - F(\gamma))dx$$

Here we have a sub-Poissonian evolution of CFs.

(iii) Stabilization via competition

[Finkelstein/K, 2007]

Consider (ii) with $\boxed{\text{m=0}}$
pure competition mechanism (i.e., without global regulation)

$$(L_0F)(\gamma) = \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} a^-(x-y)(F(\gamma\setminus x)-F(\gamma)) + z\int(F(\gamma\cup x)-F(\gamma))dx$$

Hierarchical equations for CF

do not give any a priori information

about the density of the system.

Mean density bound

Let $X_t \in \Gamma$ will be MP corresponding to L_0 and an initial measure with bounded density.

Let $\Lambda = B(0, R)$ be a ball with center at origin and radius $R > 0$. Introduce the mean density

$$\rho_t^{\Lambda} := \frac{1}{|\Lambda|} \int_{\Lambda} k_t^{(1)}(x)\, dx.$$

Our aim is to produce a time bound for this density and any such volume Λ.

Theorem

Let a^- be a continuous function of positive type on $\mathbb{R}^d$.

Then the mean density

$$\rho_t^\Lambda := \frac{1}{|\Lambda|}\int_\Lambda k_t^{(1)}(x)\,dx$$

is uniformly bounded in $t > 0$ and Λ.

Comments

We use the explicit form of the generator instead of CF equation.
Denote $\gamma_\Lambda = \gamma \cap \Lambda$.
Then

$$L_0|\gamma_\Lambda| \le -\frac{C}{|\Lambda|}|\gamma_\Lambda|^2 + z|\Lambda|$$

That gives easily for $\rho_t = \rho_t^\Lambda$

$$\frac{d}{dt}\rho_t \le z - C\rho_t^2.$$

The proof follows from an analysis of this differential inequality.

General competition models in economics

$$b(x,\gamma) \equiv \lambda > 0.$$

Then

$$(LF)(\gamma) =$$

$$\sum_{x\in\gamma} d(x,\gamma\setminus x)\big[F(\gamma\setminus x) - F(\gamma)\big]$$

$$+\lambda \int_{\mathbb{R}^d} \big[F(\gamma\cup x) - F(\gamma)\big]dx.$$

Examples

1. Surgalis process

$$d(x, \gamma \setminus x) \equiv m > 0.$$

Invariant measure: Poisson measure with intensity $\frac{\lambda}{m}$.

2. G^--dynamics

$$d(x, \gamma \setminus x) = \exp\{\sum_{y \in \gamma \setminus x} \phi(x-y)\}$$

Invariant measure: Gibbs measure, heuristically given by

$$d\mu_\phi(\gamma) = \text{"}\frac{1}{Z_\phi} \exp\{-\sum_{\{x,y\} \subset \gamma} \phi(x-y)\} d\pi_\lambda\text{"}.$$

Examples

3. "Linear part of G^{-}-dynamic"

$$d(x, \gamma \setminus x) = m + \sum_{y \in \gamma \setminus x} c(x-y), \quad m > 0, 0 \leq c \in L^1(\mathbb{R}^d)$$

$$e^x = 1 + x + \dots$$

Examples

4. "Pure local competition model"

$$d(x, \gamma \setminus x) = \sum_{y \in \gamma \setminus x} c(x-y), \quad 0 \le c \in L^1(\mathbb{R}^d).$$

Hence,

$$(L_S F)(\gamma) = \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} c(x-y) \big[F(\gamma \setminus x) - F(\gamma) \big] + \lambda \int_{\mathbb{R}^d} \big[F(\gamma \cup x) - F(\gamma) \big] dx. \quad (3)$$

Boundness of a density

Suppose that process X_t exists. Define the first correlation function via

$$\mathbb{E}\Big[\sum_{x\in X_t(\cdot)}\varphi(x)\Big]=\int_{\mathbb{R}^d}\varphi(x)k_t^{(1)}(x)dx.$$

Theorem

Let $c\in L^1(\mathbb{R}^d)$ is a continuous function of positive type on $\mathbb{R}^d$. Then the function

$$\frac{1}{|\Lambda|}\int_\Lambda k_t^{(1)}(x)dx$$

is uniformly bounded by t and Λ if its initial value is uniformly bounded by Λ.

Proof

Let $\Lambda = B(0, R)$ be a ball with center at origin and radius $R > 0$. Let $h > 0$ and $\Lambda^h = \{x : \inf_{y\in\Lambda} |x-y| < h\}$. Then for $h < R$

$$\sigma(\Lambda, h) := \frac{\left|\Lambda^h \setminus \Lambda\right|}{|\Lambda|} = \frac{\left|\Lambda^h\right|}{|\Lambda|} - 1 = \frac{(R+h)^d}{R^d} - 1$$
$$= \left(1 + \frac{h}{R}\right)^d - 1 < 2^d - 1$$

By [Lewis et al.] for any $\eta := \{x_k\}_{k=1}^n \subset \Lambda$

$$\sum_{k\neq j} c(x_k - x_j) = 2E_c(\eta) \geq \frac{n^2}{|\Lambda|} \frac{[\langle c\rangle - \delta(h)]^2}{[\langle c\rangle + \delta(h) + \sigma(\Lambda, h)\langle c\rangle]},$$

where

$$\delta(h) = 2\int_{|x|>h} c(x)\, dx.$$

Proof

Hence,

$$\begin{aligned} 2E_c(\eta) &\geq \frac{n^2}{|\Lambda|}\frac{\left[\langle c\rangle - \delta(h)\right]^2}{\left[\langle c\rangle + \delta(h) + (2^d-1)\langle c\rangle\right]} \\ &= \frac{n^2}{|\Lambda|}\frac{\left[\langle c\rangle - \delta(h)\right]^2}{\left[2^d\langle c\rangle + \delta(h)\right]}. \end{aligned}$$

Set

$$C = C(h,d) := \frac{\left[\langle c\rangle - \delta(h)\right]^2}{2^d\langle c\rangle + \delta(h)} > 0$$

if only

$$\langle c\rangle - \delta(h) = \int_{|x|\leq h} c(x)\,dx - \int_{|x|>h} c(x)\,dx \neq 0.$$

Proof

Note that for $F(\gamma) = \sum_{x\in\gamma}\varphi(x)$

$$(L_S F)(\gamma) = -\sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x} c(x-y)\right)\varphi(x) + \lambda\int_{\mathbb{R}^d}\varphi(x)\,dx.$$

Let $\varphi(x) = 1_\Lambda(x)$, $\Lambda \in B_c\left(\mathbb{R}^d\right)$. Then

$$\begin{aligned}(L_S F)(\gamma) &= -\sum_{x\in\gamma_\Lambda}\left(\sum_{y\in\gamma\setminus x} c(x-y)\right) + \lambda|\Lambda| \\ &\le -\sum_{x\in\gamma_\Lambda}\left(\sum_{y\in\gamma_\Lambda\setminus x} c(x-y)\right) + \lambda|\Lambda| \\ &= -2E_c(\gamma_\Lambda) + \lambda|\Lambda| \le -\frac{C}{|\Lambda|}|\gamma_\Lambda|^2 + \lambda|\Lambda|.\end{aligned}$$

Proof

Set

$$n_t^{\Lambda} = \mathbb{E}\left(|X_t \cap \Lambda|\right).$$

Then

$$\begin{aligned}
\frac{d}{dt} n_t^{\Lambda} &= \mathbb{E}\left(L_S \, |X_t \cap \Lambda|\right) \\
&\leq \mathbb{E}\left(\lambda \, |\Lambda| - \frac{C}{|\Lambda|} \, |X_t \cap \Lambda|^2\right) \\
&\leq \lambda \, |\Lambda| - \frac{C}{|\Lambda|} \Big(\mathbb{E}\left(|X_t \cap \Lambda|\right)\Big)^2 \\
&= \lambda \, |\Lambda| - \frac{C}{|\Lambda|} \left(n_t^{\Lambda}\right)^2.
\end{aligned}$$

Proof

Since

$$n_t^\Lambda = \int_\Lambda k_t^{(1)}(x)\,dx$$

we have for

$$\rho_t = \rho_t^\Lambda := \frac{1}{|\Lambda|}\int_\Lambda k_t^{(1)}(x)\,dx$$

that the following inequality holds

$$\frac{d}{dt}\rho_t \le \lambda - C\rho_t^2.$$

Therefore, if we consider Cauchy problem

$$\begin{cases} \dfrac{d}{dt}g(t) = \lambda - Cg^2(t) \\ g(0) = g_0 \end{cases}$$

and $\rho_0 \le g_0$ then $\rho_t \le g(t)$.

Proof

One has

$$\frac{\sqrt{C}dg\left(t\right)}{\sqrt{C}g\left(t\right)+\sqrt{\lambda}}-\frac{\sqrt{C}dg\left(t\right)}{\sqrt{C}g\left(t\right)-\sqrt{\lambda}}=2\sqrt{C\lambda}dt;$$

$$\ln\frac{\left|\sqrt{C}g\left(t\right)+\sqrt{\lambda}\right|}{\left|\sqrt{C}g\left(t\right)-\sqrt{\lambda}\right|}-\ln\tilde{D}=2\sqrt{c\lambda}t,\quad \tilde{D}>0;$$

$$\frac{\sqrt{C}g\left(t\right)+\sqrt{\lambda}}{\sqrt{C}g\left(t\right)-\sqrt{\lambda}}=De^{2\sqrt{c\lambda}t},\quad D>0;$$

$$\sqrt{C}g\left(t\right)+\sqrt{\lambda}=De^{2\sqrt{C\lambda}t}\sqrt{C}g\left(t\right)-De^{2\sqrt{C\lambda}t}\sqrt{\lambda};$$

$$g\left(t\right)=\frac{De^{2\sqrt{C\lambda}t}\sqrt{\lambda}+\sqrt{\lambda}}{De^{2\sqrt{C\lambda}t}\sqrt{C}-\sqrt{C}}=\sqrt{\frac{\lambda}{C}}\left(1+\frac{2}{De^{2\sqrt{C\lambda}t}-1}\right).$$

Proof

Then

$$g_0 = g(0) = \sqrt{\frac{\lambda}{C}} \left(1 + \frac{2}{D-1}\right).$$

Let g_0 be such that $D = 1 + \varepsilon$. Thus, if for any Λ

$$\rho_0^\Lambda = \rho_0 \leq \sqrt{\frac{\lambda}{C}} \left(1 + \frac{2}{\varepsilon}\right)$$

one has that

$$\rho_t \leq \sqrt{\frac{\lambda}{C}} \left(1 + \frac{2}{(1+\varepsilon)\, e^{2\sqrt{C\lambda}t} - 1}\right).$$

Note that for $t \geq 0$

$$(1+\varepsilon)\, e^{2\sqrt{c\lambda}t} - 1 \geq \varepsilon > 0.$$

As a result, for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\rho_t^\Lambda = \rho_t \leq \sqrt{\frac{\lambda}{C}} \left(1 + \frac{2}{\varepsilon}\right).$$

Heterogeneous Contact Models: random mortality

Lattice case: Joo/Lebowitz, Phys.Rev.E72, 2005

Mortality rate $m \to m(x, \omega) \geq 0$
Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L^{a^+} k_t(x) - V(x) k_t(x)$$

where

$$L^{a^+} f(x) = \int a^+(x-y)[f(y) - f(x)] dy$$

$$V(x, \omega) = m(x, \omega) - \langle a^+ \rangle$$

Parabolic Anderson problem for pure jump generator (CTRW in continuum)

Random establishment

$$a^+(x-y) \to a^+(x-y)b(x,\omega)$$

Generator

$$(LF)(\gamma) = \sum_{x\in\gamma} m(x)[F(\gamma\setminus x) - F(\gamma)] +$$

$$\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)b(x,\omega)[F(\gamma\cup x) - F(\gamma)]dx$$

Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L_b^{a^+} k_t(x) - V(x) k_t(x)$$

$$L_b^{a^+} f(x) = b(x,\omega) \int a^+(x-y)[f(y) - f(x)]dy$$

$$V(x,\omega) = m(x,\omega) - b(x,\omega)\langle a^+ \rangle$$

$L_b^{a^+}$ is symmetric in

$$L^2(\mathbb{R}^d, b^{-1}(x)dx)$$

(quenched random measure?).

Random fecundity

Random fecundity rate

$$\mathbb{R}^d \ni y \mapsto \varkappa(y, \omega) \geq 0$$

Birth rate

$$b(x, \gamma, \omega) = \sum_{y \in \gamma} a^+(x - y)\varkappa(y, \omega)$$

Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L_{\varkappa}^{a^+} k_t(x) - V(x) k_t(x)$$

$$L_{\varkappa}^{a^+} f(x) = \int a^+(x-y)\varkappa(y,\omega)[f(y)-f(x)]dy$$

$$V(x,\omega) = m(x,\omega) - \langle a^+(x-\cdot)\varkappa(\cdot,\omega)\rangle$$

$L_{\varkappa}^{a^+}$ is symmetric in

$$L^2(\mathbb{R}^d, \varkappa(x,\omega)dx)$$

CTRW in random environment

Two types of quenched jump generators:

$$L_b^{a^+} f(x) = b(x,\omega) \int a^+(x-y)[f(y)-f(x)]dy$$

$$L_\varkappa^{a^+} f(x) = \int a^+(x-y)\varkappa(y,\omega)[f(y)-f(x)]dy$$

Feynman-Kac formula

Quenched CTRW:

$$L_{b,\varkappa}^{a^+,\omega} \to \xi_t^\omega$$

Density of population:

$$k_t(x) = \mathbb{E}_x[k_0(\xi_t^\omega)e^{-\int_0^t V(\xi_s^\omega)ds}]$$

Quenched vs. annealed

(cf., e.g., Donsker/Varadhan, Gaertner/Molchanov, Sznitman, ...)

Other aspects of IBM

– multi-type systems
(Finkelshtein/K; F/K/Skorokhod),

– mutation-selection models in genetics
(K/Minlos/Pirogov; K/Kuna/Ohlerich)

– scaling limits
(Finkelshtein/K/Kuna/Kutovyi; Finkelshtein/K/Lytvynov; Finkelshtein/K/Kutovyi)

– Kawasaki dynamics in continuum
(K/Lytvynov/Roeckner;K/Kuna/Oliveira/Streit)

– plankton dynamics
(K/Kutovyi/Struckmeier)

– stochastic evolutions in evolving random environments
(Boldrighini/K/Minlos/Pellegrinotti; Struckmeier)

– spectral analysis of Markov generators
(K/Lytvynov; K/Minlos; K/Zhizhina; K/Kuna/Ohlerich)